

with an action for the ϕ field plus a new field Φ of mass Λ :

$$- \int d^d x \left[\frac{1}{2} \left(\partial_\mu \phi \partial^\mu \phi + m^2 \phi^2 \right) - \frac{1}{2} \left(\partial_\mu \Phi \partial^\mu \Phi + \Lambda^2 \Phi^2 \right) \right]$$

Then, the two-point correlation function is given by

$$\langle (\phi + \Phi)(x) (\phi + \Phi)(y) \rangle = -i\hbar \int \frac{d^4 p}{(2\pi)^4} \exp(-ip \cdot (x - y)) \left[\frac{1}{p^2 + m^2 - i\epsilon} - \frac{1}{p^2 + \Lambda^2 - i\epsilon} \right]$$

Furthermore, we replace potentials $V(\phi)$ with $V(\phi + \Phi)$, *e.g.*

$$\lambda \phi^4 \mapsto \lambda (\phi + \Phi)^4$$

and compute correlation functions of $\phi + \Phi$ rather than just ϕ .

The Pauli-Villars regularization method is a technical improvement on the primitive cutoff regularization scheme we have used so far. For example, since momentum integrals now run over all of momentum space, Dirac delta functions are no longer Λ -regulated, and so momentum conservation should be conserved exactly, not just approximately.

On the other hand, the Pauli-Villars turns out not to be nearly so efficient at handling more complicated quantum field theories. For example, Pauli-Villars regularization breaks gauge symmetries in theories such as electromagnetism. To understand this, recall that giving a mass to the gauge field breaks the gauge symmetry. Since Pauli-Villars adds a propagator term with a nonzero mass (Λ), a Pauli-Villars regularization necessarily breaks the gauge symmetry. That, by itself, does not necessarily mean Pauli-Villars is less useful than other regularizations, but it does turn out to be one reason why Pauli-Villars regularization is often inconvenient and inefficient.

7.3 Dimensional regularization

Most modern workers do not typically use either a cutoff regularization or Pauli-Villars, but instead use a different method, known as *dimensional regularization*. It will turn out that this method is computationally far more efficient in nontrivial theories than either cutoff regularization or Pauli-Villars, and is also compatible with far more symmetries. The disadvantage to dimensional regularization will be a lack of concrete intuition for why precisely it works.

In dimensional regularization, we *formally* evaluate integrals in d dimensions instead of 4 dimensions, where d need not be an integer. For example, the integral

$$\int \frac{d^4 p}{(p^2 + m^2)^n}$$

is replaced by an integral we shall write

$$\int \frac{d^d p}{(p^2 + m^2)^n}$$

We shall see that integrals which diverge for integer d , will give finite results for noninteger d . In effect, dimensional regularization consists of performing a sort of analytic continuation in the dimension, thus the name.

Integrals in d dimensions are defined to obey the following axioms:

1. **Linearity:** For any complex numbers a, b ,

$$\int d^d p [af(p) + bg(p)] = a \int d^d p f(p) + b \int d^d p g(p)$$

2. **Scaling:** For any complex number s ,

$$\int d^d p f(sp) = s^{-d} \int d^d p f(p)$$

3. **Translation invariance:** For any vector q ,

$$\int d^d p f(p+q) = \int d^d p f(p)$$

Furthermore, when d is an integer, the integrals above should match ordinary integrals. Note in particular translation invariance of integrals in dimensional regularization. As we observed in section **** CITE ***, translation invariance is *not* a property of all integrals in cutoff regularization, specifically, it is not a property of linearly divergent integrals, and this is essential to understand anomalies in cutoff regularization. We shall see in section **** CITE **** that anomalies appear in dimensional regularization in a very different fashion.

Given the axioms above, it is possible to systematically develop a general theory of integration in d dimensions, see for example [Collins][chapter 4]. However, we will adopt a different approach to dimensionally-regularized integrals here. We shall first evaluate examples of divergent loop integrals in arbitrary integral dimension d , then rewrite those formulas in such a way as to make sense for noninteger d .

Now, let us work through a particular example. Consider the integral

$$I_d(q, n) = \int \frac{d^d p}{(p^2 + 2p \cdot q + m^2)^n}$$

(in Minkowski metric) for some q and some integer n . Let us work out how dimensional regularization applies to this integral. The basic idea will be to rewrite this as an integral that only depends on p^2 , removing the angular dependence, so that the integral over momentum space consists of an angular piece (which can be factored out) and an integral over the magnitude of p , which we can turn into an integral that formally does not require d to be integral.

First, let us Wick rotate the integral from Minkowski space to Euclidean space, to get

$$I_d(q, n) = i \int \frac{d^d p}{(p^2 + 2p \cdot q + m^2)^n}$$

where we have used $p_M^0 = +ip_E^0$.

First, let us complete the square in the denominator.

$$p^2 + 2p \cdot q + m^2 = (p + q)^2 + (m^2 - q^2)$$

Then, shift the momentum by an amount q : $p' = p + q$. The integral becomes

$$I_d(q, n) = i \int \frac{d^d p}{(p^2 + (m^2 - q^2))^n}$$

Note that at this point in the evaluation, we are using the translation-invariance axiom of dimensionally-regularized integrals. (As we saw in section *** CITE ***, in cutoff regularization not all integrals are translation-invariant.)

Next, we need to evaluate this integral in d dimensions. Since the integrand only depends upon the magnitude of p , we can write

$$\int d^d p = \int d\Omega_d \int_0^\infty p^{d-1} dp$$

where Ω_d is the d -dimensional solid angle. Since the integrand only depends upon the magnitude, it is independent of Ω_d , so we can evaluate it separately. To do so, we can use a trick:

$$\begin{aligned} (\sqrt{\pi})^d &= \left(\int_{-\infty}^\infty e^{-x^2} \right)^d \\ &= \int d^d x \exp\left(-\sum_{i=1}^d x_i^2\right) \\ &= \int d\Omega_d \int_0^\infty dx x^{d-1} e^{-x^2} \end{aligned}$$

We can evaluate the integral over x 's using the gamma function discussed in appendix **** CITE *****.

$$\begin{aligned} \int_0^\infty dx x^{d-1} e^{-x^2} &= \frac{1}{2} \int_0^\infty d(x^2) (x^2)^{d/2-1} e^{-x^2} \\ &= \frac{1}{2} \Gamma(d/2) \end{aligned}$$

To briefly review, the gamma function $\Gamma(x)$ is a function which for complex numbers x such that $\text{Re } x > 0$ is given by the expression (due to Euler)

$$\Gamma(x) = \int_0^\infty e^{-t} t^{x-1} dt$$

More generally, the gamma function is defined almost everywhere in the complex plane, though for $\text{Re } x \leq 0$ one must work a little harder to define it. It diverges for x an integer less than or equal to zero, *i.e.* $x = 0, -1, -2, -3, \dots$, but otherwise Properties of the gamma function include:

$$\begin{aligned} \Gamma(1/2) &= \sqrt{\pi} \\ \Gamma(1) &= 1 \\ \Gamma(2) &= 1 \\ \Gamma(3) &= 2 \\ \Gamma(x+1) &= x\Gamma(x) \end{aligned}$$

from which one finds that $\Gamma(n) = (n-1)!$ for n a positive integer. Thus, the gamma function is a generalization of the factorial function – the factorial is only defined over the integers, whereas the gamma function is defined for complex numbers.

***** NOTE TO SELF: At this point in my notes, there was an aside on the gamma function, qftpart3.pdf p 3 – what to include ?? I've put a little bit above, I'm just wondering if maybe more should be included.

Thus, we see that

$$(\sqrt{\pi})^d = \left(\int d\Omega_d \right) \frac{1}{2} \Gamma(d/2)$$

and so

$$\int d\Omega_d = \frac{2\pi^{d/2}}{\Gamma(d/2)}$$

In particular, the expression above makes sense for non-integer d as well as integer d , so we can use it to “analytically continue” to non-integer dimensions d .

Let us briefly check that the expression above for the solid angle is sensible. For $d = 2$, Ω_d should be the number of radians in a circle, and indeed

$$\frac{2\pi^{2/2}}{\Gamma(2/2)} = 2\pi$$

Similarly, for $d = 3$, Ω_d should count the solid angles in a sphere, and indeed

$$\frac{2\pi^{3/2}}{\Gamma(3/2)} = \frac{2\pi^{3/2}}{(1/2)\Gamma(1/2)} = \frac{4\pi^{3/2}}{\sqrt{\pi}} = 4\pi$$

Returning to our integral $I_d(q, n)$, we have

$$\begin{aligned} I_d(q, n) &= i \int \frac{d^d p}{(p^2 + (m^2 - q^2))^n} \\ &= i \int d\Omega_d \int_0^\infty \frac{p^{d-1} dp}{(p^2 + (m^2 - q^2))^n} \\ &= i \frac{2\pi^{d/2}}{\Gamma(d/2)} \frac{1}{2} \int_0^\infty \frac{(p^2)^{d/2-1} d(p^2)}{(p^2 + (m^2 - q^2))^n} \end{aligned}$$

Define

$$x \equiv \frac{m^2 - q^2}{p^2 + m^2 - q^2}$$

so

$$\begin{aligned} p^2 + m^2 - q^2 &= \frac{m^2 - q^2}{x} \\ d(p^2) &= -\frac{m^2 - q^2}{x^2} dx \end{aligned}$$

Thus,

$$\begin{aligned} I_d(q, n) &= i \frac{\pi^{d/2}}{\Gamma(d/2)} \int_0^1 dx \left(\frac{m^2 - q^2}{x^2} \right) (m^2 - q^2)^{d/2-1} \left(\frac{1}{x} - 1 \right)^{d/2-1} \left(\frac{m^2 - q^2}{x} \right)^{-n} \\ &= i \frac{\pi^{d/2}}{\Gamma(d/2)} (m^2 - q^2)^{d/2-n} \int_0^1 dx x^{n-d/2-1} (1-x)^{d/2-1} \end{aligned}$$

In appendix**** CITE ****, the *beta function* $B(a, b)$ is defined to be a ratio of gamma functions

$$B(a, b) = \frac{\Gamma(a)\Gamma(b)}{\Gamma(a+b)}$$

and it is shown there that

$$B(a, b) = \int_0^1 dx x^{a-1} (1-x)^{b-1}$$

Thus, we see that

$$\begin{aligned} I_d(q, n) &= \int \frac{d^d p}{(p^2 + 2p \cdot q + m^2)^n} \quad (\text{Minkowski}) \\ &= i \frac{\pi^{d/2}}{\Gamma(d/2)} (m^2 - q^2)^{d/2-n} B(n - d/2, d/2) \\ &= i \frac{\pi^{d/2}}{\Gamma(d/2)} (m^2 - q^2)^{d/2-n} \frac{\Gamma(n - d/2)\Gamma(d/2)}{\Gamma(n)} \\ &= i \pi^{d/2} (m^2 - q^2)^{d/2-n} \frac{\Gamma(n - d/2)}{\Gamma(n)} \end{aligned} \tag{7.2}$$

Note that although the integral above was originally defined for integer d , the result of the computation above is an expression that makes sense for arbitrary, not-necessarily-integer d .

Aside: The careful reader will complain that when $d/2 - n$ is irrational, then the expression above does not make sense, since in that case

$$(m^2 - q^2)^{d/2-n} \equiv \exp\left((d/2 - n) \ln(m^2 - q^2)\right)$$

and $m^2 - q^2$ is not unitless, so we cannot compute the \ln . However, when we apply dimensional regularization to specific theories, we shall introduce an additional mass parameter Λ , and that mass parameter will appear in such a way so as to solve this problem – we will only ever compute \ln 's of unitless quantities such as $(m^2 - q^2)/\Lambda^2$.

As a consistency check, let us compare singularities. The original integral, for large $|p|$, is proportional to

$$\int^\infty \frac{r^{d-1} dr}{r^{2n}}$$

and so has a divergence at the upper end of the integral when $d - 1 - 2n \geq -1$, or equivalently $n \leq d/2$. We derived a closed-form expression for the integral, proportional to $\Gamma(n - d/2)$, and

since gamma functions diverge at integers less than or equal to zero, we see that for $n - d/2$ an integer, the divergences of the original integral and our expression match. However, our expression is not only defined more generally, but when $n - d/2$ is not an integer, our expression converges, regardless of whether $n \leq d/2$.

We originally advertised dimensional regularization as a simplification, but the reader may complain that we have had to do a great deal of work so far. However, the work above only has to be done once – once one has a table of integrals of the form above, the workload drops greatly. To that end, here is a collection of commonly-used results:

$$\int \frac{d^d p}{(2\pi)^d} \frac{1}{(p^2 + m^2)^n} = i \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(n - d/2)}{\Gamma(n)} \left(\frac{1}{m^2}\right)^{n-d/2} \quad (7.3)$$

$$\int \frac{d^d p}{(2\pi)^d} \frac{(p^2)^k}{(p^2 + m^2)^n} = i \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(n - d/2 - k)\Gamma(d/2 + k)}{\Gamma(d/2)\Gamma(n)} \left(\frac{1}{m^2}\right)^{n-d/2-k} \quad (7.4)$$

$$\int \frac{d^d p}{(2\pi)^d} \frac{p^\mu p^\nu}{(p^2 + m^2)^n} = i \frac{1}{(4\pi)^{d/2}} \frac{g^{\mu\nu}}{2} \frac{\Gamma(n - d/2 - 1)}{\Gamma(n)} \left(\frac{1}{m^2}\right)^{n-d/2-1} \quad (7.5)$$

$$\int \frac{d^d p}{(2\pi)^d} \frac{p^\mu p^\nu p^\rho p^\sigma}{(p^2 + m^2)^n} = i \frac{1}{(4\pi)^{d/2}} \frac{\Gamma(n - d/2 - 2)}{\Gamma(n)} \left(\frac{1}{m^2}\right)^{n-d/2-2} \cdot \frac{1}{4} (g^{\mu\nu} g^{\rho\sigma} + g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho}) \quad (7.6)$$

Next, let us further simplify the expression for $I_d(q, n)$. To this end, we will use the fact that

$$\Gamma(\epsilon) = \frac{1}{\epsilon} + \psi(1) - \frac{\epsilon}{2} \left(\psi^{(1)}(1) - \psi(1)^2 - \frac{\pi^2}{3} \right) + \mathcal{O}(\epsilon^2)$$

and more generally,

$$\Gamma(-n + \epsilon) = \frac{(-)^n}{n!} \left(\frac{1}{\epsilon} + \psi(n + 1) - \frac{\epsilon}{2} \left(\psi^{(1)}(n + 1) - \psi(n + 1)^2 - \frac{\pi^2}{3} \right) + \mathcal{O}(\epsilon^2) \right)$$

for n a positive integer, where $\psi(x)$ is the *digamma function*, defined by

$$\psi(x) = \frac{d}{dx} \ln \Gamma(x).$$

and $\psi^{(1)}(x)$ is a polygamma function, defined as the first derivative of the digamma function. The *Euler-Mascheroni constant* γ is defined to be $\gamma = -\psi(1)$. Derivations of the expansions above and other information on the digamma and polygamma functions can be found in appendix *** FILL IN ***.

Let us apply this to simplify our expression for $I_d(q, n)$. Suppose that the dimension d is “close to” 4, and that n is an integer such that $2 - n > 0$. Define $\epsilon = 4 - d$, then

$$\Gamma(n - d/2) = \Gamma(n - 2 + \epsilon/2) = \frac{(-)^{2-n}}{(2-n)!} \left(\frac{2}{\epsilon} + \psi(3 - n) + \mathcal{O}(\epsilon) \right)$$

From this we find

$$\begin{aligned} I_d(q, n) &= \int \frac{d^d p}{(p^2 + 2p \cdot q + m^2)^n} \\ &= i\pi^{d/2} (m^2 - q^2)^{d/2-n} \frac{1}{\Gamma(n)} \frac{(-)^{2-n}}{(2-n)!} \left(\frac{2}{\epsilon} + \psi(3-n) + \mathcal{O}(\epsilon) \right) \end{aligned}$$

Next, let us apply this to some specific examples. First, however, with a bit of foresight, we should slightly modify the action so as to get more sensible units. We shall rewrite the interaction term in $\lambda\phi^4$ theory so that the coupling constant has its $d = 4$ mass dimensions for all d ; in particular, for $\lambda\phi^4$ near four dimensions, this will insure that λ is always dimensionless. We will see later that this has the effect of sanitizing units in our later expressions.

In addition, it is also sometimes convenient to arrange for the coupling constant to be dimensionless. In $\lambda\phi^4$ theory in four dimensions, λ is automatically dimensionless, but this need not be true in other cases. Therefore, in such other cases, sometimes one adds factors of Λ so as to manufacture a dimensionless constant, *e.g.* $g = \Lambda^n \bar{g}$ where g is the original coupling constant and \bar{g} is dimensionless.

We argued previously in section *** CITE *** that if we demand that the action be scale-invariant, then the real scalar field ϕ has units of mass dimension $d/2 - 1$, so classically ϕ^4 has mass dimension $r(d/2 - 1)$. Thus, if we add a scale factor Λ so that the action becomes

$$\int d^d x \left[-\frac{1}{2} (\partial_\mu \phi \partial^\mu \phi + m^2 \phi^2) - \lambda \Lambda^{4-d} \phi^4 \right]$$

then we see that the coupling constant has classical mass dimension zero. In other words, we have made the mass scale explicit. We will see that this will have the effect of sanitizing the units in the expressions we shall derive.

Now, in $\lambda\phi^4$ theory “near” $d = 4$, consider the diagram

$$\text{Diagram: a loop with two external lines labeled 1 and 2} = \left(-i \frac{\lambda \Lambda^{4-d}}{\hbar} \right) (4)(3) \int d^d z D_F(x_1, z) D_F(z, x_2) D_F(z, z)$$

The $D_F(z, z) = D_F(0, 0)$ factor is divergent. It is given by

$$\begin{aligned} \Lambda^{4-d} D_F(z, z) &= -i\hbar \Lambda^{4-d} \int \frac{d^d p}{(2\pi)^d} \frac{1}{p^2 + m^2} \\ &= -\frac{i\hbar}{(2\pi)^d} \Lambda^{4-d} I_d(0, 1) \\ &= -\frac{i\hbar}{(2\pi)^d} \Lambda^{4-d} (i\pi)^{d/2} (m^2)^{d/2-1} \frac{\Gamma(1-d/2)}{\Gamma(1)} \\ &= \hbar (4\pi)^{-d/2} (m^2)^{d/2-1} \Lambda^{4-d} \Gamma(1-d/2) \end{aligned} \tag{7.7}$$

where to clean up the units we have included the Λ^{4-d} factor. For d “near” 4, $\epsilon = 4 - d$, we have

$$\begin{aligned}\Lambda^{4-d}D_F(z, z) &= -\hbar(4\pi)^{-d/2}(m^2)^{d/2-1}\Lambda^{4-d}\left(\frac{2}{\epsilon} + \psi(2) + \mathcal{O}(\epsilon)\right) \\ &= -\hbar\frac{m^2}{(4\pi)^2}\left(\frac{2}{\epsilon} + \psi(2) + \ln\left(4\pi\frac{\Lambda^2}{m^2}\right) + \mathcal{O}(\epsilon)\right)\end{aligned}\quad (7.8)$$

using the fact that

$$\begin{aligned}\frac{(m^2)^{d/2-1}}{(4\pi)^{d/2}}\Lambda^{4-d} &= \frac{(m^2)^{1-\epsilon/2}}{(4\pi)^{2-\epsilon/2}}\Lambda^{2(2-d/2)} \\ &= \frac{m^2}{(4\pi)^2}\left(4\pi\frac{\Lambda^2}{m^2}\right)^{\epsilon/2} \\ &= \frac{m^2}{(4\pi)^2}\left(1 + \frac{\epsilon}{2}\ln\left(4\pi\frac{\Lambda^2}{m^2}\right) + \mathcal{O}(\epsilon^2)\right)\end{aligned}$$

Note that the divergent factor above could be obtained more simply from the amputated diagram

$$\begin{aligned}\text{Diagram 1} &= \left(-i\frac{\lambda\Lambda^{4-d}}{\hbar}\right)(4)(3)\int d^dz\delta^d(x_1 - z)\delta^d(z - x_2)D_F(z, z) \\ &= \left(-i\frac{\lambda\Lambda^{4-d}}{\hbar}\right)(4)(3)\delta^d(x_1 - x_2)D_F(0, 0)\end{aligned}$$

As a related exercise, let us compute the two-loop “double scoop” diagram

$$\begin{aligned}x \text{---} \text{Diagram 2} \text{---} y &= (144)\left(-i\frac{\lambda\Lambda^{4-d}}{\hbar}\right)^2\int d^dz_1d^dz_2D_F(x, z_1)D_F(z_1, y)D_F(z_1, z_2)^2D_F(z_2, z_2) \\ &= (144)\left(-i\frac{\lambda\Lambda^{4-d}}{\hbar}\right)^2(-i\hbar)^5\int d^dz_1d^dz_2\int\frac{d^dp_1}{(2\pi)^d}\cdots\frac{d^dp_5}{(2\pi)^d} \\ &\quad \frac{e^{-ip_1\cdot(x-z_1)}e^{-ip_2\cdot(y-z_1)}e^{-i(p_3+p_4)\cdot(z_1-z_2)}}{(p_1^2+m^2)\cdots(p_5^2+m^2)} \\ &= (144)\left(-i\frac{\lambda\Lambda^{4-d}}{\hbar}\right)^2(-i\hbar)^5\int\frac{d^dp_1}{(2\pi)^d}\cdots\frac{d^dp_5}{(2\pi)^d}(2\pi)^d\delta^d(-p_1-p_2+p_3+p_4) \\ &\quad \cdot(2\pi)^d\delta^d(p_3+p_4)\frac{e^{-ip_1\cdot x}e^{-ip_2\cdot y}}{(p_1^2+m^2)\cdots(p_5^2+m^2)} \\ &= (144)\left(-i\frac{\lambda\Lambda^{4-d}}{\hbar}\right)^2(-i\hbar)^5\int\frac{d^dp_1}{(2\pi)^d}\frac{e^{-ip_1\cdot(x-y)}}{(p_1^2+m^2)^2}\end{aligned}$$

$$\begin{aligned} & \cdot \int \frac{d^d p_3}{(2\pi)^d} \frac{1}{(p_3^2 + m^2)} \frac{1}{(p_3^2 + m^2)} \\ & \cdot \int \frac{d^d p_5}{(2\pi)^d} \frac{1}{p_5^2 + m^2} \end{aligned}$$

The second divergent integral we evaluated in equation (7.8). The first is given by

$$\begin{aligned} \Lambda^{4-d} \int \frac{d^d p_3}{(2\pi)^d} \frac{1}{(p_3^2 + m^2)^2} &= \frac{\Lambda^{4-d}}{(2\pi)^d} I_d(0, 2) \\ &= i \frac{1}{(4\pi)^{d/2}} \left(\frac{m^2}{\Lambda^2} \right)^{d/2-2} \frac{\Gamma(2-d/2)}{\Gamma(2)} \\ &= \frac{i}{(4\pi)^2} \left(4\pi \frac{\Lambda^2}{m^2} \right)^{\epsilon/2} \frac{1}{\Gamma(2)} \left(\frac{2}{\epsilon} + \psi(1) + \mathcal{O}(\epsilon) \right) \\ &= \frac{i}{(4\pi)^2} \frac{1}{\Gamma(2)} \left(\frac{2}{\epsilon} + \psi(1) + \ln \left(4\pi \frac{\Lambda^2}{m^2} \right) + \mathcal{O}(\epsilon) \right) \end{aligned}$$

Putting this together, we see that the double scoop diagram is given by

$$\begin{aligned} (144) & \left(-i \frac{\lambda \Lambda^{4-d}}{\hbar} \right)^2 \int d^d z_1 d^d z_2 D_F(x, z_1) D_F(z_1, y) D_F(z_1, z_2)^2 D_F(z_2, z_2) \\ &= (144) \left(-i \frac{\lambda}{\hbar} \right)^2 (-i\hbar)^4 \int \frac{d^d p_1}{(2\pi)^d} \frac{e^{-ip_1 \cdot (x-y)}}{(p_1^2 + m^2)^2} \\ & \quad \cdot \frac{i}{(4\pi)^2} \frac{1}{\Gamma(2)} \left(\frac{2}{\epsilon} + \psi(1) + \ln \left(4\pi \frac{\Lambda^2}{m^2} \right) + \mathcal{O}(\epsilon) \right) \\ & \quad \cdot (-\hbar) \frac{m^2}{(4\pi)^2} \left(\frac{2}{\epsilon} + \psi(2) + \ln \left(4\pi \frac{\Lambda^2}{m^2} \right) + \mathcal{O}(\epsilon) \right) \\ &= (144) \left(-i \frac{\lambda}{\hbar} \right)^2 (-i\hbar)^4 \int \frac{d^d p_1}{(2\pi)^d} \frac{e^{-ip_1 \cdot (x-y)}}{(p_1^2 + m^2)^2} \frac{i}{(4\pi)^2} \frac{1}{\Gamma(2)} (-\hbar) \frac{m^2}{(4\pi)^2} \\ & \quad \cdot \left[\left(\frac{2}{\epsilon} \right)^2 + \left(\frac{2}{\epsilon} \right) \left(\psi(1) + \psi(2) + 2 \ln \left(4\pi \frac{\Lambda^2}{m^2} \right) \right) + \mathcal{O}(1) \right] \end{aligned} \quad (7.9)$$

Next, let us apply dimensional regularization to the diagram

$$\begin{aligned} \text{Diagram} &= \frac{1}{2!} \left(-i \frac{\lambda \Lambda^{4-d}}{\hbar} \right)^2 (4!)^2 \\ & \quad \cdot \int d^d z_1 d^d z_2 D_F(x_1, z_1) D_F(x_2, z_1) D_F(x_3, z_2) D_F(x_4, z_2) D_F(z_1, z_2)^2 \end{aligned}$$

In fact, to be more efficient, let us apply dimensional regularization to the amputated diagram

$$\text{Diagram} = \frac{1}{2!} \left(-i \frac{\lambda \Lambda^{4-d}}{\hbar} \right)^2 (4!)^2$$

$$\cdot \int d^d z_1 d^d z_2 \delta^d(x_1 - z_1) \delta^d(x_2 - z_1) \delta^d(x_3 - z_2) \delta^d(x_4 - z_2) D_F(z_1, z_2)^2$$

Note that the Dirac delta functions above are *not* Λ -regulated Dirac delta functions, but rather are ordinary Dirac delta functions, since we are not using momentum cutoff regularization. We compute this diagram to be

$$\begin{aligned} & \frac{1}{2!} \left(-i \frac{\lambda \Lambda^{4-d}}{\hbar} \right)^2 (4!)^2 \delta^d(x_1 - x_2) \delta^d(x_3 - x_4) D_F(x_1, x_3)^2 \\ &= \frac{1}{2!} \left(-i \frac{\lambda \Lambda^{4-d}}{\hbar} \right)^2 (4!)^2 \delta^d(x_1 - x_2) \delta^d(x_3 - x_4) \\ & \quad \cdot \int \frac{d^d p_1}{(2\pi)^d} \frac{d^d p_2}{(2\pi)^d} \left(\frac{-i\hbar}{p_1^2 + m^2} \right) \left(\frac{-i\hbar}{p_2^2 + m^2} \right) e^{-i(p_1 + p_2) \cdot (x_1 - x_3)} \end{aligned}$$

Define $p = p_1$, $q = p_1 + p_2$, then this becomes

$$\begin{aligned} & \frac{1}{2!} \left(-i \frac{\lambda \Lambda^{4-d}}{\hbar} \right)^2 (4!)^2 \delta^d(x_1 - x_2) \delta^d(x_3 - x_4) D_F(x_1, x_3)^2 \\ &= \frac{1}{2!} \left(-i \frac{\lambda \Lambda^{4-d}}{\hbar} \right)^2 (4!)^2 \delta^d(x_1 - x_2) \delta^d(x_3 - x_4) \\ & \quad \cdot \int \frac{d^d p}{(2\pi)^d} \frac{d^d q}{(2\pi)^d} \left(\frac{-i\hbar}{p^2 + m^2} \right) \left(\frac{-i\hbar}{(q - p)^2 + m^2} \right) e^{-iq \cdot (x_1 - x_3)} \\ &= \frac{1}{2!} \left(-i \frac{\lambda \Lambda^{4-d}}{\hbar} \right)^2 (4!)^2 \delta^d(x_1 - x_2) \delta^d(x_3 - x_4) \int \frac{d^d q}{(2\pi)^d} e^{-iq \cdot (x_1 - x_3)} \\ & \quad \cdot \int \frac{d^d p}{(2\pi)^d} \left(\frac{-i\hbar}{p^2 + m^2} \right) \left(\frac{-i\hbar}{(q - p)^2 + m^2} \right) \end{aligned}$$

The p integral is logarithmically divergent, so let us apply dimensional regularization.

To do so, we would like to apply the results for the integral $I_d(q, \alpha)$ that we computed previously, but this integral is not yet of the desired form. In order to put it in the desired form, we will use a trick due to Feynman. *Feynman's trick* is to use the identity:

$$\frac{1}{ab} = \int_0^1 \frac{dz}{(az + b(1-z))^2}$$

To derive this identity, note that

$$\frac{1}{ab} = \frac{1}{b-a} \left(\frac{1}{a} - \frac{1}{b} \right) = \frac{1}{b-a} \int_a^b \frac{dx}{x^2}$$

and then define z by $x = az + b(1-z)$. Applying this to the present case,

$$\begin{aligned} \frac{1}{p^2 + m^2} \frac{1}{(p-q)^2 + m^2} &= \int_0^1 \frac{dz}{((p^2 + m^2)z + ((p-q)^2 + m^2)(1-z))^2} \\ &= \int_0^1 \frac{dz}{(p^2 z + (p-q)^2(1-z) + m^2)^2} \\ &= \int_0^1 \frac{dz}{(p^2 + m^2 - 2p \cdot q(1-z) + q^2(1-z))^2} \end{aligned}$$

Define $p' = p - q(1 - z)$, then the denominator in the expression above becomes

$$(p')^2 + m^2 + q^2 \left((1 - z) - (1 - z)^2 \right) = (p')^2 + m^2 + q^2 z(1 - z)$$

Thus, after shifting the p integral, applying our earlier result for $I_d(q, \alpha)$, and adding a factor of Λ^{4-d} to clean up the units, we can write

$$\begin{aligned} \Lambda^{4-d} \int d^d p \frac{1}{p^2 + m^2} \frac{1}{(p - q)^2 + m^2} &= \Lambda^{4-d} \int_0^1 dz \int \frac{d^d p}{(p^2 + m^2 + q^2 z(1 - z))} \\ &= \int_0^1 dz (+i) \pi^{d/2} \left(\frac{m^2 + q^2 z(1 - z)}{\Lambda^2} \right)^{d/2-2} \frac{\Gamma(2 - d/2)}{\Gamma(2)} \\ &= +i \frac{\Gamma(2 - d/2)}{\Gamma(2)} \int_0^1 dz \pi^{d/2} \left(\frac{m^2 + q^2 z(1 - z)}{\Lambda^2} \right)^{d/2-2} \end{aligned}$$

In the present case, let us assume d is “near” 4, and define $\epsilon = 4 - d$, as previously. In this case, $d/2 - 2 = -\epsilon/2$, so

$$\begin{aligned} &\int_0^1 dz \pi^{d/2} \left(\frac{m^2 + q^2 z(1 - z)}{\Lambda^2} \right)^{d/2-2} \\ &= \int_0^1 dz \pi^2 \exp \left(-\frac{\epsilon}{2} \ln \left(\pi \frac{m^2 + q^2 z(1 - z)}{\Lambda^2} \right) \right) \\ &= \pi^2 - \frac{\epsilon}{2} \pi^2 \int_0^1 dz \ln \left(\pi \frac{m^2 + q^2 z(1 - z)}{\Lambda^2} \right) + \mathcal{O}(\epsilon^2) \end{aligned} \quad (7.10)$$

using the fact that $A^\epsilon = \exp(\epsilon \ln A)$. If we only keep track of leading $1/\epsilon$ effects, then we see that

$$\Lambda^{4-d} \int d^d p \frac{1}{p^2 + m^2} \frac{1}{(p - q)^2 + m^2} = i\pi^2 \left(\frac{2}{\epsilon} + \mathcal{O}(1) \right) \quad (7.11)$$

However, we can also extract the next order if we work harder, using the integral

$$\int_0^1 dz \ln \left(1 + \frac{4}{a} z(1 - z) \right) = -2 + \sqrt{1 + a} \ln \left(\frac{\sqrt{1 + a} + 1}{\sqrt{1 + a} - 1} \right), \quad a > 0 \quad (7.12)$$

This can be proven by factoring

$$\ln(a + 4z - 4z^2) = \ln(\sqrt{1 + a} + (2z - 1)) + \ln(\sqrt{1 + a} - (2z - 1))$$

and integrating each term separately. Thus,

$$\begin{aligned} &\int_0^1 dz \ln \left(\pi \frac{m^2 + q^2 z(1 - z)}{\Lambda^2} \right) \\ &= \ln \left(\pi \frac{m^2}{\Lambda^2} \right) + \int_0^1 dz \ln \left(1 + \frac{q^2}{m^2} z(1 - z) \right) \\ &= \ln \left(\pi \frac{m^2}{\Lambda^2} \right) - 2 + \sqrt{1 + \frac{4m^2}{q^2}} \ln \left(\frac{\sqrt{q^2 + 4m^2} + \sqrt{q^2}}{\sqrt{q^2 + 4m^2} - \sqrt{q^2}} \right) \end{aligned}$$

Thus,

$$\begin{aligned}
 & \Lambda^{4-d} \int d^d p \frac{1}{p^2 + m^2} \frac{1}{(p-q)^2 + m^2} \\
 &= +i \frac{\Gamma(2-d/2)}{\Gamma(2)} \left[\pi^2 - \frac{\epsilon}{2} \pi^2 \left(\ln \left(\pi \frac{m^2}{\Lambda^2} \right) - 2 \right. \right. \\
 & \qquad \qquad \qquad \left. \left. + \sqrt{1 + \frac{4m^2}{q^2}} \ln \left(\frac{\sqrt{q^2 + 4m^2} + \sqrt{q^2}}{\sqrt{q^2 + 4m^2} - \sqrt{q^2}} \right) \right) + \mathcal{O}(\epsilon^2) \right] \quad (7.13)
 \end{aligned}$$

**** NOTE TO SELF: there's a potential problem with the expression above – it's only valid for positive q^2 , but at the end we integrate over q 's. I'd say it's still valid in momentum space onshell, except that in my conventions, on-shell means q^2 is negative not positive. *Maybe* I can analytically continue to get results valid more generally? (Or maybe not.)

Finally, putting this all together (and truncating the $\mathcal{O}(1)$ terms for brevity), we see that the amputated diagram is given by

$$\begin{aligned}
 & \frac{1}{2!} \left(-i \frac{\lambda \Lambda^{4-d}}{\hbar} \right)^2 (4!)^2 \delta^d(x_1 - x_2) \delta^d(x_3 - x_4) D_F(x_1, x_3)^2 \\
 &= \frac{1}{2!} \left(-i \frac{\lambda}{\hbar} \right)^2 (4!)^2 \delta^d(x_1 - x_2) \delta^d(x_3 - x_4) \int \frac{d^4 q}{(2\pi)^4} e^{-iq \cdot (x_1 - x_3)} \\
 & \qquad \qquad \qquad \cdot (+i) \frac{\pi^2}{(2\pi)^4} (-i\hbar)^2 \left(\frac{2}{\epsilon} + \mathcal{O}(1) \right) \\
 &= (i) \frac{(4!)^2}{2} \frac{\lambda^2}{16\pi^2} \delta^d(x_1 - x_2) \delta^d(x_3 - x_4) \delta^d(x_1 - x_3) \left(\frac{2}{\epsilon} + \mathcal{O}(1) \right)
 \end{aligned}$$

Then, for later reference, it is straightforward to show – either by repeating the computation above or by multiplying in the external leg propagators – that the regularized unamputated diagram is given by

$$\begin{aligned}
 \text{Diagram} &= (i) \frac{(4!)^2}{2} \frac{\lambda^2}{16\pi^2} \left(\frac{2}{\epsilon} + \mathcal{O}(1) \right) \int \frac{d^4 p_1}{(2\pi)^4} \cdots \frac{d^4 p_4}{(2\pi)^4} \left(\frac{-i\hbar}{p_1^2 + m^2} \right) \cdots \left(\frac{-i\hbar}{p_4^2 + m^2} \right) \\
 & \qquad \qquad \qquad \cdot e^{-ip_1 \cdot x_1} \cdots e^{-ip_4 \cdot x_4} (2\pi)^4 \delta^4(p_1 + p_2 + p_3 + p_4)
 \end{aligned}$$

**** NOTE TO SELF: What sort of Dirac delta function should I be using, to be consistent with dimensional regularization? Do I want

$$\delta^4(x) = \int \frac{d^4 p}{(2\pi)^4} e^{-ip \cdot x} \quad \text{or} \quad \delta^d(x) = \int \frac{d^d p}{(2\pi)^d} e^{-ip \cdot x}$$

COMMENT explicitly on this choice – it only affects subleading ϵ 's, but those are very important for anyone who goes past leading order.

A note on notation, before proceeding. When we introduced cutoff regularization, we denoted a cutoff-regularized field ϕ by ϕ_Λ , to distinguish it from a classical field. To be consistent, here we should probably denote dimensionally-regularized fields by $\phi_{\Lambda,\epsilon}$. However, for reasons of brevity, in the rest of this text dimensional regularization will always be assumed explicitly (unless otherwise stated), and so we will simply denote all (dimensionally-regularized) fields by ϕ , the same as the classical field.

7.4 Dimensional regularization, gamma matrices, and chiral anomalies

In d dimensions, $g^{\mu\nu}$ obeys $g_{\mu\nu}g^{\mu\nu} = d$, so when deriving expressions for integrals, one should replace

$$p^\mu p^\nu \mapsto \frac{1}{d} p^2 g^{\mu\nu}$$

We think of the gamma matrices in dimensional regularization as a set of d matrices obeying

$$\{\gamma^\mu, \gamma^\nu\} = -2g^{\mu\nu} I$$

and

$$\text{Tr } I = 4$$

(for dimensional regularization “near” four dimensions). Furthermore, we assume that the traces are meromorphic in d , that they still possess the same cyclic property as in $d = 4$, and match classical results in the special case that $d = 4$.

Also note that this means that although we are varying d , the gamma matrices are tied to four dimensions – if we were to dimensionally regularize a theory in two dimensions, for example, then we would impose a different constraint on $\text{Tr } I$, *etc.* If we were to set $d = 2$ in this theory, we would *not* get gamma matrices for a two-dimensional theory. This is unlike the previous section, where we considered dimensional regularization of bosonic integrals. There, integer values of d would correspond to classical theories in those dimensions. By contrast, when setting up gamma matrices in dimensional regularization, the regularized gamma matrices are intrinsically tied to the dimension we started with.

Some of the standard identities involving gamma matrices are modified:

$$\begin{aligned} \gamma^\alpha \gamma^\mu \gamma_\alpha &= (2 - \epsilon) \gamma^\mu \\ \gamma^\alpha \gamma^\mu \gamma^\nu \gamma_\alpha &= 4g^{\mu\nu} I + \epsilon \gamma^\mu \gamma^\nu \\ \gamma^\alpha \gamma^\mu \gamma^\nu \gamma^\lambda \gamma_\alpha &= 2\gamma^\lambda \gamma^\nu \gamma^\mu - \epsilon \gamma^\mu \gamma^\nu \gamma^\lambda \end{aligned}$$

where $\epsilon = 4 - d$.

Trace identities are largely unmodified. For example,

$$\text{Tr} (\gamma^\mu \gamma^\nu) = -4g^{\mu\nu} \quad (7.14)$$

$$\text{Tr} (\gamma^\alpha \gamma^\beta \gamma^\lambda \gamma^\delta) = 4 (g^{\alpha\beta} g^{\lambda\delta} - g^{\alpha\lambda} g^{\beta\delta} + g^{\alpha\delta} g^{\beta\lambda}) \quad (7.15)$$

$$\begin{aligned} & \text{Tr} (\gamma^\alpha \gamma^\beta \gamma^\lambda \gamma^\delta \gamma^\rho \gamma^\sigma) \\ &= 4 \left(-g^{\alpha\beta} g^{\lambda\delta} g^{\rho\sigma} + g^{\alpha\beta} g^{\lambda\rho} g^{\delta\sigma} - g^{\alpha\beta} g^{\lambda\sigma} g^{\delta\rho} + g^{\lambda\delta} g^{\alpha\rho} g^{\beta\sigma} - g^{\lambda\delta} g^{\alpha\sigma} g^{\beta\rho} \right. \\ & \quad + g^{\rho\sigma} g^{\alpha\lambda} g^{\beta\delta} - g^{\rho\sigma} g^{\alpha\delta} g^{\beta\lambda} + g^{\alpha\rho} g^{\beta\lambda} g^{\delta\sigma} - g^{\beta\rho} g^{\alpha\lambda} g^{\delta\sigma} - g^{\alpha\rho} g^{\beta\delta} g^{\lambda\sigma} \\ & \quad \left. - g^{\alpha\sigma} g^{\beta\lambda} g^{\delta\rho} + g^{\beta\rho} g^{\alpha\delta} g^{\lambda\sigma} + g^{\alpha\sigma} g^{\beta\delta} g^{\lambda\rho} + g^{\beta\sigma} g^{\alpha\lambda} g^{\delta\rho} - g^{\beta\sigma} g^{\alpha\delta} g^{\lambda\rho} \right) \quad (7.16) \end{aligned}$$

in all d . Each of the identities above can be derived by first using the cyclic property of the trace to move a gamma matrix from one side to the other, then systematically using the Dirac algebra to move it back into its original position.

Furthermore, we need to generalize γ^5 to d dimensions. Doing so requires a certain amount of care. In particular, we previously defined $\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3$, but in d dimensions this is no longer the product of all the gamma matrices. One way to try to proceed would be to define γ^5 to be a matrix with the property that

$$\{\gamma^5, \gamma^\mu\} = 0$$

for all μ in all dimensions d , but we will see in exercise *** CITE **** that this leads to a contradiction.

Instead, we will proceed as follows. We define

$$\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3$$

which formally looks the same as our previous definition in four dimensions in section *** CITE **, but which now has the consequence that although γ^5 anticommutes with γ^μ for $\mu = 0, 1, 2, 3$, it *commutes* with γ^μ for other values of μ . (See [Collins][section 13.2] and references therein for alternate approaches to this problem.)

Applying the Dirac algebra to the definition above, one immediate consequence is that

$$(\gamma^5)^2 = 1$$

for all d , just as in $d = 4$.

Another consequence of this definition is as follows. Write $\gamma^\mu = \gamma^\mu_{\parallel} + \gamma^\mu_{\perp}$, or equivalently $\not{p} = \not{p}_{\parallel} + \not{p}_{\perp}$, where \parallel indicates components in the original four dimensions, and \perp indicates all other components, so that, for example, $\gamma^0_{\perp} = 0$, $\gamma^0_{\parallel} = \gamma^0$, and

$$\not{p}\gamma^5 = -\gamma^5 \not{p}_{\parallel} + \gamma^5 \not{p}_{\perp}$$

Then, for a trace to be nonzero, it must contain an even number of \parallel components:

$$\begin{aligned} \text{Tr} \left(\gamma_{\parallel}^{\mu_1} \cdots \gamma_{\parallel}^{\mu_m} \gamma_{\perp}^{\nu_1} \cdots \gamma_{\perp}^{\nu_n} \right) &= \text{Tr} \left((\gamma^5)^2 \gamma_{\parallel}^{\mu_1} \cdots (\gamma^5)^2 \gamma_{\parallel}^{\mu_m} (\gamma^5)^2 \gamma_{\perp}^{\nu_1} \cdots (\gamma^5)^2 \gamma_{\perp}^{\nu_n} \right) \\ &= \text{Tr} \left((\gamma^5 \gamma_{\parallel}^{\mu_1} \gamma^5) \cdots (\gamma^5 \gamma_{\parallel}^{\mu_m} \gamma^5) (\gamma^5 \gamma_{\perp}^{\nu_1} \gamma^5) \cdots (\gamma^5 \gamma_{\perp}^{\nu_n} \gamma^5) \right) \\ &= (-)^m \text{Tr} \left(\gamma_{\parallel}^{\mu_1} \cdots \gamma_{\parallel}^{\mu_m} \gamma_{\perp}^{\nu_1} \cdots \gamma_{\perp}^{\nu_n} \right) \end{aligned}$$

and so we see that if m is odd, then

$$\text{Tr} \left(\gamma_{\parallel}^{\mu_1} \cdots \gamma_{\parallel}^{\mu_m} \gamma_{\perp}^{\nu_1} \cdots \gamma_{\perp}^{\nu_n} \right) = 0$$

Another useful identity is

$$\text{Tr} \gamma^5 = 0 \tag{7.17}$$

in all d . We can prove this as follows. Let p, q be four-vectors, then

$$\begin{aligned} \text{Tr} \left(\gamma^5 \not{p}_{\parallel} \not{q}_{\parallel} \right) &= -\text{Tr} \left(\gamma^5 \not{q}_{\parallel} \not{p}_{\parallel} \right) \text{ using cyclic property of trace} \\ &= -\text{Tr} \left(\gamma^5 \left(-2q_{\parallel} \cdot p_{\parallel} - \not{p}_{\parallel} \not{q}_{\parallel} \right) \right) \\ &= \text{Tr} \left(\gamma^5 \not{p}_{\parallel} \not{q}_{\parallel} \right) + 2q_{\parallel} \cdot p_{\parallel} \text{Tr} \gamma^5 \end{aligned}$$

Taking, for example, $p = q$ spacelike, so that $p_{\parallel} \cdot q_{\parallel} \neq 0$, we find the desired result.

One can also show

$$\text{Tr} \left(\gamma^5 \gamma^{\mu} \gamma^{\nu} \right) = 0 \tag{7.18}$$

for all μ, ν , in all d . Some other handy identities are

$$\text{Tr} \left(\gamma^5 \gamma_{\perp}^{\mu_1} \cdots \gamma_{\perp}^{\mu_{2m}} \gamma_{\parallel}^{\nu_1} \gamma_{\parallel}^{\nu_2} \right) = 0 \tag{7.19}$$

for $m \geq 0$ in all d ,

$$\text{Tr} \left(\gamma^5 \gamma_{\perp}^{\mu_1} \cdots \gamma_{\perp}^{\mu_{2m}} \right) = 0 \tag{7.20}$$

for $m \geq 0$ in all d , and

$$\text{Tr} \left(\gamma_{\perp}^{\alpha} \gamma_{\perp}^{\beta} \gamma_{\parallel}^{\nu_1} \cdots \gamma_{\parallel}^{\nu_{2n}} \right) = -g^{\alpha\beta} \text{Tr} \left(\gamma_{\parallel}^{\nu_1} \cdots \gamma_{\parallel}^{\nu_{2n}} \right) \tag{7.21}$$

for $n \geq 0$ in all d .

Some useful consequences of the identities above include, for example,

$$\begin{aligned} \text{Tr} \left(\gamma^5 \gamma^{\alpha} \gamma^{\beta} \gamma^{\lambda} \gamma^{\delta} \right) &= \text{Tr} \left(\gamma^5 \gamma_{\parallel}^{\alpha} \gamma_{\parallel}^{\beta} \gamma_{\parallel}^{\lambda} \gamma_{\parallel}^{\delta} \right) \\ &= \begin{cases} -4i \epsilon^{\alpha\beta\lambda\delta} & \alpha, \beta, \lambda, \delta \in \{0, 1, 2, 3\} \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{7.22}$$

and

$$\begin{aligned}
\text{Tr} \left(\gamma^5 \gamma^\alpha \gamma^\beta \gamma^\lambda \gamma^\delta \gamma^\rho \gamma^\sigma \right) &= \text{Tr} \left(\gamma^5 \gamma_{\parallel}^\alpha \gamma_{\parallel}^\beta \gamma_{\parallel}^\lambda \gamma_{\parallel}^\delta \gamma_{\parallel}^\rho \gamma_{\parallel}^\sigma \right) \\
&\quad + \text{Tr} \left(\gamma^5 \gamma_{\perp}^\alpha \gamma_{\perp}^\beta \gamma_{\parallel}^\lambda \gamma_{\parallel}^\delta \gamma_{\parallel}^\rho \gamma_{\parallel}^\sigma \right) + \text{permutations} \\
&= \text{Tr} \left(\gamma^5 \gamma_{\parallel}^\alpha \gamma_{\parallel}^\beta \gamma_{\parallel}^\lambda \gamma_{\parallel}^\delta \gamma_{\parallel}^\rho \gamma_{\parallel}^\sigma \right) \\
&\quad - g^{\alpha\beta} \text{Tr} \left(\gamma^5 \gamma_{\parallel}^\lambda \gamma_{\parallel}^\delta \gamma_{\parallel}^\rho \gamma_{\parallel}^\sigma \right) - \text{permutations} \tag{7.23}
\end{aligned}$$

Now, let us apply this to the study of the chiral anomaly in a dimensionally-regularized theory. Let us assume the same Lagrangian density as in section **** CITE ***, where we studied anomalies in theories with a cutoff regularization.

Briefly, in that section we learned that the Ward-Takahashi identities for the vector and axial symmetries could not all simultaneously be satisfied; due to an ambiguity caused by lack of translation invariance in the linearly divergent integrals, one must work harder to uniquely define the effect of the cutoff regularization, and we made the choice of preserving the vector symmetries while letting the axial symmetry be violated.

The argument for the anomaly in section *** CITE *** revolved around lack of translation-invariance in linearly divergent integrals, in cutoff regularization. In dimensional regularization, on the other hand, all integrals are automatically translation invariant, including the linearly divergent ones. Instead, the anomaly will turn out to arise from the difficulties in defining γ^5 in a dimensionally-regularized theory.

Let us work through the details of the anomaly computation in the dimensionally-regularized theory. In section *** CITE ****, the relevant part of a triangle Feynman diagram was labelled $S^{\mu\lambda\nu}(q_1, q_2)$ in (5.44). For completeness, we repeat it here, in dimensional regularization:

$$S^{\mu\lambda\nu}(q_1, q_2) \equiv \int \frac{d^d p}{(2\pi)^d} \left(\frac{1}{(p - q_1)^2} \frac{1}{(p + q_2)^2} \frac{1}{p^2} \text{Tr} \left[(\not{p} - \not{q}_1) \gamma^\mu \gamma^5 (\not{p} + \not{q}_2) \gamma^\lambda \not{p} \gamma^\nu \right] \right)$$

The quantity $T^{\mu\lambda\nu}(q_1, q_2)$ was defined in (5.45) to be the sum of two sets of S 's corresponding to two Feynman diagram contributions:

$$T^{\mu\lambda\nu}(q_1, q_2) \equiv S^{\mu\lambda\nu}(q_1, q_2) + S^{\mu\nu\lambda}(q_2, q_1)$$

If both the vector and axial anomalies were anomaly-free, then the Ward-Takahashi identities (5.46, 5.47, 5.48) would be satisfied. For completeness, we repeat them here. The Ward identity for the axial current, expressing $\partial_\mu \langle J_5^\mu \rangle = 0$, is

$$(q_{1\mu} + q_{2\mu}) T^{\mu\lambda\nu}(q_1, q_2) = 0$$

The Ward identity for the vector current, expressing $\partial_\mu \langle J^\mu \rangle = 0$, can also be checked using the same Feynman diagrams, as the triangle diagram is roughly a contribution to the correlation function $\langle J_5^\mu J^\lambda J^\nu \rangle$. The statement that the two copies of J be divergence-free is

$$\begin{aligned}
q_{1\nu} T^{\mu\lambda\nu}(q_1, q_2) &= 0 \\
q_{2\lambda} T^{\mu\lambda\nu}(q_1, q_2) &= 0
\end{aligned}$$

In section *** CITE ***, it was argued that

$$(q_{1\mu} + q_{2\mu}) S^{\mu\lambda\nu}(q_1, q_2) = 0$$

by using the identity (5.49), namely

$$(\not{q}_1 + \not{q}_2) \gamma^5 = -(\not{p} - \not{q}_1) \gamma^5 - \gamma^5 (\not{p} + \not{q}_2)$$

to simplify the integrand. In dimensional regularization, identity (5.49) is replaced by

$$(\not{q}_1 + \not{q}_2) \gamma^5 = -(\not{p} - \not{q}_1) \gamma^5 - \gamma^5 (\not{p} + \not{q}_2) + 2\gamma^5 \not{p}_\perp$$

The first two terms are the same as before; the third term is new, and reflects the technical issues in defining γ^5 in dimensional regularization. The contribution of the first two terms to $(q_{1\mu} + q_{2\mu}) S^{\mu\lambda\nu}$ vanishes, by essentially the same argument² as in section *** CITE ****.

Thus, we see that in dimensional regularization,

$$\begin{aligned} & (q_{1\mu} + q_{2\mu}) S^{\mu\lambda\nu}(q_1, q_2) \\ &= \int \frac{d^d p}{(2\pi)^d} \left(\frac{1}{(p - q_1)^2} \frac{1}{(p + q_2)^2} \frac{1}{p^2} \text{Tr} [(\not{p} - \not{q}_1)(2\gamma^5 \not{p}_\perp)(\not{p} + \not{q}_2)\gamma^\lambda \not{p}\gamma^\nu] \right) \end{aligned}$$

Next, let us combine denominators with Feynman's trick:

$$\begin{aligned} \frac{1}{(p - q_1)^2} \frac{1}{(p + q_2)^2} \frac{1}{p^2} &= 2 \int_0^1 dx \int_0^{1-x} dy \left[x(p - q_1)^2 + y(p + q_2)^2 + p^2(1 - x - y) \right]^{-3} \\ &= 2 \int_0^1 dx \int_0^{1-x} dy \left[p^2 - 2xp \cdot q_1 + 2yp \cdot q_2 + xq_1^2 + yq_2^2 \right]^{-3} \end{aligned}$$

Next, let us use translation invariance of the (dimensionally-regulated) integral to replace

$$p \mapsto p' = p + yq_2 - xq_1$$

so that we can write

$$\begin{aligned} & (q_{1\mu} + q_{2\mu}) S^{\mu\lambda\nu}(q_1, q_2) \\ &= 2 \int_0^1 dx \int_0^{1-x} dy \int \frac{d^d p}{(2\pi)^d} \left[p^2 + xq_1^2 + yq_2^2 - (yq_2 - xq_1)^2 \right]^{-3} \\ & \quad \cdot \text{Tr} [(\not{p} - y \not{q}_2 + x \not{q}_1 - \not{q}_1)(2\gamma^5 \not{p}_\perp)(\not{p} - y \not{q}_2 + x \not{q}_1 + \not{q}_2)\gamma^\lambda (\not{p} - y \not{q}_2 + x \not{q}_1)\gamma^\nu] \end{aligned}$$

In writing the above, we have assumed, without meaningful loss of generality, that $q_{1,2}$ are ordinary four-vectors, *i.e.* $\not{q}_{1\perp} = 0 = \not{q}_{2\perp}$. Furthermore, we shall also assume that λ and ν are also ordinary four-vector indices, so that $\gamma_\perp^\lambda = 0 = \gamma_\perp^\nu$.

²The only point in the argument that is a concern is the trace identities; however, we see from identity (7.22) that the trace identities used are unchanged in dimensional regularization.

Next, we shall simplify the trace above. In light of identity (7.23), since there is already one \not{p}_\perp factor, the only nonzero contribution will arise from terms involving precisely one additional \perp factor, with all other factors \parallel . Thus, we find

$$\begin{aligned}
& \text{Tr} \left[(\not{p} - y \not{q}_2 + x \not{q}_1 - \not{q}_1)(2\gamma^5 \not{p}_\perp)(\not{p} - y \not{q}_2 + x \not{q}_1 + \not{q}_2)\gamma^\lambda(\not{p} - y \not{q}_2 + x \not{q}_1)\gamma^\nu \right] \\
&= \text{Tr} \left[(\not{p}_\perp)(2\gamma^5 \not{p}_\perp)(\not{p}_\parallel - y \not{q}_2 + x \not{q}_1 + \not{q}_2)\gamma^\lambda(\not{p}_\parallel - y \not{q}_2 + x \not{q}_1)\gamma^\nu \right] \\
&\quad + \text{Tr} \left[(\not{p}_\parallel - y \not{q}_2 + x \not{q}_1 - \not{q}_1)(2\gamma^5 \not{p}_\perp)(\not{p}_\perp)\gamma^\lambda(\not{p}_\parallel - y \not{q}_2 + x \not{q}_1)\gamma^\nu \right] \\
&\quad + \text{Tr} \left[(\not{p}_\parallel - y \not{q}_2 + x \not{q}_1 - \not{q}_1)(2\gamma^5 \not{p}_\perp)(\not{p}_\parallel - y \not{q}_2 + x \not{q}_1 + \not{q}_2)\gamma^\lambda(\not{p}_\perp)\gamma^\nu \right] \\
&= 2(p_\perp)^2(4i)(p_\parallel - yq_2 + xq_1 + q_2)_\alpha(p_\parallel - yq_2 + xq_1)_\beta \epsilon^{\alpha\lambda\beta\nu} \\
&\quad + 2(p_\perp)^2(4i)(p_\parallel - yq_2 + xq_1)_\alpha(p_\parallel - yq_2 + xq_1 - q_1)_\beta \epsilon^{\lambda\alpha\nu\beta} \\
&\quad + 2(p_\perp)^2(4i)(p_\parallel - yq_2 + xq_1 + q_2)_\alpha(p_\parallel - yq_2 + xq_1 - q_1)_\beta \epsilon^{\alpha\lambda\nu\beta} \\
&= 8i(p_\perp)^2 \left(q_{2\alpha}(p_\parallel - yq_2 + xq_1)_\beta \epsilon^{\alpha\lambda\beta\nu} - q_{1\beta}(p_\parallel - yq_2 + xq_1)_\alpha \epsilon^{\lambda\alpha\nu\beta} \right. \\
&\quad \left. + q_{2\alpha}(p_\parallel - yq_2 + xq_1)_\beta \epsilon^{\alpha\lambda\nu\beta} - q_{1\beta}(p_\parallel - yq_2 + xq_1)_\alpha \epsilon^{\alpha\lambda\nu\beta} - q_{2\alpha}q_{1\beta} \epsilon^{\alpha\lambda\nu\beta} \right) \\
&= -8i(p_\perp)^2 q_{2\alpha}q_{1\beta} \epsilon^{\alpha\lambda\nu\beta}
\end{aligned}$$

***** NOTE: I need to include some Λ factors, look at the expression below. Go back and include them.

Putting this all together, we find

$$\begin{aligned}
& (q_{1\mu} + q_{2\mu}) S^{\mu\lambda\nu}(q_1, q_2) \\
&= -16iq_{2\alpha}q_{1\beta} \epsilon^{\alpha\lambda\nu\beta} \int_0^1 dx \int_0^{1-x} dy \int \frac{d^d p}{(2\pi)^d} (p_\perp)^2 \left[p^2 + xq_1^2 + yq_2^2 - (yq_2 - xq_1)^2 \right]^{-3}
\end{aligned}$$

Inside the symmetric integration, we can replace

$$(p_\perp)^2 \mapsto \frac{d-4}{d} p^2 \quad (7.24)$$

so that the integral becomes

$$\begin{aligned}
& (q_{1\mu} + q_{2\mu}) S^{\mu\lambda\nu}(q_1, q_2) \\
&= -16iq_{2\alpha}q_{1\beta} \epsilon^{\alpha\lambda\nu\beta} \frac{(d-4)}{d} \int_0^1 dx \int_0^{1-x} dy \int \frac{d^d p}{(2\pi)^d} p^2 \left[p^2 + xq_1^2 + yq_2^2 - (yq_2 - xq_1)^2 \right]^{-3} \\
&= -16iq_{2\alpha}q_{1\beta} \epsilon^{\alpha\lambda\nu\beta} \frac{(d-4)}{d} \int_0^1 dx \int_0^{1-x} dy \frac{i}{(4\pi)^d} \frac{\Gamma(3-d/2-1)\Gamma(d/2+1)}{\Gamma(d/2)\Gamma(3)} \\
&\quad \cdot \left(xq_1^2 + yq_2^2 - (yq_2 - xq_1)^2 \right)^{d/2+1-3} \\
&= -\frac{16}{(4\pi)^d} q_{2\alpha}q_{1\beta} \epsilon^{\alpha\lambda\nu\beta} \frac{\Gamma(3-d/2)\Gamma(d/2+1)}{\Gamma(d/2+1)\Gamma(3)} \\
&\quad \cdot \int_0^1 dx \int_0^{1-x} dy \left(xq_1^2 + yq_2^2 - (yq_2 - xq_1)^2 \right)^{d/2-2}
\end{aligned}$$

where in the next-to-last line we evaluated the integral using equation **** CITE ****.

The expression above has the property that it has a finite limit as $d \rightarrow 4$; taking that limit, we find

$$\begin{aligned} (q_{1\mu} + q_{2\mu}) S^{\mu\lambda\nu}(q_1, q_2) &= -\frac{16}{(4\pi)^4} q_{2\alpha} q_{1\beta} \epsilon^{\alpha\lambda\nu\beta} \frac{\Gamma(3-2)}{\Gamma(3)} \cdot \frac{1}{2} \\ &= -\frac{1}{64\pi^4} q_{2\alpha} q_{1\beta} \epsilon^{\alpha\lambda\nu\beta} \end{aligned}$$

and hence

$$\begin{aligned} (q_{1\mu} + q_{2\mu}) T^{\mu\lambda\nu}(q_1, q_2) &= -\frac{1}{64\pi^4} q_{2\alpha} q_{1\beta} \epsilon^{\alpha\lambda\nu\beta} - \frac{1}{64\pi^4} q_{1\alpha} q_{2\beta} \epsilon^{\alpha\nu\lambda\beta} \\ &= -\frac{1}{32\pi^4} q_{1\beta} q_{2\alpha} \epsilon^{\alpha\beta\lambda\nu} \end{aligned}$$

***** ALMOST matches earlier result, but not quite – off by a sign, and a factor of $16\pi^2$.

***** Modulo factors*****, this matches the result (5.50), and so repeating the result of the analysis from that section we can immediately read off that

$$\partial_\mu \langle J_5^\mu(x) \rangle = -\frac{\hbar}{32\pi^4} \partial_\mu A_\nu(x) \partial_\rho A_\lambda(x) \epsilon^{\rho\mu\lambda\nu}$$

***** I've modified the factors above to be consistent with my work in this section.

**** NEED to track down this difference in factors. Where is it coming from? Is there an error in dim'l reg' computations, or in the earlier computation of chiral anomaly?

Let us also check whether the vector Ward identities are satisfied, beginning with the statement

$$q_{1\nu} T^{\mu\lambda\nu}(q_1, q_2) = 0$$

It is straightforward to check that the analysis of section *** CITE *** for this case can be repeated without any significant modification, modulo the fact that all integrals in dimensional regularization are translation-invariant. Back in section *** CITE ****, the only nonzero contribution to $q_{1\nu} S^{\mu\lambda\nu}(q_1, q_2)$ arose from the lack of translation-invariance in cutoff-regularized linearly divergent integrals; here, translation invariance implies that

$$q_{1\nu} S^{\mu\lambda\nu}(q_1, q_2) = 0$$

and hence the vector Ward identity above is satisfied. Similarly, the remaining vector Ward identity is also satisfied.

Now, let us review what we have found, and compare to our earlier results in section *** CITE **** for the same anomaly, in cutoff regularization. There, there was an ambiguity in the specification of the loop integrals, arising because of a lack of translation invariance in linearly divergent integrals in cutoff regularization. Because of that ambiguity, one could choose whether to satisfy axial or vector current conservation – both could not simultaneously hold, but which

held was determined by how the ambiguity was resolved. By contrast, all integrals are translation-invariant in dimensional regularization, even the linearly divergent ones. The anomaly arises here because of difficulties in defining γ^5 in dimensional regularization. Taking that into account, we find that vector Ward identities are automatically satisfied in dimensional regularization, and the axial Ward identity is broken. There is no longer a choice of which symmetry must be broken (reflecting the fact that dimensional regularization does not break gauge symmetries, a fact we shall return to later in section *** CITE ****).

Ultimately, in cutoff regularization and dimensional regularization, we follow two different computational methods, but arrive at the same final result: the axial Ward identity cannot hold, and axial current conservation is violated as

$$\partial_\mu \langle J_5^\mu(x) \rangle \propto F_{\mu\nu} F_{\rho\sigma} \epsilon^{\mu\nu\rho\sigma}$$

***** Also mention relation to path integrals, if in the fermion section I discussed Fujikawa's perspective.

7.5 An alternative method: zeta function regularization

So far in this text we have seen several regularization schemes: for most of the text so far we have used cutoff regularization, and this chapter has introduced Pauli-Villars regularization and dimensional regularization. There exist additional regularization schemes beyond these. One further regularization method is called *lattice regularization* and will be a powerful tool for understanding the gauge theories that will be discussed later. We shall discuss lattice regularization in chapter **** FILL IN ****.

In this section, we shall outline yet another regularization method, known as *zeta function regularization*. Zeta function regularization is not used as commonly as the other methods we have discussed, but it does pop up sufficiently often to warrant specific mention.

**** For more material, see the discussion in Berline-Getzler-Vergne, *Heat kernels and Dirac operators*, section 9.6, pp 300-, “zeta function of a laplacian.”

**** Also see Nakahara section 1.4, 1.5 for some discussion, examples.

When encountering infinite products (as in operator determinants) and infinite sums, a powerful technique called zeta function regularization is often used to make sense of them. The upshot is relatively simple, though I'm going to take the time to explain the background behind the method.

At its most basic level, the idea is that given divergent sums like $\sum_{n>0} n$, for example, we're going to use properties of zeta functions to *formally* make sense of them. [Such prescriptions aren't unique, since the sum is divergent; zeta functions will give a prescription.] In principle the idea can be applied to operator determinants. Suppose the eigenfunctions and eigenvalues of an operator A are given by

$$Af_n(x) = a_n f_n(x)$$

Construct

$$\zeta_A(s) = \sum_n \frac{1}{a_n^s}$$

the *zeta function associated to operator A*. It can be shown that

$$\det A = \prod_n a_n = \exp -\zeta'_A(0)$$

so if one knows that infinite sum $\zeta_A(s)$, and can make sense of it, then you can get a finite expression for $\det A$. (See Ramond's book for further comments on this, and also P. di Francesco, P. Mathieu, D. Senechal, *Conformal Field Theory*, section 6.4.)

So far I've briefly outlined why being able to make divergent sums finite might be a good thing, but how exactly do we do it? First, recall the ordinary Riemann zeta function is defined as

$$\zeta(s) = \sum_{n>0} n^{-s}$$

(see appendix **** CITE ****) but this only converges for $s \geq 2$. Unfortunately, we sometimes run across sums such as $\sum_{n>0} n$, which formally looks like $\zeta(-1)$, a divergent sum. Now, the trick is we're going to replace the ordinary Riemann zeta function with the *Hurwitz zeta function* (also called the *generalized Riemann zeta function*). That by itself *still* will not converge at the desired places, *but* we can analytically continue it to something that does, and we'll use that last something to formally make sense of divergent sums. It sounds complicated, but the final result is actually very simple.

The Hurwitz zeta function, or generalized Riemann zeta function, is defined by

$$\zeta(s, q) = \sum_{n=0}^{\infty} (q + n)^{-s}$$

so that $\zeta(s, 1)$ is the same as the ordinary Riemann zeta function $\zeta(s)$. (Also, we now think of s as a complex number, not necessarily an integer.) This converges so long as $\Re s > 1$ and q is not a negative integer or zero, but unfortunately we will typically want to understand cases in which $\Re s < 0$.

The next step is to analytically continue the Hurwitz zeta function, to a function we shall denote $\tilde{\zeta}(s, q)$. This is discussed in appendix B.3.2; the pertinent result is that

$$\tilde{\zeta}(-n, q) = -\frac{B_{n+1}(q)}{n+1}$$

where n is a positive integer, and B_n is the n th Bernoulli polynomial.

In any event, now we finally have a sophisticated way of making sense of divergent sums. For example:

$$\begin{aligned} \sum_{n>0} n &= \tilde{\zeta}(-1, 1) \\ \sum_{n>0} (n - 1/2) &= \tilde{\zeta}(-1, 1/2) \end{aligned}$$

and furthermore the results are finite numbers. For example, using $B_2(x) = x^2 - x + 1/6$, you can calculate using the expression above that

$$\sum_{n>0} (n - 1/2) = 1/24$$

(Note as always that strictly speaking the expression on the left diverges, it's only because of this choice of regularization that we can get a finite number out of it.)

For more information on zeta function regularization, see for example

- E. Elizalde, *Ten physical applications of spectral zeta functions*, Springer-Verlag, Berlin, 1995.

**** More refs?

7.6 Further reading

*** In 7650 at Utah, regularization and renormalization were one long section spread over 2 weeks. Reading: PS sections 8, 10.1-10.2, 10.4-10.5; Ryder 9.1-9.3, 9.A, 9.B. Skim PS 6.3, 7.1, 7.5. Math track: read Munkres 2-9, 2-10, 3-1, 3-5 over the course of the two weeks.

7.7 Notes

1. Several weeks ago, when discussing path integrals, I said that it was misleading to believe that a path integral weights paths by the value of the classical action. At the time, I pointed out that (a) this sounds like putting the cart before the horse, since we're supposed to get classical physics from quantum physics not vice-versa, and (b) the fields one sums over are typically not differentiable, so any action phrased in terms of derivatives of fields should be viewed with a bit of suspicion. At the time, I said that later we would learn that talking about a classical action is misleading, that the classical action would eventually have to be replaced by a 'regularized' action. Well, we've now gotten to the point that I can explain what I meant.
2. To "regularize" a theory means to replace classical expressions with alternative expressions in which the divergences of Feynman diagrams are rendered finite. In general, there are many different ways to regularize theories. This week, you'll learn about Pauli-Villars regularization and dimensional regularization. Later, we'll learn more. (For example, when discussing nonabelian gauge theories, we'll see that replacing spacetime with a lattice yields another regularization, known as a lattice regularization, which turns out to be important for computational work.)
3. The idea behind Pauli-Villars regularization is that since the divergences you've seen so far arise from momenta going to infinity (so-called "ultraviolet" or "UV" divergences), one

should impose a cutoff on those momenta. Intuitively, this is akin to saying there's a shortest distance in spacetime, and nothing smaller. Pauli-Villars is comparatively intuitive, and we will typically use the language of "imposing a cutoff," but as a practical computational matter Pauli-Villars is often very cumbersome. (For the simple examples we'll see this week they'll be comparable, but when discussing nonabelian gauge theories Pauli-Villars becomes grotesque.) This effectively truncates the infinitely-jagged fields counted in the classical path integral.

4. Dimensional regularization is, computationally, much more efficient than Pauli-Villars in general. The drawback is that it is far less intuitive. The idea is a bit more abstract: we replace the divergent integrals with other integrals in which we've formally allowed the dimension to stop being integral. (More properly, we work in spherical coordinates, split off the angular part, then rethink the radial integral with gamma functions, which allows us to play some games that formally resemble making the dimension non-integral.) How literally should one interpret the idea of changing the dimension? (ie, is the dimension really a fractal dimension or some such?) Nobody knows; everyone treats it as an efficient computational technique, the deeper understanding, if there is one, does not yet exist.
5. In condensed matter systems there is often already a natural cutoff, provided by the lattice size in the crystal. So, one might naively think that much of this discussion is irrelevant for condensed matter physics. However, we saw last week that renormalization is teaching us that physics is scale-dependent (the "renormalization group"), and that lesson is very applicable to condensed matter physics.
6. What's the difference between "regularization" and "renormalization"? There's really a two-step process here. First, we must alter the action in some way – impose a cutoff, play formal games with the dimension, whatever – so as to make the infinities in Feynman diagrams finite (reflecting the fact that the path integral can probably only be rigorously defined after such regularization), then we must alter the coupling constants, masses, and wavefunction scales so as to insure that physical quantities remain finite when we remove the regularization. The first step is "regularization," the second step is "renormalization." Neither step is unique – there are several different regularizations, and there are also several ways to renormalize. One way is to alter the original ("bare") masses, coupling constants, and wavefunction scales so as correlation functions match certain fixed target values. If we're renormalizing QED, then, well, we can measure the electron mass and couplings, so we should renormalize in such a way as to insure that they pop out. Another way is to merely subtract off whatever is becoming divergent. If, for example, adding $\delta\lambda = \epsilon^{-1}$ to λ will remove a pole from a loop amplitude, then that's one thing we could do. But, if that makes the amplitude finite, then so will $\delta\lambda = \epsilon^{-1} + 1$. The choice of what to subtract off is called a "subtraction scheme," and the obvious minimal choice is known as "minimal subtraction." Subtraction schemes differ from one another in what finite parts to subtract, known as "finite renormalizations."
7. A practical aside. Peskin-Schroeder and Ryder take two slightly different approaches to dimensional regularization: PS Wick rotates to Euclidean space and calculates integrals there (as we do here), whereas Ryder works in Minkowski space.

8. **Overlapping divergences.** If you think through the details of how to carry out renormalization to higher loops, you'll quickly see there's a potential rough spot. Some higher-loop diagrams have divergent bits that are easily recognized as being cancelled by lower-loop counterterms – for ex, if in the middle of some complicated diagram there's a one-loop propagator correction, then, the corresponding one-loop counterterm will make that part of the complicated higher-loop diagram finite. However, it's possible to have situations in which, for example, a 2-loop diagram looks like a pair of divergent one-loop diagrams, except that there's a common propagator between them. These are called overlapping divergences. In the limit that either of the loop momenta become large, the corresponding subdiagram shrinks to zero size, and a one-loop counterterm can help cancel the divergence. Residual divergences not taken care of by the lower-loop counterterms should be cancelled by a higher-order counterterm. ***** ADD MATERIAL ON OVERLAPPING DIVERGENCES *****

7.8 Exercises

1. In this problem we shall generalize Feynman's trick, namely the identity

$$\frac{1}{ab} = \int_0^1 dx (ax + b(1-x))^{-2}$$

as discussed in the text.

- (a) Show that

$$\frac{1}{abc} = 2 \int_0^1 dx \int_0^{1-x} dy (ax + by + c(1-x-y))^{-3}$$

- (b) Use induction to show that

$$\frac{1}{a_1 a_2 \cdots a_n} = (n-1)! \int_0^1 dx_1 \int_0^{1-x_1} dx_2 \cdots \int_0^{1-x_1-\cdots-x_{n-2}} dx_{n-1} \\ (a_1 x_1 + a_2 x_2 + a_{n-1} x_{n-1} + a_n (1-x_1-x_2-\cdots-x_{n-1}))^{-n}$$

2. In this problem we shall study another generalization of Feynman's trick.

- (a) For real numbers α, β , show that

$$\frac{1}{a^\alpha b^\beta} = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 dx \frac{x^{\alpha-1} (1-x)^{\beta-1}}{(ax + b(1-x))^{\alpha+\beta}}$$

Hint: make the change of variables

$$z = \frac{ax}{ax + b(1-x)}$$

(b) For real numbers $\alpha_1, \dots, \alpha_n$, show that

$$\frac{1}{a_1^{\alpha_1} \dots a_n^{\alpha_n}} = \frac{\Gamma(\alpha_1 + \dots + \alpha_n)}{\Gamma(\alpha_1) \dots \Gamma(\alpha_n)} \int dx_1 \dots dx_n \delta(1 - x_1 - \dots - x_n) \frac{x_1^{\alpha_1-1} x_2^{\alpha_2-1} \dots x_n^{\alpha_n-1}}{(a_1 x_1 + a_2 x_2 + \dots + a_n x_n)^{\alpha_1 + \dots + \alpha_n}}$$

Hint: use the identity

$$\frac{\Gamma(\alpha)}{a^\alpha} = \int_0^\infty dt t^{\alpha-1} e^{-at}$$

to get an expression for

$$\frac{\Gamma(\alpha_1) \dots \Gamma(\alpha_n)}{a_1^{\alpha_1} \dots a_n^{\alpha_n}}$$

then multiply the other side by

$$1 = \int_0^\infty ds \delta\left(s - \sum t_i\right)$$

make the change of variables $t_i = sx_i$, and do the s integral.

**** The further generalization above is proven in Nayak's lecture notes, p 60 – compare this with that. Above based on Sredniki (14.1). Seems like the first two problems should be truncated to something more efficient.

3. (**** OMIT for fall 2008 ****) **Pauli-Villars regularization** The basic idea behind Pauli-Villars regularization of scalar field theory is to replace the propagator

$$D_F(x - y) = -i\hbar \int \frac{d^4 p}{(2\pi)^4} \frac{1}{p^2 + m^2} \exp(-ip \cdot (x - y))$$

by the propagator

$$-i\hbar \int \frac{d^4 p}{(2\pi)^4} \left(\frac{1}{p^2 + m^2} - \frac{1}{p^2 + \Lambda^2} \right) \exp(-ip \cdot (x - y))$$

(where Λ^2 is assumed to be much larger than m^2). When $\Lambda^2 \rightarrow \infty$, the second propagator reduces to the former.

- (a) We can interpret the procedure above as adding a second scalar field, of mass Λ and all same couplings as the first, but with a wrong-sign kinetic term. Comment on the meaning of that sign.
- (b) Show that

$$\frac{1}{p^2 + m^2} - \frac{1}{p^2 + \Lambda^2} = \frac{1}{p^2 + m^2} \frac{\Lambda^2 - m^2}{p^2 + \Lambda^2}$$

Thus, by increasing the number of powers of p in the denominator, we make divergent loop integrals more nearly convergent.

- (c) The one-loop correction to the propagator in $\lambda\phi^4$ theory in four dimensions has a divergent loop integral which for large momenta takes the form

$$\sim \int \frac{d^4p}{p^2}$$

The Pauli-Villars regulator discussed so far improves that to

$$\sim \int \frac{d^4p}{p^4}$$

but that's still log-divergent.

In such cases, we instead make the replacement

$$\begin{aligned} \frac{1}{p^2 + m^2} &\mapsto \frac{1}{p^2 + m^2} - \frac{1}{p^2 + \Lambda_1^2} \frac{\Lambda_2^2 - m^2}{\Lambda_2^2 - \Lambda_1^2} - \frac{1}{p^2 - \Lambda_2^2} \frac{m^2 - \Lambda_1^2}{\Lambda_2^2 - \Lambda_1^2} \\ &= \frac{1}{p^2 + m^2} \frac{\Lambda_1^2 - m^2}{p^2 + \Lambda_1^2} \frac{\Lambda_2^2 - m^2}{p^2 + \Lambda_2^2} \end{aligned}$$

which makes the integral convergent. Compute the mass renormalization of $\lambda\phi^4$ in four dimensions, to first order in λ , using minimal subtraction and the Pauli-Villars regulator above. After performing the substitution in the second form, for simplicity take $\Lambda_1 = \Lambda_2$. You will find that calculations are greatly simplified if you use the Feynman parameter trick discussed above, and also feel free to use equation (7.2).
 (**** PROBLEM: result converges as $\Lambda \rightarrow \infty$, instead of diverging ****)

Although Pauli-Villars appears more intuitive than dimensional regularization (it imposes a cutoff on momenta, which seems very natural), as a practical computational matter it can be much more cumbersome in general. For example in gauge theories, Pauli-Villars regularization automatically destroys gauge invariance, as the second “fake” photon one adds is massive. Although the language of Pauli-Villars regularization is still sometimes used, as a practical matter nowadays everyone uses dimensional regularization, so we shall stick to that from now on.

4. Show that no quadratic, quartic, *etc* divergences are encountered in dimensional regularization, in the sense that

$$\int d^d p (p^2)^N = 0 \text{ for } N > -d/2$$

by taking the limit $m \rightarrow 0$ of the integral $\int d^d p (p^2 + m^2)^N$, evaluated with dimensional regularization.

5. (a) Show that

$$\int \frac{d^d p}{(2\pi)^d} \frac{(p^2)^k}{(p^2 + m^2)^n} = i \frac{(m^2)^{d/2+k-n}}{(4\pi)^{d/2}} \frac{\Gamma(n - d/2 - k)\Gamma(d/2 + k)}{\Gamma(d/2)\Gamma(n)}$$

(b) Show that

$$\int \frac{d^d p}{(2\pi)^d} \frac{p^\mu p^\nu}{(p^2 + m^2)^n} = i \frac{(m^2)^{d/2+1-n}}{(4\pi)^{d/2}} \frac{g^{\mu\nu}}{2} \frac{\Gamma(n - d/2 - 1)}{\Gamma(n)}$$

Hint: use Lorentz invariance to argue that

$$\int \frac{d^d p}{(2\pi)^d} \frac{p^\mu p^\nu}{(p^2 + m^2)^n} = \int \frac{d^d p}{(2\pi)^d} \frac{A g^{\mu\nu} p^2}{(p^2 + m^2)^n}$$

for some constant A , then solve for A and use previous results to evaluate the integral.

(c) Show that

$$\int \frac{d^d p}{(2\pi)^d} \frac{p^\mu p^\nu p^\rho p^\sigma}{(p^2 + m^2)^n} = i \frac{(m^2)^{d/2+2-n}}{(4\pi)^{d/2}} \frac{\Gamma(n - d/2 - 2)}{\Gamma(n)} \left(\frac{1}{4}\right) (g^{\mu\nu} g^{\rho\sigma} + g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho})$$

Hint: use Lorentz invariance to argue that

$$\int \frac{d^d p}{(2\pi)^d} \frac{p^\mu p^\nu p^\rho p^\sigma}{(p^2 + m^2)^n} = \int \frac{d^d p}{(2\pi)^d} \frac{A(p^2)^2 (g^{\mu\nu} g^{\rho\sigma} + g^{\mu\rho} g^{\nu\sigma} + g^{\mu\sigma} g^{\nu\rho})}{(p^2 + m^2)^n}$$

for some constant A , then solve for A and use previous results to evaluate the integral.

6. Gamma matrices in dimensional regularization In a dimensionally-regularized theory ‘near’ four dimensions, show that the gamma matrices have the properties below. Take $d = 4 - \epsilon$.

(a)

$$\gamma^\alpha \gamma^\mu \gamma_\alpha = (2 - \epsilon) \gamma^\mu$$

(b)

$$\gamma^\alpha \gamma^\mu \gamma^\nu \gamma_\alpha = 4g^{\mu\nu} I + \epsilon \gamma^\mu \gamma^\nu$$

(c)

$$\gamma^\alpha \gamma^\mu \gamma^\nu \gamma^\lambda \gamma_\alpha = 2\gamma^\lambda \gamma^\nu \gamma^\mu - \epsilon \gamma^\mu \gamma^\nu \gamma^\lambda$$

7. Properties of γ^5 in dimensional regularization In this problem, assume that for all d , for all γ^μ ,

$$\{\gamma^5, \gamma^\mu\} = 0 \tag{7.25}$$

We will show that this leads to a contradiction.

(a) Starting from

$$d \text{Tr} \gamma^5 \gamma^\mu \gamma^\nu = -\text{Tr} \gamma^5 \gamma^\mu \gamma^\nu \gamma_\lambda \gamma^\lambda = \text{Tr} \gamma^5 \gamma^\lambda \gamma^\mu \gamma^\nu \gamma_\lambda$$

use the identity (7.25) and the Dirac algebra in d dimensions to show that

$$(d - 2) \text{Tr} \gamma^5 \gamma^\mu \gamma^\nu = 0$$

This implies that $\text{Tr} \gamma^5 \gamma^\mu \gamma^\nu = 0$ for $d \neq 2$, and as we assume the trace is meromorphic in d , that implies the trace must vanish for all d .

(b) Starting from

$$d \operatorname{Tr} \gamma^5 \gamma^\alpha \gamma^\beta \gamma^\mu \gamma^\nu = -\operatorname{Tr} \gamma^5 \gamma^\alpha \gamma^\beta \gamma^\mu \gamma^\nu \gamma_\lambda \gamma^\lambda = +\operatorname{Tr} \gamma^5 \gamma^\lambda \gamma^\alpha \gamma^\beta \gamma^\mu \gamma^\nu \gamma_\lambda$$

use the identity (7.25) and the Dirac algebra in d dimensions to show that

$$(d-4) \operatorname{Tr} \gamma^5 \gamma^\alpha \gamma^\beta \gamma^\mu \gamma^\nu = 0$$

Thus, the trace must vanish for $d \neq 4$, and as the trace is assumed to be meromorphic in d , the trace must vanish for all d . But this contradicts the results of section *** CITE ***, where we saw explicitly that in four dimensions, the trace above is nonzero. Thus, we have a contradiction, and identity (7.25) cannot hold for all γ^μ in all dimensions d .

8. **Trace identities in dimensional regularization** Using gamma matrices in dimensional regularization, show the following identities.

(a) Show that

$$\operatorname{Tr} (\gamma^\mu \gamma^\nu) = -4g^{\mu\nu}$$

in all d . (Hint: This problem and the next two can be solved by first cyclically permuting the last factor through the trace to the front, then using the Dirac algebra to move it to its original position.)

(b) Show that

$$\operatorname{Tr} (\gamma^\alpha \gamma^\beta \gamma^\lambda \gamma^\delta) = 4(g^{\alpha\beta} g^{\lambda\delta} - g^{\alpha\lambda} g^{\beta\delta} + g^{\alpha\delta} g^{\beta\lambda})$$

in all d .

(c) Show that

$$\begin{aligned} & \operatorname{Tr} (\gamma^\alpha \gamma^\beta \gamma^\lambda \gamma^\delta \gamma^\rho \gamma^\sigma) \\ &= 4 \left(-g^{\alpha\beta} g^{\lambda\delta} g^{\rho\sigma} + g^{\alpha\beta} g^{\lambda\rho} g^{\delta\sigma} - g^{\alpha\beta} g^{\lambda\sigma} g^{\delta\rho} + g^{\lambda\delta} g^{\alpha\rho} g^{\beta\sigma} - g^{\lambda\delta} g^{\alpha\sigma} g^{\beta\rho} \right. \\ & \quad \left. + g^{\rho\sigma} g^{\alpha\lambda} g^{\beta\delta} - g^{\rho\sigma} g^{\alpha\delta} g^{\beta\lambda} + g^{\alpha\rho} g^{\beta\lambda} g^{\delta\sigma} - g^{\beta\rho} g^{\alpha\lambda} g^{\delta\sigma} - g^{\alpha\rho} g^{\beta\delta} g^{\lambda\sigma} \right. \\ & \quad \left. - g^{\alpha\sigma} g^{\beta\lambda} g^{\delta\rho} + g^{\beta\rho} g^{\alpha\delta} g^{\lambda\sigma} + g^{\alpha\sigma} g^{\beta\delta} g^{\lambda\rho} + g^{\beta\sigma} g^{\alpha\lambda} g^{\delta\rho} - g^{\beta\sigma} g^{\alpha\delta} g^{\lambda\rho} \right) \end{aligned}$$

in all d .

(d) Show that

$$\operatorname{Tr} (\gamma^5 \gamma^\mu \gamma^\nu) = 0$$

for all μ, ν , in all d .

(e) Show that

$$\operatorname{Tr} (\gamma^5 \gamma_\perp^{\mu_1} \cdots \gamma_\perp^{\mu_{2m}} \gamma_\parallel^{\nu_1} \gamma_\parallel^{\nu_2}) = 0$$

for $m \geq 0$, in all d , by induction on m .

(f) Show that

$$\operatorname{Tr} (\gamma^5 \gamma_\perp^{\mu_1} \cdots \gamma_\perp^{\mu_{2m}}) = 0$$

for $m \geq 0$ in all d , by induction on m .

(g) Show that

$$\text{Tr} \left(\gamma_{\perp}^{\alpha} \gamma_{\perp}^{\beta} \gamma_{\parallel}^{\nu_1} \cdots \gamma_{\parallel}^{\nu_{2n}} \right) = -g^{\alpha\beta} \text{Tr} \left(\gamma_{\parallel}^{\nu_1} \cdots \gamma_{\parallel}^{\nu_{2n}} \right)$$

for $n \geq 0$ in all d .

9. **Casimir effect** In deriving the Hamiltonian in canonical quantization, we would often get an infinite constant, $\sim \int d^d p(1)$, which we have so far ignored. Sometimes, however, we can extract some physics from that constant. We will consider, for simplicity, a free massless real scalar field in $1 + 1$ dimensions between parallel plates, and we'll see that quantum effects cause there to be a force on the plates. (This *Casimir effect* has been measured experimentally. *** REFS? ***)

**** See Itzykson-Zuber section 3-2-4 pp 138-142 for a more detailed discussion, that should be translated into this problem here. In add'n, Radovanovic's book section 9 page 53 has a good problem.

Consider three parallel plates, at positions $x = 0, d, L$, with the property that the scalar field is forced to vanish at the position of each plate. One can show that the energies of field modes between plates separated by distance d is $\sim n\pi/d$. The constant term appearing in the Hamiltonian is then

$$\sim f(d) = \sum_{n=1}^{\infty} \frac{\hbar n\pi}{2d}$$

so the total contribution from the regions between all the plates is

$$E = f(d) + f(L - d)$$

Both terms in the expression above are infinite, but by now you should have an idea how to handle infinities: we regularize. Define

$$\tilde{f}(d) = \sum_{n=1}^{\infty} \frac{\hbar n\pi}{2d} \exp(-an\pi/d)$$

where a is a small positive number. Note $\tilde{f} \rightarrow f$ in the limit $a \rightarrow 0$.

(a) Show that

$$\tilde{f}(d) = \frac{\hbar\pi}{2d} \frac{\exp(a\pi/d)}{(\exp(a\pi/d) - 1)^2}$$

- (b) Estimate $\tilde{f}(d)$ for small a ; compute the leading two terms.
- (c) The force between the plates is $-\partial E/\partial d$. Calculate it, to leading orders. You should find that the result is well-defined (and nonzero) in the limit we remove the regulator, *i.e.* the limit $a \rightarrow 0$.
- (d) Now, let us repeat that analysis with zeta function regularization. Use zeta function regularization to calculate E and $-\partial E/\partial d$, and compare your result to the result you just obtained in the previous part using a different method.

10. When phonons are described in terms of vibrations of ions sitting at lattice points, there is a natural momentum cutoff: momenta are only integrated over a single Brillouin zone, not all of \mathbf{R}^3 . When phonons are described with a continuum model, that fact becomes more obscure, and a UV cutoff must be imposed by hand. In this problem you will compute an estimate for such a cutoff.

Consider a cubical Bravais lattice in three dimensions, with lattice spacing a in each direction. Compute

$$\int_B \frac{d^3k}{(2\pi)^3} (1)$$

for this case. Now, let's approximate the Brillouin zone by a solid sphere, of radius ω_D/c (the cutoff frequency, known in this context as the Debye frequency). Compute

$$\int^{\omega_D/c} \frac{d^3k}{(2\pi)^3} (1)$$

over this volume. By setting the two integrals equal, derive an approximate expression for ω_D in terms of the volume of a cell of the Bravais lattice. (See Fetter-Walecka section 44 for more information.)