Quantum electrodynamics (QED)

based on S-58

Quantum electrodynamics is a theory of photons interacting with the electrons and positrons of a Dirac field:

$$\mathcal{L} = -\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + i\overline{\Psi}\partial\!\!\!/\Psi - m\overline{\Psi}\Psi + e\overline{\Psi}\gamma^\mu\Psi A_\mu$$

$e = -0.302822$

$\alpha = e^2/4\pi = 1/137.036$

Noether current of the lagrangian for a free Dirac field $j^\mu(x) = e\overline{\Psi}(x)\gamma^\mu\Psi(x)$

$$\partial_\mu j^\mu(x) = \delta\mathcal{L}(x) - \frac{\delta S}{\delta\varphi_a(x)}\delta\varphi_a(x)$$

we want the current to be conserved and so we need to enlarge the gauge transformation also to the Dirac field:

$$A^\mu(x) \to A^\mu(x) - \partial^\mu\Gamma(x)\ ,$$

$$\Psi(x) \to \exp[-ie\Gamma(x)]\Psi(x)\ ,$$

$$\overline{\Psi}(x) \to \exp[+ie\Gamma(x)]\overline{\Psi}(x)\ .$$

global symmetry is promoted into local

$\Psi \to e^{-i\alpha}\Psi$

$\overline{\Psi} \to e^{+i\alpha}\overline{\Psi}$

symmetry of the lagrangian and so the current is conserved no matter if equations of motion are satisfied

We can write the QED lagrangian as:

$$\mathcal{L} = -\tfrac{1}{4}F^{\mu\nu}F_{\mu\nu} + i\overline{\Psi}\not{D}\Psi - m\overline{\Psi}\Psi$$

$$D_\mu \equiv \partial_\mu - ieA_\mu$$

covariant derivative

(the covariant derivative of a field transforms as the field itself)

$$\Psi(x) \to \exp[-ie\Gamma(x)]\Psi(x)$$

$$D_\mu\Psi(x) \to \exp[-ie\Gamma(x)]D_\mu\Psi(x)$$

and so the lagrangian is manifestly gauge invariant!

Proof:

$$\begin{aligned} D_\mu\Psi &\to \Big(\partial_\mu - ie[A_\mu - \partial_\mu\Gamma]\Big)\Big(\exp[-ie\Gamma]\Psi\Big) \\ &= \exp[-ie\Gamma]\Big(\partial_\mu\Psi - ie(\partial_\mu\Gamma)\Psi - ie[A_\mu - \partial_\mu\Gamma]\Psi\Big) \\ &= \exp[-ie\Gamma]\Big(\partial_\mu - ieA_\mu\Big)\Psi \\ &= \exp[-ie\Gamma]D_\mu\Psi\ . \end{aligned}$$

$$\Psi(x) \rightarrow \exp[-ie\Gamma(x)]\Psi(x)$$
$$D_\mu\Psi(x) \rightarrow \exp[-ie\Gamma(x)]D_\mu\Psi(x)$$

We can also define the transformation rule for D:

$$D_\mu \rightarrow e^{-ie\Gamma} D_\mu \, e^{+ie\Gamma}$$

then

$$D_\mu\Psi \rightarrow \left(e^{-ie\Gamma} D_\mu e^{+ie\Gamma}\right)\left(e^{-ie\Gamma}\Psi\right)$$
$$= e^{-ie\Gamma} D_\mu \Psi \, ,$$

as required.

Now we can express the field strength in terms of D's:

$$D_\mu \equiv \partial_\mu - ieA_\mu$$

$$[D^\mu, D^\nu]\Psi(x) = -ieF^{\mu\nu}(x)\Psi(x)$$

$$F^{\mu\nu} = \frac{i}{e}[D^\mu, D^\nu]$$

$$F^{\mu\nu} = \frac{i}{e}[D^\mu, D^\nu]$$

$$D_\mu \to e^{-ie\Gamma} D_\mu \, e^{+ie\Gamma}$$

Then we simply see:

$$\begin{aligned} F^{\mu\nu} &\to \frac{i}{e}\left[e^{-ie\Gamma} D^\mu \, e^{+ie\Gamma}, e^{-ie\Gamma} D^\nu \, e^{+ie\Gamma}\right] \\ &= e^{-ie\Gamma}\left(\frac{i}{e}[D^\mu, D^\nu]\right)e^{+ie\Gamma} \\ &= e^{-ie\Gamma} F^{\mu\nu} e^{+ie\Gamma} \\ &= F^{\mu\nu} \, . \end{aligned}$$

no derivatives act on exponentials

the field strength is gauge invariant as we already knew

Nonabelian symmetries

based on S-24

Let's generalize the theory of two real scalar fields:

$$\mathcal{L} = -\tfrac{1}{2}\partial^\mu\varphi_1\partial_\mu\varphi_1 - \tfrac{1}{2}\partial^\mu\varphi_2\partial_\mu\varphi_2 - \tfrac{1}{2}m^2(\varphi_1^2+\varphi_2^2) - \tfrac{1}{16}\lambda(\varphi_1^2+\varphi_2^2)^2$$

to the case of N real scalar fields:

$$\mathcal{L} = -\tfrac{1}{2}\partial^\mu\varphi_i\partial_\mu\varphi_i - \tfrac{1}{2}m^2\varphi_i\varphi_i - \tfrac{1}{16}\lambda(\varphi_i\varphi_i)^2$$

the lagrangian is clearly invariant under the SO(N) transformation:

orthogonal matrix with det = 1

$$\varphi_i(x) \to R_{ij}\varphi_j(x)$$

$$R^T = R^{-1}$$

$$\det R = +1$$

lagrangian has also the $\mathbf{Z}_2$ symmetry, $\varphi_i(x) \to -\varphi_i(x)$, that enlarges SO(N) to O(N) infinitesimal SO(N) transformation:

$$R^T = R^{-1}$$

$$R^T_{ij} = \delta_{ij} + \theta_{ji}$$

$$R^{-1}_{ij} = \delta_{ij} - \theta_{ij}$$

antisymmetric

$$R_{ij} = \delta_{ij} + \theta_{ij} + O(\theta^2)$$

real

$$Im(R^{-1}R)_{ij} = Im\sum_k R_{ki}R_{kj} = 0$$

(N^2 linear combinations of Im parts = 0)

there are $\frac{1}{2}N(N-1)$ linearly independent real antisymmetric matrices, and we can write:

$$\theta_{jk} = -i\theta^a(T^a)_{jk}$$

hermitian, antisymmetric, NxN generator matrices of SO(N)

or $R = e^{-i\theta^a T^a}$.

The commutator of two generators is a lin. comb. of generators:

$$[T^a, T^b] = if^{abc}T^c$$

we choose normalization: $\mathrm{Tr}(T^aT^b) = 2\delta^{ab}$

$$f^{abd} = -\tfrac{1}{2}i\,\mathrm{Tr}\left([T^a, T^b]T^d\right)$$

structure constants of the SO(N) group

e.g. SO(3):

$$(T^a)_{ij} = -i\varepsilon^{aij}$$

$$[T^a, T^b] = i\varepsilon^{abc}T^c$$

Levi-Civita symbol

$$\varepsilon^{123} = +1$$

consider now a theory of N complex scalar fields:

$$\mathcal{L} = -\partial^\mu \varphi_i^\dagger \partial_\mu \varphi_i - m^2 \varphi_i^\dagger \varphi_i - \tfrac{1}{4}\lambda(\varphi_i^\dagger \varphi_i)^2$$

the lagrangian is clearly invariant under the U(N) transformation:

$$\varphi_i(x) \to U_{ij}\varphi_j(x)$$

group of unitary NxN matrices

$$U^\dagger = U^{-1}$$

we can always write $U_{ij} = e^{-i\theta}\widetilde{U}_{ij}$ so that $\det \widetilde{U} = +1$.

SU(N) - group of special unitary NxN matrices

U(N) = U(1) x SU(N)

actually, the lagrangian has larger symmetry, SO(2N):

$$\varphi_j = (\varphi_{j1} + i\varphi_{j2})/\sqrt{2}$$

$$\varphi_j^\dagger \varphi_j = \tfrac{1}{2}(\varphi_{11}^2 + \varphi_{12}^2 + \ldots + \varphi_{N1}^2 + \varphi_{N2}^2)$$

infinitesimal SU(N) transformation:

hermitian $U^\dagger = U^{-1}$

$$\tilde{U}_{ij} = \delta_{ij} - i\theta^a (T^a)_{ij} + O(\theta^2)$$

traceless

$$\det \tilde{U} = +1$$

$$\ln \det A = \operatorname{Tr} \ln A$$

or $\tilde{U} = e^{-i\theta^a T^a}$.

there are N^2-1 linearly independent traceless hermitian matrices:

$$[T^a, T^b] = i f^{abc} T^c$$

$$\operatorname{Tr}(T^a T^b) = \frac{1}{2}\delta^{ab}$$

e.g. SU(2) - 3 Pauli matrices

SU(3) - 8 Gell-Mann matrices

the structure coefficients are $f^{abc} = 2\varepsilon^{abc}$, the same as for SO(3)

Nonabelian gauge theory

based on S-69

Consider a theory of N scalar or spinor fields that is invariant under:

$$\phi_i(x) \to U_{ij}\phi_j(x)$$

for SU(N): a special unitary NxN matrix
for SO(N): a special orthogonal NxN matrix

In the case of U(1) we could promote the symmetry to local symmetry but we had to include a gauge field $A_\mu(x)$ and promote ordinary derivative to covariant derivative:

$$\phi(x) \to U(x)\phi(x) \qquad U(x) = \exp[-ie\Gamma(x)] \qquad D_\mu = \partial_\mu - ieA_\mu$$

$$D_\mu \to U(x)D_\mu U^\dagger(x)$$

then the kinetic terms and mass terms: $-(D_\mu\varphi)^\dagger D^\mu\varphi$, $m^2\varphi^\dagger\varphi$, $i\overline{\Psi}\not{D}\Psi$ and $m\overline{\Psi}\Psi$, are gauge invariant. The transformation of covariant derivative in general implies that the gauge field transforms as:

$$A_\mu(x) \to U(x)A_\mu(x)U^\dagger(x) + \frac{i}{e}U(x)\partial_\mu U^\dagger(x)$$

for U(1): $A_\mu(x) \to A_\mu(x) - \partial_\mu\Gamma(x)$

Now we can easily generalize this construction for SU(N) or SO(N):

an infinitesimal SU(N) transformation:

from 1 to N^2-1

from 1 to N

$$U_{jk}(x) = \delta_{jk} - ig\theta^a(x)(T^a)_{jk} + O(\theta^2)$$

gauge coupling constant

generator matrices
(hermitian and traceless):

$$[T^a, T^b] = if^{abc}T^c$$

$$\mathrm{Tr}(T^a T^b) = \tfrac{1}{2}\delta^{ab}$$

structure constants
(completely antisymmetric)

the SU(N) gauge field is a traceless hermitian NxN matrix transforming as:

$$A_\mu(x) \to U(x)A_\mu(x)U^\dagger(x) + \frac{i}{g}U(x)\partial_\mu U^\dagger(x)$$

$$U(x) = \exp[-ig\Gamma^a(x)T^a]$$

the covariant derivative is:

$$D_\mu = \partial_\mu - igA_\mu(x)$$

NxN identity matrix

or acting on a field:

$$(D_\mu \phi)_j(x) = \partial_\mu \phi_j(x) - igA_\mu(x)_{jk}\phi_k(x)$$

using covariant derivative we get a gauge invariant lagrangian

We define the field strength (kinetic term for the gauge field) as:

$$F_{\mu\nu}(x) \equiv \frac{i}{g}[D_\mu, D_\nu]$$

$$= \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu]$$

a new term

it transforms as:

$$D_\mu \to U(x) D_\mu U^\dagger(x)$$

$$F_{\mu\nu}(x) \to U(x) F_{\mu\nu}(x) U^\dagger(x)$$

not gauge invariant separately

and so the gauge invariant kinetic term can be written as:

$$\mathcal{L}_{\text{kin}} = -\frac{1}{2}\text{Tr}(F^{\mu\nu}F_{\mu\nu})$$

we can expand the gauge field in terms of the generator matrices:

$$A_\mu(x) = A^a_\mu(x) T^a$$

that can be inverted:

$$\mathrm{Tr}(T^a T^b) = \frac{1}{2}\delta^{ab}$$

$$A^a_\mu(x) = 2\,\mathrm{Tr}\,A_\mu(x) T^a$$

similarly:

$$F_{\mu\nu}(x) = F^a_{\mu\nu} T^a \;,$$

$$F^a_{\mu\nu}(x) = 2\,\mathrm{Tr}\,F_{\mu\nu} T^a \;.$$

$$F_{\mu\nu}(x) \equiv \frac{i}{g}[D_\mu, D_\nu]$$
$$= \partial_\mu A_\nu - \partial_\nu A_\mu - ig[A_\mu, A_\nu] \longrightarrow F^c_{\mu\nu} T^c = (\partial_\mu A^c_\nu - \partial_\nu A^c_\mu) T^c - ig A^a_\mu A^b_\nu [T^a, T^b]$$

$$[T^a, T^b] = i f^{abc} T^c \longrightarrow = (\partial_\mu A^c_\nu - \partial_\nu A^c_\mu + g f^{abc} A^a_\mu A^b_\nu) T^c \;.$$

thus we have:

$$F^c_{\mu\nu} = \partial_\mu A^c_\nu - \partial_\nu A^c_\mu + g f^{abc} A^a_\mu A^b_\nu$$

$$\mathcal{L}_{\text{kin}} = -\tfrac{1}{2}\text{Tr}(F^{\mu\nu}F_{\mu\nu})$$

$$F_{\mu\nu}(x) = F^a_{\mu\nu}T^a$$

$$\text{Tr}(T^aT^b) = \tfrac{1}{2}\delta^{ab}$$

the kinetic term can be also written as:

$$\mathcal{L}_{\text{kin}} = -\tfrac{1}{4}F^{c\mu\nu}F^c_{\mu\nu}$$

$$F^c_{\mu\nu} = \partial_\mu A^c_\nu - \partial_\nu A^c_\mu + g f^{abc} A^a_\mu A^b_\nu$$

Example, quantum chromodynamics - QCD:

$$\mathcal{L} = i\overline{\Psi}_{iI} \not{D}_{ij} \Psi_{jI} - m_I \overline{\Psi}_I \Psi_I - \tfrac{1}{2}\text{Tr}(F^{\mu\nu}F_{\mu\nu})$$

flavor index: up, down, strange, charm, top, bottom

$$(D_\mu)_{ij} = \delta_{ij}\partial_\mu - igA^a_\mu T^a_{ij}$$

1, ... , 8 gluons (massles spin 1 particles)

color index: 1,2, 3

in general, scalar and spinor fields can be in different representations of the group, T^a_{R} ; gauge invariance requires that the gauge fields transform independently of the representation.