QCD as a Gauge Theory

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May 30 - June 16, 2023
Newport News, VA
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The Gauge Principle

Historical roots of gauge invariance

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CHINESE JOURNAL OF PHYSICS VOL. 30, NO. 7 DECEMBER 1992

Gauge Theory-Past, Present, and Future?

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(Received November 3, 1992)

The Gauge Principle

Historical roots of gauge invariance

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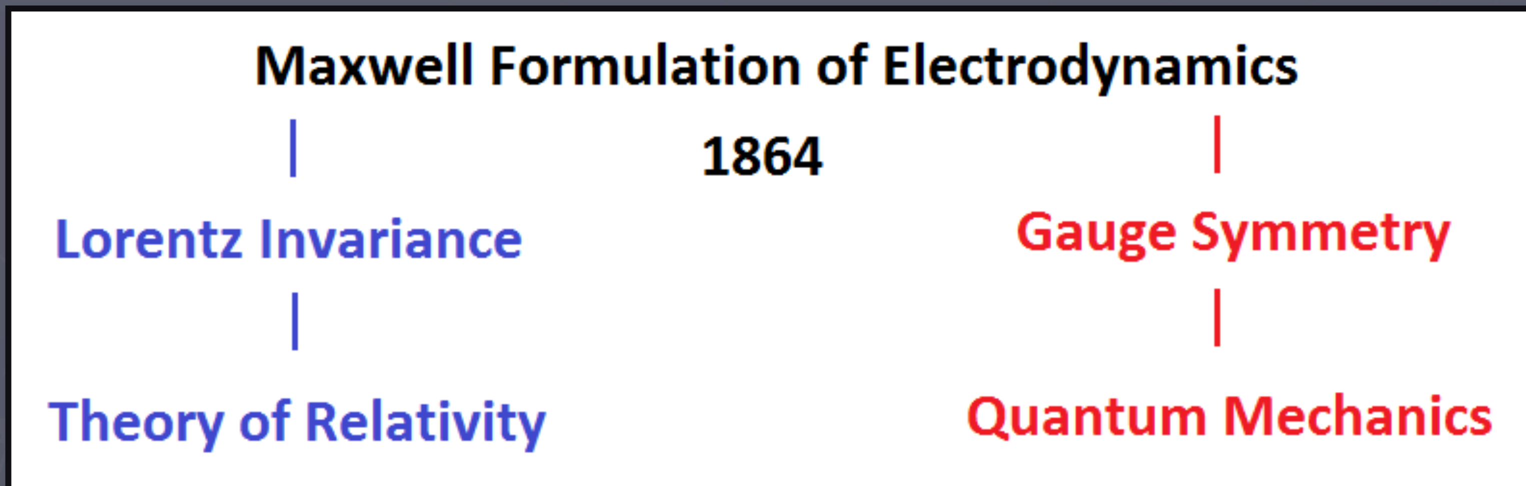
ABSTRACT
Gauge invariance is the basis of the modern theory of electroweak and strong interactions (the so called Standard Model). The roots of gauge invariance go back to the year 1820 when electromagnetism was discovered and the first electrodynamic theory was proposed. Subsequent developments led to the discovery that different forms of the vector potential result in the same observable forces. The partial arbitrariness of the vector potential **A** brought forth various restrictions on it. $\nabla \cdot \mathbf{A} = 0$ was proposed by J. C. Maxwell; $\partial_\mu A^\mu = 0$ was proposed L. V. Lorenz in the middle of 1860's . In most of the modern texts the latter condition is attributed to H. A. Lorentz, who half a century later was the key figure in the final formulation of classical electrodynamics. In 1926 a relativistic equation for charged spinless particles was formulated by E. Schrödinger, O. Klein, and V. Fock. The latter discovered that this equation is invariant with respect to multiplication of the wave function by a phase factor $exp(ie\chi/\hbar c)$ with the accompanying additions to the scalar potential of $-\partial\chi/c\partial t$ and to the vector potential of $\nabla\chi$. In 1929 H. Weyl proclaimed this invariance as a general principle and called it Eichinvarianz in German and gauge invariance in English. The present era of non-abelian gauge theories started in 1954 with the paper by C. N. Yang and R. L. Mills.

The gauge principle

Maxwell's formulation of electrodynamics was perhaps the first field theory and a gauge theory in physics.



In 1929, Weyl showed that electrodynamics was invariant under the gauge transformation of the gauge field and the wave-function of the charged field.

In 1954, Yang and Mills studied the gauge principle in non-abelian field theories.

The gauge principle

U(1) and SU(2) local gauge invariance:

On the invariant form of the wave equation and the equations of motion for a charged point mass.

V. Fock. 1926.

Z.Phys. 39 (1926) 226-232, Surveys High Energ.Phys. 5 (1986) 245-251

PHYSICAL REVIEW VOLUME 96, NUMBER 1 OCTOBER 1, 1954

Conservation of Isotopic Spin and Isotopic Gauge Invariance*

SU(2) local

C. N. YANG † AND R. L. MILLS
Brookhaven National Laboratory, Upton, New York

(Received June 28, 1954)

It is pointed out that the usual principle of invariance under isotopic spin rotation is not consistant with the concept of localized fields. The possibility is explored of having invariance under local isotopic spin rotations. This leads to formulating a principle of isotopic gauge invariance and the existence of a **b** field which has the same relation to the isotopic spin that the electromagnetic field has to the electric charge. The **b** field satisfies nonlinear differential equations. The quanta of the **b** field are particles with spin unity, isotopic spin unity, and electric charge $\pm e$ or zero.

Quantum electrodynamics

The free Dirac Lagrangian is:

$$\boxed{\mathcal{L} = \overline{\psi}\,[i\gamma^\mu\,\partial_\mu - m]\,\psi}$$

It is invariant under abelian global U(1) gauge transformation:

$$\boxed{\begin{aligned}\psi(x) &\rightarrow \psi^{'}(x) = \exp(-i\alpha)\psi(x)\\ \overline{\psi}(x) &\rightarrow \overline{\psi'}(x) = \overline{\psi}(x)\exp(+i\alpha)\end{aligned}}$$

where α is a real constant. Find conserved current.

We demand generalizing it into a local gauge symmetry by demanding the invariance even if $\alpha=\alpha(x)$. The mass term is still invariant but not the kinetic energy term.

Quantum electrodynamics

$$\overline{\psi}(x)\partial_\mu\psi(x) \to \overline{\psi}(x)\partial_\mu\psi(x) - i\overline{\psi}(x)(\partial_\mu\alpha(x))\psi(x)$$

The invariance can be reinstated by introducing a gauge field and a gauge covariant derivative:

$$D_\mu\psi(x) = \left(\partial_\mu + ieA_\mu(x)\right)\psi(x)$$

e is the fermion coupling to the gauge field.

The starting point now is:

$$\mathcal{L} = \overline{\psi}\left[i\gamma^\mu D_\mu - m\right]\psi$$

This is invariant under simultaneous gauge transformations of the fermion and the gauge fields:

Quantum electrodynamics

$$\psi(x) \rightarrow \psi'(x) = \exp(-i\alpha(x))\psi(x)$$

$$\overline{\psi}(x) \rightarrow \overline{\psi}'(x) = \overline{\psi}(x)\exp(+i\alpha(x))$$

$$A_\mu(x) \rightarrow A'_\mu(x) = A_\mu(x) + \frac{1}{e}\partial_\mu\alpha(x)$$

This invariance is achieved because:

$$D_\mu\psi(x) \rightarrow [D_\mu\psi(x)]' = \exp(-i\alpha(x))\; D_\mu\psi(x)$$

and hence:

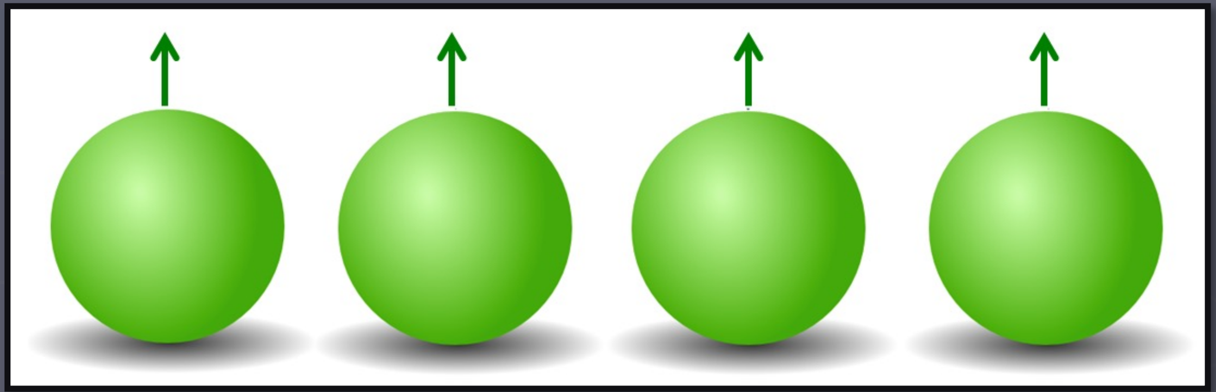
$$\overline{\psi}(x)D_\mu\psi(x) \rightarrow \overline{\psi}'(x)[D_\mu\psi(x)]' = \overline{\psi}(x)D_\mu\psi(x)$$

The gauge field A = vector potential of Maxwell equations.

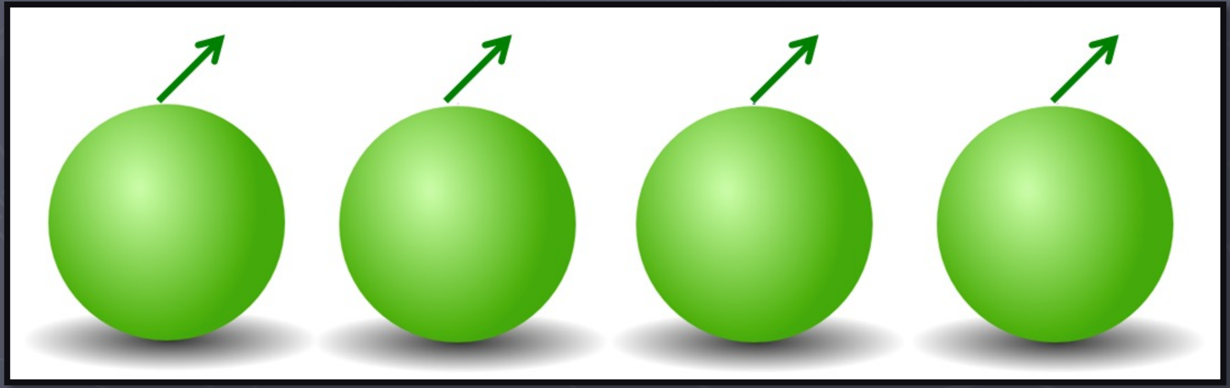
$$\boxed{\mathcal{L} = \overline{\psi}\left[i\gamma^\mu D_\mu - m\right]\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}} \qquad F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$$

Quantum electrodynamics

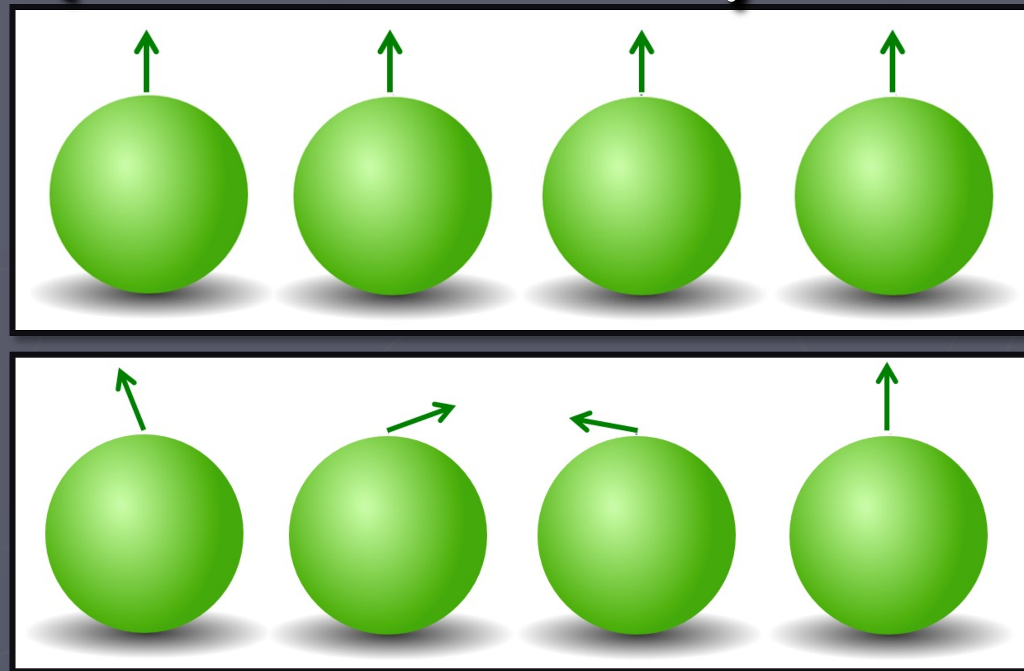
The wave-function of an electron carries a phase with it.



In global gauge transformation, this phase is changed by the same amount at every space-time point and the Lagrangian is invariant.

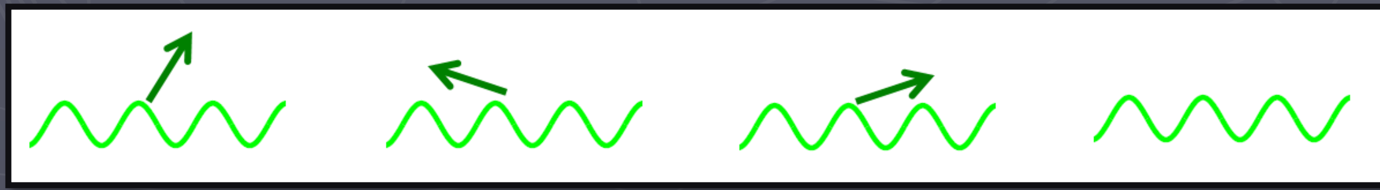


Quantum electrodynamics



In local gauge transformation, this phase is changed by arbitrarily different amounts at every space-time point.

Gauge field with compensating transformation is introduced to obtain local gauge invariance.



SU(N) gauge theory

The **free field** is described by the **Dirac Lagrangian**:

$$\boxed{\mathcal{L} = \overline{\psi}\left[i\gamma^\mu \partial_\mu - m\right]\psi}$$

$$\boxed{\psi(x) = \begin{pmatrix} \psi_1(x) \\ \psi_2(x) \\ \cdots \\ \psi_n(x) \end{pmatrix}}$$

It is invariant under the **SU(N)** transformations, with θ^a real constants. **Find conserved current.**

$$\psi(x) \to \psi'(x) = \exp(-i\frac{\tau^a\theta^a}{2})\psi(x)$$

$$\overline{\psi}(x) \to \overline{\psi}'(x) = \overline{\psi}(x)\exp(+i\frac{\tau^a\theta^a}{2})$$

SU(N) gauge theory

$$\psi(x) \to \psi'(x) = \exp(-i\frac{\tau^a\theta^a}{2})\psi(x)$$
$$\overline{\psi}(x) \to \overline{\psi}'(x) = \overline{\psi}(x)\exp(+i\frac{\tau^a\theta^a}{2})$$

Here $\frac{1}{2}\tau^a(a=1,2,3, \ldots , N^2-1)$ are n x n matrices in n-dimensional representation of the SU(N) group.

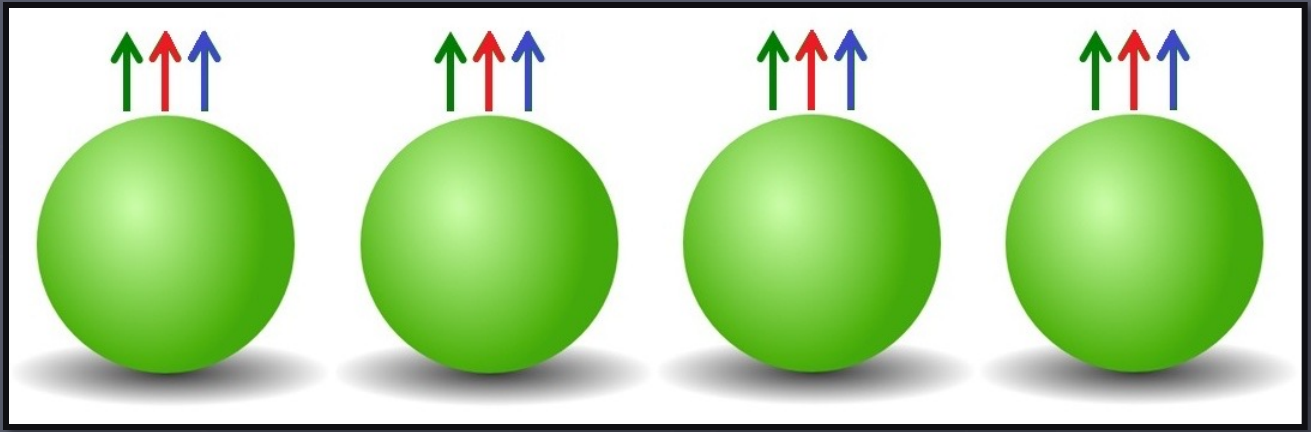
These matrices obey the following Lie algebra:

$$\left[\frac{\tau^a}{2}, \frac{\tau^b}{2}\right] = i f_{abc}\frac{\tau^c}{2}$$

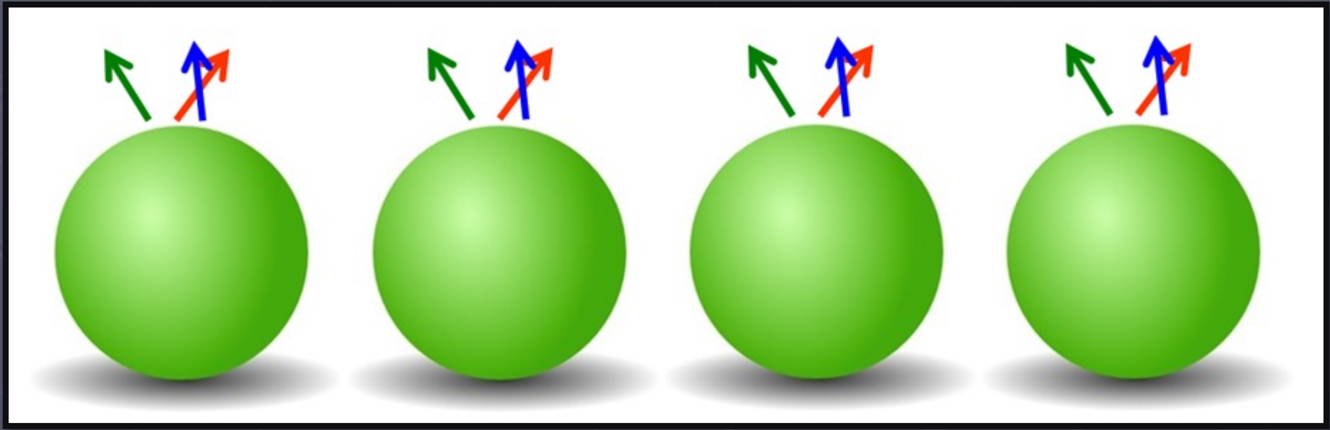
The structure functions f_{abc} uniquely determine the group SU(N) in operation.

SU(3) gauge theory

There are different color phases associated with quarks



In global gauge transformations, they are transformed independently of x and hence the Lagrangian is invariant.



SU(N) gauge theory

We want to enforce the invariance of the Lagrangian under local gauge transformations:

$$\psi(x) \to \psi'(x) = U(\theta^a(x))\psi(x) = \exp(-i\frac{\tau^a\theta^a(x)}{2})\psi(x)$$

$$\overline{\psi}(x) \to \overline{\psi}'(x) = \overline{\psi}(x)U^{-1}(\theta^a(x)) = \overline{\psi}(x)\exp(+i\frac{\tau^a\theta^a(x)}{2})$$

We now need (N^2-1) gauge fields & the covariant derivative is:

$$\mathcal{L} = \overline{\psi}\left[i\gamma^\mu D_\mu - m\right]\psi$$

$$D_\mu\psi(x) = \left(\partial_\mu + ig\frac{\tau^a A_\mu^a(x)}{2}\right)\psi(x)$$

SU(N) gauge theory

The gauge fields must transform as:

$$\frac{\tau^a A_\mu^a(x)}{2} \rightarrow$$

$$\frac{\tau^a A_\mu^a(x)'}{2} = U(\theta(x))\left(\frac{\tau^a A_\mu^a(x)}{2}\right)U^{-1}(\theta(x)) + \frac{i}{g}\left[\partial_\mu U(\theta(x))\right]U^{-1}(\theta(x))$$

so that:

$$D_\mu \psi(x) \rightarrow [D_\mu \psi(x)]' = \exp(-i\frac{\tau^a \theta^a(x)}{2})\,[D_\mu \psi(x)]$$

which ensures the invariance of the Lagrangian.

SU(N) gauge theory

$$\frac{\tau^a A^a_\mu(x)}{2} \to \frac{\tau^a A^a_\mu(x)'}{2} = U(\theta(x))\left(\frac{\tau^a A^a_\mu(x)}{2}\right)U^{-1}(\theta(x)) + \frac{i}{g}\,[\partial_\mu U(\theta(x))]U^{-1}(\theta(x))$$

proof:

$$\left(\partial_\mu + ig\frac{\tau^a A^a_\mu(x)'}{2}\right)(U(\theta)\psi) = U(\theta)\,\partial_\mu\psi + (\partial_\mu U(\theta))\,\psi$$

$$+\, ig\; U(\theta(x))\left(\frac{\tau^a A^a_\mu(x)}{2}\right)U^{-1}(\theta(x))\,(U(\theta)\psi)$$

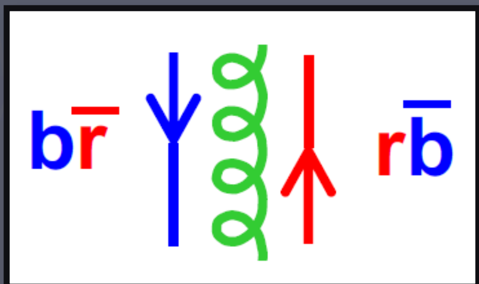
$$-\; [\partial_\mu U(\theta(x))]\,U^{-1}(\theta(x))\,(U(\theta)\psi)$$

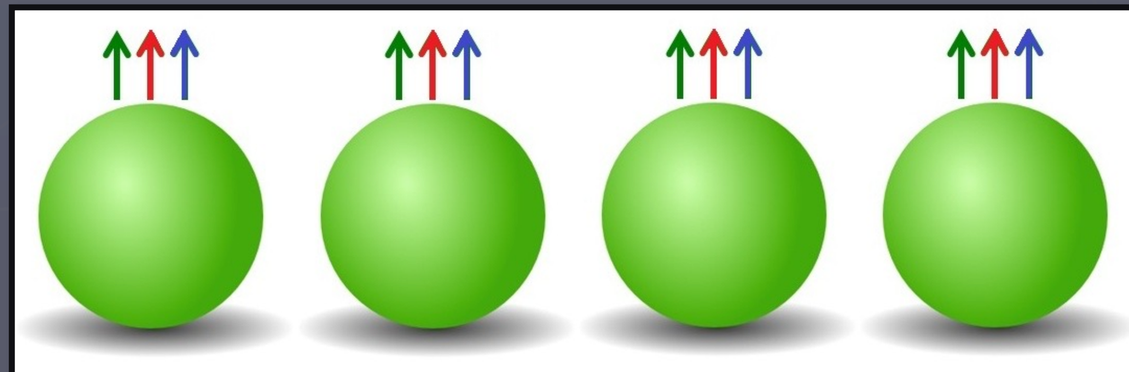
$$=\; U(\theta)\,\partial_\mu\psi + ig\; U(\theta(x))\left(\frac{\tau^a A^a_\mu(x)}{2}\right)\psi$$

$$=\; U(\theta)\left(\partial_\mu + ig\frac{\tau^a A^a_\mu(x)}{2}\right)\psi$$

SU(3) gauge theory

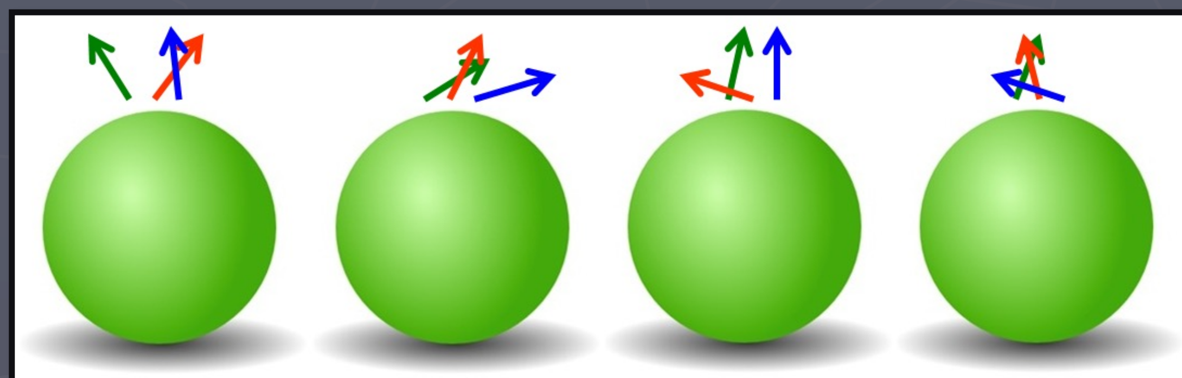
Initially:





Compensating fields restore gauge invariance

Local Gauge transformation:



$$r\bar{g},\ r\bar{b},\ g\bar{r},\ g\bar{b},\ b\bar{r},\ b\bar{g},\ \frac{1}{\sqrt{2}}(r\bar{r}-g\bar{g}),\ \frac{1}{\sqrt{6}}(r\bar{r}+g\bar{g}-2b\bar{b})$$

Recalling quantum electrodynamics

A candidate for the dynamics of the gauge field A in QED is the vector potential of the Maxwell's equations:

$$\mathcal{L} = \overline{\psi}\left[i\gamma^\mu D_\mu - m\right]\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

$$F_{\mu\nu}(x) = \partial_\mu A_\nu(x) - \partial_\nu A_\mu(x)$$

$F_{\mu\nu}$ is invariant under the gauge transformations:

$$\begin{aligned} A_\mu(x) \rightarrow A'_\mu(x) &= A_\mu(x) + \frac{i}{e}(\partial_\mu U(\alpha))U^{-1}(\alpha) \\ &= A_\mu(x) + \frac{1}{e}\partial_\mu \alpha(x) \end{aligned}$$

Recalling quantum electrodynamics

We can easily check the following:

$$\boxed{\left(D_\mu D_\nu - D_\nu D_\mu\right)\ \psi = ieF_{\mu\nu}\ \psi}$$

It goes beyond U(1):

$$\begin{aligned}
& \left(D_\mu D_\nu - D_\nu D_\mu\right)\ \psi \\
= \quad & \left(\left[\partial_\mu + ieA_\mu\right]\left[\partial_\nu + ieA_\nu\right] - \left[\partial_\nu + ieA_\nu\right]\left[\partial_\mu + ieA_\mu\right]\right)\psi \\
= \quad & \left(\cancel{\partial_\mu\partial_\nu} + ie\partial_\mu A_\nu + \cancel{ieA_\nu\partial_\mu} + \cancel{ieA_\mu\partial_\nu} - e^2\cancel{A_\mu A_\nu}\right)\ \psi \\
- \quad & \left(\cancel{\partial_\nu\partial_\mu} + ie\partial_\nu A_\mu + \cancel{ieA_\mu\partial_\nu} + \cancel{ieA_\nu\partial_\mu} - e^2\cancel{A_\nu A_\mu}\right)\ \psi \\
= \quad & ie\left(\partial_\mu A_\nu - \partial_\nu A_\mu\right)\psi \\
= \quad & ieF_{\mu\nu}\ \psi
\end{aligned}$$

Recalling quantum electrodynamics

We now want to show the following:

$$\boxed{[(D_\mu D_\nu - D_\nu D_\mu)\psi]' = \exp\left(-i\alpha(x)\right)(D_\mu D_\nu - D_\nu D_\mu)\psi}$$

We start from:

$$\begin{aligned}
(D_\mu D_\nu \psi)' &= D'_\mu D'_\nu \psi' \\
&= D'_\mu \left[D'_\nu \exp\left(-i\alpha(x)\right)\psi\right] \\
&= D'_\mu \left[\exp\left(-i\alpha(x)\right) D_\nu \psi\right] \\
&= \exp\left(-i\alpha(x)\right)\left[D_\mu D_\nu \psi\right]
\end{aligned}$$

This proves the above identity.

Recalling quantum electrodynamics

The two identities are:

$$\boxed{\left(D_\mu D_\nu - D_\nu D_\mu\right)\ \psi = ieF_{\mu\nu}\ \psi}$$

$$\boxed{[(D_\mu D_\nu - D_\nu D_\mu)\psi]' = \exp\left(-i\alpha(x)\right)\left(D_\mu D_\nu - D_\nu D_\mu\right)\psi}$$

$$\boxed{[(D_\mu D_\nu - D_\nu D_\mu)\psi]' = ieF'_{\mu\nu}\ \psi'}$$

Combining these expressions:

$$\boxed{F'_{\mu\nu}\psi' = \exp(-i\alpha(x))(F_{\mu\nu}\psi)}$$

Recalling quantum electrodynamics

$$F'_{\mu\nu}\psi' = \exp(-i\alpha(x))(F_{\mu\nu}\psi)$$

As ψ is an arbitrary spinor and $\exp(-i\alpha(x))$ & $F_{\mu\nu}$ commute, the later being a function, we obtain the gauge invariance of the field tensor:

$$F'_{\mu\nu} = F_{\mu\nu}$$

So we construct the Lorentz invariant quantity: $F_{\mu\nu}F^{\mu\nu}$

And the Lagrangian is:

$$\mathcal{L} = \overline{\psi}\left[i\gamma^{\mu}D_{\mu} - m\right]\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu}$$

Dynamics of the field tensor

How do we define $F_{\mu\nu}$ for SU(N)?

The following definition does not work:

$$\boxed{F^a_{\mu\nu}(x) = \partial_\mu A^a_\nu(x) - \partial_\nu A^a_\mu(x)}$$

We can check that with this definition,

$\boxed{F^a_{\mu\nu} F_a^{\mu\nu}}$ is not gauge invariant.

Thus we follow the procedure just outlined. Use the definition of $F_{\mu\nu}$ as follows and deduce what it is:

$$\boxed{(D_\mu D_\nu - D_\nu D_\mu)\,\psi = ig\frac{\tau^a}{2} F^a_{\mu\nu}\psi}$$

Non abelian field tensor

$$
\begin{aligned}
& \left(D_\mu D_\nu - D_\nu D_\mu\right)\,\psi \\
= \; & \left(\left[\partial_\mu + ig\frac{\tau^a}{2}A^a_\mu\right]\left[\partial_\nu + ig\frac{\tau^b}{2}A^b_\nu\right] - \left[\partial_\nu + ig\frac{\tau^b}{2}A^b_\nu\right]\left[\partial_\mu + ig\frac{\tau^a}{2}A^a_\mu\right]\right)\psi \\
= \; & \left(\cancel{\partial_\mu\partial_\nu\,\psi} + \cancel{ig\frac{\tau^a}{2}A^a_\mu\partial_\nu\,\psi} + \cancel{ig\frac{\tau^b}{2}A^b_\nu\partial_\mu\,\psi} + ig\frac{\tau^b}{2}(\partial_\mu A^b_\nu)\,\psi - g^2A^a_\mu A^b_\nu\,\frac{\tau^a}{2}\frac{\tau^b}{2}\,\psi\right) \\
- \; & \left(\cancel{\partial_\nu\partial_\mu\,\psi} + \cancel{ig\frac{\tau^b}{2}A^b_\nu\partial_\mu\,\psi} + \cancel{ig\frac{\tau^a}{2}A^a_\mu\partial_\nu\,\psi} + ig\frac{\tau^a}{2}(\partial_\nu A^a_\mu)\,\psi - g^2A^b_\nu A^a_\mu\,\frac{\tau^b}{2}\frac{\tau^a}{2}\,\psi\right) \\
= \; & ig\left(\frac{\tau^a}{2}\partial_\mu A^a_\nu - \frac{\tau^a}{2}\partial_\nu A^a_\mu + igA^a_\mu A^b_\nu\left[\frac{\tau^a}{2},\frac{\tau^b}{2}\right]\right)\psi \\
= \; & ig\left(\frac{\tau^a}{2}\partial_\mu A^a_\nu - \frac{\tau^a}{2}\partial_\nu A^a_\mu - gf_{abc}A^a_\mu A^b_\nu\frac{\tau^c}{2}\right)\psi \\
= \; & ig\left(\frac{\tau^a}{2}\partial_\mu A^a_\nu - \frac{\tau^a}{2}\partial_\nu A^a_\mu - gf_{abc}A^b_\mu A^c_\nu\frac{\tau^a}{2}\right)\psi \\
= \; & ig\frac{\tau^a}{2}\boxed{\left(\partial_\mu A^a_\nu - \partial_\nu A^a_\mu - gf_{abc}A^b_\mu A^c_\nu\right)}\,\psi = ig\frac{\tau^a}{2}\boxed{F^a_{\mu\nu}}
\end{aligned}
$$

Dynamics of the field tensor

Is $F_{\mu\nu}^a$ gauge invariant? No.

Show:

$$\frac{\tau^a}{2} F'^a_{\mu\nu} = U \frac{\tau^a}{2} F^a_{\mu\nu} U^{-1}$$

Show that $F_{\mu\nu}^a F^{\mu\nu a}$ is gauge invariant.

SU(N) Lagrangian:

$$\mathcal{L} = \overline{\psi} \left[i\gamma^\mu D_\mu - m \right] \psi - \frac{1}{4} F^a_{\mu\nu} F_a^{\mu\nu}$$

$$F^a_{\mu\nu} F_a^{\mu\nu} \sim -g f^{abc} \partial_\mu A^a_\nu A^{b\mu} A^{c\nu} - \frac{g^2}{4} f^{abc} f^{ade} A^b_\mu A^c_\nu A^{d\mu} A^{e\nu}$$

These terms correspond to self coupling of the gauge field, triple and quartic gluon interactions.

The QCD Lagrangian

Thus the QCD Lagrangian can be written as (modulus gauge fixing term and ghosts)

$$\mathcal{L}_{QCD} = -\frac{1}{4} F^a_{\mu\nu} F^{\mu\nu}_a + \sum_{f=1}^{N_f} \overline{\psi}_f \left[i\gamma^\mu D_\mu - m_f \right] \psi_f$$

$$\mathcal{L}_{QCD} = -\frac{1}{2} Tr(F_{\mu\nu} F^{\mu\nu}) + \sum_{f=1}^{N_f} \overline{\psi}_f \left[i\gamma^\mu D_\mu - m_f \right] \psi_f$$

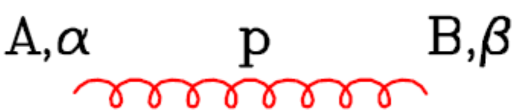
$$F_{\mu\nu}(x) = \frac{\lambda^a}{2} F^a_{\mu\nu}(x), \quad D_\mu q_f(x) = \left(\partial_\mu + ig \frac{\lambda^a A^a_\mu(x)}{2} \right) \psi_f(x)$$

Gluon Fields: Massless, spin 1 bosons, color octet because the group contains eight generators, flavor singlet (i.e., gluons are flavor blind, not distinguishing between flavors of whatever they interact with).

Feynman rules in covariant gauges

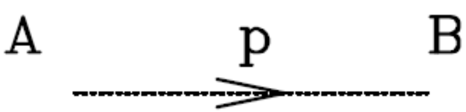
The propagators:

Gluon Propagator:



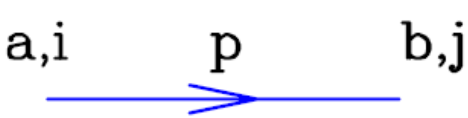
$$\delta^{AB}\left[-g^{\alpha\beta}+(1-\lambda)\,\frac{p^{\alpha}p^{\beta}}{p^{2}+i\epsilon}\right]\frac{i}{p^{2}+i\epsilon}$$

Ghost Propagator:



$$\delta^{AB}\,\frac{i}{(p^{2}+i\epsilon)}$$

Quark Propagator:

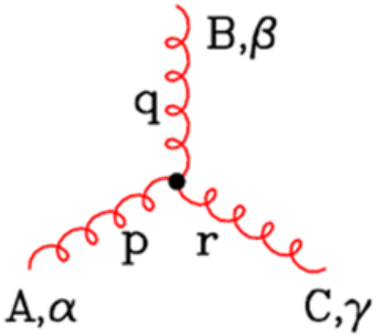


$$\delta^{ab}\,\frac{i}{(\not{p}-m+i\epsilon)_{ji}}$$

Feynman rules in covariant gauges

The gluon self interactions:

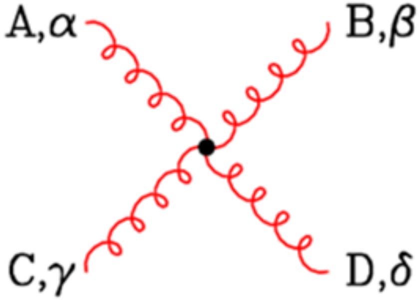
Triple Gluon Vertex:



$$-g\ f^{ABC}[(p-q)^{\gamma}g^{\alpha\beta}+(q-r)^{\alpha}g^{\beta\gamma}+(r-p)^{\beta}g^{\gamma\alpha}]$$

(all momenta incoming)

Four Gluon Vertex:

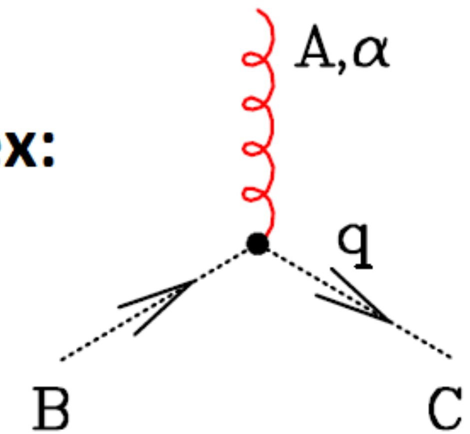


$$-ig^2\ f^{XAC}f^{XBD}\ [g^{\alpha\beta}g^{\gamma\delta}-g^{\alpha\delta}g^{\beta\gamma}]$$
$$-ig^2\ f^{XAD}f^{XBC}\ [g^{\alpha\beta}g^{\gamma\delta}-g^{\alpha\gamma}g^{\beta\delta}]$$
$$-ig^2\ f^{XAB}f^{XCD}\ [g^{\alpha\gamma}g^{\beta\delta}-g^{\alpha\delta}g^{\beta\gamma}]$$

Feynman rules in covariant gauges

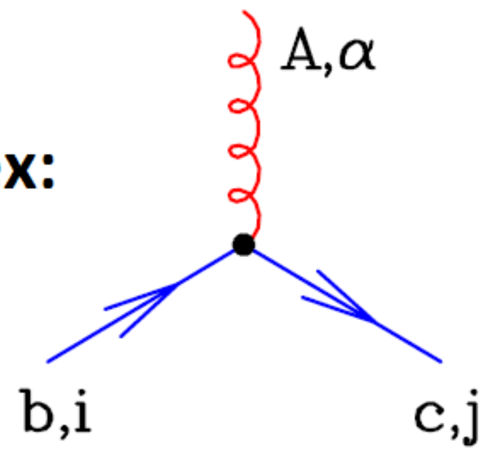
The gluon vertices with ghosts and quarks:

Ghost-Gluon Vertex:



$$g\ \ f^{ABC} q^{\alpha}$$

Quark-Gluon Vertex:



$$-ig\ \ (t^{A})_{cb}\ \ (\gamma^{\alpha})_{ji}$$

Higher orders and infinities

- Electrodynamics was born in 1920s through the works of Born, Heisenberg, Dirac, Pauli, Feynman, Schwinger, etc.
- Works of F. Bloch, A. Nordsieck (1937) and V. Weiskopf (1939) revealed QED calculations worked only at first order in perturbation theory. Infinites emerged at higher orders.
- In 1940s, precise measurements of the levels of Hydrogen atom, Lamb shift and the magnetic moment of the electron exposed discrepancies between experiment and tree level theory.
- Works of Feynman, Schwinger and Tomonaga [1943-1949] introduced the concept of renormalization to solve the problems of QED. Nobel prize of 1965.

Richard Feynman and other physicists gathered in June 1947 at Shelter Island, New York, several months before the meeting at the Pocono Manor Inn in which Feynman introduced his diagrams. Standing are Willis Lamb *(left)* and John Wheeler. Seated, from left to right, are Abraham Pais, Richard Feynman, Hermann Feshbach and Julian Schwinger. (Photograph courtesy of the Emilio Segrè Visual Archives, American Institute of Physics.)

Higher orders and infinities

Lamb Shift:

According to Dirac and Schrodinger, the atomic states with the same n and j quantum numbers but different l quantum numbers ought to be degenerate.

A famous experiment of Retherford and Lamb in 1947 showed that the states

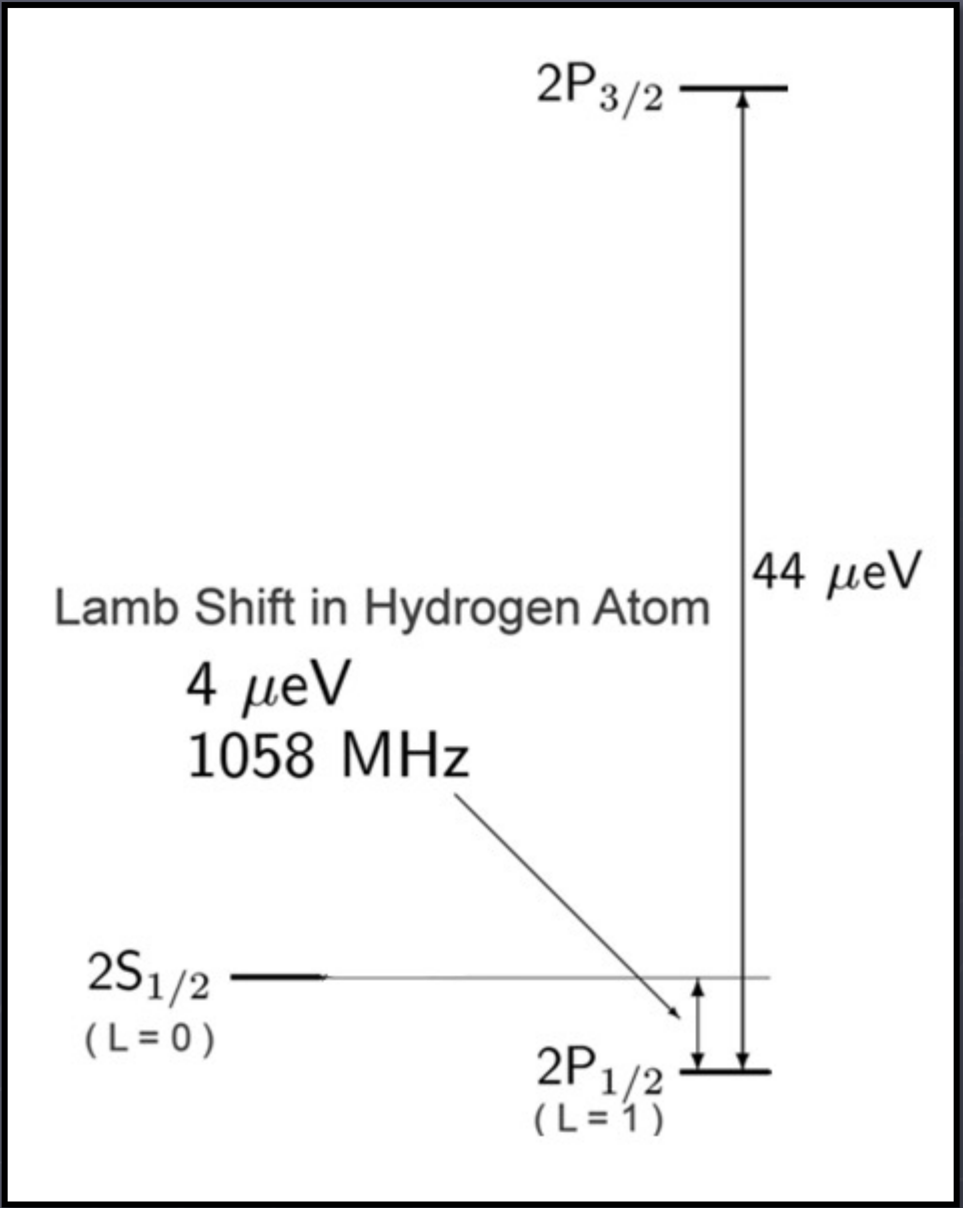
$$\boxed{2s_{1/2}(n=2, l=0, j=1/2) \qquad 2p_{1/2}(n=2, l=1, j=1/2)}$$

of the Hydrogen atom were not degenerate.

This effect is explained by perturbative QED.

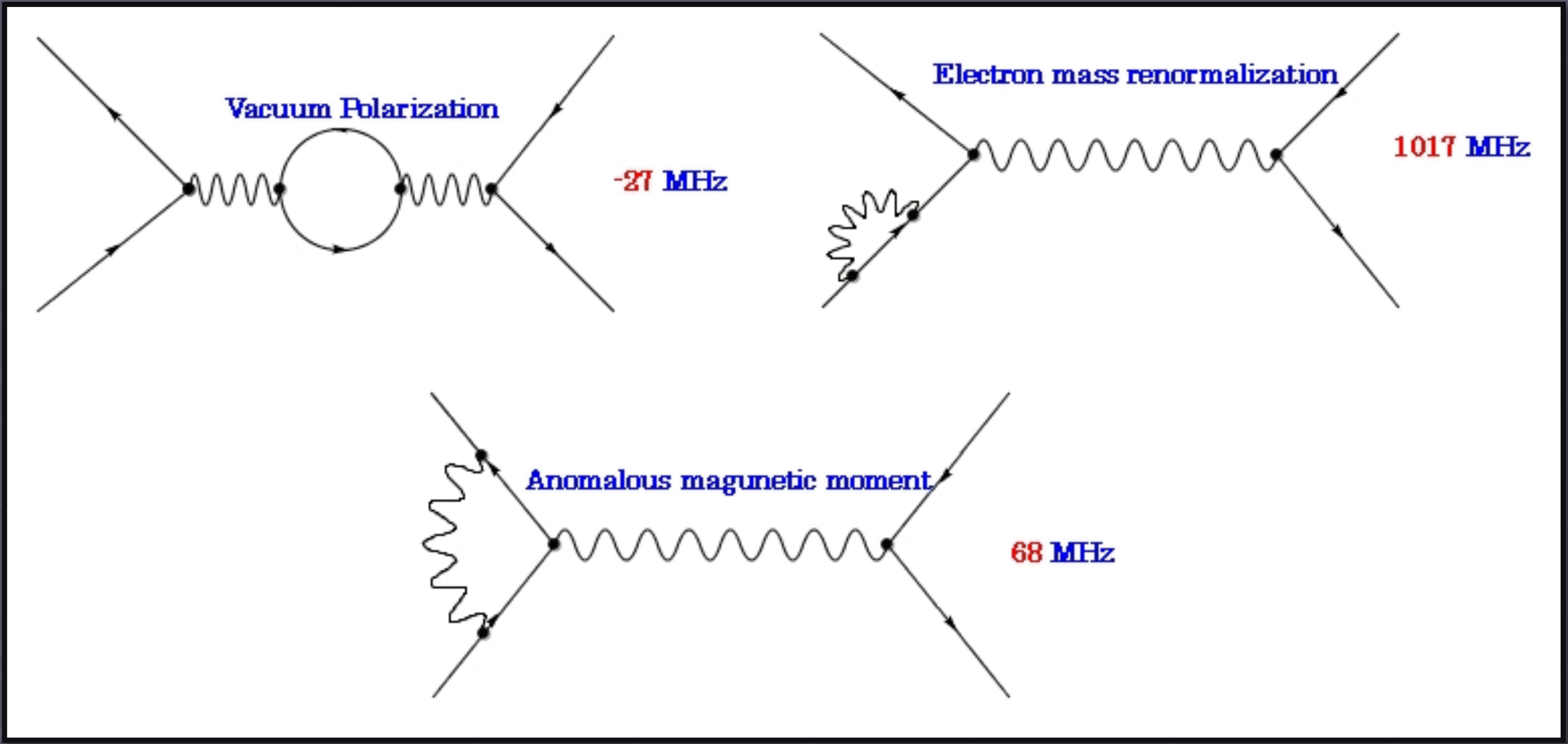
Higher orders and infinities

Lamb Shift:

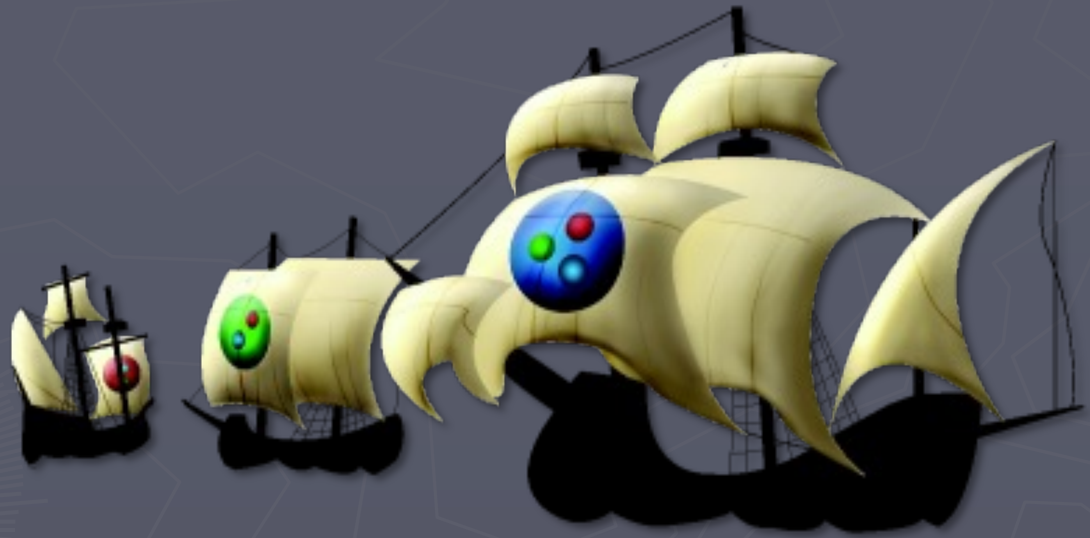


Higher orders and infinities

Lamb Shift:

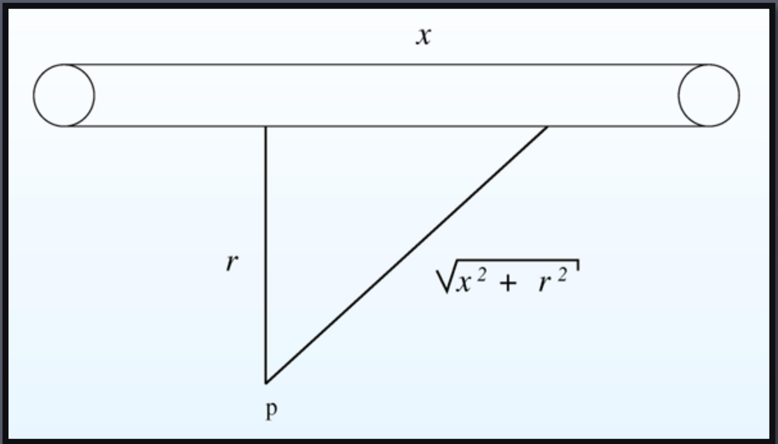


Renormalization in One Hour



Infinities in electromagnetism

Electric potential due to an infinite line charge:



$$V(r) = \frac{\lambda}{4\pi\epsilon_0}\int_{-\infty}^{+\infty}\frac{1}{\sqrt{x^2+r^2}}dx = \frac{\lambda}{4\pi\epsilon_0}U(r)$$

$$U(r) = \int_{-\infty}^{+\infty}\frac{1}{\sqrt{x^2+r^2}}dx$$

This function U does not have dimensions. Such f functions have possibilities:

$$\begin{array}{lcll} f(r) & = & c & \text{a finite constant} \\ f(r) & \Rightarrow & c & \text{an infinite constant} \\ f(r) & = & f(r/L) & \end{array}$$

L is a problem related scale.

$U(r)$ is divergent and is translation invariant

Infinities in electromagnetism

Regularize potential: (i) Cut off method

$$V_{reg}(r/L) = \frac{\lambda}{4\pi\epsilon_0}\int_{-L}^{+L}\frac{1}{\sqrt{x^2+r^2}}dx = \frac{\lambda}{4\pi\epsilon_0}\mathrm{Log}\left[\frac{L+\sqrt{L^2+r^2}}{-L+\sqrt{L^2+r^2}}\right]$$

Electric field intensity:

$$E = -\frac{\partial V_{reg}(r/L)}{\partial r} = \frac{\lambda}{2\pi\epsilon_0 r}\frac{L}{\sqrt{L^2+r^2}} \overset{L\to\infty}{=} \frac{\lambda}{2\pi\epsilon_0 r}$$

Repeat steps for potential difference between two points.

This regularized potential has following properties:.

$V_{reg}(r/L)$ is convergent

$V_{reg}(r/L)$ is NOT translation invariant

Infinities in electromagnetism

Regularize potential: (i) Dimensional Regularization

$$V(r) = \frac{\lambda}{4\pi\epsilon_0}\int_{-\infty}^{+\infty}\frac{1}{\sqrt{x^2+r^2}}dx$$

Evaluate integral in D dimensions:

$$dx \;\to\; \mu^{1-D}\; d^D x$$

$$\mu^{1-D}\; d^D x = \mu^{1-D}\; d\Omega_D\; x^{D-1}\; dx$$
$$d\Omega_D = \frac{2\pi^{D/2}}{\Gamma(D/2)}$$

Substitute:

$$D = 1 - 2\epsilon$$

Infinities in electromagnetism

Regularize potential: (i) Dimensional Regularization

$$V_{reg}(r/\mu) = \frac{\lambda}{4\pi\epsilon_0} \frac{\Gamma\left(\frac{1-D}{2}\right)}{\left(\frac{r}{\mu}\sqrt{\pi}\right)^{1-D}} = \frac{\lambda}{4\pi^{1+\epsilon}\epsilon_0} \left[\left(\frac{\mu}{r}\right)^{2\epsilon} \Gamma(\epsilon)\right]$$

Electric field is:

$$E = -\frac{\partial V_{reg}(r/\mu)}{\partial r} = \frac{\lambda}{2\pi^{1+\epsilon}\epsilon_0 r} \left[\left(\frac{\mu}{r}\right)^{2\epsilon} \epsilon\Gamma(\epsilon)\right] \overset{\epsilon\to 0}{=} \frac{\lambda}{2\pi\epsilon_0 r}$$

This regularized potential has the following properties:

$V_{reg}(r/\mu)$ is convergent
$V_{reg}(r/\mu)$ IS translation invariant

Infinities in electromagnetism

"Renormalization Schemes"

- Expand $V_{reg}(r/\mu)$ in powers of ε:

$$V_{reg}(r/\mu) = \frac{\lambda}{4\pi\epsilon_0}\left[\frac{1}{\epsilon} - \gamma - \log\pi + \log\frac{\mu^2}{r^2} + \mathcal{O}(\epsilon)\right]$$

- In MS (minimal subtraction) scheme (Electric potential?)

$$V_{MS}(r/\mu) = \frac{\lambda}{4\pi\epsilon_0}\left[\quad - \gamma - \log\pi + \log\frac{\mu^2}{r^2} + \mathcal{O}(\epsilon)\right]$$

- In $\overline{\text{MS}}$ (modified minimal subtraction) scheme

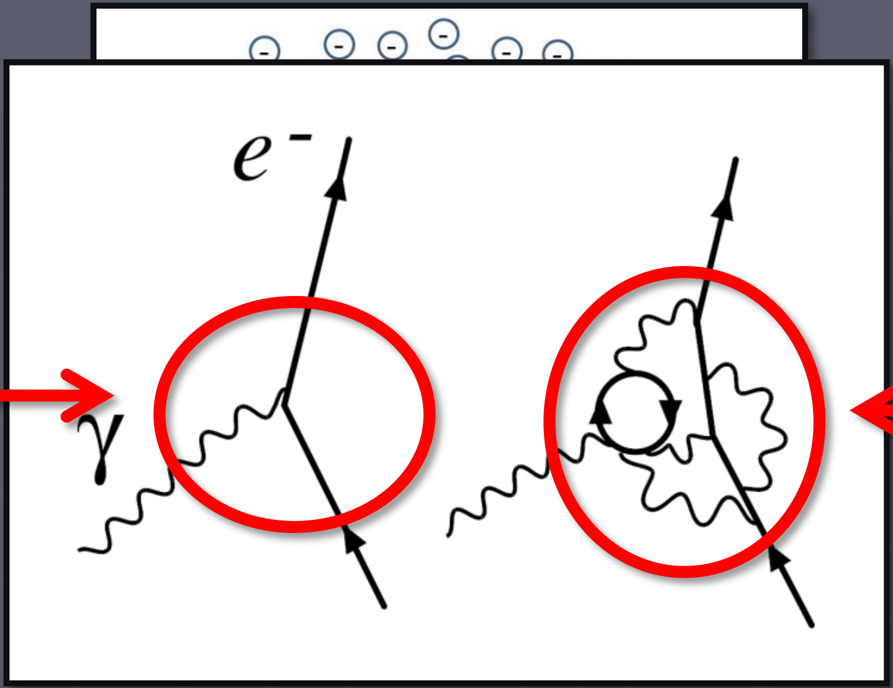
$$V_{\overline{MS}}(r/\mu) = \frac{\lambda}{4\pi\epsilon_0}\left[\qquad \log\frac{\mu^2}{r^2} + \mathcal{O}(\epsilon)\right]$$

- V depends on an additional length scale!

Infinities in QED

- Charge or coupling renormalization:

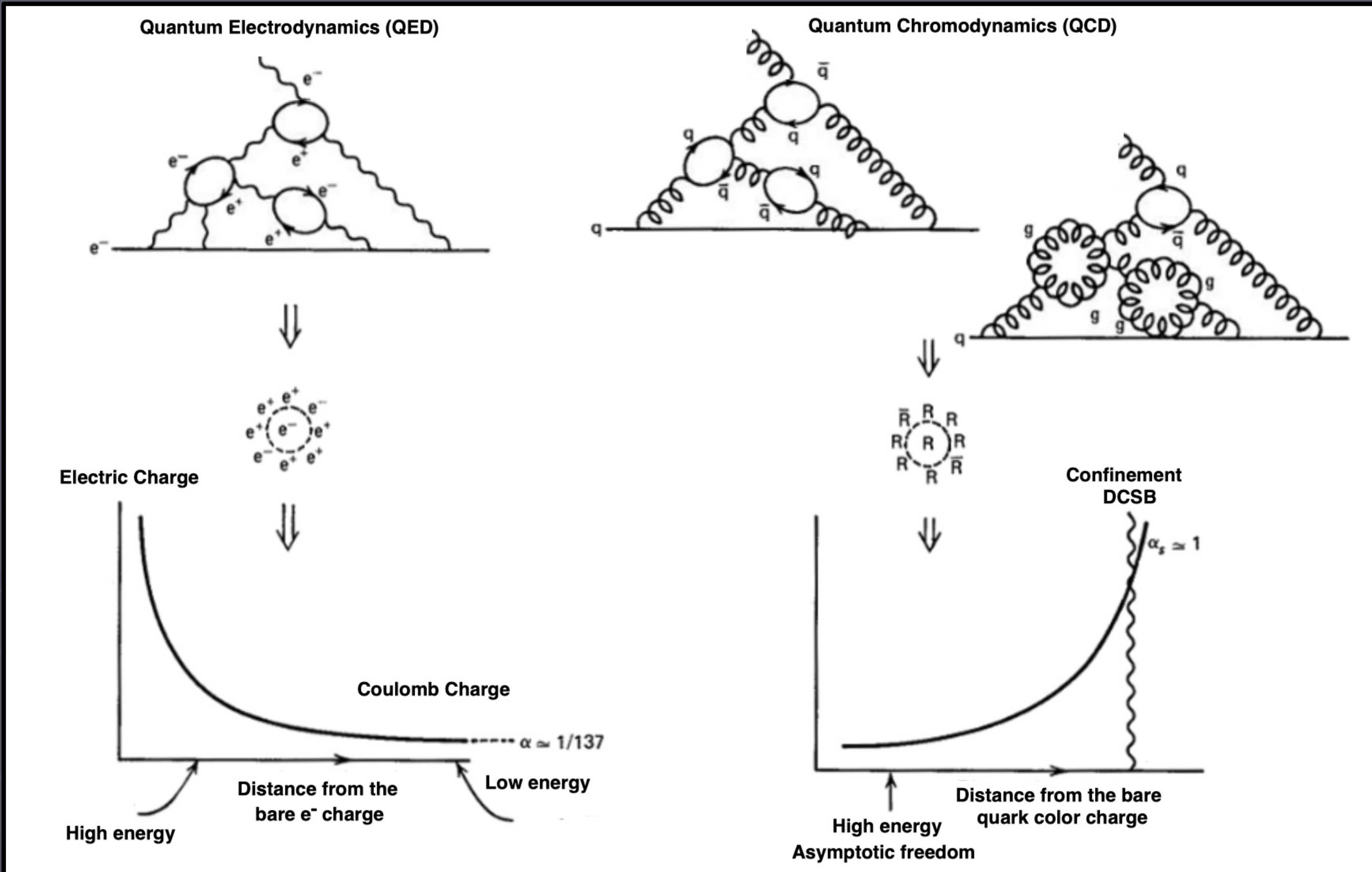
Bare charge



Physical Charge

- Loops introduce divergences which need to be regularized and renormalized.
- Quantities such as coupling run with momenta.

QED vs QCD



Quantum Electrodynamics (QED)
Quantum Chromodynamics (QCD)
Electric Charge
Coulomb Charge
α ≃ 1/137
Distance from the bare e⁻ charge
High energy
Low energy
Confinement
DCSB
$\alpha_s \simeq 1$
Distance from the bare quark color charge
High energy
Asymptotic freedom

Dimensional regularization

One-loop calculations in QED and QCD involve integrals which diverge. An example of a divergent integral is:

$$\mathcal{I}(\Delta) = \int \frac{d^4k}{(2\pi)^4} \frac{1}{(k^2 - \Delta + i\varepsilon)^2} = \infty$$

Introduce some regulator. Regularized diagrams converge.

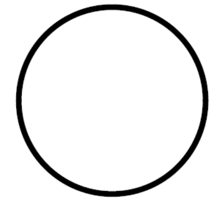
There exist many regularization methods. Cut-off, Pauli-Villars, lattice and dimensional regularization.

The regularization should preserve as many symmetries of the theory as possible, ensuring manipulating regularized Feynman integrals are simple, etc.

The best choice is the dimensional regularization. Nobel Prize 1999 (t'Hooft and Veltman).

Massive vacuum diagram

1-loop massive integral with
dimensional regularization



One-loop massive vacuum diagram

$$\int \frac{d^d k}{D^n} \equiv i\pi^{d/2} m^{d-2n} V(n)\,, \quad D = m^2 - k^2 - i\varepsilon$$

$$V(n) = \frac{\Gamma(-d/2+n)}{\Gamma(n)}$$

$$\text{Thus for } d = 4 - 2\epsilon, \quad V(1) = \frac{\Gamma(1+\epsilon)}{\epsilon(\epsilon-1)}$$

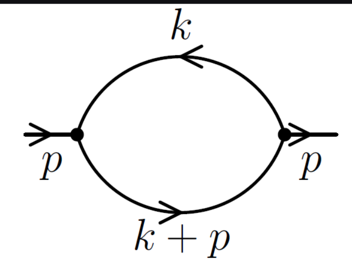
Divergence!

Massless integrals with no external momentum scale

$$\int \frac{d^d k}{(-k^2 - i\epsilon)^n} = 0$$

Massless bubble diagram

The massless propagator diagram with arbitrary powers:



One-loop massless propagator diagram

Expression:

$$\int \frac{d^d k}{D_1^{n_1} D_2^{n_2}} = i\pi^{d/2}(-p^2)^{d/2-n_1-n_2} G(n_1, n_2)\,,$$

$$D_1 = -(k+p)^2\,, \quad D_2 = -k^2$$

$$G(n_1,n_2) = \frac{\Gamma(-d/2+n_1+n_2)}{\Gamma(n_1)\Gamma(n_2)} \int_0^1 x^{d/2-n_2-1}(1-x)^{d/2-n_1-1}dx$$

Finally:

$$G(n_1,n_2) = \frac{\Gamma(-d/2+n_1+n_2)\Gamma(d/2-n_1)\Gamma(d/2-n_2)}{\Gamma(n_1)\Gamma(n_2)\Gamma(d-n_1-n_2)}$$

Massless bubble diagram

Ultraviolet divergences:

The denominator in

$$\int \frac{d^d k}{(k+p)^{2n_1} k^{2n_2}} \propto \frac{\boxed{\Gamma(-d/2+n_1+n_2)}\Gamma(d/2-n_1)\Gamma(d/2-n_2)}{\Gamma(n_1)\Gamma(n_2)\Gamma(d-n_1-n_2)}$$

behaves as $(k^2)^{n_1+n_2}$ at $k \to \infty$. Therefore, the integral diverges if $d \geq 2(n_1+n_2)$. At $d \to 4$ this means $n_1 + n_2 \leq 2$. This ultraviolet divergence shows itself as a $1/\varepsilon$ pole of the first Γ function in the numerator for $n_1 = n_2 = 1$

This Γ function depends on $n_1 + n_2$, i.e., on behaviour of the integrand at $k \to \infty$.

Massless bubble diagram

Infrared divergences:

The integral

$$\int \frac{d^d k}{(k+p)^{2n_1} k^{2n_2}} \propto \frac{\Gamma(-d/2+n_1+n_2)\boxed{\Gamma(d/2-n_1)\Gamma(d/2-n_2)}}{\Gamma(n_1)\Gamma(n_2)\Gamma(d-n_1-n_2)}$$

can also have infrared divergences. Its denominator behaves as $(k^2)^{n_2}$ at $k \to 0$, and the integral diverges in this region if $d \leq 2n_2$. At $d \to 4$ this means $n_2 \geq 2$. This infrared divergence shows itself as a $1/\varepsilon$ pole of the third Γ function in the numerator for $n_2 \geq 2$ (This Γ function depends on n_2, i.e., on the behaviour of the integrand at $k \to 0$). Similarly, the infrared divergence at $k+p \to 0$ appears, at $d \to 4$, as a pole of the second Γ function, if $n_1 \geq 2$.

What next?

- How can we regularize & renormalize QED at one loop?
- How does the coupling run in QED?
- How can we answer the same questions in QCD?
- What are the consequences of running coupling in QCD and its infrared enhancement and what are its implications for hadron physics?