# Digital quantum simulation tutorial: Schwinger model hopping propagator via phase kickback
# Quantum Computing Boot Camp 2023
# Jefferson Lab

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In this tutorial, we learn how to effect time evolution for the hopping term of a link in the Schwinger model. We will regard the Hamiltonian to simulate as $\hat{H} = x\,\sigma_1^- \sigma_2^+ U + \text{H.c.}$, where 1 and 2 refer to the fermions at the left and right sides of the link, and $U = \sum_{E=E_{min}}^{E_{max}-1} |E+1\rangle\langle E|$ is the truncated link operator.

## SVD approach to hopping term

1. Consider first the toy Hamiltonian $H' = x\,\sigma_1^- \sigma_2^+ + \text{H.c.}$ and recall $\sigma^+ = |0\rangle\langle 1|$, $\sigma^- = |1\rangle\langle 0|$. Note how $H'$ is off-diagonal on both of the matter qubits.

   - Draw points for the computational basis states in the $n_1 \times n_2$ plane, and make from them a mathematical graph showing what states are mixed by the application of $H'$. (An edge joining points $|a\rangle$ and $|b\rangle$ means that $\langle a| H' |b\rangle \neq 0$.)
   - Consider the effect of applying the single-qubit gates $X_1$ or $X_2$ on the basis states and how they permute the points in the plane:

   Come up with a gate $V'$ which, when applied to the qubits, has the effect of realigning the edge(s) in the graph of $H'$ parallel to the $n_1$ axis. How would you characterize this graph in terms of being diagonal vs. off-diagonal?

2. Now consider another toy Hamiltonian, $H'' = x\,\sigma_1^- U + \text{H.c.}$, which is off-diagonal on one matter qubit and the entire $E$ register.

   - Draw points for the computational basis states in the $n_1 \times E$ plane, and make from them a mathematical graph showing what states are mixed by the application of $H''$.
   - Let $\lambda^+ = U + |E_{min}\rangle\langle E_{max}|$ and $\lambda^- = U^\dagger + |E_{max}\rangle\langle E_{min}|$ denote cyclic incrementers on the electric field, which are unitary and therefore valid as multiqubit gates. Using $\lambda^-$, devise a gate $V''$ which, when applied to the matter qubit and $E$ register, aligns the graph of $H''$ parallel to the $n_1$ axis. You should find that the circuit commutes with the gate found at the end of step (1).

3. If all was done correctly, you should have found two unitary transformations $V'$ and $V''$, which when applied to the original $H$, transform it as

$$\begin{aligned} V'V''H(V'')^\dagger(V')^\dagger &= x\,\sigma_1^- \; |1\rangle\langle 1|_2 \,(1 - |E_{max}\rangle\langle E_{max}|) + \text{H.c.} \\ &= x\,X_1 \; |1\rangle\langle 1|_2 \,(1 - |E_{max}\rangle\langle E_{max}|). \end{aligned}$$

   The transformed Hamiltonian is off-diagonal on a single qubit alone, and can be fully diagonalized with Hadamards on the 1-mode qubit. Separating out the two terms $x\,Z_1\;|1\rangle\langle 1|_2$ and $-x\,Z_1\;|1\rangle\langle 1|_2\,|E_{max}\rangle\langle E_{max}|$ (they commute), how would you simulate each one? (What controls are needed?)