Digital quantum simulation tutorial: Schwinger model electric propagator via phase kickback
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Jefferson Lab

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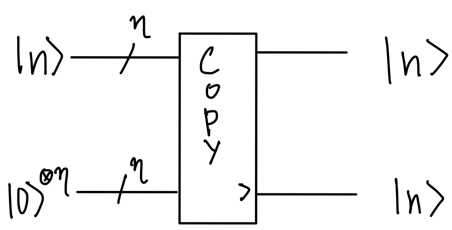
In this tutorial, we learn how to effect time evolution for the $\hat{E}^2$ operator of a link (i.e., circuitize $\exp(-i\phi\hat{E}^2)$) in the Schwinger model without applying phase rotations to the $\hat{E}$ register itself. The electric energy $\hat{E}^2$ is a rather simple function of $\hat{E}$, but this method becomes especially handy when one needs to deal with more complicated functions of $\hat{E}$.

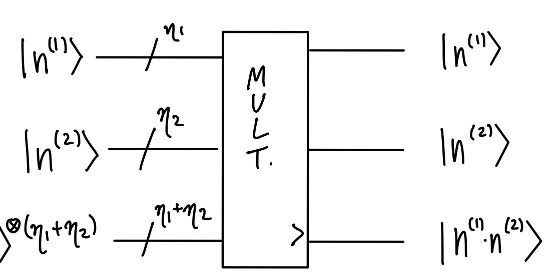
1. Consider a single qubit with computational basis states $|n\rangle$ $(n = 0, 1)$. If $Z|0\rangle = |0\rangle$ and $Z|1\rangle = -|1\rangle$, give an expression for the number operator $\hat{n}$ associated with this qubit ($\hat{n}|n\rangle = n|n\rangle$) in terms of Z.

2. Now let $|n\rangle = \otimes_{j=0}^{\eta-1}|n_j\rangle = |n_{\eta-1}n_{\eta-2}\cdots n_1 n_0\rangle$ be a nonnegative integer expressed in binary using η qubits. $(0 \le n \le 2^\eta - 1)$. If Z_j is the Z operator corresponding to the j^{th} qubit in the register, write the expression for the number operator $\hat{n}$ of the complete register in terms of the Z_j's.

3. Suppose we want to circuitize $\hat{n}$ as a Hamiltonian, i.e., find a circuit to effect $\hat{V} = \exp(-i\phi\hat{n})$ for some angle ϕ. Using the expression for $\hat{n}$ found in (2), express $\hat{V}$ as a function of the Z_j's. "Trotterize" this expression by decomposing $\hat{V}$ into a product of simpler exponentials. What kind of gates does your expression suggest for implementing $\hat{V}$? Sketch the circuit, including rotation angles. (For single-qubit rotation gates, you can use the spin-1/2 definition for rotations about the z-axis: $R_z(\phi) = \exp(-i\phi Z/2)$.)

4. Next, suppose you have access to black-box circuits COPY and MULT(IPLY) (and their inverses) defined by the transformations





   Write down a minimal circuit to effect $|n\rangle|0\rangle^{\otimes\gamma} \to |n^2\rangle \otimes |\text{other output bits}\rangle$. How many auxiliary qubits γ had to be introduced in order to obtain $|n^2\rangle$? And what exactly is stored in the "other output" bits?

5. Using the results from (3), draw or explicitly describe a circuit that will effect

$$|n^2\rangle \to e^{-i\phi n^2}|n^2\rangle .$$

6. Put together the results of (4) and (5) to give a complete circuit that will effect

$$|n\rangle|0\rangle^{\otimes\gamma} \to e^{-i\phi n^2}|n\rangle|0\rangle^{\otimes\gamma} .$$

   Remember to clean up or "uncompute" any intermediate results generated in step (4).

You should be convinced that steps (1)-(6) above have the same effect as having applied $\exp(-i\phi\hat{n}^2)$ to the $|n\rangle$ register itself. However, we instead used an ancilla register and rotations applied to the ancilla register to ultimately "kick back" the desired phase to $|n\rangle$.

7. Connecting to the lattice Schwinger model, one possibility is to take $\hat{E} \equiv \hat{n} - \Lambda$, where the cutoff Λ is related to the electric register size by $2\Lambda = 2^\eta$. Noting that $\hat{E}^2 = \widehat{n^2} - 2\Lambda\hat{n} + \Lambda^2$, explain how to simulate the electric propagator $\exp(-i\phi\hat{E}^2)$ by taking advantage of the phase kickback method.