2023/06/26
Jesse Stryker

Reference material for AM lecture on

Introduction to lattice gauge theories and Hamiltonian formulations

(pp. 5-24 of thesis; full document at https://inspirehep.net/literature/1829575)

1.1 Classical gauge fields in the continuum

Here we summarize the key features of continuum theories, especially as they pertains to the features lattice formulations must reproduce. This section also serves to set conventions. The development follows Ref. [36] and Ref. [37].

The theories of interest, such as QED or QCD, are special in that they have notions of conserved charges. In QED, conservation of charge dictates that the charge of the Universe is conserved. Particles can be created or destroyed but they must do so in such a way that equal parts of positive and negative charges are created or destroyed. In QCD, we say "color" is conserved—quarks can be "red," "green," or "blue," and antiquarks can carry the corresponding anticolors. The combination of r, g, and b quarks would give a colorless combination (no net color). A proton can be thought of as such a state. In any reaction of particles, no net color can be created or destroyed.

Not only is charge universally conserved, but where there is charge, its fingerprint is evident in the configuration of the gauge fields. In electrodynamics, this is expressed by Gauss's law,

$$\vec{\nabla} \cdot \vec{E} = \rho \ . \tag{1.1}$$

The requirements of Lorentz invariance and charge conservation constrain how the involved quantum fields may interact.

The most direct path to constructing continuum gauge field theories is to start with the principle of local gauge invariance. A Lagrangian for matter fields is first observed to have a conservation law (total number of particles minus antiparticles, say) associated with a continuous and global transformation on the fields; the symmetry transformation is then "promoted" to a local symmetry by insisting that the transformation can be done locally while still leaving the Lagrangian invariant.

Concretely, the classical Lagrange density is formed from fields $\psi_l(x)$ that can be transformed by symmetry transformations belonging to a Lie group. Local, Lie group gauge transformations on matter fields ψ_l take the form

$$\psi_l(x) \to [e^{\mathrm{i}\,\epsilon^\alpha(x) T_\alpha}]_l{}^m \psi_m(x) \tag{1.2}$$

$$\equiv \Omega(\epsilon(x))_l{}^m \psi_m(x) \ , \tag{1.3}$$

or in matrix/vector notation, $$\psi(x) \to \Omega(\epsilon(x))\,\psi(x) \ . \tag{1.4}$$

Here, l and m are generalized indices for fields that transform under the symmetry.[1] $\Omega_l{}^m$ is a square matrix-valued function of x defining how the fields are locally mixed, with $\epsilon(x)$ being real parameters for the gauge transformation function Ω. The matrices T_α are matrices in some representation of the Lie algebra. They are generators of the group and each T_α labels

[1] For example, $\psi_{\sigma\, l} = (q_{\sigma\, 1}, q_{\sigma\, 2}, q_{\sigma\, 3})$ for the theory of a single quark flavor, but with the Dirac index σ suppressed since eq. (1.3) does not mix Dirac components. A two-flavor theory of up and down quarks could have $\psi_l = (u_1, u_2, u_3, d_1, d_2, d_3)$, in which case the T_α would be block diagonal.

a distinct transformation. The generators form a Lie algebra under commutation:

$$[T_\alpha, T_\beta] = \mathrm{i}\, C^\gamma{}_{\alpha\beta} T_\gamma \qquad (\text{real } C^\gamma{}_{\alpha\beta}). \tag{1.5}$$

The $C^\gamma{}_{\alpha\beta}$ are known as structure constants. We assume the Lie algebra is a direct sum of commuting compact simple and U(1) subalgebras (cf. §15.2 of [36]).

The principle of local gauge invariance is the requirement that the Lagrange density constructed from the fields is unchanged by all such transformations in eq. (1.3). A locally gauge invariant Lagrange density will involve derivatives of the fields, but the issue arises that spatial derivatives $\partial_\mu \psi_l$ do not transform simply like ψ_l; ∂ is not a gauge covariant operator. For example,

$$\partial_\mu \psi \to \partial_\mu (\Omega\psi) = \Omega(\partial_\mu \psi) + (\partial_\mu \Omega)\psi \ . \tag{1.6}$$

The first term matches the form of eq. (1.3), but we would like to avoid the second term (which does vanish for global gauge transformations, i.e., constant Ω). A covariant derivative is formed by introducing 'gauge fields' A^α_μ (one four-vector field per generator α) as follows:

$$D_\mu \psi_l = \partial_\mu \psi_l - \mathrm{i}\, A^\alpha_\mu (T_\alpha)_l{}^m \psi_m \ . \tag{1.7}$$

This is expressed more compactly by collecting the gauge fields into a matrix-valued field,

$$A_\mu(x) \equiv A^\alpha_\mu(x) T_\alpha \ , \tag{1.8}$$

$$D_\mu \psi_l = \partial_\mu \psi_l - \mathrm{i}\, (A_\mu)_l{}^m \psi_m \tag{1.9}$$

$$\text{or just} \quad D_\mu \psi = (\partial_\mu - \mathrm{i}\, A_\mu)\psi \ . \tag{1.10}$$

The covariant derivative $D_\mu \psi_l$ is made to transform like ψ_l by having the gauge fields A^α_μ undergo simultaneous transformations:

$$A_\mu \to \Omega A_\mu \Omega^{-1} + \mathrm{i}\, \Omega (\partial_\mu \Omega^{-1}) \tag{1.11}$$

$$\Rightarrow D_\mu \psi_l \to \Omega_l{}^m (D_\mu \psi)_m \tag{1.12}$$

$$\text{or just} \quad D_\mu \psi \to \Omega D_\mu \psi \ . \tag{1.13}$$

To make A_μ a dynamical field too, we will need to couple derivatives of A_μ, but we again need to avoid the same problem of how spatial derivatives transform. The solution is to form the field strength tensor using covariant derivatives of A:

$$
\begin{aligned}
F_{\mu\nu} &= \mathrm{i}\,[D_\mu, D_\nu] && (1.14)\\
&= \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i}\,[A_\mu, A_\nu] && (1.15)\\
&= \partial_\mu A_\nu - \partial_\nu A_\mu + C^\gamma{}_{\alpha\beta} A^\alpha_\mu A^\beta_\nu T_\gamma && (1.16)\\
F^\alpha_{\mu\nu} &\equiv \partial_\mu A^\alpha_\nu - \partial_\nu A^\alpha_\mu + C^\alpha{}_{\beta\gamma} A^\beta_\mu A^\gamma_\nu && (1.17)
\end{aligned}
$$

Gauge transformations on the field strength are indeed homogeneous:

$$F_{\mu\nu} \to \Omega F_{\mu\nu} \Omega^{-1} \tag{1.18}$$

$$\text{or equivalently,} \qquad F^\alpha_{\mu\nu} \to (\Omega^A)^\alpha{}_\beta F^\beta_{\mu\nu} \quad \text{with } (\Omega^A)^\alpha{}_\beta = r\,\mathrm{tr}(T_\alpha \Omega T_\beta \Omega^{-1}) \tag{1.19}$$

The latter shows how the field strength transforms like a matter field in the adjoint representation. To summarize the gauge transformations thus far,

$$
\begin{aligned}
&\psi \to \Omega\psi && (1.20)\\
&A_\mu \to \Omega A_\mu \Omega^{-1} + \mathrm{i}\,\Omega(\partial_\mu \Omega^{-1}) \quad (= \mathrm{i}\,\Omega(\partial_\mu - \mathrm{i}\,A_\mu)\Omega^{-1}) && (1.21)\\
\Rightarrow\ &F_{\mu\nu} \to \Omega F_{\mu\nu} \Omega^{-1} && (1.22)
\end{aligned}
$$

ψ, D_μ, $F_{\mu\nu}$ all transform homogeneously, so as long as we only couple fields via these objects, it will be trivial to construct locally gauge invariant Lagrangians:

$$\mathcal{L} = \mathcal{L}(\psi, D_\mu, F_{\mu\nu}, \cdots) \tag{1.23}$$

The simplest gauge invariant and Lorentz invariant kinetic Lagrangian for the gauge fields is known as the Yang-Mills Lagrangian, with

$$
\begin{aligned}
\mathcal{L}_{YM} &\propto \mathrm{tr}(F_{\mu\nu} F^{\mu\nu}) && (1.24)\\
&= g_{\alpha\beta} F^\alpha_{\mu\nu} F^{b\ \mu\nu} && (1.25)\\
g_{\alpha\beta} &= \mathrm{tr}(T_\alpha T_\beta) && (1.26)
\end{aligned}
$$

(where $g_{\alpha\beta}$ should not be confused with the Minkowski metric tensor η). The quadratic form also has the lowest possible mass dimension (D_μ being of mass dimension 1 and $F_{\mu\nu}$ being of mass dimension 2). For the groups we consider, we can take

$$g_{\alpha\beta} = \frac{1}{2g^2}\delta_{\alpha\beta} \ . \tag{1.27}$$

The conventionally normalized continuum Lagrangian is then

$$\mathcal{L}_{YM} = -\frac{1}{4g^2}\delta_{\alpha\beta}F^{\alpha\,\mu\nu}F^{\beta}_{\mu\nu} = -\frac{1}{2g^2}\mathrm{tr}(\delta_{\alpha\beta}F_{\mu\nu}F^{\mu\nu}) \ . \tag{1.28}$$

From this point forward, we will have little to say about the continuum Lagrangians.

Going over to Hamiltonian dynamics, we need conjugate momenta to the gauge fields:

$$\frac{\partial}{\partial(\partial_\rho A^\beta_\sigma)}\left(F^\alpha_{\mu\nu}F^{\mu\nu}_\alpha\right) = 4F^{\rho\sigma}_\beta \tag{1.29}$$

$$\Rightarrow \frac{\partial\mathcal{L}_{YM}}{\partial(\partial_0 A^\alpha_j)} = \Pi^j_\alpha = -F^{0j}_\alpha/g^2 = F^{j0}/g^2 = F_{0j}/g^2 \ . \tag{1.30}$$

Note the correspondence of the momenta with the Maxwell electric fields $\vec{E} = (-\partial_t\vec{A} - \vec{\nabla}\Phi)/g$:

$$\text{(Maxwell)} \qquad A^\mu = (\Phi, \vec{A}) \tag{1.31}$$

$$\text{(Maxwell)} \qquad \Pi_i = \frac{1}{g^2}(\partial_0 A_i - \partial_i A_0) \tag{1.32}$$

$$\Rightarrow E^\alpha_i = -g\Pi^\alpha_i \ . \tag{1.33}$$

We can also identify the generalization of the Maxwell magnetic field $\vec{B} = \vec{\nabla}\times\vec{A}/g$,

$$\text{(Maxwell)} \qquad B_i = \epsilon_{ijk}(\partial^j A^k - \partial^k A^j)/g \tag{1.34}$$

$$\Rightarrow B^\alpha_i = \epsilon_{ijk}F^{\alpha jk}/g \ . \tag{1.35}$$

The conjugate momentum Π^0_α evidently vanishes, so A_0 is not actually dynamical. In the canonical formalism, it can therefore be convenient to fix the potential to Weyl gauge,

$$A_0 \equiv 0 \tag{1.36}$$

$$\Rightarrow \Pi^j_\alpha = g^{-2}\dot{A}^j_\alpha \ . \tag{1.37}$$

Spatial gauge transformations $\Omega(x) = \Omega(\vec{x})$ are still a symmetry since these preserve the gauge fixing condition. The Hamiltonian density resulting from Legendre transformation is then

$$\mathscr{H} = \frac{g^2}{2}\Pi^j_a \Pi_{a\,j} + \frac{1}{2g^2}\mathrm{tr}(F_{ij}F^{ij}) \ . \tag{1.38}$$

1.2 Quantization

The gauge fields are quantized by stipulating the equal-time commutation relations:

$$[\hat{A}^\alpha_i(\vec{x}), \hat{\Pi}^j_\beta(\vec{y})] = \mathrm{i}\,\delta^\alpha_\beta\,\delta^j_i\,\delta(\vec{x}-\vec{y}) \tag{1.39}$$

Gauge transformations must now be realized in operator form—we need symmetry operators to effect the matrix multiplications. Infinitesimally, the requirement on the operator gauge fields is (for $\Omega = 1 + \mathrm{i}\,\epsilon^\alpha T_\alpha + \cdots$)

$$\delta\hat{A}^\alpha_\mu = \partial_\mu\epsilon^\alpha - \mathrm{i}\,(\hat{A}^A_\mu)^\alpha{}_\beta\epsilon^\beta \tag{1.40}$$

$$= (\hat{D}^A_\mu\epsilon)^\alpha \tag{1.41}$$

The covariant divergences $D_i\Pi^i_\alpha$ generate the infinitesimal gauge transformations on the gauge fields:

$$\hat{\mathcal{T}}_\alpha \equiv (\hat{D}_i\hat{\Pi}^i)_\alpha \tag{1.42}$$

$$= -(\hat{D}_i\hat{E}^i)_\alpha/g \tag{1.43}$$

$$\left[(-\mathrm{i}\,)\int d^dx\ \epsilon^\alpha(\vec{x})(\hat{D}_i\hat{\Pi}^i(\vec{x}))_\alpha\ ,\ \hat{A}^\beta_j(\vec{y})\right] = \left[(-\mathrm{i}\,)\int d^dx\ \epsilon^\alpha(\vec{x})\hat{\mathcal{T}}_\alpha(\vec{x})\ ,\ \hat{A}^\beta_j(\vec{y})\right] \tag{1.44}$$

$$= \partial_j\epsilon^\beta(\vec{y}) - \mathrm{i}\,\hat{A}^A_j(\vec{y})^\beta{}_\gamma\epsilon^\gamma(\vec{y}) \tag{1.45}$$

$$= (\hat{D}^A_j\epsilon(\vec{y}))^\beta \tag{1.46}$$

(with a discarded boundary term). The $D_i\Pi^i_\alpha$ are themselves a representation of the Lie group, satisfying the exact same algebra of the generators in a distributional sense:

$$\int d^dx\,\epsilon^\alpha(x)\left[(\hat{D}_i\hat{\Pi}^i)_\alpha(x)\ ,\ (\hat{D}_j\hat{\Pi}^j)_\beta(y)\right] = \epsilon^\alpha(y)\,\mathrm{i}\,C^\gamma{}_{\alpha\beta}(\hat{D}_i\hat{\Pi}^i(y))_\gamma \tag{1.47}$$

$$\text{or just}\quad \left[(\hat{D}_i\hat{\Pi}^i)_\alpha(x)\ ,\ (\hat{D}_j\hat{\Pi}^j)_\beta(y)\right] = \mathrm{i}\,C^\gamma{}_{\alpha\beta}(\hat{D}_i\hat{\Pi}^i(y))_\gamma\,\delta(\vec{x}-\vec{y}) \tag{1.48}$$

So the unitary operator to effect the matrix transformation $\Omega_\epsilon(\vec{x}) = \exp[\mathrm{i}\,\epsilon^\alpha(\vec{x})T_\alpha]$ on $\hat{A}$ would be the quantum (Hilbert space) operator $\exp[-\mathrm{i}\int d^dx\,\epsilon^\alpha(\vec{x})\hat{\mathcal{T}}_\alpha(\vec{x})]$:

$$\hat{\Omega}(\epsilon) \equiv e^{\mathrm{i}\int d^dx\,\epsilon^\alpha(x)\hat{\mathcal{T}}_\alpha(x)} \tag{1.49}$$

$$\Rightarrow \quad \hat{\Omega}^\dagger(\epsilon)\hat{A}_j(\vec{y})\hat{\Omega}(\epsilon) = \Omega_\epsilon(\vec{y})\hat{A}_j\Omega^\dagger_\epsilon(\vec{y}) + \mathrm{i}\,\Omega_\epsilon(\vec{y})\partial_j\Omega^\dagger_\epsilon(\vec{y}) \tag{1.50}$$

where $\hat{A}_j$ is in any representation and the dagger placement is not accidental. In particular, note that $\hat{\Omega}(\epsilon)$ is a functional of the spatial gauge transformation function $\epsilon(x)$—it is not a field, it is a single quantum operator fixed by ϵ, and it lacks "group indices"—while the classical matrix $\Omega_\epsilon(\vec{x})$ does carry group indices, is still a function of space, and acts like a c-number as far as quantum operations are concerned.

Later we will consider fermionic matter. Here we have

$$\delta\hat{\psi}_l = \mathrm{i}\,\epsilon^\alpha(T_\alpha)_l{}^m\hat{\psi}_m \tag{1.51}$$

$$J^0_\alpha = \psi^\dagger T_\alpha\psi \tag{1.52}$$

$$[\hat{J}^0_\alpha(t,\vec{x}), \hat{J}^0_\beta(t,\vec{y})] = \mathrm{i}\,C^\gamma{}_{\alpha\beta}\hat{J}^0_\gamma(t,\vec{x})\delta(\vec{x}-\vec{y}) \tag{1.53}$$

$$[(-\mathrm{i}\,)\hat{J}^0_\alpha(t,\vec{x}), \hat{\psi}_l(t,\vec{y})] = \mathrm{i}\,(T_\alpha)_l{}^m\hat{\psi}_m(t,\vec{x})\delta(\vec{x}-\vec{y}) \tag{1.54}$$

Combining the above, we have the Gauss law operators that generate gauge transformations and satisfy the same Lie algebra:

$$\hat{\mathcal{G}}_\alpha(x) = (\hat{D}_i\hat{\Pi}^i(x))_\alpha + \hat{\psi}^\dagger(x)T_\alpha\hat{\psi}(x) = -\frac{1}{g}(\hat{D}_i\hat{E}^i(x))_\alpha + \hat{J}^0_\alpha(x) \tag{1.55}$$

$$\Rightarrow \int d^dx\,\epsilon^\alpha(x)\left[\hat{\mathcal{G}}_\alpha(x)\ ,\ \hat{\mathcal{G}}_\beta(y)\right] = \epsilon^\alpha(y)\,\mathrm{i}\,C^\gamma{}_{\alpha\beta}\hat{\mathcal{G}}(y)_\gamma \tag{1.56}$$

$$\text{or just} \qquad \left[\hat{\mathcal{G}}_\alpha(x)\ ,\ \hat{\mathcal{G}}_\beta(y)\right] = \mathrm{i}\,C^\gamma{}_{\alpha\beta}\hat{\mathcal{G}}(y)_\gamma\,\delta(\vec{x}-\vec{y}) \tag{1.57}$$

To summarize, using the Gauss law operators $\hat{\mathcal{G}}_\alpha(x)$ we can construct the symmetry operators

$\hat{\Theta}$ that effect the gauge transformations introduced in the classical theory as follows:

$$\hat{\Theta}(\epsilon) \equiv e^{\mathrm{i}\int d^d x\, \epsilon^\alpha(x)\hat{\mathcal{G}}_\alpha(x)} \tag{1.58}$$

$$\hat{\Theta}^\dagger(\epsilon)\hat{A}_j(\vec{x})\hat{\Theta}(\epsilon) = \Omega_\epsilon(\vec{x})\hat{A}_j(\vec{x})\Omega_\epsilon^\dagger(\vec{x}) + \mathrm{i}\,\Omega_\epsilon(\vec{x})\partial_j\Omega_\epsilon^\dagger(\vec{x}) \tag{1.59}$$

$$\hat{\Theta}^\dagger(\epsilon)\hat{F}_{\mu\nu}(\vec{x})\hat{\Theta}(\epsilon) = \Omega_\epsilon(\vec{x})\hat{F}_{\mu\nu}(\vec{x})\Omega_\epsilon^\dagger(\vec{x}) \tag{1.60}$$

$$\hat{\Theta}^\dagger(\epsilon)\hat{F}_\alpha^{\mu\nu}(\vec{x})\hat{\Theta}(\epsilon) = [\Omega_\epsilon^A(\vec{x})]_\alpha{}^\beta\hat{F}_\beta^{\mu\nu}(\vec{x}) \tag{1.61}$$

$$\hat{\Theta}^\dagger(\epsilon)\hat{\psi}_l(\vec{x})\hat{\Theta}(\epsilon) = [\Omega_\epsilon(\vec{x})]_l{}^m\hat{\psi}_m(\vec{x}) \tag{1.62}$$

1.3 Gauge fields on the lattice

For any numerical simulation, the continuum of degrees of freedom of a field theory must be truncated. This is traditionally done by defining the fields only on a discrete (d+1)-dimensional lattice of points in Euclidean spacetime. It is standard to use a Cartesian geometry with lattice spacing a, as it preserves the largest possible symmetry subgroup of the (d+1)-dimensional rotation group. A volume truncation is also necessary; it is common to use a (d+1)-dimensional box, with length L_x along d "spatial" directions and L_t along the "time" direction. For Euclidean simulations, singling out a direction to call "time" is due to the fact that the (d+1)-dimensional Euclidean path integral is identified with the partition function of the same system in d spatial dimensions, in thermal equilibrium at temperature $1/L_t$. Ground state properties are thus extracted by taking the limit of large Euclidean time extent.

The lattice is itself a regularization scheme for the continuum quantum field theory. The nonzero lattice spacing a means that particles can only be resolved with momenta below the scale π/a—an ultraviolet cutoff—and the finite volume also implies a cutoff on wavelength at the scale of the box size L_x—an infrared cutoff. These two limits are important considerations for what physics will be accessible, e.g., L_x must be larger than 1 femtometer to study the proton, while a must be much smaller than 1 femtometer.

At a more practical level, latticization renders the domain of the functional integral finite-dimensional so that the functional integral is itself well-defined and amenable to Monte Carlo integration. In addition, while the continuum limit is formally taken by sending $a \to 0$, in practice the continuum limit is approached by instead tuning parameters in the lattice action to 'critical values' at which correlation lengths diverge (in lattice units) [38]. In this way, the effects of lattice discretization are made small or negligible relative to the probed physics.

In the subsequent chapters, we will be primarily considered with the UV structure of the theory, meaning the local lattice degrees of freedom themselves. These depart from traditional lattice field theory because we are most interested in Hamiltonian mechanics, in which time remains continuous and only space is discretized, with a lattice spacing a_s. Of course, we also work with Hilbert spaces rather than classical field configurations. For gauge theories, we take the Weyl gauge $A_0 = 0$ so that gauge fields have only spatial components. The details of extrapolating to the continuum and ameliorating lattice "artifacts" will not be taken up because Hamiltonian simulation problems are still in such a nascent stage of development. It is not even known how different these processes will be when working with spatially-discretized Hamiltonians in real time, as opposed to spatiotemporally-discretized actions in imaginary time.

On the lattice, matter field transformations take the form they did in the continuum,

$$\psi(x) \to \Omega(x)\psi(x) \tag{1.63}$$

at all sites x. For derivative fields, we have to choose what we mean by 'derivative' on the lattice. A simple choice is the forward derivative $\partial_\mu^+ f(x) = (f(x+1) - f(x))/a_s$. The discrete operator ∂_μ^+ suffers a problem like that encountered in the continuum,

$$\partial_\mu^+(\Omega(x)\psi_l(x)) = \Omega(x)\partial_\mu^+\psi_l(x) + (\partial_\mu^+\Omega(x))\psi_l(x + e_\mu) \tag{1.64}$$

[cf. discussion around (1.6)], but the second term is arguably worse now, due to the dependence on the translated value $\psi_l(x + e_\mu)$. A covariant derivative is formed this time by introducing group matrices $U(x, x + e_\mu)$:

$$D_\mu\psi(x) = a_s^{-1}[U(x, x + e_\mu)\psi(x + e_\mu) - \psi(x)] \tag{1.65}$$

These matrices are prescribed the simultaneous transformations

$$U(x, x+e_\mu) \to \Omega(x) U(x, x+e_\mu) \Omega^{-1}(x+e_\mu) \ . \tag{1.66}$$

The $U(x, x+e_\mu)$ are referred to as link variables. For brevity, we can also take $U_\mu(x) \equiv U(x, x+e_\mu)$ and $U_\mu(x)^{-1} = U(x+e_\mu, x)$. The link variables are most closely related to the continuum A_μ by path-ordered products in the continuum, which have the same transformation rules:

$$\mathscr{P}\exp\left[-\mathrm{i}\int_z^x dy^\mu A_\mu(y)\right] \to \Omega(x)\mathscr{P}\exp\left[-\mathrm{i}\int_z^x dy^\mu A_\mu(y)\right]\Omega^{-1}(z) \tag{1.67}$$

$$\Rightarrow U_\mu(x, x+e_\mu) \sim \mathscr{P}\exp\left[-\mathrm{i}\int_{x+e_\mu}^x dy^\mu A_\mu(y)\right] \ . \tag{1.68}$$

Also by analogy with the continuum, the transformation rules of the path ordered product imply that gauge invariant Wilson loops from the continuum have lattice counterparts formed by multiplying together U_μ's along links to form closed contours. The smallest such loop is called a plaquette, given by

$$U_{\mu\nu}(x) = U_\mu(x) U_\nu(x+e_\mu) U(x+e_\nu)_\mu^{-1} U(x)_\nu^{-1} \tag{1.69}$$

$$U_{\mu\nu}(x) \to \Omega(x) U_{\mu\nu}(x) \Omega^{-1}(x) \tag{1.70}$$

$$\Rightarrow \mathrm{tr}(U_{\mu\nu}) \to \mathrm{tr}(U_{\mu\nu}) \ . \tag{1.71}$$

Using the plaquettes and lattice covariant derivative, the Wilson gauge action with naïve fermions is

$$\begin{aligned} S = &-\sum_{x\mu\nu} \frac{1}{2g^2 r a_s^4} \mathrm{tr}(1 - U_{\mu\nu x}) \\ &-\sum_{x\mu} \frac{1}{2a_s}\left[\bar\psi(x)\gamma^\mu U_\mu(x, x+e_\mu)\psi(x+e_\mu) - \bar\psi(x+e_\mu)\gamma^\mu U_\mu^\dagger(x, x+e_\mu)\psi(x)\right] \ . \end{aligned} \tag{1.72}$$

The continuum limit of the Wilson action reproduces the classical action. The naïve lattice action is so named because it is known to suffer from so-called fermion "doublers", but addressing them is outside the scope of this dissertation and a variety of improvements are already known and regularly employed (for example, staggered [39], clover [40], domain wall [41–44], overlap [45], Ginsparg-Wilson [46], and fixed point [47, 48] fermions).

1.4 Quantized Hamiltonian formulation

In this section we review the basic setup of Hamiltonian lattice gauge theory. This largely follows Chapter 4 of Ref. [37] on lattice gauge fields. Ref. [49] is also an instructive contemporary resource.

In Hamiltonian lattice gauge theory, we work with Hilbert spaces corresponding to elements of the gauge group G, one such space for each link. In a path integral formulation, we integrate over all classical matrices U^ρ in some irrep ρ of the gauge group. Where necessary, we take ρ to be a unitary representation. We can think of these classical matrices as positions on the group manifold G in the particular representation ρ. The local link Hilbert space $\mathcal{H}_{\text{link}}$ can then be constructed from the eigenstates of a position operator $\hat{U}^\rho$. The operator-matrix eigenvalue relation is

$$\hat{U}^\rho \left|g\right\rangle = \left|g\right\rangle D^\rho(g) \ , \quad \text{i.e.,} \quad {\hat{U}^\rho}{}_{m'}{}^{m} \left|g\right\rangle = \left|g\right\rangle D^\rho(g)_{m'}{}^{m} \tag{1.73}$$

where $D^\rho(g)$ is a Wigner matrix for g, in the representation ρ. With respect to Hilbert space operations, the eigenvalue matrix behaves as a c-number. Note that even if we had started from a path integral for a particular irrep, we can freely choose ρ in (1.73) to be any irrep we like, with the action of $\hat{U}^\rho$ defined as given.

In the path integral formulation, the Lagrangian is defined to be invariant under local gauge transformations. The generators are of course the matrices T_α introduced at the start. What are needed now are the quantum generators that will induce the right transformations on link variables.

The effect of a lattice gauge transformation $\Omega(x)$ is to rotate the link operators just like the classical transformation:

$$\hat{U}^\rho(x,i) \xrightarrow{\Omega} \hat{U}^\rho(x,i)' = \Omega^\rho(x)\hat{U}^\rho(x,i)\Omega^\rho(x+e_i)^{-1} \ . \tag{1.74}$$

We will often use L (R) to refer to the refer to the "left" ("right") end of the link, namely x ($x + e_i$) for link $U_i(x)$.

We will denote the Hilbert space symmetry transformations that perform these local gauge transformations for us by $\hat{\Theta}_L$ and $\hat{\Theta}_R$. $\hat{\Theta}_L(g)$ will be defined to shift the eigenvalue of $\hat{U}^\rho$ by left-multiplication with g:

$$\hat{U}^\rho \hat{\Theta}_L(g) \left|h\right\rangle \equiv D^\rho(gh) \hat{\Theta}_L(g) \left|h\right\rangle \tag{1.75}$$

$$\Rightarrow \hat{\Theta}_L(g)^{-1} \hat{U}^\rho \hat{\Theta}_L(g) = D^\rho(g) \hat{U}^\rho \ . \tag{1.76}$$

Similarly, $\hat{\Theta}_R(g)$ is defined so that the eigenvalue of $\hat{U}^\rho$ is right-multiplied by g^{-1}:

$$\hat{U}^\rho \hat{\Theta}_R(g) \left|h\right\rangle \equiv D^\rho(hg^{-1}) \left|h\right\rangle \tag{1.77}$$

$$\Rightarrow \hat{\Theta}_R(g)^{-1} \hat{U}^\rho \hat{\Theta}_R(g) = \hat{U}^\rho D^\rho(g^{-1}) \ . \tag{1.78}$$

Defined this way, the left and right transformations each provide representations of the group, namely

$$\hat{\Theta}_L(g_1) \hat{\Theta}_L(g_2) = \hat{\Theta}_L(g_1 g_2) \ , \tag{1.79}$$

$$\hat{\Theta}_R(g_1) \hat{\Theta}_R(g_2) = \hat{\Theta}_R(g_1 g_2) \ . \tag{1.80}$$

The generators of left and right rotations are those operators $\hat{L}_\alpha$ and $\hat{R}_\alpha$ such that

$$\hat{\Theta}_L(g) = \exp(\mathrm{i}\, \omega^\alpha \hat{L}_\alpha) \ , \tag{1.81}$$

$$\hat{\Theta}_R(g) = \exp(\mathrm{i}\, \omega^\alpha \hat{R}_\alpha) \ . \tag{1.82}$$

Since the $\hat{\Theta}_{L/R}$ have the same multiplication table as the elements of G, the generators must obey the same Lie algebra as the generators T_α:

$$[\hat{L}_\alpha, \hat{L}_\beta] = \mathrm{i}\, C^\gamma{}_{\alpha\beta} \hat{L}_\gamma \ , \tag{1.83}$$

$$[\hat{R}_\alpha, \hat{R}_\beta] = \mathrm{i}\, C^\gamma{}_{\alpha\beta} \hat{R}_\gamma \ . \tag{1.84}$$

The $\hat{\Theta}_{L/R}$ operators effectively translate the link operator through the group manifold. The operators $\hat{L}_\alpha$ and $\hat{R}_\alpha$ are then infinitesimal generators of translations along that manifold. This makes it natural to ask what the commutation relations are for $\hat{U}^\rho$ with $\hat{L}_\alpha$ and $\hat{R}_\alpha$. The properties given above are all we need to infer the commutation relations.

First, by acting $[\hat{\Theta}_L(g), \hat{U}^\rho]$ on an arbitrary state $|h\rangle$ and factoring out $\hat{U}^\rho$, we obtain the operator identity

$$[\hat{\Theta}_L(g), \hat{U}^\rho] = \hat{\Theta}_L(g)(I^\rho - D^\rho(g))\hat{U}^\rho \ . \tag{1.85}$$

Now taking $g = \exp(\mathrm{i}\,\omega^\alpha T_\alpha)$ with $\omega^\alpha \ll 1$, the left-hand side is $\mathrm{i}\,\omega^\alpha[\hat{L}_\alpha, \hat{U}^\rho] + O(\omega)^2$. On the right-hand side, we have $D^\rho(g) = \exp(\mathrm{i}\,\omega^\alpha T^\rho_\alpha)$, and since the factor $(I^\rho - D^\rho(g))$ is already $O(\omega)$, the leading order behavior is obtained by dropping $O(\omega)$ corrections to $\hat{\Theta}_L(g)$. Thus,

$$\mathrm{i}\,\omega^\alpha[\hat{L}_\alpha, \hat{U}^\rho] + O(\omega)^2 = -\mathrm{i}\,\omega^\alpha T^\rho_\alpha \hat{U}^\rho + O(\omega^2) \tag{1.86}$$

From here we can read off the canonical commutation relation $[\hat{L}_\alpha, \hat{U}^\rho] = -T^\rho_\alpha \hat{U}^\rho$.

We can then do an analogous exercise using $\hat{\Theta}_R(g)$. This time we find

$$[\hat{\Theta}_R(g), \hat{U}^\rho] = \hat{U}^\rho(D^\rho(g) - I^\rho)\hat{\Theta}_R(g) \tag{1.87}$$

Using infinitesimals as was done with $\hat{\Theta}_L$, we find $[\hat{R}_\alpha, \hat{U}^\rho] = \hat{U}^\rho T^\rho_\alpha$. The results of both calculations are summarized as

$$[\hat{L}_\alpha, \hat{U}^\rho] = -T^\rho_\alpha \hat{U}^\rho \ , \tag{1.88a}$$

$$[\hat{R}_\alpha, \hat{U}^\rho] = +\hat{U}^\rho T^\rho_\alpha \ . \tag{1.88b}$$

To be sure, T^ρ_α are matrices in group space, while $\hat{U}^\rho$ have both group space and Hilbert space structure.

So far left and right generators have been discussed as separate objects. However, they are related by parallel transport, which will be important for characterizing states later.

A right rotation $\hat{\Theta}_R(h)$ on a state can be expressed in terms of a left rotation in a trivial way by noting

$$\hat{\Theta}_R(h)\,|g\rangle = |gh^{-1}\rangle = \hat{\Theta}_L(gh^{-1}g^{-1})\,|g\rangle \ . \tag{1.89}$$

To relate right and left generators, however, we are after an operator relation and not just a ket-dependent equivalence. To proceed we will concentrate on infinitesimal transformations since that is enough to tell us about the generators.

We start by taking $h = e^{\mathrm{i}\,\eta^\alpha T_\alpha}$ for small η. Then the left-hand side of 1.89 expands out to

$$\hat{\Theta}_R(h)\,|g\rangle = (\hat{1} + \mathrm{i}\,\eta^\alpha \hat{R}_\alpha + \cdots)\,|g\rangle \ . \tag{1.90}$$

As for the right-hand side, we first need to expand the argument of $\hat{\Theta}_L$ as

$$gh^{-1}g^{-1} = 1 - \mathrm{i}\,\eta^\alpha g T_\alpha g^{-1} + \cdots \ . \tag{1.91}$$

We know that if the linear term of this group element is put into the form $\mathrm{i}\,\omega^\alpha T_\alpha$, then the linear term of the left rotation $\hat{\Theta}_L$ will simply be $\mathrm{i}\,\hat{L}_\alpha \omega^\alpha$. Here we will need the automorphism property relating the adjoint representation to any other representation (Appendix A of Ref. [37])

$$(\Omega^\rho)^{-1} T^\rho_\alpha \Omega^\rho = (\Omega^A)_\alpha{}^\beta T^\rho_\beta \tag{1.92}$$

or equivalently $\Omega^\rho T^\rho_\alpha (\Omega^\rho)^{-1} = T^\rho_\beta (\Omega^A)^\beta{}_\alpha$. The linear term of the argument to $\hat{\Theta}_L$ is therefore $-\mathrm{i}\,T_\alpha D^A(g)^\alpha{}_\beta \eta^\beta$, from which we see that $\omega^\alpha = -D^A(g)^\alpha{}_\beta \eta^\beta$. Thus, the right-hand side of eq. (1.89) is given by

$$\begin{aligned}
\hat{\Theta}_L(gh^{-1}g^{-1})\,|g\rangle &= [\hat{1} + \mathrm{i}\,\hat{L}_\alpha(-D^A(g)^\alpha{}_\beta \eta^\beta) + \cdots]\,|g\rangle && (1.93)\\
&= (\hat{1} - \mathrm{i}\,\eta^\alpha \hat{L}_\beta (\hat{U}^A)^\beta{}_\alpha + \cdots)\,|g\rangle \ . && (1.94)
\end{aligned}$$

Comparing this with eq. (1.90), we finally see the parallel transport relation of left and right generators:

$$\hat{R}_\alpha = -\hat{L}_\beta (\hat{U}^A)^\beta{}_\alpha \ . \tag{1.95}$$

We have motivated a parallel transport relationship between $\hat{L}$ and $\hat{R}$, but the real test is if $\hat{R}$ as defined has all the properties we expect. A natural starting point would be the canonical commutation relations. To show that (1.88) works out, we will need the automorphism property again as well as the fact that elements of $\hat{U}$ from any representations commute with each other. Proceeeding,

$$\begin{aligned}
[\hat{R}_\alpha, \hat{U}^\rho{}_\beta{}^\gamma] &= -[\hat{L}_\delta (\hat{U}^A)^\delta{}_\alpha, \hat{U}^\rho{}_\beta{}^\gamma] \\
&= -[\hat{L}_\delta, \hat{U}^\rho{}_\beta{}^\gamma](\hat{U}^A)^\delta{}_\alpha - \hat{L}_\delta[(\hat{U}^A)^\delta{}_\alpha, \hat{U}^\rho{}_\beta{}^\gamma] \\
&= (T^\rho_\delta)_\beta{}^\epsilon \hat{U}^\rho{}_\epsilon{}^\gamma (\hat{U}^A)^\delta{}_\alpha - 0 \\
&= (T^\rho_\delta (\hat{U}^A)^\delta{}_\alpha)_\beta{}^\epsilon \hat{U}^\rho{}_\epsilon{}^\gamma \\
&= (\hat{U}^\rho T^\rho_\alpha (\hat{U}^\rho)^{-1})_\beta{}^\epsilon \hat{U}^\rho{}_\epsilon{}^\gamma \\
&= (\hat{U}^\rho T^\rho_\alpha)_\beta{}^\gamma \qquad (1.96) \\
\Rightarrow [\hat{R}_\alpha, \hat{U}^\rho] &= \hat{U}^\rho T^\rho_\alpha \ , \qquad (1.97)
\end{aligned}$$

as required.

Another check is that left and right generators commute:

$$\begin{aligned}
-[\hat{R}_\alpha, \hat{L}_\beta] &= [\hat{L}_\gamma (\hat{U}^A)^\gamma{}_\alpha, \hat{L}_\beta] \\
&= [\hat{L}_\gamma, \hat{L}_\beta](\hat{U}^A)^\gamma{}_\alpha + \hat{L}_\delta [(\hat{U}^A)^\delta{}_\alpha, \hat{L}_\beta] \\
&= \mathrm{i}\, C^\delta{}_{\gamma\beta} \hat{L}_\delta (\hat{U}^A)^\gamma{}_\alpha + \hat{L}_\delta (T^A_\beta)^\delta{}_\gamma (\hat{U}^A)^\gamma{}_\alpha \\
&= \mathrm{i}\, C^\delta{}_{\gamma\beta} \hat{L}_\delta (\hat{U}^A)^\gamma{}_\alpha + \hat{L}_\delta (-\mathrm{i}\, C^\delta{}_{\gamma\beta})(\hat{U}^A)^\gamma{}_\alpha \\
&= 0 \ . \qquad (1.98)
\end{aligned}$$

Yet another important test is reproducing the Lie algebra. Using a couple applications of a product rule for commutators, one finds

$$\begin{aligned}
[\hat{R}_\alpha, \hat{R}_\beta] &= [\hat{R}_\alpha, -\hat{L}_\delta (\hat{U}^A)^\delta{}_\beta] \\
&= -\hat{L}_\delta [\hat{R}_\alpha, (\hat{U}^A)^\delta{}_\beta] - [\hat{R}_\alpha, \hat{L}_\delta](\hat{U}^A)^\delta{}_\beta \\
&= \hat{L}_\delta [\hat{L}_\gamma (\hat{U}^A)^\gamma{}_\alpha, (\hat{U}^A)^\delta{}_\beta] - 0 \\
&= \hat{L}_\delta \left([\hat{L}_\gamma, (\hat{U}^A)^\delta{}_\beta](\hat{U}^A)^\gamma{}_\alpha + \hat{L}_\gamma [(\hat{U}^A)^\gamma{}_\alpha, (\hat{U}^A)^\delta{}_\beta] \right) \\
&= \hat{L}_\delta [\hat{L}_\gamma, (\hat{U}^A)^\delta{}_\beta](\hat{U}^A)^\gamma{}_\alpha + 0 \ . \qquad (1.99)
\end{aligned}$$

The last expression is easily shown to equal $\mathrm{i}\, \hat{L}_\delta C^\delta{}_{\varepsilon\gamma} (\hat{U}^A)^\varepsilon{}_\beta (\hat{U}^A)^\gamma{}_\alpha$. Here we note that the invariance of the structure constants under any rotation Ω^A is expressed by

$$C^{\delta}{}_{\varepsilon\gamma}(\Omega^A)^{\varepsilon}{}_{\beta}(\Omega^A)^{\gamma}{}_{\alpha} = C^{\gamma}{}_{\beta\alpha}(\Omega^A)^{\delta}{}_{\gamma} \ , \tag{1.100}$$

giving

$$\begin{aligned}\hat{L}_{\delta}[\hat{L}_{\gamma}, (\hat{U}^A)^{\delta}{}_{\beta}](\hat{U}^A)^{\gamma}{}_{\alpha} &= \mathrm{i}\,\hat{L}_{\delta}C^{\gamma}{}_{\beta\alpha}(\hat{U}^A)^{\delta}{}_{\gamma} \\ &= \mathrm{i}\,C^{\gamma}{}_{\alpha\beta}(-\hat{L}_{\delta}(\hat{U}^A)^{\delta}{}_{\gamma}) \\ &= \mathrm{i}\,C^{\gamma}{}_{\alpha\beta}\hat{R}_{\gamma} \ . \end{aligned}\tag{1.101}$$

We have therefore seen that $[\hat{R}_{\alpha}, \hat{R}_{\beta}] = \mathrm{i}\,C^{\gamma}{}_{\alpha\beta}\hat{R}_{\gamma}$, as required.

Finally, the parallel transport relationship can be used to show that the quadratic Casimirs at either end of a link are equal. This follows by noting that the parallel transport relation is equivalently expressed by

$$\hat{R}^{\alpha} = -(\hat{U}^{A\ -1})^{\alpha}{}_{\beta}\hat{L}^{\beta} \ . \tag{1.102}$$

Therefore,

$$\hat{R}_{\alpha}\hat{R}^{\alpha} = \hat{L}_{\beta}(\hat{U}^A)^{\beta}{}_{\alpha}(\hat{U}^{A\ -1})^{\alpha}{}_{\gamma}\hat{L}^{\gamma} \tag{1.103}$$

$$= \hat{L}_{\beta}\hat{L}^{\beta} \ . \tag{1.104}$$

Considering a particular link $(x, x+e_i)$ within a Cartesian lattice, the operators associated with it are $\hat{U}^{\rho}_i(x) = \hat{U}^{\rho}(x, x+e_i)$, $\hat{L}^i_{\alpha}(x)$, and $\hat{R}^i_{\alpha}(x+e_i)$. We have fixed Weyl gauge, so $\hat{U}^{\rho}(x, x+e_0) = 1$ and only the spatial links are dynamical and affected by the residual gauge symmetry. Both the $\hat{L}_{\alpha}$ and $\hat{R}_{\alpha}$ from a given link obey the Lie algebra of the group. Summing all of them around a site then gives generators for all links joined to the site simultaneously,

$$\hat{\mathcal{T}}_{\alpha}(x) = a_s^{-d}\sum_i(\hat{L}^i_{\alpha}(x) + \hat{R}^i_{\alpha}(x)) \tag{1.105}$$

The factors a_s^{-d} are inserted to get the same mass dimension as the covariant divergences $(\hat{D}_i\hat{\Pi}^i)_{\alpha}$ from the continuum, which the above operators must correspond to given how they generate gauge transformations on all links emanating from a site. These operators satisfy lattice analogues of the continuum commutation relations:

$$[\hat{\mathcal{T}}_\alpha(\vec{x}), \hat{\mathcal{T}}_\beta(\vec{y})] = \mathrm{i}\, C^{\gamma}{}_{\alpha\beta} \hat{\mathcal{T}}_\gamma(\vec{x}) a_s^{-d} \delta_{\vec{x},\vec{y}} \ . \tag{1.106}$$

To explicitly see the correspondence with the covariant divergences $(\hat{D}_i \hat{\Pi}^i)_\alpha$ from the continuum, let us identify gauge fields $A^\rho(x, x+e_i)$ with link operators by $\hat{U}^\rho(x, x+e_i) \equiv \exp(-\mathrm{i}\, a_s \hat{A}^\rho(x, x+e_i))$. Then we have

$$\begin{aligned}
(\hat{L}^i_\alpha(x) + \hat{R}^i_\alpha(x)) &= \left(\hat{L}^i_\alpha(x) - (\hat{U}^A)^\beta{}_\alpha(x-e_i, x)\hat{L}_\beta(x-e_i)\right) \\
&= \left(\hat{L}^i_\alpha(x) - \left(\delta^\beta_\alpha - \mathrm{i}\, a_s \hat{A}^\gamma(x-e_i, x)(T^A_\gamma)^\beta{}_\alpha + O(a_s^2)\right)\hat{L}_\beta(x-e_i)\right) \\
&= \left(a_s \partial^-_i \hat{L}^i_\alpha(x) + \mathrm{i}\, a_s \hat{A}^A(x-e_i, x)^\beta{}_\alpha \hat{L}_\beta(x-e_i) + O(a_s^2)\right) \\
&= a_s \partial^-_i \hat{L}^i_\alpha(x) + a_s C^\beta{}_{\alpha\gamma} \hat{A}^\gamma(x-e_i, x)\hat{L}_\beta(x-e_i)) + O(a_s^2) \ .
\end{aligned} \tag{1.107}$$

Or, alternatively,

$$(\hat{L}^i_\alpha(x) + \hat{R}^i_\alpha(x)) = -a_s \partial^+_i \hat{R}^i_\alpha(x) - C^\beta{}_{\alpha\gamma} A^\gamma(x, x+e_i)\hat{R}^i_\alpha(x+e_i) + O(a_s^2) \ . \tag{1.108}$$

The above two results show that the lattice generators $\hat{\mathcal{T}}_\alpha$ can be identified with $\hat{D}_i \hat{\Pi}^i_\alpha$ in the continuum limit if we identify $\hat{\Pi}^i_\alpha$ with $a_s^{1-d} \hat{L}^i_\alpha$ or with $-a_s^{1-d} \hat{R}^i_\alpha$.

Summarizing these observations, $\hat{L}^i_\alpha$ and $\hat{R}^i_\alpha$ can essentially be thought of as being $\hat{\Pi}^i_\alpha$ evaluated at infinitesimal distances to either side of the site,

$$\begin{gathered}
\hat{\Pi}^i_\alpha(\vec{x} + \delta\, e_i) = a_s^{1-d} \hat{L}^i_\alpha(\vec{x}) \ , \\
\hat{\Pi}^i_\alpha(\vec{x} - \delta\, e_i) = -a_s^{1-d} \hat{R}^i_\alpha(\vec{x}) \ , \\
0 < \delta \ll 1 \ .
\end{gathered}$$

Note, however, that at nonzero a_s the conjugate momenta identified as such do not commute.

Instead, we have only that

$$\left[a_s^{1-d}\hat{L}^i_\alpha(\vec{x}), a_s^{1-d}\hat{L}^j_\beta(\vec{y})\right] = a_s \mathrm{i}\, C^\gamma{}_{\alpha\beta}(a_s^{1-d}\hat{L}^i_\gamma(\vec{x}))(a_s^{-d}\delta_{\vec{x},\vec{y}})\delta^{ij} \tag{1.109}$$

$$\sim a_s \mathrm{i}\, C^\gamma{}_{\alpha\beta}(a_s^{1-d}\hat{L}^i_\gamma(\vec{x}))\,\delta(\vec{x}-\vec{y})\delta^{ij} \ , \tag{1.110}$$

$$\left[-a_s^{1-d}\hat{R}^i_\alpha(\vec{x}),\, -a_s^{1-d}\hat{R}^j_\beta(\vec{y})\right] = a_s \mathrm{i}\, C^\gamma{}_{\alpha\beta}(a_s^{1-d}\hat{R}^i_\gamma(\vec{x}))(a_s^{-d}\delta_{\vec{x},\vec{y}})\delta^{ij} \tag{1.111}$$

$$\sim a_s \mathrm{i}\, C^\gamma{}_{\alpha\beta}(a_s^{1-d}\hat{R}^i_\gamma(\vec{x}))\,\delta(\vec{x}-\vec{y})\delta^{ij} \ . \tag{1.112}$$

These commutators are suppressed by a factor of a_s, so the requirement that $[\hat{\Pi}_\alpha(\vec{x}), \hat{\Pi}_\beta(\vec{y})] = 0$ is recovered in the continuum limit.

For matter fields, gauge transformations at a site should take the form

$$\hat{\Theta}^{-1}(g)\hat{\psi}_l\hat{\Theta}(g) = D^\rho(g)_l{}^m\hat{\psi}_m \tag{1.113}$$

$$\hat{\Theta}^{-1}(g)\hat{\psi}\hat{\Theta}(g) = D^\rho(g)\hat{\psi} \ . \tag{1.114}$$

If $\hat{J}^0_\alpha(x)$ are Noether charge densities such that

$$[\hat{J}^0_\alpha(x), \hat{\psi}_l(y)] = -(T^\rho_\alpha)_l{}^m\hat{\psi}_m(x)\,\delta_{\vec{x},\vec{y}}/a_s^d \ , \tag{1.115}$$

$$[\hat{J}^0_\alpha(\vec{x}), \hat{J}^0_\beta(\vec{y})] = \mathrm{i}\, C^\gamma{}_{\alpha\beta}\hat{J}^0_\gamma(\vec{x})\delta_{\vec{x},\vec{y}}/a_s^d \ , \tag{1.116}$$

then complete generators of local gauge transformations are given by the lattice Gauss law operators $\hat{\mathcal{G}}_\alpha$:

$$\hat{\mathcal{G}}_\alpha(x) = a_s^{-d}\sum_{i=1}^{d}(\hat{L}_{\alpha,i}(x) + \hat{R}_{\alpha,i}(x)) + \hat{J}^0_\alpha(x) \ . \tag{1.117}$$

Like in the continuum, examples of such a charge density would be $\psi^\dagger(x)T^\rho_\alpha\psi(x)$ for fermionic fields. Then the symmetry operator associated to an aggregate gauge transformation parametrized by $\epsilon^\alpha(\vec{x})$ would be

$$\hat{\Theta}[\epsilon] = \exp\left[\mathrm{i}\sum_{\vec{x}} a_s^d\,\epsilon^\alpha(\vec{x})\hat{\mathcal{G}}_\alpha(\vec{x})\right] = \prod_{\vec{x}}\exp\left[\mathrm{i}\, a_s^d\,\epsilon^\alpha(\vec{x})\hat{\mathcal{G}}_\alpha(\vec{x})\right] \ . \tag{1.118}$$

When we speak of a theory being gauge invariant, we mean foremost that its Hamiltonian commutes with the Gauss law operators. The Gauss law operators then give a collection of constants of motion, and we will always consider “allowed” or “physical” lattice states to be those that are invariant under gauge transformations, with

$$\hat{\mathcal{G}}_\alpha(x)\,|\text{phys}\rangle = 0 \tag{1.119}$$

being the lattice realization of $\hat{D}_i \hat{E}^i_\alpha\,|\text{phys}\rangle = g\hat{J}^0_\alpha\,|\text{phys}\rangle$ from the continuum.[2] However, one can in principle change these constants of motion to describe static charge sources, giving rise to charge superselection sectors that are dynamically isolated from each other.

For calculations, one must eventually choose a basis, and in Hamiltonian lattice gauge theory it is common to use one that diagonalizes electric fields rather than their conjugate gauge fields. This is because the Gauss law constraints given above are expressed in terms of electric fields, so characterizing and working in the subspace of allowed states is easier. The quantum numbers characterizing a link state correspond to some complete set of commuting operators (CSCO) for the generators.

For SU(2), the familiar CSCO is $\{\vec{J}\cdot\vec{J}, J_3\}$; $\vec{J}\cdot\vec{J}$ has angular momentum eigenvalues $j(j+1)$ for half-integers j and completely characterizes any irreducible representation of SU(2), while J_3 can have eigenvalues $m = j, j-1, \cdots, -j$. Recalling that a link comes with mutually commuting left and right electric fields,

$$\{\hat{L}_\alpha \hat{L}^\alpha, \hat{L}_3, \hat{R}_\alpha \hat{R}^\alpha, \hat{R}_3\} \tag{1.120}$$

are all mutually commuting, but we also have that the quadratic Casimir is a link invariant, $\hat{L}_\alpha \hat{L}^\alpha = \hat{R}_\alpha \hat{R}^\alpha$. Putting this all together, a basis of SU(2) irreducible representation or “irrep” states can be labeled as

$$|j, m_L, m_R\rangle\ . \tag{1.121}$$

For SU(3), an additional Casimir invariant would be available, $d^{\alpha\beta\gamma}T_\alpha T_\beta T_\gamma$. A CSCO is then given by the two Casimir operators, along with isospin $t_1^2+t_2^2+t_3^2$, isospin projection t_3,

[2] A proper physical Hilbert space should generally respect more symmetries than just gauge constraints, such as translational invariance, parity, etc., but this dissertation is primarily concerned with the gauge constraints.

and hypercharge t_8. The latter three, not being invariants, have distinct quantum numbers on each side of a link, so that states would be of the form

$$|p, q, I_L, I_{3,L}, Y_L, I_R, I_{3,R}, Y_R\rangle \tag{1.122}$$

[with p and q labeling the irreducible representation of SU(3).] We will not have much to say about SU(3) in this dissertation, but details of the formalism have been discussed and worked out, e.g., in Refs. [25,50,51].