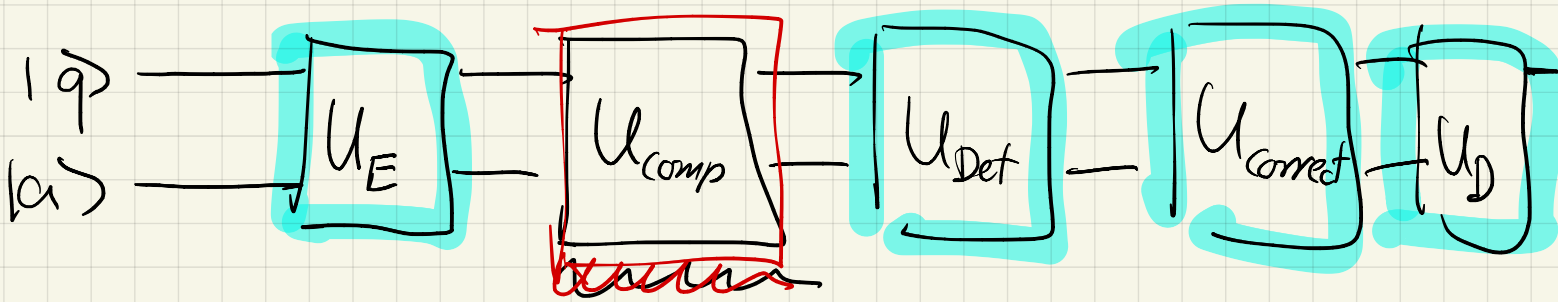
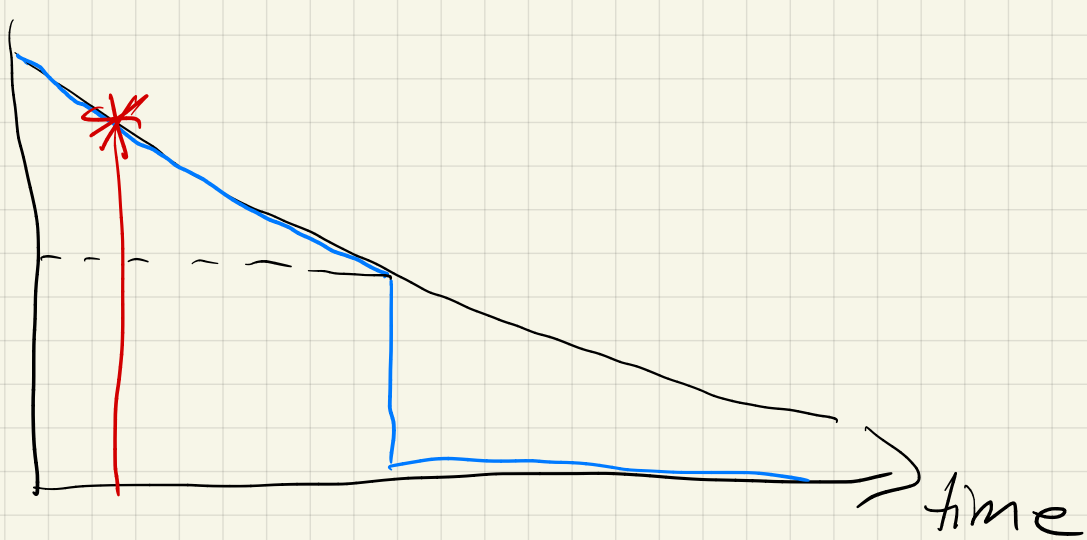
Noise in Quantum Computers & its mitigation

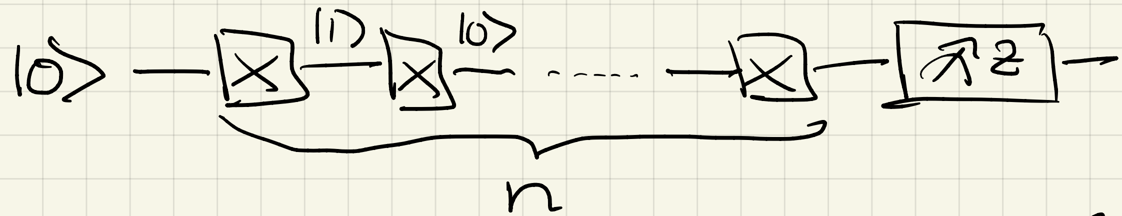
- Noise can and will affect the results that come out of a real quantum computer

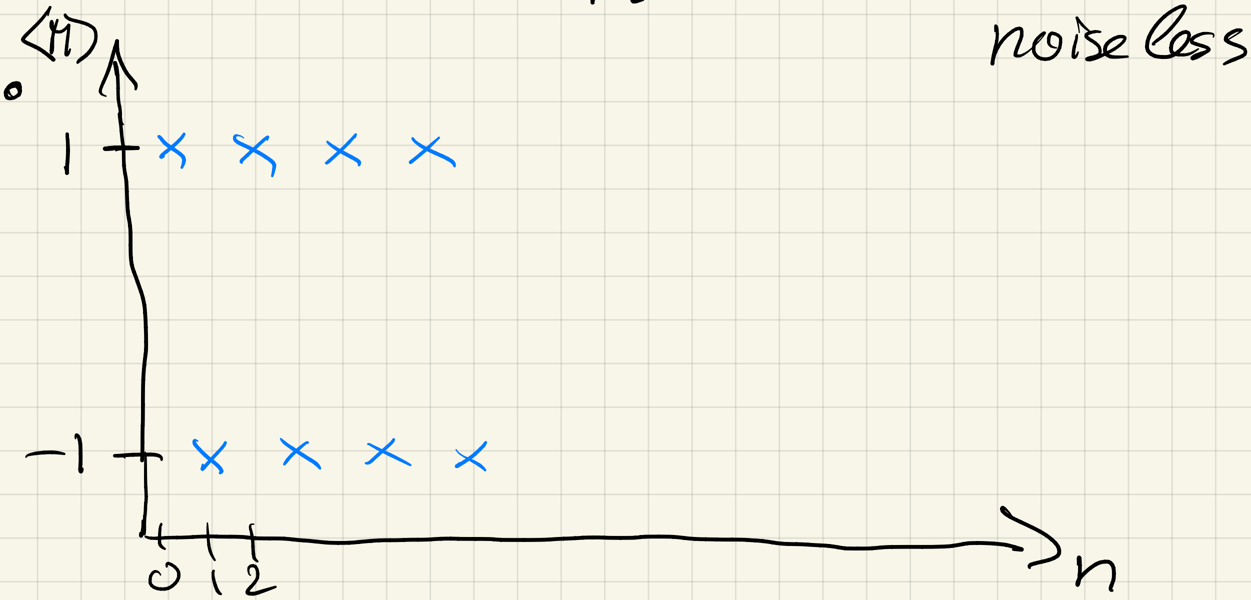
  Noisy Intermediate Scale Quantum (NISQ)



- Adding Error Correcting operations adds more gates ⇒ adds more noise
- Given noise levels of current QCs add more noise than we can remove
- But there is a gate noise threshold, once below, we can drive error to zero



- Error correction: Correct errors while comp. is running
- Error mitigation: Use noisy measurements to extract limit noise results
- 

- 

- 4 General Types of noise:
  1) Coherent error:
     The operations performed are not exact, so instead of applying x-gate, applies some some different gate.
  2) Sampling error:

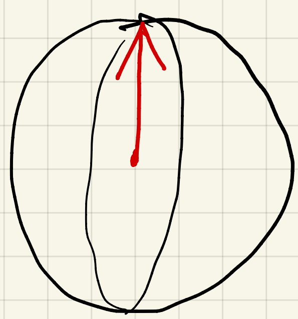
Quantum mechanics is probabilistic, and therefore it has statistical uncertainties.

3) Measurement error:
The probability to measure a pure state $|0\rangle$ or $|1\rangle$ is not 100%.

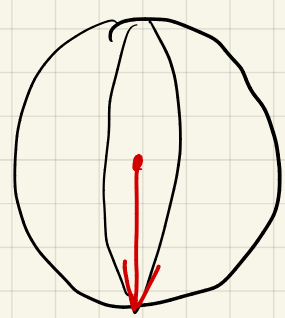
4) Incoherent error:
The Quantum computer is coupled to the environment which will lead to decoherence.

Coherent errors

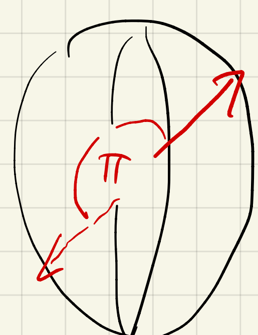
- How does a QC implement X-gate
- Know any 1-qubit state lies on Bloch sphere



$|q\rangle = |0\rangle$

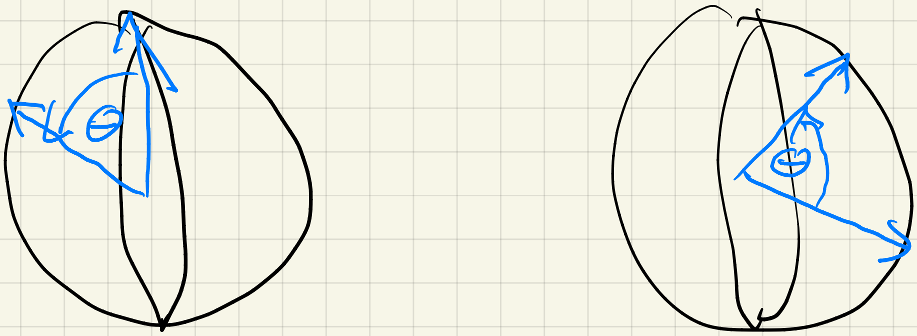


$|q\rangle = |1\rangle$



$|q\rangle = \alpha|0\rangle + \beta|1\rangle$

- Any 1-qubit gate applies some transformation on the Block sphere

- 

- $\hat{X}|0\rangle = |1\rangle$, $\hat{X}|1\rangle = |0\rangle$

  $\hat{X}(\alpha|0\rangle + \beta|1\rangle) = \alpha|1\rangle + \beta|0\rangle$

- $\hat{X} = i R_x(\pi)$

- So when tell a QC to apply X-gate, reall say "apply R_x rotation by angle π"

- $\tilde{X} = i R_x(\pi + \varepsilon) = i R_x(\varepsilon) R_x(\pi) = R_x(\varepsilon)\hat{X}$

$$\boxed{\tilde{X} = R_x(\varepsilon)\hat{X}}$$

- Use simple math $\qquad\qquad \sigma_i^2 = \mathbb{1}$

$$R_x(\varepsilon) = \exp\left(\frac{-i\varepsilon}{2}\hat{\sigma}_x\right)$$

$$= \mathbb{1} - \frac{i\varepsilon}{2}\hat{\sigma}_x + \frac{1}{2!}\left(\frac{-i\varepsilon}{2}\right)^2\hat{\sigma}_x^2 + \frac{1}{3!}\left(\frac{-i\varepsilon}{2}\right)^3\hat{\sigma}_x^3 + \dots$$

$$= \mathbb{1} - \frac{i\varepsilon}{2}\hat{\sigma}_x + \frac{1}{2!}\left(\frac{-i\varepsilon}{2}\right)^2 \mathbb{1} + \frac{1}{3!}\left(\frac{-i\varepsilon}{2}\right)^3 \hat{\sigma}_x + \dots$$

$$= \mathbb{1}\left[1 + \frac{1}{2!}\left(-\frac{i\varepsilon}{2}\right)^2 + \frac{1}{4!}\left(-\frac{i\varepsilon}{2}\right)^4 + \dots\right]$$

$$+ \hat{\sigma}_x\left[-\frac{i\varepsilon}{2} + \frac{1}{3!}\left(-\frac{i\varepsilon}{2}\right)^3 + \dots\right]$$

$$= \cos\left(\frac{\varepsilon}{2}\right)\mathbb{1} - i\sin\left(\frac{\varepsilon}{2}\right)\hat{\sigma}_x$$

$$\xrightarrow{\varepsilon \to 0} \mathbb{1}$$

$$\tilde{X}^n = R_x(\varepsilon)\hat{X}\, R_x(\varepsilon)\hat{X} \times \dots$$

$$= R_x^n(\varepsilon)\,\hat{X}^n$$

$$= R_x(n\varepsilon)\,\hat{X}^n$$

$$= \left[\cos\left(\frac{n\varepsilon}{2}\right)\mathbb{1} - i\sin\left(\frac{n\varepsilon}{2}\right)\hat{\sigma}_x\right]\hat{\sigma}_x^n$$

$$= \cos\left(\frac{n\varepsilon}{2}\right)\hat{\sigma}_x^n - i\sin\left(\frac{n\varepsilon}{2}\right)\sigma_x^{n+1}$$

$$\hat{\tilde{X}} = \cos\frac{\varepsilon}{2}\,\hat{\sigma}_x - i\sin\left(\frac{\varepsilon}{2}\right)\mathbb{1}$$

$$\tilde{X}|0\rangle = \cos\left(\frac{\varepsilon}{2}\right)|1\rangle - i\sin\left(\frac{\varepsilon}{2}\right)|0\rangle$$

$$\frac{N_0}{N} = \sin^2\frac{\varepsilon}{2} \qquad \frac{N_0}{N} - \frac{N_1}{N} = \sin^2\frac{\varepsilon}{2} - \cos^2\frac{\varepsilon}{2}$$

$$\frac{N_1}{N} = \cos^2\frac{\varepsilon}{2} \qquad \xrightarrow{\varepsilon \to 0} -1$$

2) Sampling error

- Know QM is probabilistic
- Say I create $|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$

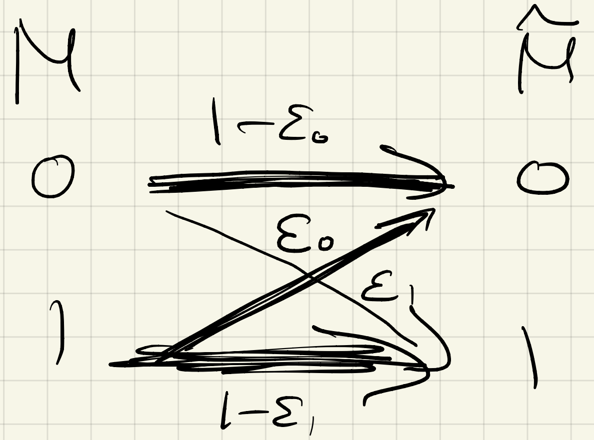
  $|\langle 0|\psi\rangle|^2 = \frac{1}{2} \qquad \langle 1|\psi\rangle = \frac{1}{2}$
- But if I do single measurement
- Probabilities are recovered statistically
- uncertainty scales as $1/\sqrt{N}$
- In general

$$|\psi\rangle = \alpha|0\rangle + \sqrt{1-\alpha^3}\,|1\rangle$$

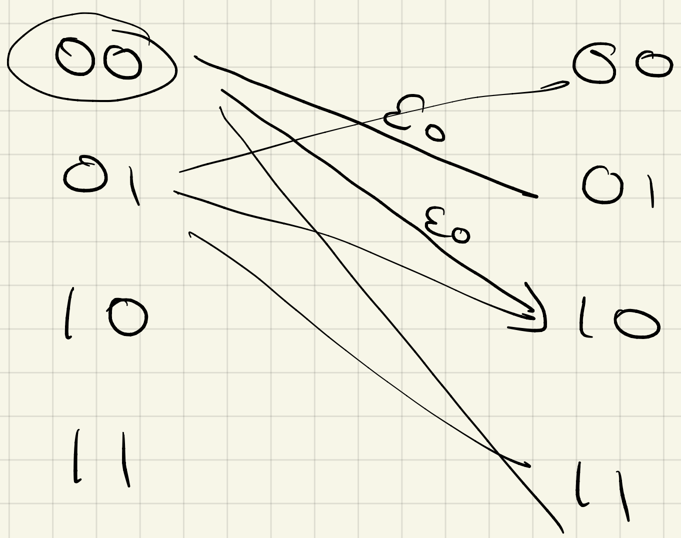
$$\Rightarrow \sigma = \sqrt{\frac{\alpha^2(1-\alpha^2)}{N}}$$

$$|\psi\rangle = |0\rangle$$

3) Measurement error



$2n_a$



Incoherent error

- Arise from interaction with environment
- $|\psi\rangle$ and $|e\rangle$ undergo unitary evolution together
- $$|\psi\rangle \otimes |e\rangle = \sum_k |\psi\rangle \otimes |e_k\rangle \xrightarrow{U} \sum_{k=1}^{n_e} \left[E_k|\psi\rangle\right] \otimes |e_k\rangle, \quad \sum_k E_k^\dagger E_k = 1$$

$$U = \begin{pmatrix} E_0 & & & & \\ & E_1 & & & \\ & & E_2 & & \\ & & & \ddots & \\ & & & & E_\ell \end{pmatrix}$$

- $\rho = |\psi\rangle\langle\psi| \otimes |e\rangle\langle e|$

$$\xrightarrow{U} \sum_{k,\ell} \left[E_k|\psi\rangle\langle\psi|E_\ell^\dagger\right] \otimes |e_k\rangle\langle e_\ell|$$

- $\rho_\psi = \mathrm{Tr}[\rho]$

$$\boxed{|\psi\rangle\langle\psi| \longrightarrow \sum_k \left[E_k|\psi\rangle\langle\psi|E_k^\dagger\right] = \sum_k E_k \rho E_k^\dagger}$$

Krauss representation, E_k Krauss ops.

$$|\psi\rangle\langle\psi| \xrightarrow{U} U|\psi\rangle\langle\psi|U^T$$

$$= \sum E_k|\psi\rangle\langle\psi|E_k$$

- All possible operations on $|\psi\rangle$ are spanned by B_i

example : single qubit, $\underbrace{\mathbb{1}, \hat{\sigma}_x, \hat{\sigma}_y, \hat{\sigma}_z}$

n_q^2 operators in $\{B_i\}$

- $E_k = \sum_i c_{ik} B_i$, $Tr[B_i B_j] = d\,\delta_{ij}$

$$\sum_k E_k \rho E_k^\dagger = \sum_{ij} \underbrace{c_{ik} c_{jk}^*} B_i \rho B_j^\dagger$$
$$= \sum_{ij} d_{ij} B_i \rho B_j^\dagger$$

$$\rho \longrightarrow \sum_{ij} d_{ij} B_i \rho B_j^\dagger$$

$$\underline{d_{ij}} = \sum_m \underline{u_{im}}\, \underline{p_m}\, \underline{u_{jm}^*} \qquad M = A M_D A^\dagger$$

$$A_m = \sum_i u_{im} B_i$$

$$d_{ij} = \sum_m u_{im} p_m u_{jm}^*$$
$$= \sum_{m=1}^{d^2} p_m A_m \rho A_m^\dagger$$

$$\boxed{\rho \longrightarrow \sum_{m=1}^{d^2} p_m A_m \rho A_m^\dagger \qquad \sum_m p_m = 1}$$

$$\{A_m\} = \{\mathbb{1}, \hat{X}, \hat{Y}, \hat{Z}\}$$

$$\rho \longrightarrow p_1 \rho + p_x \underbrace{X \rho X}_{\text{bit flip}} + p_y \underbrace{Y \rho Y}_{\text{bit-phase flip}} + p_z \underbrace{Z \rho Z}_{\text{phase flip}}$$

- Depolarizing error!

All noise channels are equally likely

- $p_x = p_y = p_z = \frac{p}{4} \Rightarrow p_1 = 1 - \frac{3}{4}$

$$\rho \xrightarrow{dep} \left(1-\frac{3}{4}\right)\rho + \frac{p}{4}\left[X\rho X + Y\rho Y + Z\rho Z\right]$$

$$= (1-p)\rho + \frac{p}{4}\left[\mathbb{1}\rho\mathbb{1} + X\rho X + Y\rho Y + Z\rho Z\right]$$

$$\rho = \begin{pmatrix} a & b \\ c & 1-a \end{pmatrix}$$

$$\mathbb{1}\rho\mathbb{1} + X\rho X + Y\rho Y + Z\rho Z = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$$

$$\Rightarrow \boxed{\rho \xrightarrow{dep} (1-p)\rho + \frac{p}{2}\mathbb{1}}$$

$$\mathrm{Tr}\rho \longrightarrow (1-p)\,\mathrm{Tr}(\rho) + p$$

Mitigating depolarizing noise in CNOT gates

$$\rho \longrightarrow \sum_{k,l} p_{kl}\, A_k A_l \rho A_l^{+} A_k^{+}$$

$$A_k = \mathbb{1}, \sigma_x, \sigma_y, \sigma_z$$

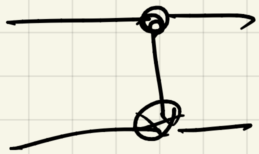
- Depolarizing noise: p_{kl} all equal

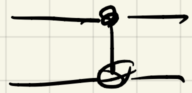
$$\rho \longrightarrow (1-\varepsilon)\rho + \varepsilon \frac{1}{2^{n_q}}\mathbb{1}$$

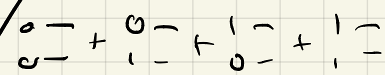
- Combine with CNOT

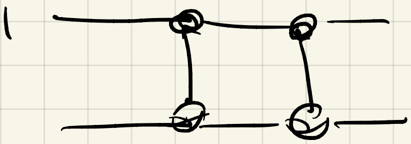
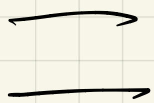
$$\rho \longrightarrow U_c \rho U_c$$

- $$\rho \longrightarrow (1-\varepsilon)\underbrace{U_c \rho U_c}_{\mathrm{Tr}[\,]=1} + \varepsilon \underbrace{\left(\frac{\mathbb{1}_{4\times 4}}{4}\right)}_{\mathrm{Tr}[\,]=1}$$

 $=$ $\begin{cases} \text{} \end{cases}$

 prob $(1-\varepsilon)$

 prob $\frac{\varepsilon}{4}$

-  $=$ 

$$\mathrm{CNOT}^2 = \mathbb{1} \qquad U_c = U_c^\dagger$$

- $$\rho \xrightarrow{2\times} (1-\varepsilon)\, U_c \left[(1-\varepsilon) U_c \rho U_c + \frac{\varepsilon}{4}\mathbb{1}\right] U_c + \frac{\varepsilon}{4}\mathbb{1}$$

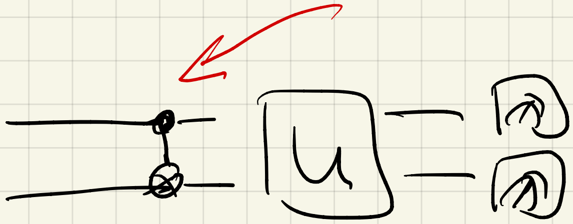
$$= (1-\varepsilon)^2 \rho + (1-\varepsilon)\varepsilon\left(\mathbb{1}/4\right) + \varepsilon\left(\mathbb{1}/4\right)$$

$$= (1-\varepsilon)^2 \rho + \left[1-(1-\varepsilon)^2\right]\left(\mathbb{1}/4\right)$$

- $$\rho \xrightarrow{3\times} = (1-\varepsilon)^3 U_c \rho U_c + \left[1-(1-\varepsilon)^3\right]\left(\mathbb{1}/4\right)$$

- $$\rho \xrightarrow{2r+1} (1-\varepsilon)^{2r+1} U_c \rho U_c + \left[1-(1-\varepsilon)^{2r+1}\right]\left(\mathbb{1}/4\right)$$

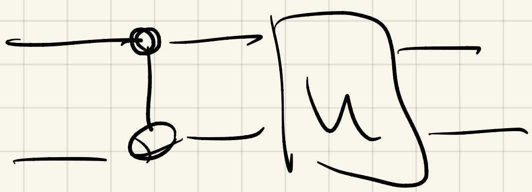
$$\simeq \underbrace{(1-r\varepsilon) U_c \rho U_c + r\varepsilon\left(\mathbb{1}/4\right)}$$



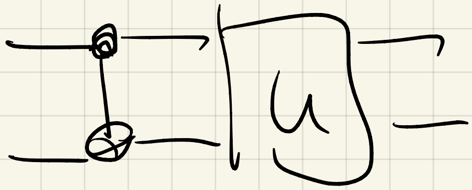
$$\rho \to \rho(r) = U\left[(1-r\varepsilon)U_c \rho U_c + r\varepsilon\,(\mathbb{1}/4)\right]U^\dagger$$

$$= (1-r\varepsilon)\,U U_c \rho U_c U^\dagger + r\varepsilon\,(\mathbb{1}/4)$$

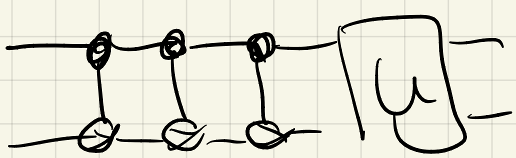
$$\mathrm{Tr}[O\rho(r)] = (1-r\varepsilon)\,\boxed{\mathrm{Tr}[O\, U U_c \rho U_c U^\dagger]}$$
$$+ r\varepsilon$$

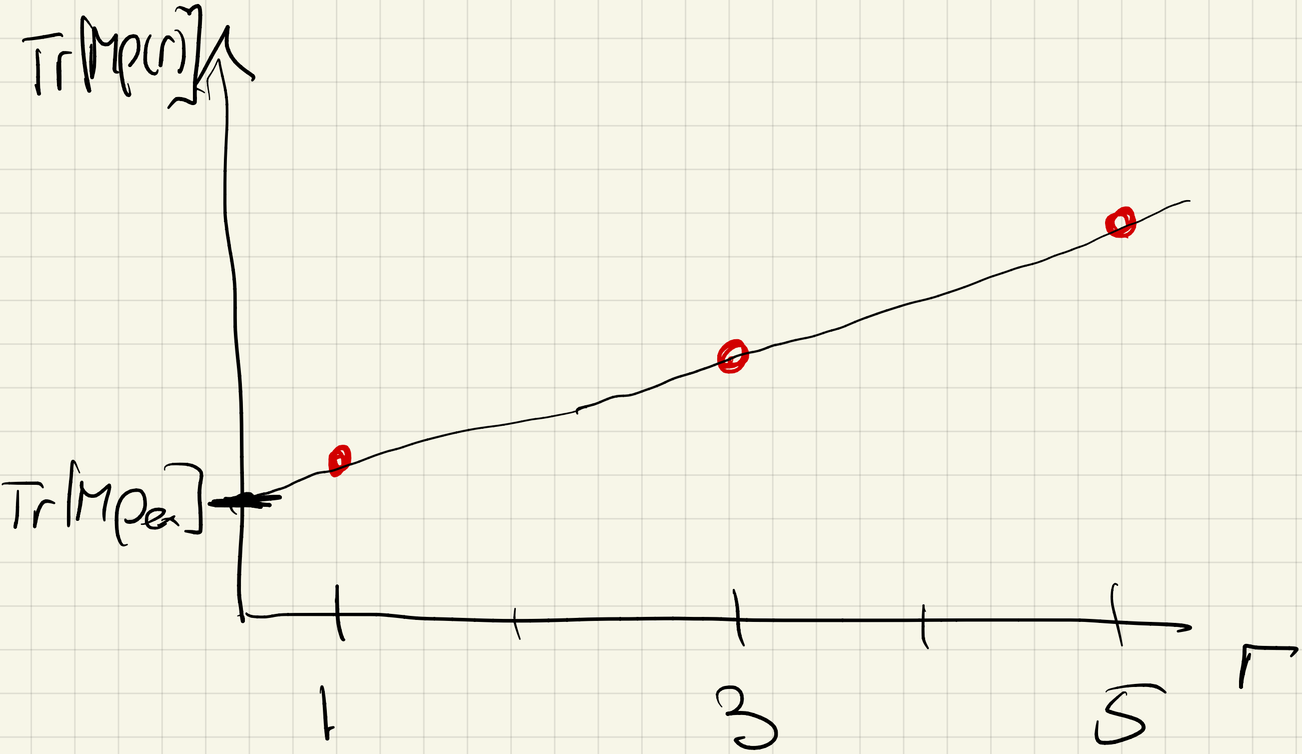


$r=1$



$r=3$





- Zero Nois Extrapolation

1) Measure a noisy circuit at different noise levels

2) Extrapolate the results at diffrent levels to zero.

$$\boxed{\mathrm{Tr}\left[O\rho(r)\right]} = a(r)\langle O\rangle_{ex} + b(r)\langle O\rangle_{err},$$

$$+\,c(1)\langle O\rangle_{err_2}$$

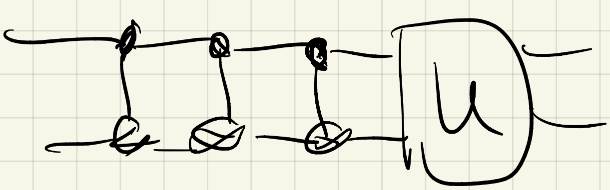
$$\mathrm{Tr}[O\rho(1)] = a_1\langle O\rangle_{ex} + b_1\langle O\rangle_{err}$$

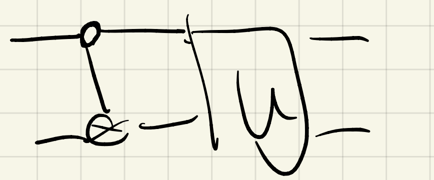
$$\mathrm{Tr}[O\rho(3)] = a_3\langle O\rangle_{ex} + b_3\langle O\rangle_{err}$$

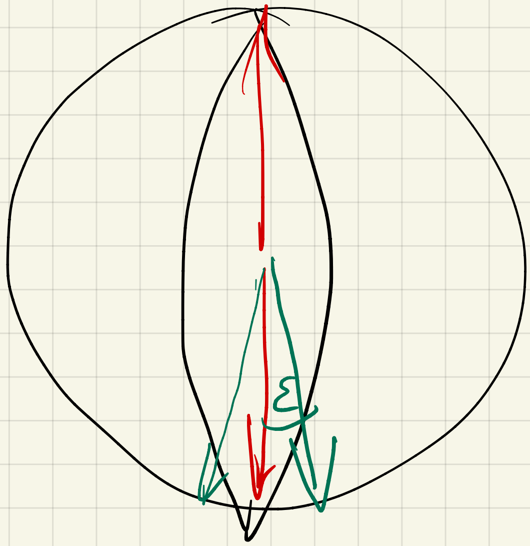
$$b_1\,\mathrm{Tr}[O\rho(3)] - b_3\,\mathrm{Tr}[O\rho(1)]$$

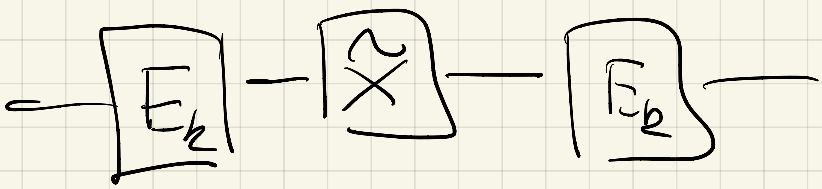
$$= [b_1 a_3 - b_3 a_1]\langle O\rangle_{ex}$$

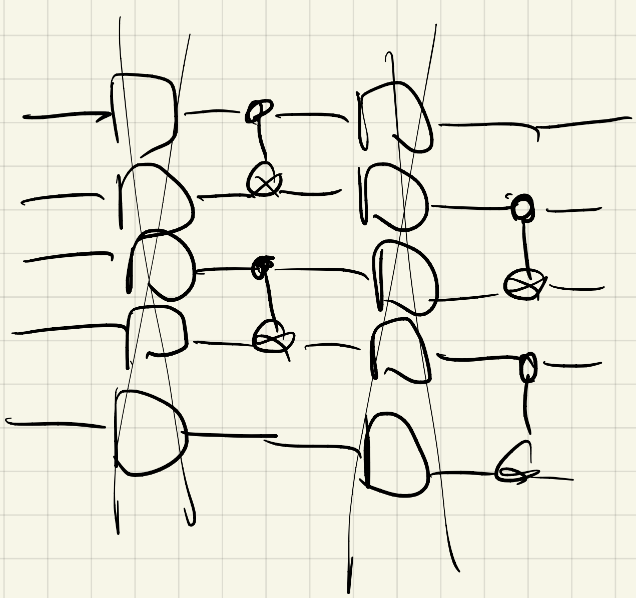
$$\Rightarrow \langle O\rangle_{ex} = \frac{b_1\,\mathrm{Tr}[O\rho(3)] - b_3\,\mathrm{Tr}[O\rho(1)]}{a_1 b_3 - a_3 b_1}$$






ε


E_k
$\tilde{X}$
E_b



$$\rho \longrightarrow p_1 \rho + p_x X\rho X + p_y Y\rho Y + p_z Z\rho Z$$

$$\boxed{\underline{p_1}, \underline{p_x}, \underline{p_y}, \underline{p_z}}$$

Randomized compiling