Observables for scattering on targets with any spin

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Motivation

Matrix elements for Operators of composite particles with arbitrary spin

- Covariant decomposition of matrix element in independent non-perturbative objects

$$e.g.\ \text{one-photon-exch.:}\quad \langle p', s' \,|j^\mu|\, p, s\rangle = \bar{u}_{(p',s')}\Gamma^\mu_{(p',p)}u_{(p,s)} \underset{\text{spin}1/2}{\longrightarrow} \bar{u}_{(p',s')}\left[F_{1\,(t^2)}\gamma^\mu - F_{2\,(t^2)}\frac{i}{2m}\sigma^{\mu\nu}q_\nu\right]u_{(p,s)}$$

Spin-j fields embedded in objects with $> 2j+1$ components

- Polarization four vector (ε) for spin 1 → $p_\mu \epsilon^\mu(p,s) = 0$
- Rarita Schwinger for spin 3/2 → $\gamma^\mu \psi_\mu(p,s) = 0$
- Need for constraints, subsidiary conditions

Use $(2j+1)$-component spinors

- Via SL(2,C) fundamental rep tensor products [Zwanziger 60s, Polyzou '18]
- Weinberg's construction [64-65] (not yet applied in this context)

Advantages of Weinberg's construction

- Systematic approach, *e.g.*, for any spin j
- Covariant "multipole" physical interpretation
- For parity conserving interactions a generalized Dirac algebra is obtained
- "Basic" construction and implementation. From su(2) $\rightarrow$ su(N) algebra
- Easy to switch between forms of dynamics (instant form, light front)
- Use only exact degrees of freedom (chiral reps), no need for constraints
- No kinematic singularities (improved analyticity properties of operators)

Outline

- ~~Review of~~ Flying over Weinberg's formalism
- Building uponWeinberg's formalism: Bilinear Calculus
- Simplification (I): Algorithm for Construction of t-tensors
- Simplification (II): Algebra of t-tensors

See in Back-up slides

- Examples: Spin 1/2, Spin 1
- Different forms of dynamics: Canonical (Instant Form), Light Front

Weinberg's "Feynman rules for Any Spin" [1964]

- Algebra for Generators of the Lorentz group

$$[\mathbb{J}_l, \mathbb{J}_m] = i\epsilon_{lmn}\mathbb{J}_n \ , \quad [\mathbb{J}_l, \mathbb{K}_m] = i\epsilon_{lmn}\mathbb{K}_n \ , \quad [\mathbb{K}_l, \mathbb{K}_m] = -i\epsilon_{lmn}\mathbb{J}_n$$

- Two independent su(2) subalgebras $\rightarrow$ irreps (j_A, j_B)

$$\mathbb{A}_m = \frac{1}{2}(\mathbb{J}_m + i\mathbb{K}_m) \quad , \quad \mathbb{B}_m = \frac{1}{2}(\mathbb{J}_m - i\mathbb{K}_m)$$

$$[\mathbb{A}_l, \mathbb{A}_m] = i\epsilon_{lmn}\mathbb{A}_n \ , \quad [\mathbb{B}_l, \mathbb{B}_m] = i\epsilon_{lmn}\mathbb{B}_n \ , \quad [\mathbb{A}_l, \mathbb{B}_m] = 0$$

- Simplest irreps that contain spin-j $\rightarrow$ $(2j + 1$ components$)$
  - Right-handed $(j, 0)$: $\mathbb{K}_m \rightarrow -i\mathbb{J}_m$
  - Left-handed $(0, j)$: $\mathbb{K}_m \rightarrow +i\mathbb{J}_m$

[Wigner(1939)]

Causal chiral fields (massive, left- right-handed)

- Lorentz invariant S-matrix using a Hamiltonian density built up from causal fields

$$U_{[\Lambda,a]}\psi_{\sigma\,(x)}U^{-1}_{[\Lambda,a]}=\sum_{\sigma'}\left(D^{(j)}_{[\Lambda^{-1}]}\right)_{\sigma\sigma'}\psi_{\sigma'\,(\Lambda x+a)}$$

- No EoM for chiral fields (only obey KG eq.)

- Spinors appearing in the fields (not invariants, depend on choice boost)

Canonical $\rightarrow$

$$D^{(j)}_{[L(p)]}=e^{-\hat{p}\cdot\vec{J}^{(j)}\theta}$$

$$\bar{D}^{(j)}_{[L(p)]}=e^{+\hat{p}\cdot\vec{J}^{(j)}\theta}$$

Propagator of chiral fields

- Numerator (invariant)

$$\Pi^{(j)}_{\sigma\sigma'}(\vec{p},\omega) = m^{2j} D^{(j)}_{\sigma\sigma'}[L(\vec{p})] \left(D^{(j)}_{\sigma'\sigma''}[L(\vec{p})] \right)^{\dagger} = m^{2j} \left(e^{-2\hat{p}\cdot\vec{J}^{(j)}\theta} \right)_{\sigma\sigma'}$$

$$\bar{\Pi}^{(j)}_{\sigma\sigma'}(\vec{p},\omega) = m^{2j} \bar{D}^{(j)}_{\sigma\sigma'}[L(\vec{p})] \left(\bar{D}^{(j)}_{\sigma'\sigma''}[L(\vec{p})] \right)^{\dagger} = m^{2j} \left(e^{2\hat{p}\cdot\vec{J}^{(j)}\theta} \right)_{\sigma\sigma'}$$

- Introduction of 2j-rank t-tensors

$$\Pi^{(j)}_{\sigma\sigma'}(\vec{p},\omega) = t^{\mu_1\mu_2\ldots\mu_{2j}}_{\sigma\sigma'} p_{\mu_1} p_{\mu_2} \ldots p_{\mu_{2j}}$$

totally symmetric

$$\bar{\Pi}^{(j)}_{\sigma\sigma'}(\vec{p},\omega) = \bar{t}^{\mu_1\mu_2\ldots\mu_{2j}}_{\sigma\sigma'} p_{\mu_1} p_{\mu_2} \ldots p_{\mu_{2j}}$$

covariantly traceless

$$g_{\mu_k\mu_l} t^{\mu_1\ldots\mu_k\ldots\mu_l\ldots\mu_{2j}}_{\sigma\sigma'} = 0$$

- Central roll of t-tensors

$$D^{(j)}_{[L(p)]} = t^{\mu_1\mu_2\ldots\mu_{2j}} \tilde{p}_{\mu_1} \tilde{p}_{\mu_2} \ldots \tilde{p}_{\mu_{2j}}$$

used to construct boosts/spinors

$$\bar{D}^{(j)}_{[L(p)]} = \bar{t}^{\mu_1\mu_2\ldots\mu_{2j}} \tilde{p}_{\mu_1} \tilde{p}_{\mu_2} \ldots \tilde{p}_{\mu_{2j}}$$

Instant form dynamics (Canonical)

$\tilde{p}^{\mu}$ not four-vectors

$$\tilde{p}^{\mu}{}_{\mathrm{C}} = \sqrt{\frac{1}{2m(m+p^0)}}(p^0 + m, \vec{p}\,)$$

Bi-Spinors (direct sum representation $(j,0)\bigoplus(0,j)$)

- For Parity conserving interactions the direct sum of both chiral representations is used (like the spin 1/2 case)
- Boosts and bispinor (Weyl rep.)

$$u^{(j)}_{(p,s)} = \mathcal{D}^{(j)}_{[L_p]}\mathring{u}^{(j)}_s = \begin{pmatrix} D^{(j)}_{[L_p]} & 0 \\ 0 & \bar{D}^{(j)}_{[Lp]} \end{pmatrix} \mathring{u}^{(j)}_s = \begin{pmatrix} \Pi^{(j)}_{(\tilde{p})} & 0 \\ 0 & \bar{\Pi}^{(j)}_{(\tilde{p})} \end{pmatrix} \mathring{u}^{(j)}_s$$

$$\mathring{u}^{(j)}_s = \begin{pmatrix} \mathring{\phi}^{(j)}_s \\ \mathring{\phi}^{(j)}_s \end{pmatrix} \quad , \quad \mathring{\phi}^{(j)}_s = m^j \begin{pmatrix} \vdots \\ 1 \\ \vdots \end{pmatrix} \quad \text{(1 in the s-th position)}$$

- Adjoint bispinor ($\mathcal{D}^{\dagger}_{[\Lambda]} = \beta\mathcal{D}^{-1}_{[\Lambda]}\beta$)

$$\bar{u}^{(j)}_{(p,s)} = u^{(j)}_{(p,s)}{}^{\dagger}\beta = \mathring{u}^{(j)\dagger}_s\mathcal{D}^{(j)}_{[L_p]}{}^{\dagger}\beta = \mathring{u}^{(j)\dagger}_s \begin{pmatrix} 0 & \Pi^{(j)}_{(\tilde{p})} \\ \bar{\Pi}^{(j)}_{(\tilde{p})} & 0 \end{pmatrix} \quad ; \quad \beta = \begin{pmatrix} 0 & \mathbf{1}^{(j)} \\ \mathbf{1}^{(j)} & 0 \end{pmatrix}$$

Dirac Eq. & Gamma matrices

- The bispinor satisfy the Dirac eq.

$$\left(\gamma^{\mu_1\cdots\mu_{2j}} p_{\mu_1}\cdots p_{\mu_{2j}} - m^{2j}\right) u^{(j)}_{(p,s)} = 0$$

$$\bar{u}^{(j)}_{(p,s)} \left(\gamma^{\mu_1\cdots\mu_{2j}} p_{\mu_1}\cdots p_{\mu_{2j}} - m^{2j}\right) = 0$$

- The gamma matrices appear from

$$\begin{pmatrix} D^{(j)}_{[L_p]} & 0 \\ 0 & \bar{D}^{(j)}_{[Lp]} \end{pmatrix} \begin{pmatrix} 0 & D^{(j)}_{[L_p]} \\ \bar{D}^{(j)}_{[Lp]} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \left(\Pi^{(j)}_{(\tilde{p})}\right)^2 \\ \left(\bar{\Pi}^{(j)}_{(\tilde{p})}\right)^2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \Pi^{(j)}_{(p)} \\ \bar{\Pi}^{(j)}_{(p)} & 0 \end{pmatrix} = \gamma^{\mu_1\cdots\mu_{2j}} p_{\mu_1}\cdots p_{\mu_{2j}}$$

$$\gamma^{\mu_1\cdots\mu_{2j}} = \begin{pmatrix} 0 & t^{\mu_1\cdots\mu_{2j}} \\ \bar{t}^{\mu_1\cdots\mu_{2j}} & 0 \end{pmatrix} \quad ; \quad \beta = \gamma^{0\cdots 0} \quad ; \quad \gamma_5 = \begin{pmatrix} -\mathbf{1}^{(j)} & 0 \\ 0 & \mathbf{1}^{(j)} \end{pmatrix}$$

Dirac Bilinear Calculus Generalization

Generalized Bilinears

- $\bar{u}^{(j)}_{(p_f,s_f)}\Gamma u^{(j)}_{(p_i,s_i)} = \frac{1}{(m_f m_i)^{2j}} \overset{\circ}{u}{}^{(j)\dagger}_{s_f} \begin{pmatrix} 0 & t^{\beta_1\cdots}\tilde{p}^f_{\beta_1\cdots} \\ \bar{t}^{\beta_1\cdots}(\tilde{p}^f_{\beta_1\cdots})^* & 0 \end{pmatrix} \Gamma \begin{pmatrix} t^{\alpha_1\cdots}\tilde{p}^i_{\beta_1\cdots} & 0 \\ 0 & \bar{t}^{\bar{\alpha}_1\cdots}(\tilde{p}^i_{\alpha_1\cdots})^* \end{pmatrix} \overset{\circ}{u}{}^{(j)}_{s_i}$

- Dirac basis: $4(2j+1)^2$ ind. elements

  $\Gamma = \mathbf{1}$ (1) , $\gamma^{\mu_1\cdots\mu_{2j}}$ $(2j+1)^2$, $\gamma_5\gamma^{\mu_1\cdots\mu_{2j}}$ $(2j+1)^2$, γ_5 (1)

  $[\gamma^{\mu_1\cdots\mu_{2j}},\gamma^{\nu_1\cdots\nu_{2j}}]$ $2\sum_{n=1,3,\cdots}^{2j}(2n+1)$

  $\{\gamma^{\mu_1\cdots\mu_{2j}},\gamma^{\nu_1\cdots\nu_{2j}}\}_{\text{traceless}}$ $2\sum_{n=0,2,\cdots}^{2j}(2n+1)$

  Examples

  - Spin-1/2 (16): $\mathbf{1}(1)$, $\gamma^\mu(4)$, $[\gamma^\mu,\gamma^\nu](6)$, $(\{\gamma^\mu,\gamma^\nu\}-2g^{\mu\nu})\,(0)$, $\gamma^\mu\gamma_5(4)$, $\gamma_5(1)$
  - Spin-1 (36): $\mathbf{1}(1)$, $\gamma^{\mu\nu}(9)$, $[\gamma^{\mu_1\mu_2},\gamma^{\mu_3\mu_4}](6)$, $\{\gamma^{\mu_1\mu_2},\gamma^{\mu_3\mu_4}\}_{\text{trless}}(10)$, $\gamma^{\mu\nu}\gamma_5(9)$, $\gamma_5(1)$

- Matrix elements of Operators, covariant Density matrices, Amplitudes, $\cdots$

Dirac Bilinear Calculus Generalization: On-Shell Identities

Generalized Gordon Identities

- Dirac Equation $\left(\gamma^{\mu_1\cdots\mu_{2j}} p_{\mu_1}\dots p_{\mu_{2j}} - m^{2j}\right) u_p^s = 0$ leads to On-Shell (Gordon) identities

$$u_{p'}^{s'}\left(\Gamma\right) u_p^s = \frac{1}{2\bar{m}^{2j}} u_{p'}^{s'} \left(\left\{\not{P}^{(j)}, \Gamma\right\} + \frac{1}{2}\left[\not{\Delta}^{(j)}, \Gamma\right]\right) u_p^s$$

$$0 = u_{p'}^{s'} \left(\frac{1}{2}\left\{\not{\Delta}^{(j)}, \Gamma\right\} + \left[\not{P}^{(j)}, \Gamma\right]\right) u_p^s$$

$$\bar{m}^{2j} = \frac{1}{2}\left(m'^{2j} + m^{2j}\right)$$

$$P_{\mu_1\dots\mu_{2j}} = \frac{\bar{m}^{2j}}{2}\left(\frac{p'_{\mu_1}\cdots p'_{\mu_{2j}}}{m'^{2j}} + \frac{p_{\mu_1}\cdots p_{\mu_{2j}}}{m^{2j}}\right)$$

$$\Delta_{\mu_1\dots\mu_{2j}} = \bar{m}^{2j}\left(\frac{p'_{\mu_1}\cdots p'_{\mu_{2j}}}{m'^{2j}} - \frac{p_{\mu_1}\cdots p_{\mu_{2j}}}{m^{2j}}\right)$$

$$\rightarrow \quad P^{\mu_1\cdots\mu_{2j}}_{(p',p)} = P^{\mu_1\cdots\mu_{2j}}_{(p,p')}$$

$$\Delta^{\mu_1\cdots\mu_{2j}}\left(p',p\right) = -\Delta^{\mu_1\cdots\mu_{2j}}\left(p,p'\right)$$

$$P^{\mu_1\cdots\mu_{2j}}\Delta_{\mu_1\dots\mu_{2j}} = 0$$

- Gordon identity separates general bilinears into convection and magnetization currents.
  Useful to reduce independent Dirac structures

Simplification (I): Algorithm for construction of t-tensors

Insightful construction for the t-tensors

- The 0-th degree polynomial in the J's is always $t^{0\dots0} = \mathbf{1}$

- The linear polynomials are the Rotation Group Generators $t^{0\dots i\dots0} = \frac{2}{2j} J_i = \frac{1}{j} J_i$

- From pairwise symmetrizations of the rotation generators

$$t^{0\dots m\dots0\dots n\dots0} = t^{mn0\dots0} = \frac{1}{\frac{(2j)!}{2!(2j-2)!}} \left(\{J_m, J_n\} - \frac{1}{3}\delta_{mn} \sum_{r=1}^{3} \{J_r, J_r\} \right) + \frac{1}{3} t^{0\dots0} \delta_{mn}$$

$$= \frac{j}{(2j-1)} \left(\{t^{m0\dots0}, t^{n0\dots0}\} - \frac{1}{j} \delta_{mn} t^{0\dots0} \right)$$

Simplification (I): Algorithm for construction of t-tensors

- Continues for higher orders
  - Matrices have more and more off-diagonal elements

$$t^{lmn0\ldots0} = t^{0\ldots0l0\ldots0m0\ldots0n0\ldots0} = \frac{j}{(2j-2)}\frac{1}{3}\Big(\{t^{l0\ldots0}, t^{mn0\ldots0}\} + \{t^{m0\ldots0}, t^{nl0\ldots0}\} + \{t^{n0\ldots0}, t^{lm0\ldots0}\}$$
$$-\frac{2}{j}\{\delta_{lm}t^{n0\ldots0} + \delta_{ln}t^{m0\ldots0} + \delta_{mn}t^{l0\ldots0}\}\Big)$$

- Construction stops after j steps (Cayley-Hamilton) $(J-s)(J-s-1)\ldots(J+s) = 0$

- t-tensors contain a basis for su(N=2j+1) (Universal Enveloping Algebra)

- A basis to decompose operators with physical interpretation for each term. Multipole expansion → mono-, di-, quadrupole, ...

$$\hat{O} = Tr\,[O]\,\mathbf{1} + Tr\,[OJ_i]\,J_i + Tr\,[OJ_{ij}]\,J_{ij} + \cdots = \langle O\rangle\,\mathbf{1} + O_iJ_i + O_{ij}J_{ij} + \cdots$$

t^{μ}-tensor for Spin 1/2

0-th order terms in $J_i^{(1/2)}$: $t^0 = \mathbf{1}$

Linear terms in $J_i^{(1/2)}$: $t^i = \frac{1}{1/2} J_i^{(1/2)} = \sigma_i$
(Pauli matrices)

$$J_1^{(1/2)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \;,\; J_2^{(1/2)} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \;,\; J_3^{(1/2)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Quadratic terms in $J_i^{(1/2)}$

$$(J^{(1/2)} - \tfrac{1}{2}\mathbf{1})(J^{(1/2)} + \tfrac{1}{2}\mathbf{1}) = 0 \implies (J^{(1/2)})^2 = c_0\mathbf{1} + c_2 J^{(1/2)}$$

$t^{\mu\nu}$-tensor for Spin 1

0-th order terms in $J_i^{(1)}$: $t^{00} = \mathbf{1}$

Linear terms in $J_i^{(1)}$: $t^{0i} = J_i^{(1)}$

$$t^{01} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} , \; t^{02} = \frac{i}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} , \; t^{03} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Quadratic terms in $J_i^{(1)}$: $t^{ij} = \{J_i^{(1)}, J_j^{(1)}\} - \mathbf{1}\delta_{ij}$

$$t^{11} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} , \; t^{22} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} , \; t^{33} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$t^{12} = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} , \; t^{13} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} , \; t^{23} = \frac{i}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

Cubic terms in $J_i^{(1)}$: $(J^{(1)} - 1)(J^{(1)})(J^{(1)} + 1) = 0 \implies (J^{(1)})^3 = c_0\mathbf{1} + c_2 J^{(1)} + c_3 (J^{(1)})^2$

Simplification (II): Algebra of t-tensors

Reduction for Cubic Monomials

- Monomials always appear in bilinear calculus with an alternating "barring" pattern

$$t^{\mu_1\cdots\mu_{2j}}\bar{t}^{\nu_1\cdots\nu_{2j}}t^{\rho_1\cdots\rho_{2j}} = \frac{1}{[(2j)!]^2}\underset{\{\nu_1\ldots\nu_{2j}\}}{\mathcal{S}}\underset{\{\rho_1\ldots\rho_{2j}\}}{\mathcal{S}}\left(\bar{\mathcal{C}}^{\mu_1\nu_1\rho_1\beta_1}\bar{\mathcal{C}}^{\mu_2\nu_2\rho_2\beta_2}\cdots\bar{\mathcal{C}}^{\mu_{2j}\nu_{2j}\rho_{2j}\beta_{2j}}\right)t_{\beta_1\cdots\beta_{2j}}$$

- The coefficient tensors: $\bar{\mathcal{C}}^{\mu\rho\alpha\beta} = g^{\mu\rho}g^{\alpha\beta} - g^{\mu\alpha}g^{\rho\beta} + g^{\mu\beta}g^{\rho\alpha} + i\epsilon^{\mu\rho\alpha\beta}$
  (Invariant tensors)

- Compare with:

$$\mathrm{Tr}\left\{\gamma^\mu\gamma^\rho\gamma^\alpha\gamma^\beta\right\} = 4\left(g^{\mu\rho}g^{\alpha\beta} - g^{\mu\alpha}g^{\rho\beta} + g^{\mu\beta}g^{\rho\alpha}\right)$$

$$\mathrm{Tr}\left\{\gamma^\mu\gamma^\rho\gamma^\alpha\gamma^\beta\gamma_5\right\} = 4i\varepsilon^{\mu\rho\alpha\beta}$$

- Trading matrix multiplication by number multiplication

Reduction for Quadratic Monomials

- Monomials always appear in bilinear calculus with an alternating "barring" pattern

$$t^{\mu_1\cdots\mu_{2j}}\bar{t}^{\nu_1\cdots\nu_{2j}}\left(t^{\rho_1\cdots\rho_{2j}}\eta_{\rho_1}\cdots\eta_{\rho_{2j}}\right)=\frac{1}{(2j)!}\underset{\{\nu_1\ldots\nu_{2j}\}}{\mathcal{S}}\left(\bar{\mathcal{C}}^{\mu_1\nu_1\rho_1\beta_1}\eta_{\rho_1}\bar{\mathcal{C}}^{\mu_2\nu_2\rho_2\beta_2}\eta_{\rho_2}\cdots\bar{\mathcal{C}}^{\mu_{2j}\nu_{2j}\rho_{2j}\beta_{2j}}\eta_{\rho_{2j}}\right)t_{\beta_1\cdots\beta_{2j}}$$

- The condition $t^{\rho_1\cdots\rho_{2j}}\eta_{\rho_1}\cdots\eta_{\rho_{2j}}=\mathbf{1}_{(2j+1)\times(2j+1)}$ defines η_ρ

  in Lorentz coordinates $t^{0\cdots0}=\mathbf{1}$, thus $\eta^\mu\to\eta^\mu_{\mathrm{C}}=(1,0,0,0)$

- General result $(\bar{\mathcal{D}}^{\mu\rho\alpha}_{(\eta)}\equiv\bar{\mathcal{C}}^{\mu\rho\sigma\alpha}\eta_\sigma-g^{\mu\rho}\eta^\alpha=-g^{\rho\alpha}\eta^\mu+g^{\mu\alpha}\eta^\rho+i\epsilon^{\mu\rho\sigma\alpha}\eta_\sigma)$

$$t^{\mu_1\cdots\mu_{2j}}\bar{t}^{\rho_1\cdots\rho_{2j}}=\tfrac{1}{(2j)!}\underset{\{\rho_1\ldots\rho_{2j}\}}{\mathcal{S}}\left[\sum_{n=0}^{2j}\left(\prod_{l=1}^{n}\bar{\mathcal{D}}^{\mu_l\rho_l\alpha_l}\prod_{k=n+1}^{2j}g^{\mu_k\rho_k}\eta^{\alpha_k}+\underbrace{\cdots}_{\text{choices for }l,k}\right)\right]t_{\alpha_1\cdots\alpha_{2j}}$$

Covariant (sl(2,C)) Multipole expansion

Spin 1 Example: EM Current

Using spinor representation: $\langle p', s' | j^\mu(0) | p, s \rangle = \mathring{\phi}^{(1)}_{s'} \Gamma^\mu_{(p',p)} \mathring{\phi}^{(1)}_s$

$$m^2 \Gamma^\mu_{(p',p)} = 2P^\mu \left[P^2 \mathbf{1} G_C\left(Q^2\right) - \Delta^\rho \Delta^\sigma \left(t_{\rho\sigma} - \frac{1}{3} g_{\rho\sigma} \mathbf{1} \right) G_Q\left(Q^2\right) \right]$$
$$- i\epsilon^{\mu\rho\sigma\lambda} \left[\Delta_\rho P_\sigma \left(t_{\lambda\nu} - \frac{1}{3} g_{\lambda\nu} \mathbf{1} \right) n_t^\nu G_M\left(Q^2\right) \right]$$

$P = \frac{1}{2}(p' + p)$

$\Delta = p' - p \quad (\Delta^2 = -Q^2)$

$n_t^\nu = (1, 0, 0, 0)$

Using polarization vectors: $\langle p', s' | j^\mu(0) | p, s \rangle = \varepsilon^{*\,\alpha}_{s'}(p') \Gamma^\mu_{\alpha\beta}(P, \Delta) \varepsilon^\beta_s(p)$

[Wang & Lorcé (2022)]

$$\Gamma^{\mu\alpha\beta} = 2P^\mu \left(\Pi^{\alpha\beta} G_C\left(Q^2\right) - \frac{\Delta^\rho \Delta^\sigma \left(\Sigma_{\rho\sigma}\right)^{\alpha\beta}}{2m^2} \frac{P^2}{m^2} G_Q\left(Q^2\right) \right)$$
$$- i\epsilon^{\mu\rho\sigma\lambda} \left(\frac{\Delta_\rho P_\sigma \left(\Sigma_\lambda\right)^{\alpha\beta}}{\sqrt{P^2}} G_M\left(Q^2\right) \right)$$

Summary

- Weinberg's construction allows for an efficient and manifestly covariant calculation of currents for any spin
- Central (and multifaceted) role for the covariant t-tensors
- Simple algorithm. Only need to know the matrices for the Generators of rotations in the representation of interest.
- Covariant $sl(2,\mathrm{C})$-multipole basis for operators.
  more transparent physical interpretation
- Universality of the method for any spin.
  intuition on spin 1/2 can be carried over to higher spin
- No need to work with explicit representations of spinors (Dirac matrices)
  Everything reduces to Lorentz covariant t-matrix algebra ($\mathcal{C}^{\mu\nu\rho\lambda}$, just numbers)

Many applications and extensions possible

- Local operators parameterizations: Generalized Form Factors
  (two independent four-vectors)
- Bilocal operators parameterizations
  (more than two independent four-vectors)
- Transition matrix elements
- Use in χEFT's for high energy processes

Thanks!

Questions?

Backup Slides

Properties of the t-tensors

- Each $t^{\mu_1 \dots \mu_{2j}}$ is a 2j-rank tensor

- Symmetric and (covariantly) traceless $$g_{\mu_k \mu_l} t_{\sigma\sigma'}^{\mu_1 \dots \mu_k \dots \mu_l \dots \mu_{2j}} = 0$$

- Transform covariantly $$\left(D^{(j)}_{[\Lambda]}\right)_{\sigma\delta} t_{\delta\delta'}^{\mu_1 \dots \mu_{2j}} \left(D^{(j)\dagger}_{[\Lambda]}\right)_{\delta'\sigma'} = \Lambda_{\nu_1}{}^{\mu_1} \dots \Lambda_{\nu_{2j}}{}^{\mu_{2j}} \; t_{\sigma\sigma'}^{\nu_1 \dots \nu_{2j}}$$

- Right chiral (t) and left chiral ($\bar{t}$) are related by charge conjugation (+ for even (− for odd) spacelike indices) $$\bar{t}_{\sigma\sigma'}^{\mu_1 \mu_2 \dots \mu_{2j}} = (\pm) t_{\sigma\sigma'}^{\mu'_1 \mu'_2 \dots \mu'_{2j}}$$

Spin 1/2 Example: Spinors

Right Chiral Rep

- $t^0 = \mathbf{1} \quad , \quad t^i = \sigma_i$
- t^μ Transform Covariantly: $\quad D^{(1/2)}_{[\Lambda]} t^\mu D^{(1/2)\dagger}_{[\Lambda]} = \Lambda_\rho{}^\mu t^\rho$
- Propagator (Lorentz invariant): $\Pi^{(1/2)}(p) = t^\mu p_\mu = \begin{pmatrix} E - p_z & -(p_x - i p_y) \\ -(p_x + i p_y) & E + p_z \end{pmatrix}$
  $p_\mu = (E_p, \vec{p})$
- Boost/spinors (Canonical): $D^{(1/2)}_{\mathrm{IF}} = t^\mu \tilde{p}^{\mathrm{C}}_\mu = \frac{1}{\sqrt{2m\,(m + p_0)}} \begin{pmatrix} m + p^- & -p_\ell \\ -p_r & m + p^+ \end{pmatrix}$
  $\tilde{p}^\mu_{\mathrm{C}} = \sqrt{\frac{m}{2(m+p^0)}}(p^0 + m, \vec{p}\,)$

Similarly for the Left Chiral Rep, only change is: $J_i^{(1/2)} \rightarrow \bar{J}^\mu = (1, -\vec{J}^{\,(1/2)})$

Spin 1 Example: Spinors

Right Chiral Rep

- $t^{00} = \mathbf{1}$, $t^{0i} = t^{i0} = J_i^{(1)}$, $t^{ij} = \{J_1^{(1)}, J_1^{(1)}\} - \mathbf{1}\delta_{ij}$
- $t^{\mu\nu}$ Transform covariantly $D^{(1)}_{[\Lambda]} t^{\mu\nu} D^{(1)}{}^{\dagger}_{[\Lambda]} = \Lambda_\rho{}^\mu \Lambda_\sigma{}^\nu t^{\rho\sigma}$
- Propagator ($p_\mu = (E_p, \vec{p})$): $\Pi^{(1)}(p) = t^{\mu\nu} p_\mu p_\nu = \begin{pmatrix} (p^-)^2 & -\sqrt{2} p_\ell p^- & p_\ell^2 \\ \sqrt{2} p_r p^+ & p^+ p^- + p_{\mathrm{T}}^2 & \sqrt{2} p_\ell p^- \\ p_r^2 & \sqrt{2} p_r p^- & (p^+)^2 \end{pmatrix}$
- Boost/spinors ($t^{\mu\nu} \tilde{p}_\mu \tilde{p}_\nu$)

Canonical: $D^{(1)}_{\mathrm{IF}} = \dfrac{1}{2m\,(m+p_0)} \begin{pmatrix} (m+p^-)^2 & -\sqrt{2} p_\ell (m+p^-) & p_\ell^2 \\ -\sqrt{2} p_r (m+p^-) & 2(m^2 + m p_0 + p_{\mathrm{T}}^2) & -\sqrt{2} p_\ell (m+p^+) \\ p_r^2 & -\sqrt{2} p_r (m+p^+) & (m+p^+)^2 \end{pmatrix}$

$$\tilde{p}^\mu_{\mathrm{C}} = \sqrt{\frac{m}{2(m+p^0)}}\,(p^0 + m, \vec{p}\,)$$

Similarly for the Left Chiral Rep, only change is: $J_i^{(1)} \to \bar{J}^\mu = (1, -\vec{J}^{(1)})$

Canonical Space-Time Parameterization

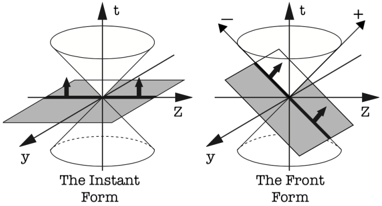
Parameterizations (Foliations) of space-time → Specify equal time surfaces

Canonical or Instant time: $x^0 = t$ [Wigner(1939)]

- Defined by rotationless boosts from rest: $\overset{\circ}{p}{}^{\mu} = (m, 0, 0, 0)$

  to final momentum: $p^\mu = (E_p, \vec{p}) = (\sqrt{m^2 + \vec{p}^{\,2}}, \vec{p})$

$$\Lambda^{\text{IF}} = \exp\left(i\vec{\mathbb{K}} \cdot \vec{\phi}\right) = \exp\left(i\phi\vec{\mathbb{K}} \cdot \hat{\phi}\right)$$



- Then, $p^\mu = (E, \vec{p}) = (\Lambda^{\text{IF}})^\mu{}_\nu \overset{\circ}{p}{}^{\nu}$

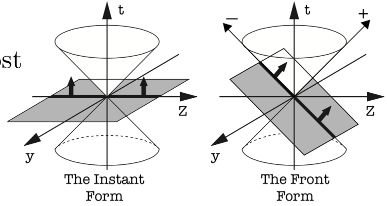
  implies, $\cosh(\phi) = \frac{E}{m}$, $\hat{\phi}_j \sinh(\phi) = \frac{p_j}{m}$

Leading to the well known result: $(\Lambda^{\text{IF}})^\mu{}_\nu = \begin{pmatrix} \frac{E}{m} & \frac{\vec{p}}{m} \\ \frac{\vec{p}}{m} & \delta_{ij} + \frac{p_i p_j}{(E+m)m} \end{pmatrix}$

Light-Front Space-Time Parameterization

Light Front time: $x^+ = t + z$ Dirac(1949)

$p^+ = E_p + p_z \; , \;\; p^- = E_p - p_z$



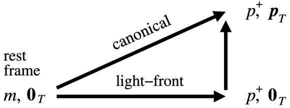
- Defined by a longitudinal boost followed by a transverse boost

$$\Lambda^{\text{LF}}_{\text{def.}} = \exp\left[i\vec{\mathbb{G}} \cdot \vec{\text{v}}_{\text{T}}\right] \cdot \exp\left[i\mathbb{K}_3 \eta\right]$$

- LF Boost Generators (light front along $z-$axis),

$$\mathbb{G}_1 = \mathbb{G}_x = \mathbb{K}_x - \mathbb{J}_y \; , \quad \mathbb{G}_2 = \mathbb{G}_y = \mathbb{K}_y + \mathbb{J}_x \; , \quad \mathbb{K}_3 = \mathbb{K}_z$$

- Comparing the action of both boosts on the same rest momentum one finds the LF boost parameters



$$e^\eta = \frac{p^+}{m} \; , \quad \vec{\text{v}}_T = \frac{\vec{p}_T}{p^+} \rightarrow \Lambda^{\text{LF}} = \exp\left[i\frac{\eta}{p^+ - m}\vec{p}_T \cdot \vec{\mathbb{G}} + i\eta\mathbb{K}_3\right]$$

Propagators - Spinors - t-tensors

The boosts/spinors for the most used forms of dynamics

- In general

$$D^{(j)}_{[L(p)]} = t^{\mu_1\mu_2\ldots\mu_{2j}}\tilde{p}_{\mu_1}\tilde{p}_{\mu_2}\ldots\tilde{p}_{\mu_{2j}}$$

$$\bar{D}^{(j)}_{[L(p)]} = \bar{t}^{\mu_1\mu_2\ldots\mu_{2j}}\tilde{p}_{\mu_1}\tilde{p}_{\mu_2}\ldots\tilde{p}_{\mu_{2j}}$$

Instant form dynamics (Canonical)

$$\tilde{p}^{\mu}_{\mathrm{C}} = \sqrt{\frac{1}{2m(m+p^0)}}(p^0+m,\vec{p}\,)$$

Light-Front dynamics (Light Cone time)

$$\tilde{p}^{\mu}_{\mathrm{LF}} = \sqrt{\frac{1}{4mp^+}}(p^+ + m, p_\ell, ip_\ell, p^+ - m)$$

$\tilde{p}^{\mu}$ not four-vectors, but same for any spin. Left/right related by complex conjugation.

(Helicity spinor also recovered with specific parameters)

Spin 1/2 Bilinears

Final evaluations recover the results of [Lorcé(2017)]

- Scalar $\bar{u}^{(1/2)}_{(p_f,s_f)} u^{(1/2)}_{(p_i,s_i)} = \widetilde{N}\phi^\dagger_{s_f}\left[4P^2\mathbf{1}_2 + 4mP_\lambda\left(\sigma^\lambda + \bar{\sigma}^\lambda\right) - \frac{i}{2}\left(\sigma_\lambda + \bar{\sigma}_\lambda\right)\varepsilon^{\lambda\beta\alpha\rho}\Delta_\beta P_\alpha\left(\sigma_\rho - \bar{\sigma}_\rho\right)\right]\phi_{s_i}$

$$= \widetilde{N}\phi^\dagger_{s_f}\left[4\left(P^2 + mP^0\right) + 2i\varepsilon^{0\beta\alpha\rho}\Delta_\beta P_\alpha\sigma_\rho\right]\phi_{s_i}$$

- Pseudoscalar

$$\bar{u}^{(1/2)}_{(p_f,s_f)}\gamma_5 u^{(1/2)}_{(p_i,s_i)} = \widetilde{N}\phi^\dagger_{s_f}\left[m\Delta_\lambda\left(\sigma^\lambda - \bar{\sigma}^\lambda\right) + \left(P_\mu\left(\sigma^\mu + \bar{\sigma}^\mu\right)\right)\left(\Delta_\nu\left(\sigma^\nu - \bar{\sigma}^\nu\right)\right) - \left(\Delta_\mu\left(\sigma^\mu + \bar{\sigma}^\mu\right)\right)\left(P_\nu\left(\sigma^\nu - \bar{\sigma}^\nu\right)\right)\right]\phi_{s_i}$$

$$= \widetilde{N}\phi^\dagger_{s_f}\left[2\Delta^0\vec{P}\cdot\vec{\sigma} - 2\left(P^0 + m\right)\vec{\Delta}\cdot\vec{\sigma}\right]\phi_{s_i}$$

$$\widetilde{N} = \widetilde{N}_f\widetilde{N}_i = \frac{1}{2m}\left[\left(p^0 + m\right)^2 - \left(\frac{1}{2}\Delta\right)^2\right]^{-1}$$

Spin 1/2 Example: Bilinears

Bilinears

- Vector

$$\bar{u}^{(j)}_{(p_f,s_f)}\gamma^{\mu}u^{(j)}_{(p_i,s_i)} = \tilde{N}\phi_{s_f} + \left[\frac{1}{2}\Delta^2\left(\sigma^{\mu}+\bar{\sigma}^{\mu}\right) - \frac{1}{2}\Delta_{\lambda}\left(\sigma^{\lambda}+\bar{\sigma}^{\lambda}\right)\Delta^{\mu}\right.$$
$$+ 4mP^{\mu}\mathbf{1}_2 + 2P_{\lambda}\left(\sigma^{\lambda}+\bar{\sigma}^{\lambda}\right)P^{\mu}$$
$$\left. i\varepsilon^{\mu\beta\alpha\rho}\Delta_{\beta}\left(m\left(\sigma_{\alpha}+\bar{\sigma}_{\alpha}\right)+P_{\alpha}\right)\left(\sigma_{\rho}-\bar{\sigma}_{\rho}\right)\right]\phi_{s_i}$$
$$= \tilde{N}\phi^{\dagger}_{s_f}\left[\left(4\left(P^0+m\right)P^{\mu}+\Delta^2 g^{0\mu}-\Delta^0\Delta^{\mu}\right)\mathbf{1}_2\right.$$
$$\left. +2i\varepsilon^{0\mu\beta\rho}\Delta_{\beta}\sigma_{\rho} + i\varepsilon^{\mu\beta\alpha\rho}\Delta_{\beta}P_{\alpha}\left(\sigma_{\rho}+\bar{\sigma}_{\rho}\right)\right]\phi_{s_i}$$

- Pseudovector

$$\bar{u}^{(1/2)}_{(p_f,s_f)}\gamma^{\mu}\gamma_5 u^{(1/2)}_{(p_i,s_i)} = \tilde{N}\phi^{\dagger}_{s_f}\left[-\left(4P^{\mu}P_{\alpha}-\Delta^{\mu}\Delta_{\alpha}\right)\left(\sigma^{\alpha}-\bar{\sigma}^{\alpha}\right)\right.$$
$$\left. +\left(P^2-\frac{1}{4}\Delta^2\right)\left(\sigma^{\mu}-\bar{\sigma}^{\mu}\right) - i\varepsilon^{\mu\alpha\beta\rho}\Delta_{\alpha}P_{\beta}\left(\sigma_{\rho}+\bar{\sigma}_{\rho}\right)\right]\phi_{s_i}$$