

Towards accessing $\gamma^* \gamma^* \rightarrow \pi\pi$ from lattice QCD

Andrew W. Jackura

University of California, Berkeley

Department of Physics

10th workshop of the APS Topical Group on Hadronic Physics

Minneapolis, Minnesota

April 12-14th 2023



 ajackura@berkeley.edu

 [ajackura.github.io](https://github.com/ajackura)

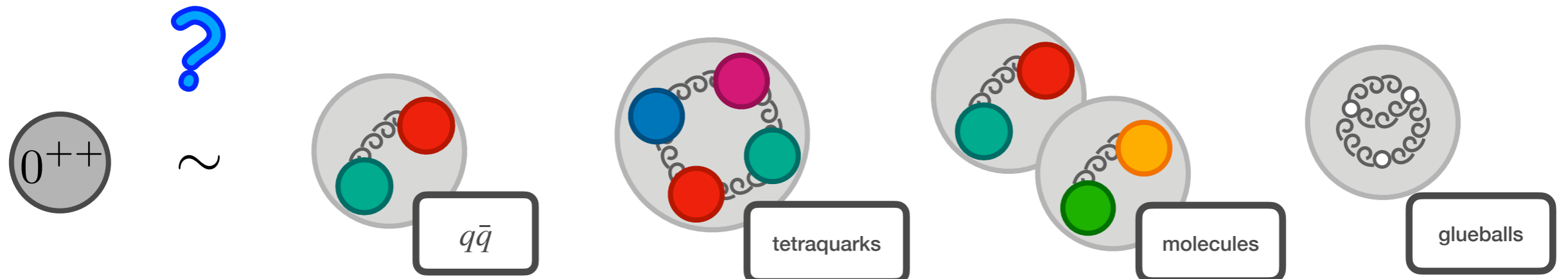
Few-Hadron Matrix Elements from QCD

What is the nature of hadrons in terms of their quarks and gluons in QCD?

- Can we discriminate against preferred compositions?
- Can we explore these features for resonances?

scalar mesons

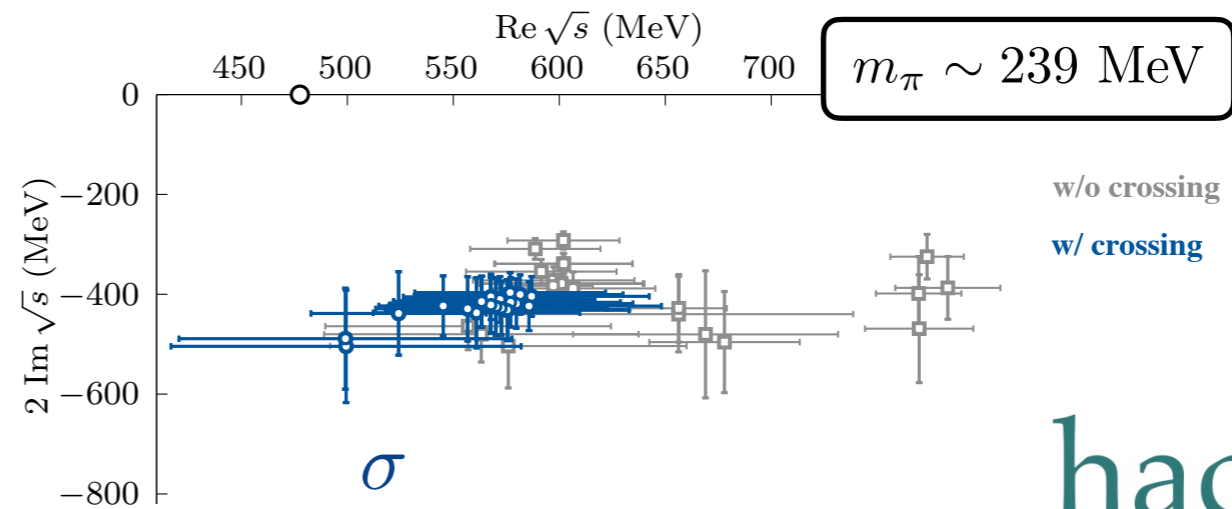
$$[\pi\pi]_{JPC} = 0^{++}$$



Few-Hadron Matrix Elements from QCD

What is the nature of hadrons in terms of their quarks and gluons in QCD?

- Can we discriminate against preferred compositions?
- Can we explore these features for resonances?

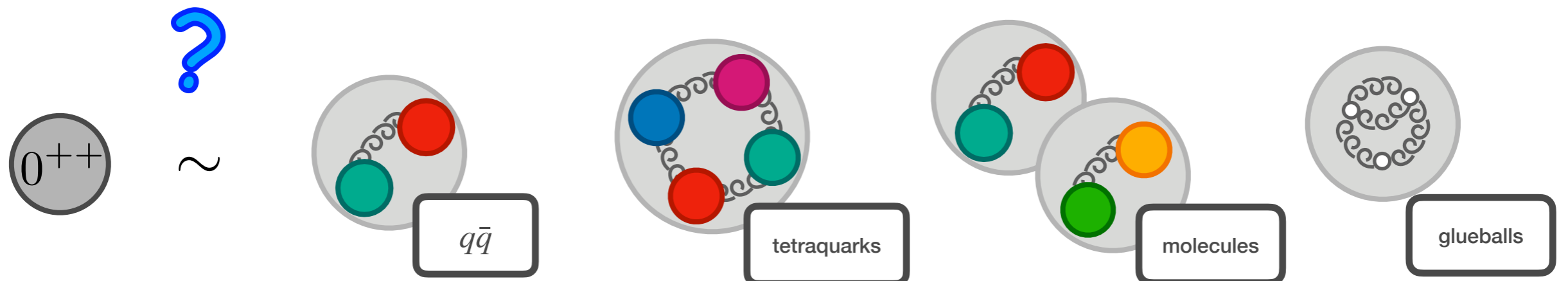


had spec

scalar mesons

$$[\pi\pi]_{JPC} = 0^{++}$$

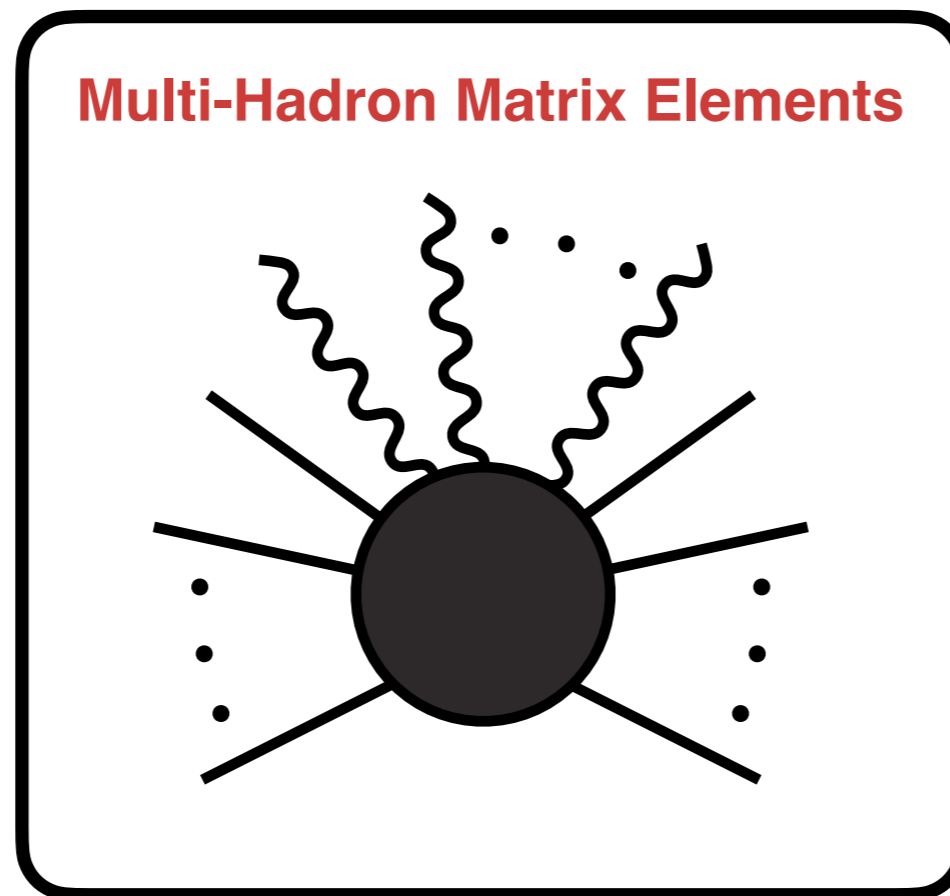
A. Rodas, J. Dudek, R. Edwards,
arXiv:2304.03762



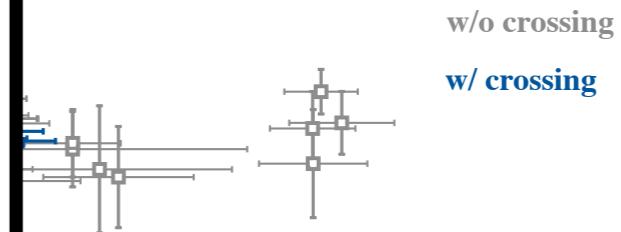
Few-Hadron Matrix Elements from QCD

What is the nature of hadrons in terms of their quarks and gluons in QCD?

- Can we discriminate against preferred compositions?
- Can we explore these features for resonances?



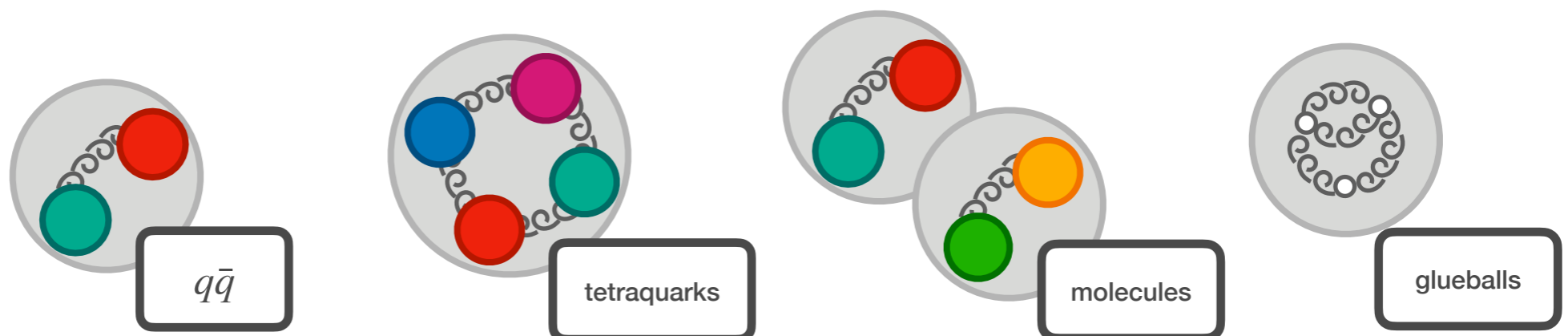
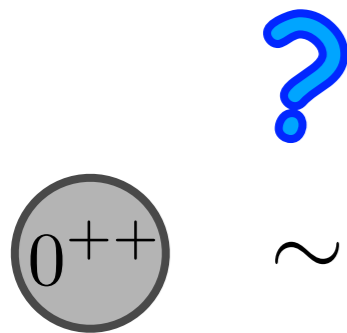
(MeV)
650 700 $m_\pi \sim 239$ MeV



had spec

scalar mesons

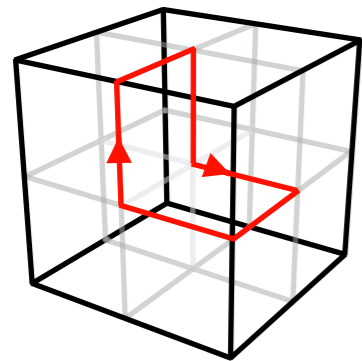
$$[\pi\pi]_{JPC} = 0^{++}$$



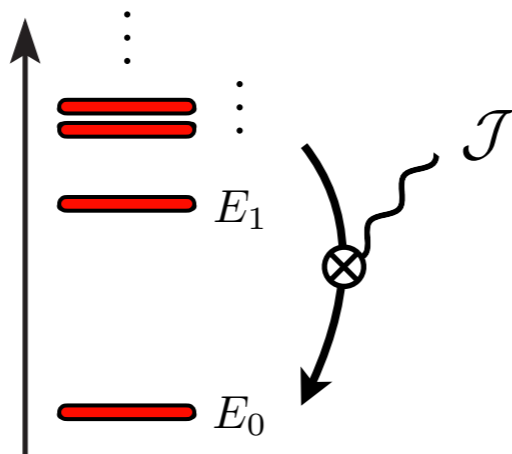
Few-Hadron Matrix Elements from QCD

A path toward understanding the substructure is to probe a hadron

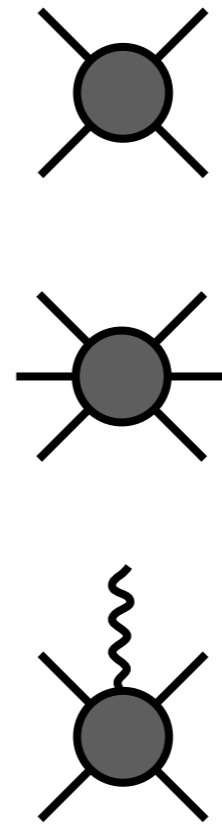
- Connect QCD to few-hadron matrix elements
- Tools: *Lattice QCD*, *Scattering Theory*, & *Effective Field Theory*



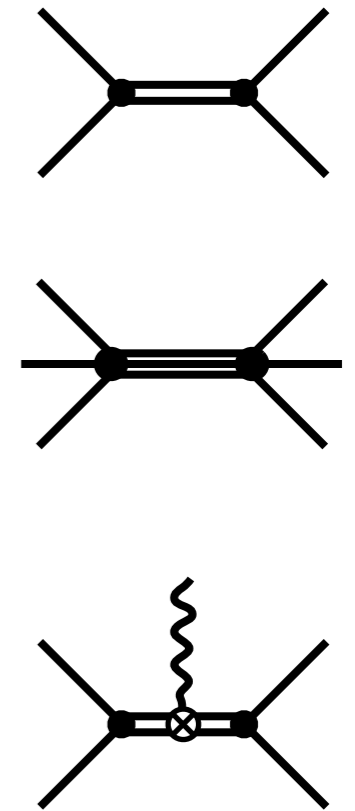
Lattice QCD



Spectra &
Matrix Elements



Scattering &
Transition Amplitudes

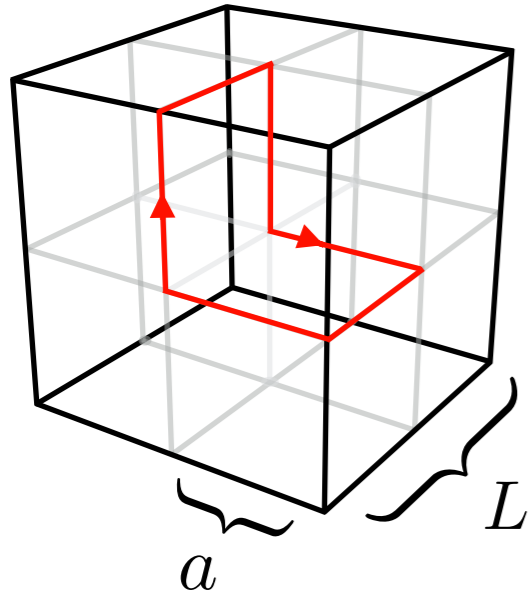


Bound & Resonant
State Properties

Connecting Resonances to QCD

Lattice QCD offers a systematic approach to compute hadrons from QCD

- Numerically evaluate QCD path integral via Monte Carlo sampling



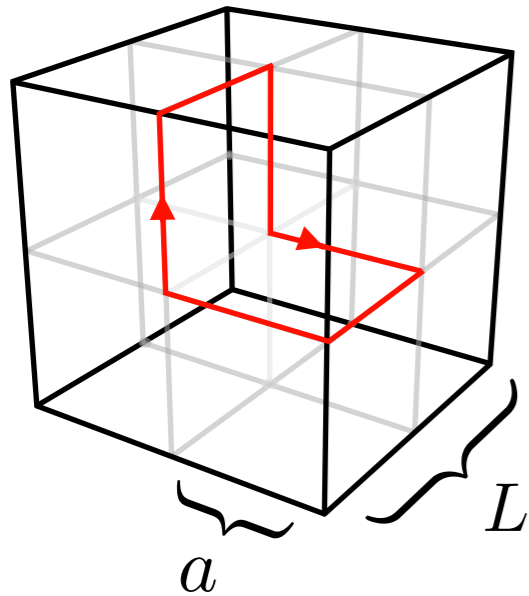
$$Z_{\text{QCD}} = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}A_{\mu} e^{iS_{\text{QCD}}(\psi, \bar{\psi}, A_{\mu})}$$

- Euclidean spacetime, $t \rightarrow -i\tau$
- Finite volume, L
- Discrete spacetime, a
- Heavier than physical quark mass, $m > m_{\text{phys}}$.

Connecting Resonances to QCD

Lattice QCD offers a systematic approach to compute hadrons from QCD

- Numerically evaluate QCD path integral via Monte Carlo sampling

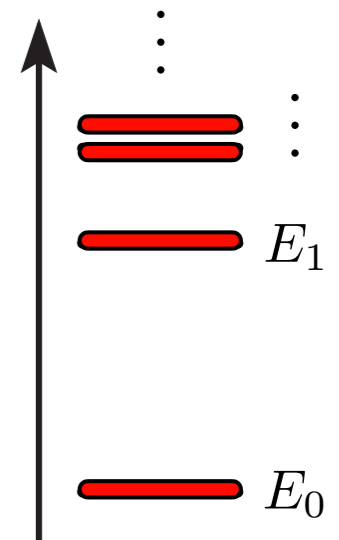


$$Z_{\text{QCD}} = \int \mathcal{D}\psi \mathcal{D}\bar{\psi} \mathcal{D}A_\mu e^{iS_{\text{QCD}}(\psi, \bar{\psi}, A_\mu)}$$

- Euclidean spacetime, $t \rightarrow -i\tau$
- Finite volume, L
- Discrete spacetime, a
- Heavier than physical quark mass, m

Correlation functions yield discrete spectrum

$$\langle \mathcal{O}(\tau) \mathcal{O}^\dagger(0) \rangle = \sum_{\mathbf{n}} |\langle 0 | \mathcal{O} | \mathbf{n} \rangle|^2 e^{-E_{\mathbf{n}} \tau}$$



Connecting Resonances to QCD

Exploit S matrix unitarity to construct analytic amplitude

If we restrict our energies of interest...

...can formally isolate long-range from short-distance behavior



Connecting Resonances to QCD

Exploit S matrix unitarity to construct analytic amplitude

If we restrict our energies of interest...

...can formally isolate long-range from short-distance behavior



$$i\mathcal{M} = \text{[Black circle with four external lines]} \supset \text{[Circle with two external lines]} = \text{[Circle with two external lines and a teal wavy line]} + \text{[Circle with two external lines and a red vertical line]}$$
$$\text{[Circle with two external lines and a teal wavy line]} = \text{[Teal square with four external lines]} + \text{[Circle with two external lines and a red vertical line]}$$


A red arrow labeled E points from the black circle to the teal square.


Connecting Resonances to QCD

Exploit S matrix unitarity to construct analytic amplitude

If we restrict our energies of interest...

...can formally isolate long-range from short-distance behavior


known (*on-shell kinematic functions*)


unknown (*encapsulating strong dynamics*)

$$i\mathcal{M} = \text{[Diagram of a vertex with four external lines and an arrow labeled } E \text{]} \supset \text{[Diagram of a box with a loop and external lines, containing the equation } i\mathcal{M} = i\mathcal{K} \cdot \frac{1}{1 - \rho \cdot i\mathcal{K}} \text{]}$$

Connecting Resonances to QCD

Connect finite-volume observables to amplitudes via non-perturbative mappings

$$\mathcal{C}_L - \mathcal{C}_\infty \supset \left(\text{Diagram with } V \right) - \left(\text{Diagram with } \infty \right) \\ \sim \left[\frac{1}{L^3} \sum_{\mathbf{k}} - \int_{\mathbf{k}} \right] \frac{i}{E - 2\omega_{\mathbf{k}}}$$

Connecting Resonances to QCD

Connect finite-volume observables to amplitudes via non-perturbative mappings

$$\mathcal{C}_L - \mathcal{C}_\infty \supset \text{[Diagram: Circle with 'V' and two external nodes]} - \text{[Diagram: Circle with '\infty' and two external nodes]}$$

$$\sim \left[\frac{1}{L^3} \sum_{\mathbf{k}} - \int_{\mathbf{k}} \right] \frac{i}{E - 2\omega_{\mathbf{k}}} = \text{[Diagram: Circle with 'V' and two external nodes, with a red vertical line through the center]} + \mathcal{O}(e^{-mL})$$

$$\rho + \mathcal{F}_L$$

Geometric function
 – characterizes finite-volume distortions


Poisson Summation Formula

$$\left[\frac{1}{L^3} \sum_{\mathbf{k}} - \int_{\mathbf{k}} \right] f(\mathbf{k}) = \mathcal{O}(e^{-mL})$$

Connecting Resonances to QCD

Connect finite-volume observables to amplitudes via non-perturbative mappings

$$C_L - C_\infty \sim \sum_{\mathbf{n}} \frac{\mathcal{R}_L}{E - E_{\mathbf{n}}} + \mathcal{O}(e^{-mL})$$


$$(1 + \mathcal{K} \cdot \mathcal{F}_L) \Big|_{E=E_{\mathbf{n}}} = 0$$

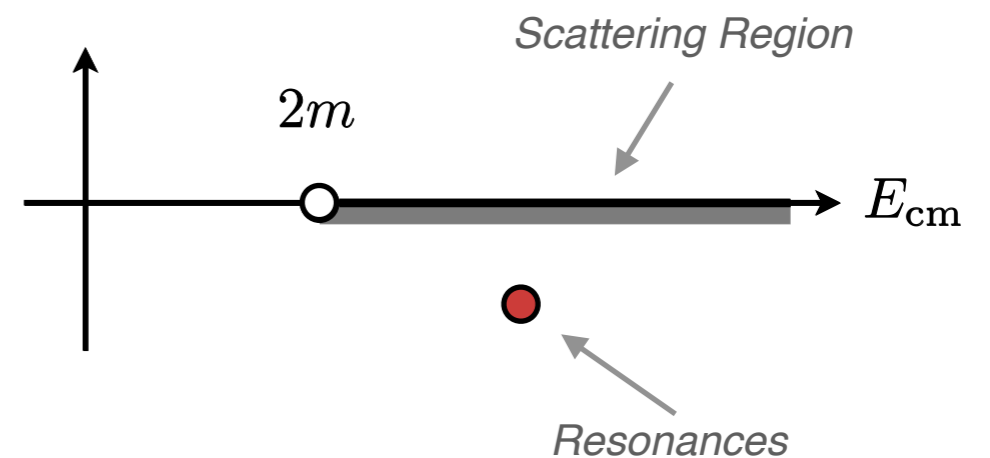
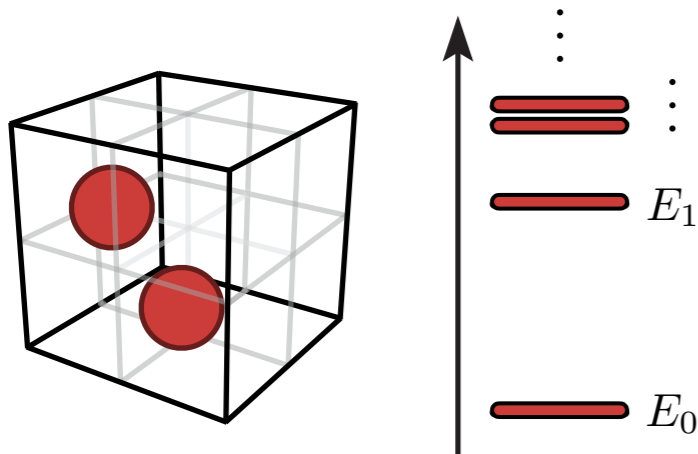
Connecting Resonances to QCD

Connect finite-volume observables to amplitudes via non-perturbative mappings

$$C_L - C_\infty \sim \sum_{\mathbf{n}} \frac{\mathcal{R}_L}{E - E_{\mathbf{n}}} + \mathcal{O}(e^{-mL})$$

$$(1 + \mathcal{K} \cdot \mathcal{F}_L) \Big|_{E=E_{\mathbf{n}}} = 0$$

$$i\mathcal{M} = i\mathcal{K} \cdot \frac{1}{1 - \rho \cdot i\mathcal{K}}$$

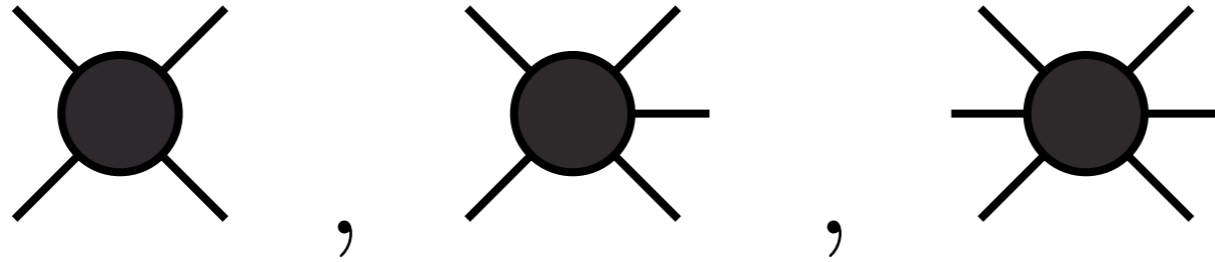


M. Lüscher
 Commun.Math.Phys. **105**, 153 (1986)
 Nucl.Phys. **B354**, 531 (1991)

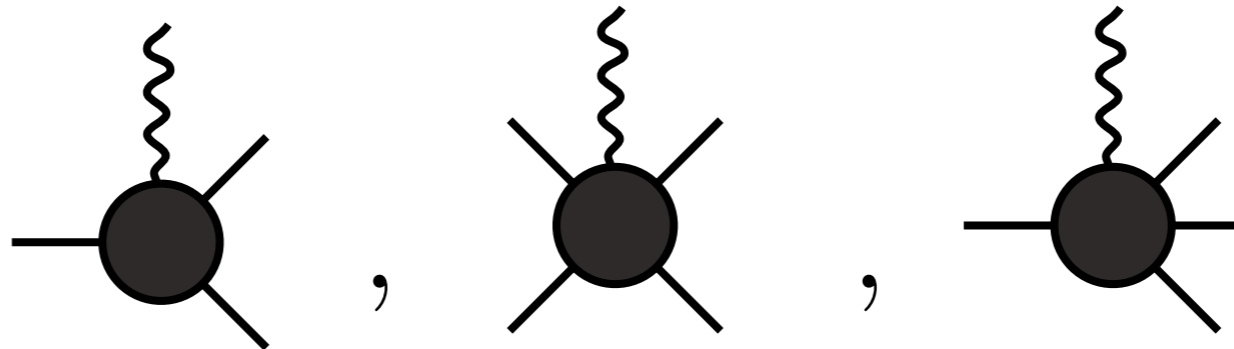
Many others...

Few-Hadron Matrix Elements

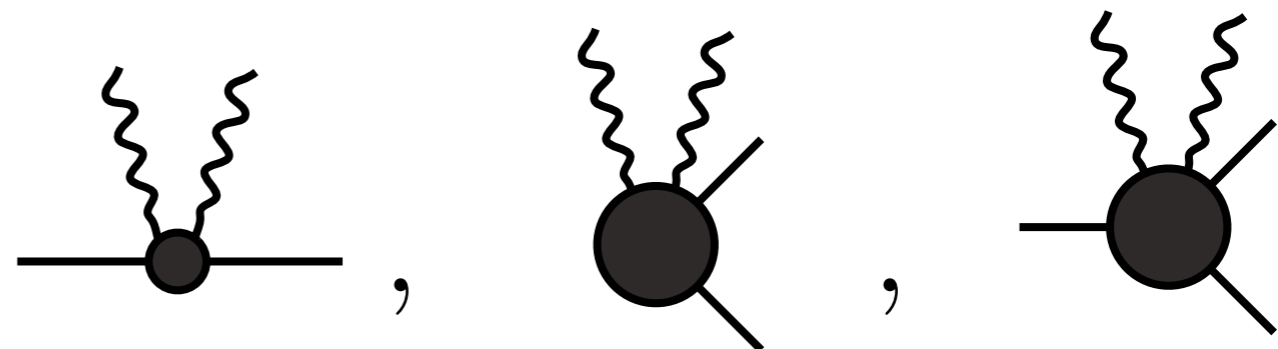
Hadronic scattering



Electroweak induced transitions

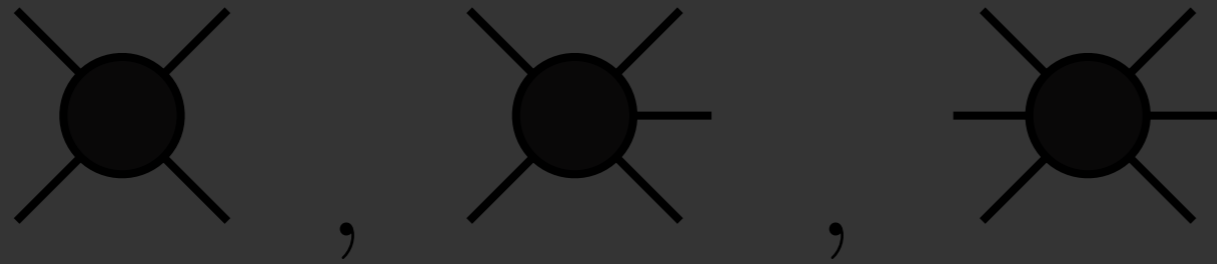


Long-range transitions

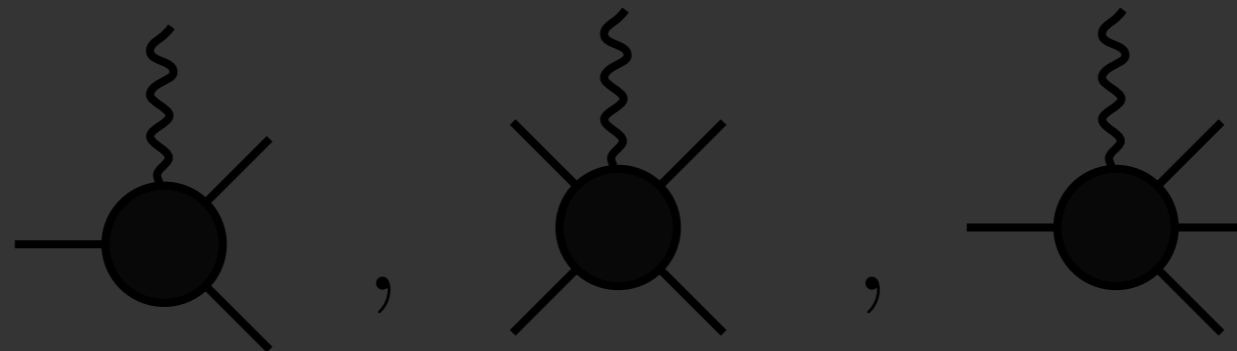


Few-Hadron Matrix Elements

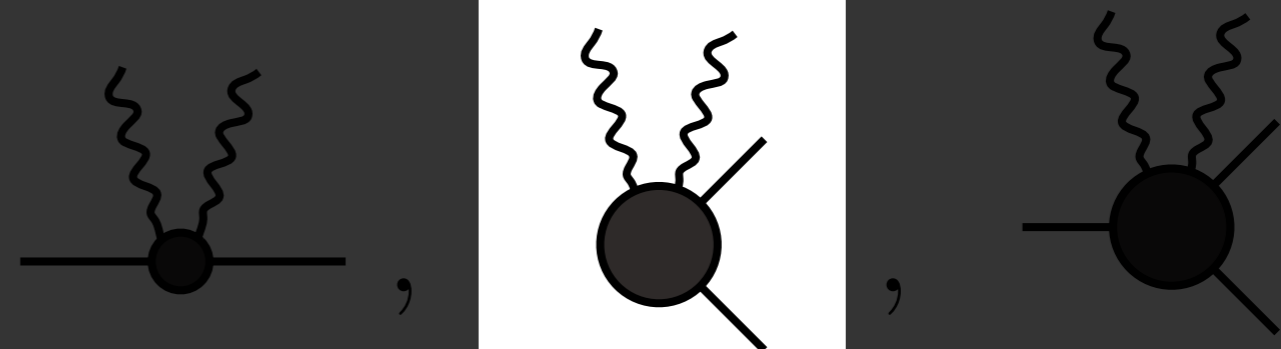
Hadronic scattering



Electroweak induced transitions



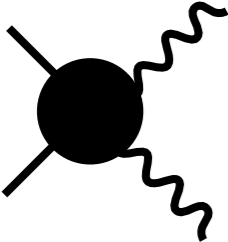
Long-range transitions



Two-current systems

Coupling two currents to hadronic systems

- Restrict kinematic region to two-pions — Isolate singularity structure


$$= \mathcal{T}^{\mu\nu} = i \int d^4x e^{-iq \cdot x} \langle \pi\pi | \mathsf{T} \mathcal{J}_M^\mu(t) \mathcal{J}^\nu(0) | \Omega \rangle_\infty$$

F. Ortega-Gama, K. Sherman, AJ, R. Briceño,
Phys. Rev. D **105** (2022)

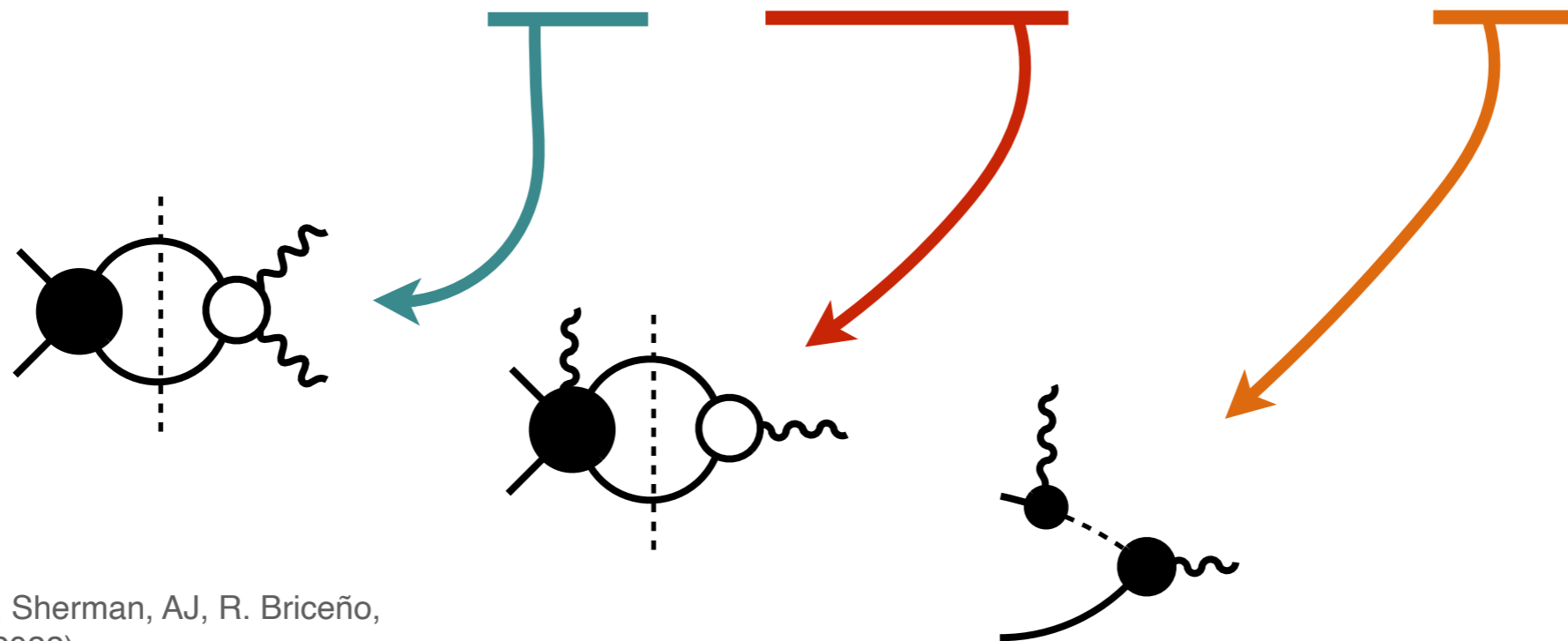
AJ, R. Briceño, A. Rodas, J. Guerrero
Phys. Rev. D **107** (2023)

Two-current systems

Coupling two currents to hadronic systems

- Restrict kinematic region to two-pions — Isolate singularity structure

$$\begin{aligned}
 \text{Diagram} &= \mathcal{T}^{\mu\nu} = i \int d^4x e^{-iq \cdot x} \langle \pi\pi | \mathcal{T} \mathcal{J}_M^\mu(t) \mathcal{J}^\nu(0) | \Omega \rangle_\infty \\
 &= \dots \\
 &= \mathcal{M} \cdot \mathcal{B}^{\mu\nu} + \mathcal{W}_{\text{df}}^\mu \cdot \mathcal{A}^\nu + \mathcal{W}_{\text{df}}^\nu \cdot \mathcal{A}^\mu + \text{poles}
 \end{aligned}$$



F. Ortega-Gama, K. Sherman, AJ, R. Briceño,
Phys. Rev. D **105** (2022)

AJ, R. Briceño, A. Rodas, J. Guerrero
Phys. Rev. D **107** (2023)

Two-current systems

Coupling two currents to hadronic systems

- Restrict kinematic region to two-pions — Isolate singular

$$\begin{aligned}
 \text{Diagram} &= \mathcal{T}^{\mu\nu} = i \int d^4x e^{-iq \cdot x} \langle \pi\pi | \dots | \pi\pi \rangle \\
 &= \dots \\
 &= \mathcal{M} \cdot \mathcal{B}^{\mu\nu} + \mathcal{W}_{\text{df}}^\mu \cdot \mathcal{A}^\nu
 \end{aligned}$$

F. Ortega-Gama, K. Sherman, AJ, R. Briceño,
Phys. Rev. D **105** (2022)

AJ, R. Briceño, A. Rodas, J. Guerrero
Phys. Rev. D **107** (2023)

Computable from Lattice QCD

Single hadron form-factors

$$f = \text{Diagram}$$

Hadronic scattering amplitude

$$\mathcal{M} = \text{Diagram}$$

Local transition amplitude

$$\mathcal{W}_{\text{df}} = \text{Diagram} - \sum \left\{ \text{Diagram} \right\}$$

AJ, R. Briceño, F. Ortega-Gama, K. Sherman,
Phys. Rev. D **103** 114512 (2021)

A. Baroni, R. Briceño, M. Hansen, F. Ortega-Gama,
Phys. Rev. D **100** 034511 (2019)

Two-current systems

Coupling two currents to hadronic systems

- Non-local matrix elements — Need to correct for Euclidean/Minkowski time

$$M_L^{\mu\nu}(\tau) = \int_L d^3\mathbf{x} e^{iq\cdot x} \langle \pi\pi, L | T_E \mathcal{J}_E^\mu(\tau, \mathbf{x}) \mathcal{J}^\nu(0) | \Omega \rangle_L$$

Two-current systems

Coupling two currents to hadronic systems

- Non-local matrix elements — Need to correct for Euclidean/Minkowski time

$$M_L^{\mu\nu}(\tau) = \int_L d^3\mathbf{x} e^{i\mathbf{q}\cdot\mathbf{x}} \langle \pi\pi, L | T_E \mathcal{J}_E^\mu(\tau, \mathbf{x}) \mathcal{J}^\nu(0) | \Omega \rangle_L$$

$$\mathcal{T}^{\mu\nu}(\tau) \sim \int d\tau e^{-q\tau} M_L^{\mu\nu}(\tau) \sim \int_0^\infty d\tau e^{(E_{\pi\pi} - (q_0 + E_n))\tau} c_n^{\mu\nu} \longrightarrow \infty$$

$E_{\pi\pi} > q_0 + E_n$

Two-current systems

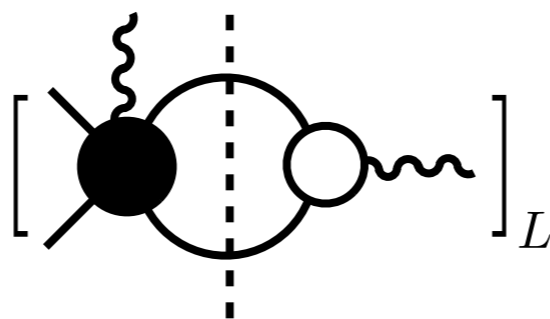
Coupling two currents to hadronic systems

- Non-local matrix elements — Need to correct for Euclidean/Minkowski time

$$M_L^{\mu\nu}(\tau) = \int_L d^3\mathbf{x} e^{i\mathbf{q}\cdot\mathbf{x}} \langle \pi\pi, L | T_E \mathcal{J}_E^\mu(\tau, \mathbf{x}) \mathcal{J}^\nu(0) | \Omega \rangle_L$$

$$\mathcal{T}^{\mu\nu}(\tau) \sim \int d\tau e^{-q\tau} M_L^{\mu\nu}(\tau) \sim \int_0^\infty d\tau e^{(E_{\pi\pi} - (q_0 + E_n))\tau} c_n^{\mu\nu} \longrightarrow \infty$$

$E_{\pi\pi} > q_0 + E_n$



Euclidean and finite-volume artifacts are directly intertwined, and are due to intermediate states going on-shell

Two-current systems

Coupling two currents to hadronic systems

- Non-local matrix elements — Need to correct for Euclidean/Minkowski time

(1) Define subtracted function — safe integration

$$\Delta M_L^{\mu\nu, >}(\tau) \sim M_L^{\mu\nu}(\tau) - \sum_n^N e^{(E_{\pi\pi} - (q_0 + E_n))\tau} c_n$$

(2) Reconstruct FV poles

$$\mathcal{T}_L^{\mu\nu, >}(\tau) \sim \sum_n^N \frac{c_n}{E_n - (E_{\pi\pi} - q_0)}$$

Two-current systems

Coupling two currents to hadronic systems

- Non-local matrix elements — Need to correct for Euclidean/Minkowski time

(3) Remove FV artifacts

$$iC_L^{\mu\nu} = \text{[Feynman diagrams]} + [q_1, \mu \leftrightarrow q_2, \nu]$$

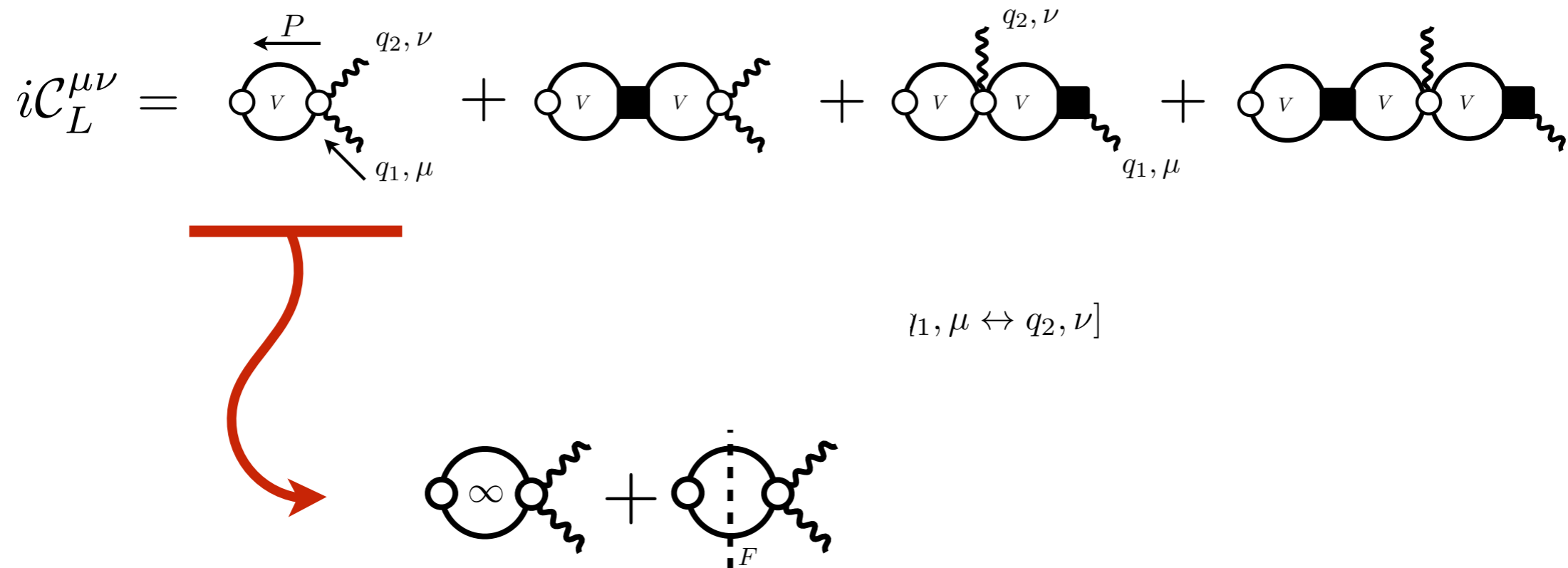
The equation shows the decomposition of the non-local matrix element $iC_L^{\mu\nu}$ into a sum of seven Feynman diagrams. The diagrams are arranged in two rows. The first row contains four diagrams: a vertex correction diagram with momentum P and external momenta q_1, μ and q_2, ν ; a self-energy diagram on the first vertex; a self-energy diagram on the second vertex; and a diagram with a gluon exchange between the two vertices. The second row contains three diagrams: a self-energy diagram on the first vertex with the external momenta swapped; a self-energy diagram on the second vertex with the external momenta swapped; and a contact term $[q_1, \mu \leftrightarrow q_2, \nu]$.

Two-current systems

Coupling two currents to hadronic systems

- Non-local matrix elements — Need to correct for Euclidean/Minkowski time

(3) Remove FV artifacts



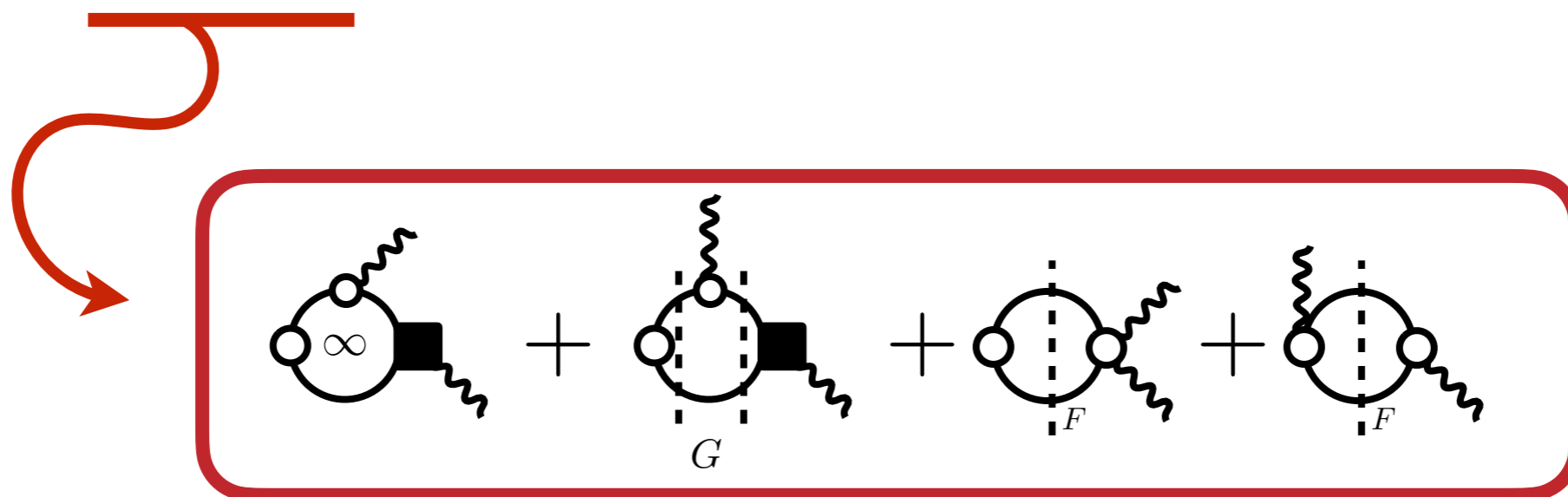
Two-current systems

Coupling two currents to hadronic systems

- Non-local matrix elements — Need to correct for Euclidean/Minkowski time

(3) Remove FV artifacts

$$iC_L^{\mu\nu} =$$



Two-current systems

Coupling two currents to hadronic systems

- Non-local matrix elements — Need to correct for Euclidean/Minkowski time

(3) Remove FV artifacts

$$\begin{aligned}
 i\mathcal{C}_L^{\mu\nu} = & \text{Diagram 1} + \text{Diagram 2} + \text{Diagram 3} + \text{Diagram 4} \\
 & + \text{Diagram 5} + \text{Diagram 6} + [q_{1,\mu} \leftrightarrow q_{2,\nu}]
 \end{aligned}$$

$$= -L^3 \frac{Z_n}{E - E_n} \sqrt{\frac{\mathcal{R}}{2E_n L^3}} \cdot \left(\mathcal{T}_{\text{df}}^{\mu\nu} - \Delta \mathcal{T}_{L,\text{df}}^{\mu\nu} \right)$$

Two-current systems

Coupling two currents to hadronic systems

- Non-local matrix elements — Need to correct for Euclidean/Minkowski time

(4) Combine all pieces

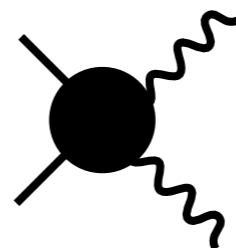
$$\sqrt{\frac{\mathcal{R}}{2E_{\pi\pi}L^3}} \cdot \mathcal{T}_{\text{df}}^{\mu\nu} = \int d\tau e^{\omega\tau} M_L^{\mu\nu,>}(\tau) + \left[\mathcal{T}_L^{\mu\nu,<} + \sqrt{\frac{\mathcal{R}}{2E_{\pi\pi}L^3}} \cdot \Delta\mathcal{T}_{L,\text{df}}^{\mu\nu} \right]$$



$$\mathcal{T}_{\text{df}}^{\mu\nu} = \mathcal{M} \cdot \mathcal{B}^{\mu\nu} + \mathcal{W}_{\text{df}}^{\mu} \cdot \mathcal{A}^{\nu} + \mathcal{W}_{\text{df}}^{\nu} \cdot \mathcal{A}^{\mu}$$



$$\mathcal{T}^{\mu\nu} = \mathcal{T}_{\text{df}}^{\mu\nu} + \text{poles} =$$

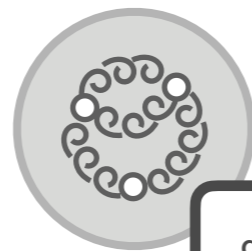
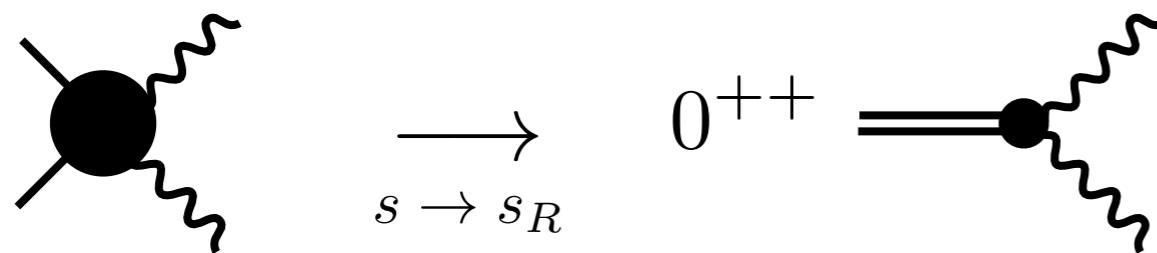


Two-current systems

Coupling two currents to hadronic systems

- Sensitivity to potential glueball components

(5) Analytically continue to resonance pole



glueballs

Access to two-photon resonance coupling

Summary

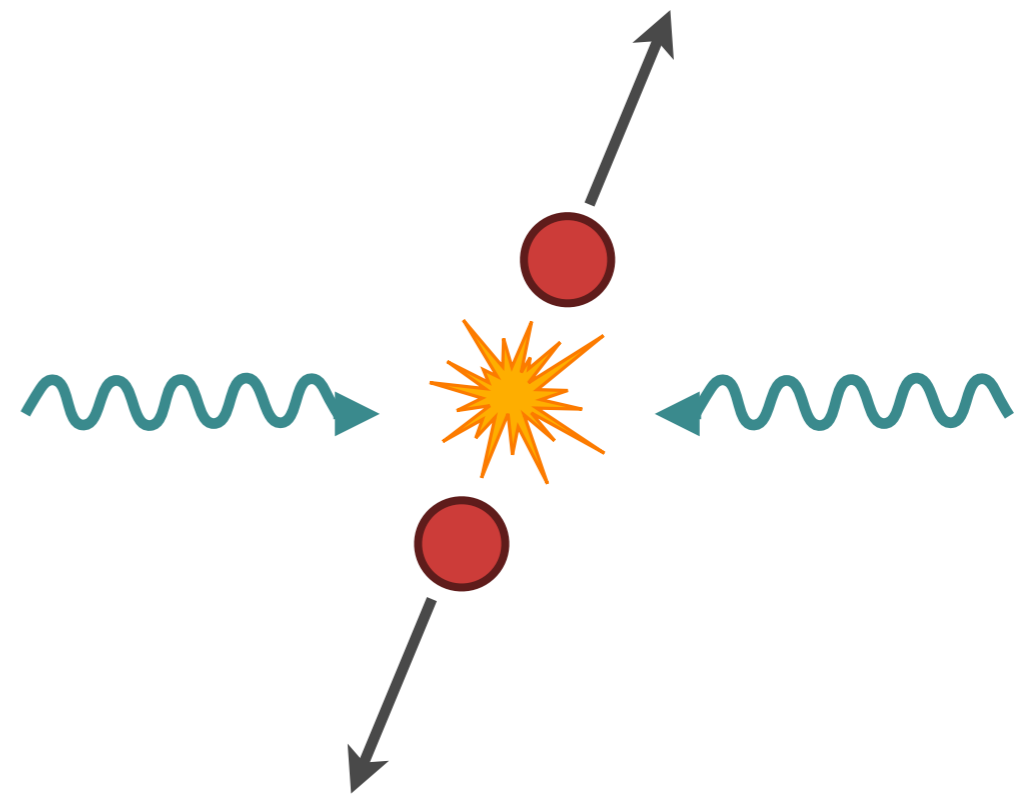
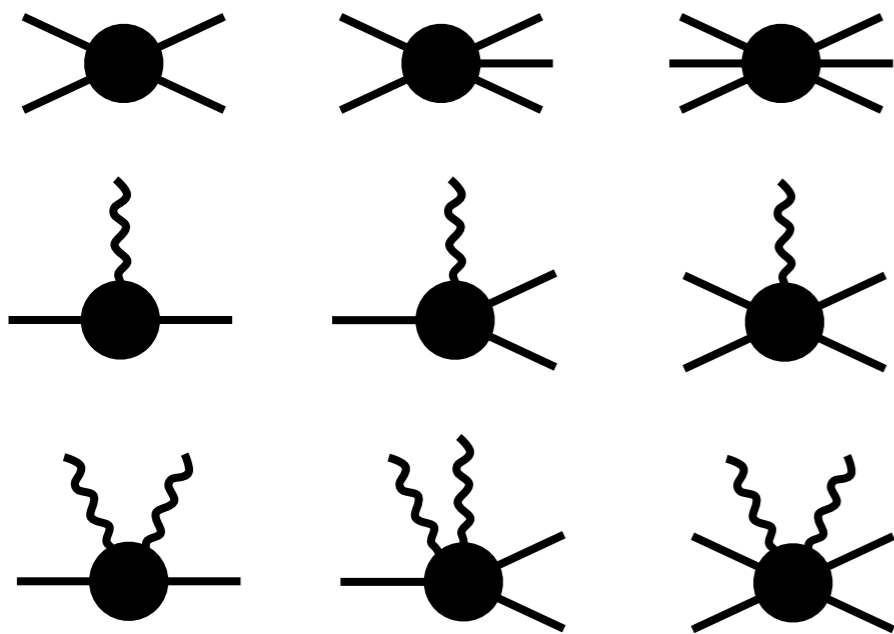
Few-body interactions play a key role in many outstanding problems in nuclear & hadron physics

Lattice QCD, EFTs, & Scattering theory combined provide useful tools to extract physics from QCD

- Rapid development in formalisms relating lattice QCD observables to amplitudes
- Scattering phenomenology is advancing in tandem

Latest developments in three-body scattering & two-body matrix elements

- First applications appearing in literature
- Can address increasingly complicated processes

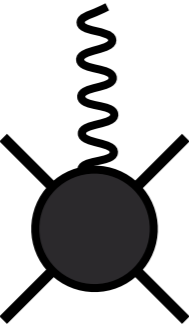


Much more to come!

AJ acknowledges financial support from The Gordon and Betty Moore Foundation and the American Physical Society to present this work at the GHP 2023 workshop.

Resonance Structure & Electroweak Probes

Scattering theory of $\mathbf{2} + \mathcal{J} \rightarrow \mathbf{2}$ amplitudes

$$i\mathcal{W} = \text{Diagram}$$


Resonance Structure & Electroweak Probes

Scattering theory of $\mathbf{2} + \mathcal{J} \rightarrow \mathbf{2}$ amplitudes

$$i\mathcal{W} = \text{Diagram} = \Sigma \left\{ \text{Diagram} \right\} + i\mathcal{W}_{\text{df}}$$

The diagram on the left shows a central black circle with four external lines: two solid lines on the left and two solid lines on the right, and a wavy line extending upwards. The diagram in the curly braces is identical but includes a small circle with a cross inside, located at the top-left vertex where the wavy line meets the circle.

Resonance Structure & Electroweak Probes

Scattering theory of $\mathbf{2} + \mathcal{J} \rightarrow \mathbf{2}$ amplitudes

$$i\mathcal{W} = \text{Diagram} = \Sigma \left\{ \text{Diagram} \right\} + i\mathcal{W}_{\text{df}}$$

The diagram on the left is a black circle with four external lines (two incoming, two outgoing) and a wavy line (representing a probe \mathcal{J}) attached to the top. The diagram in the curly braces is identical but has a small circle with a cross inside the wavy line, representing a resonance.

After considerable manipulations...

$$\mathcal{W}_{\text{df}} = \mathcal{M} \cdot (\mathcal{A} + f \cdot \mathcal{G}) \cdot \mathcal{M}$$

Resonance Structure & Electroweak Probes

Scattering theory of $2 + \mathcal{J} \rightarrow 2$ amplitudes

$$i\mathcal{W} = \text{Diagram} = \Sigma \left\{ \text{Diagram} \right\} +$$

After considerable manipulations...

$$\mathcal{W}_{\text{df}} = \mathcal{M} \cdot (\mathcal{A} + f \cdot \mathcal{G}) \cdot \mathcal{M}$$

Unknown short-distance function

- Constrain using Lattice QCD
- Constrained by Ward-Takahashi identity

Computable from Lattice QCD

Single hadron form-factors

$$f = \text{Diagram}$$

Hadronic scattering amplitude

$$\mathcal{M} = \text{Diagram}$$

Known Kinematics

Triangle diagram

Contains normal and anomalous singularities from intermediate on-shell particles

$$\mathcal{G} = \text{Diagram}$$

Resonance Structure & Electroweak Probes

Scattering theory of $\mathbf{2} + \mathcal{J} \rightarrow \mathbf{2}$ amplitudes

$$i\mathcal{W} = \text{Diagram} = \Sigma \left\{ \text{Diagram} \right\} + \mathcal{M} \cdot (\mathcal{A} + f \cdot \mathcal{G}) \cdot \mathcal{M}$$

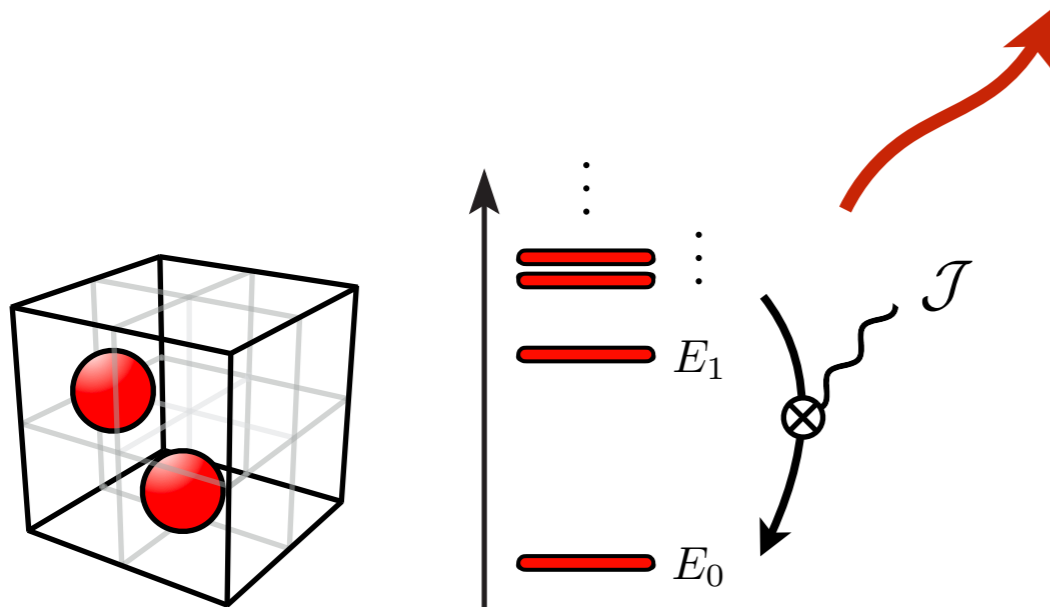
The diagram on the left is a black circle with four external lines (two incoming, two outgoing) and a wavy line (representing a gauge boson) attached to the top. The diagram inside the curly braces is identical but has a small circle with a cross inside, representing a resonance, attached to the wavy line.

Resonance Structure & Electroweak Probes

Scattering theory of $\mathbf{2} + \mathcal{J} \rightarrow \mathbf{2}$ amplitudes

$$i\mathcal{W} = \text{Diagram} = \Sigma \left\{ \text{Diagram} \right\} + \mathcal{M} \cdot (\mathcal{A} + f \cdot \mathcal{G}) \cdot \mathcal{M}$$

$$\langle \mathbf{2} | \mathcal{J} | \mathbf{2} \rangle_L \sim \sqrt{\mathcal{R}_L} \cdot (\mathcal{A} + f \cdot \mathcal{G}_L) \cdot \sqrt{\mathcal{R}_L}$$



R. Briceño, M. Hansen,
Phys. Rev. D **94** 13008 (2016)

A. Baroni, R. Briceño, M. Hansen, F. Ortega-Gama,
Phys. Rev. D **100** 034511 (2019)

Resonance Structure & Electroweak Probes

Scattering theory of $2 + \mathcal{J} \rightarrow 2$ amplitudes

- First class of resonance structure functions — form factors

$$i\mathcal{W} = \begin{array}{c} \text{Diagram 1: A central black circle with four external lines (two incoming, two outgoing) and a wavy line (photon) attached to the top.} \\ \text{Diagram 2: Two black circles connected by a double line. The left circle has two external lines, and the right circle has two external lines. A wavy line (photon) is attached to the double line between the circles, with a cross in the center of the wavy line.} \end{array} \sim \frac{g_f}{E_f - E_R} \cdot F_R(Q^2) \cdot \frac{g_i}{E_i - E_R}$$

AJ, R. Briceño, F. Ortega-Gama, K. Sherman,
Phys. Rev. D **103** 114512 (2021)

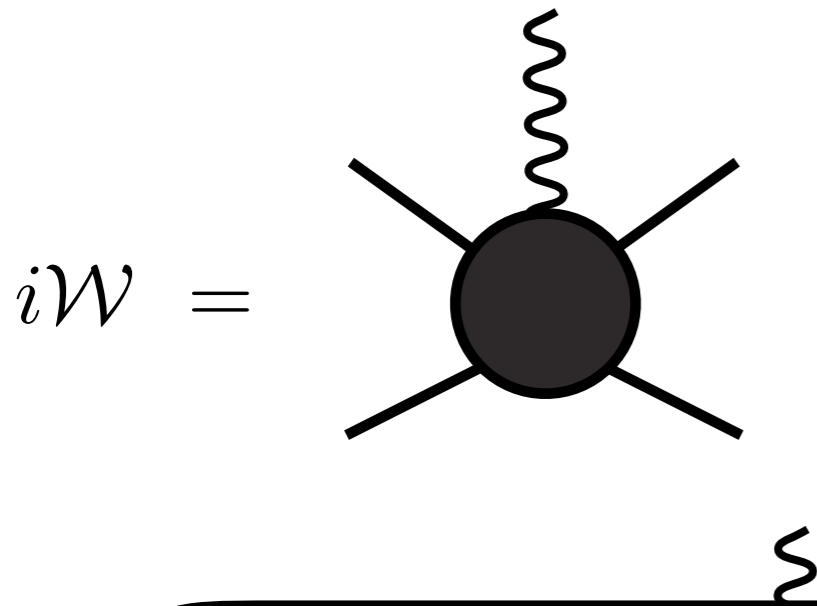
AJ, R. Briceño, M. Hansen,
Phys. Rev. D **100** 114505 (2019)

AJ, R. Briceño, M. Hansen,
Phys. Rev. D **101** 094508 (2020)

Resonance Structure & Electroweak Probes

Scattering theory of $\mathbf{2} + \mathcal{J} \rightarrow \mathbf{2}$ amplitudes

- First class of resonance structure functions — form factors



Fix K matrix from Lattice QCD

Determine poles and couplings

Compute single hadron matrix element / form factor

Fix A matrix from Lattice QCD

$$F_R(Q^2) = g_i g_f (\mathcal{A} + f \cdot \mathcal{G}) \Big|_{E_f = E_i = E_R}$$

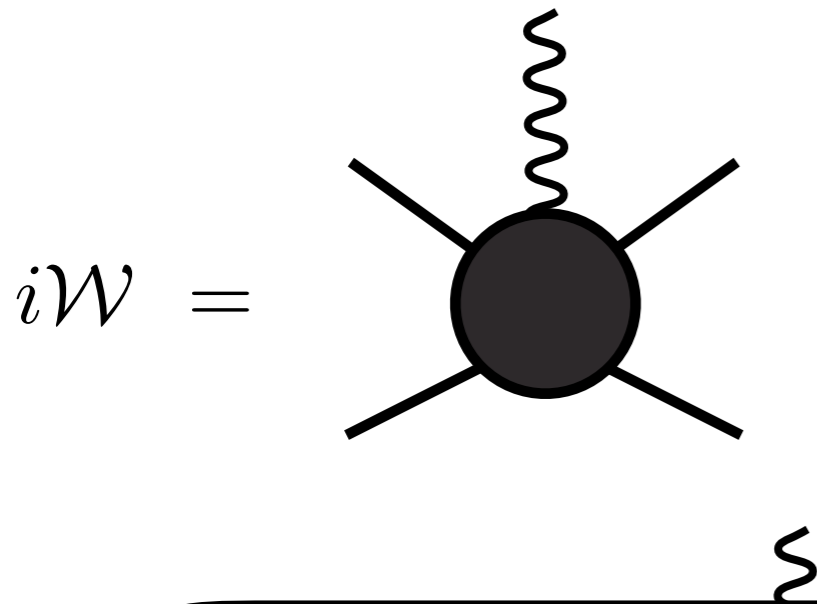
AJ, R. Briceño,
Phys. Rev. D

AJ, R. Briceño,
Phys. Rev. D

Resonance Structure & Electroweak Probes

Scattering theory of $\mathbf{2} + \mathcal{J} \rightarrow \mathbf{2}$ amplitudes

- First class of resonance structure functions — form factors



Access to resonance charge radii

$$\langle r^2 \rangle = -6 \left. \frac{d}{dQ^2} F(Q^2) \right|_{Q^2=0}$$

Fix K matrix from Lattice QCD

Determine poles and couplings

Compute single hadron matrix element / form factor

Fix A matrix from Lattice QCD

$$F_R(Q^2) = g_i g_f (\mathcal{A} + f \cdot \mathcal{G}) \Big|_{E_f = E_i = E_R}$$

AJ, R. Briceño,
Phys. Rev. D

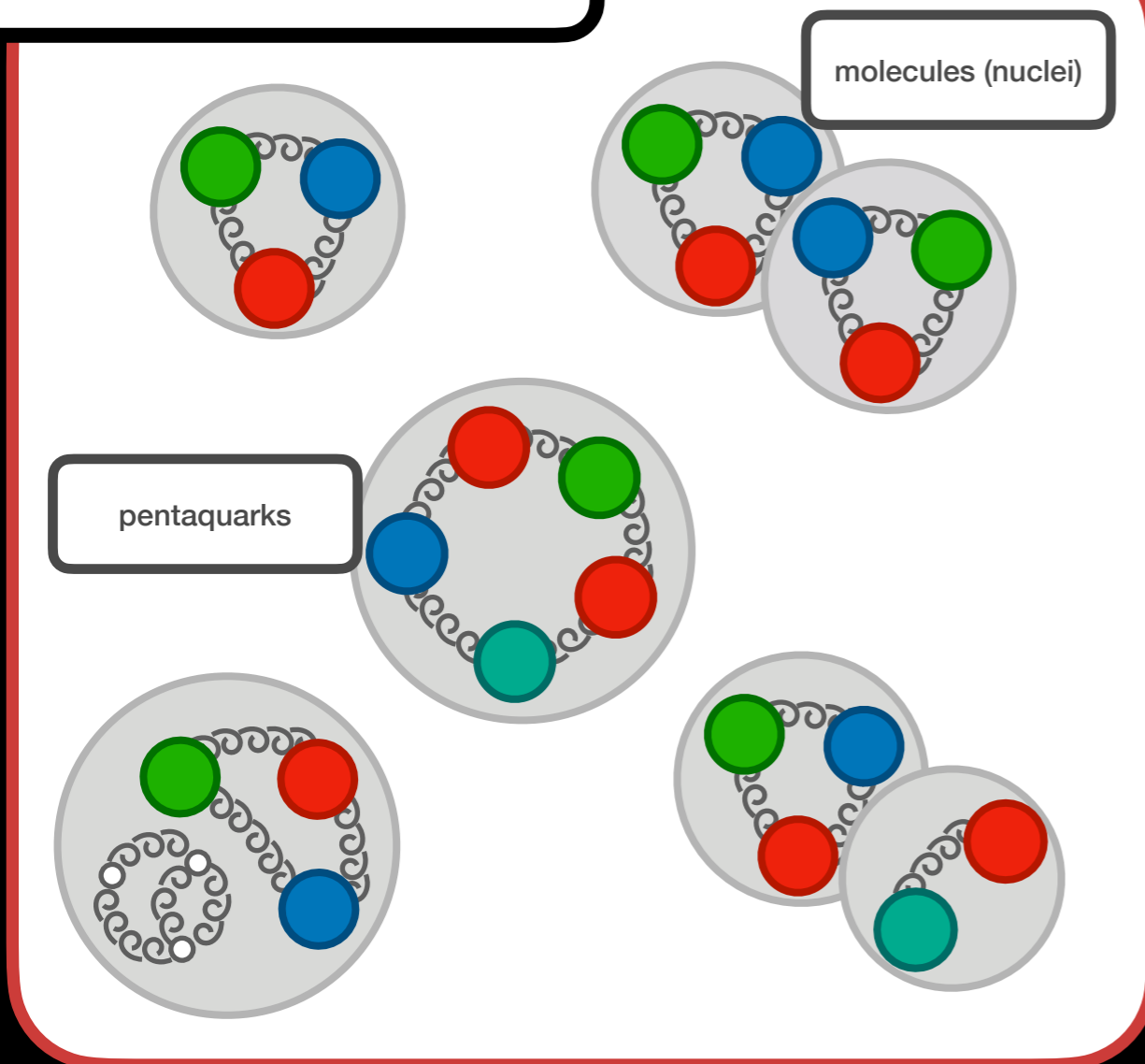
AJ, R. Briceño,
Phys. Rev. D

Resonance Structure & Electroweak Probes

First class of structural observables of unstable hadrons

- Potential to discriminate compact / extended objects

Baryons (fermions)



Mesons (bosons)

