Consistent large transverse momentum matching in TMDs with "Hadron Structure Oriented" approach

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GHP Workshop 2023

Based on

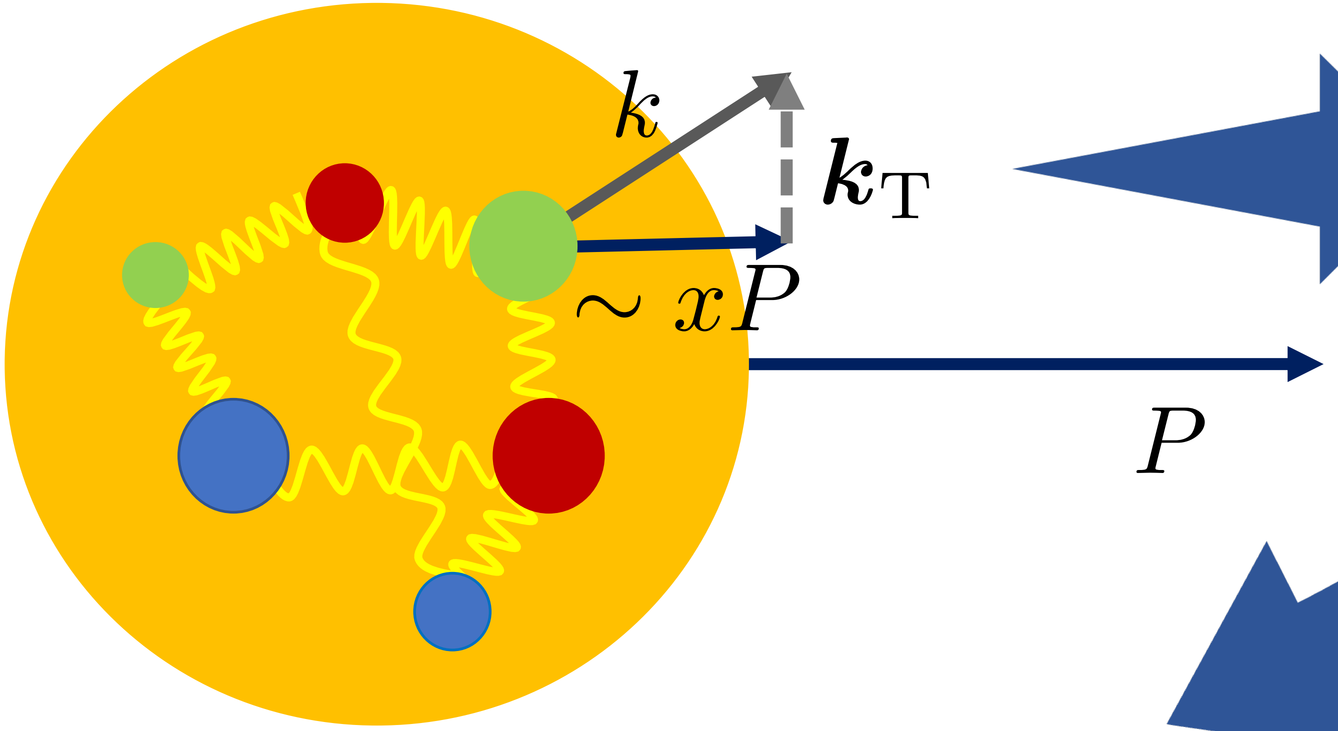
- **The resolution to the problem of consistent large transverse momentum in TMDs**
(arXiv:2303.04921)

(J. O. Gonzalez-Hernandez, T. Rainaldi, T. C. Rogers)

- **Combining nonperturbative transverse momentum dependence with TMD evolution**
(PhysRevD.106.034002)

(J. O. Gonzalez-Hernandez, T. C. Rogers, N. Sato)

Why TMDs ?



$$f_{j/h}\left(x, \boldsymbol{k}_\mathrm{T}; \mu, \sqrt{\zeta}\right)$$

Hadronic structure
(nonperturbative effects)

Predictions
(evolution & universality)

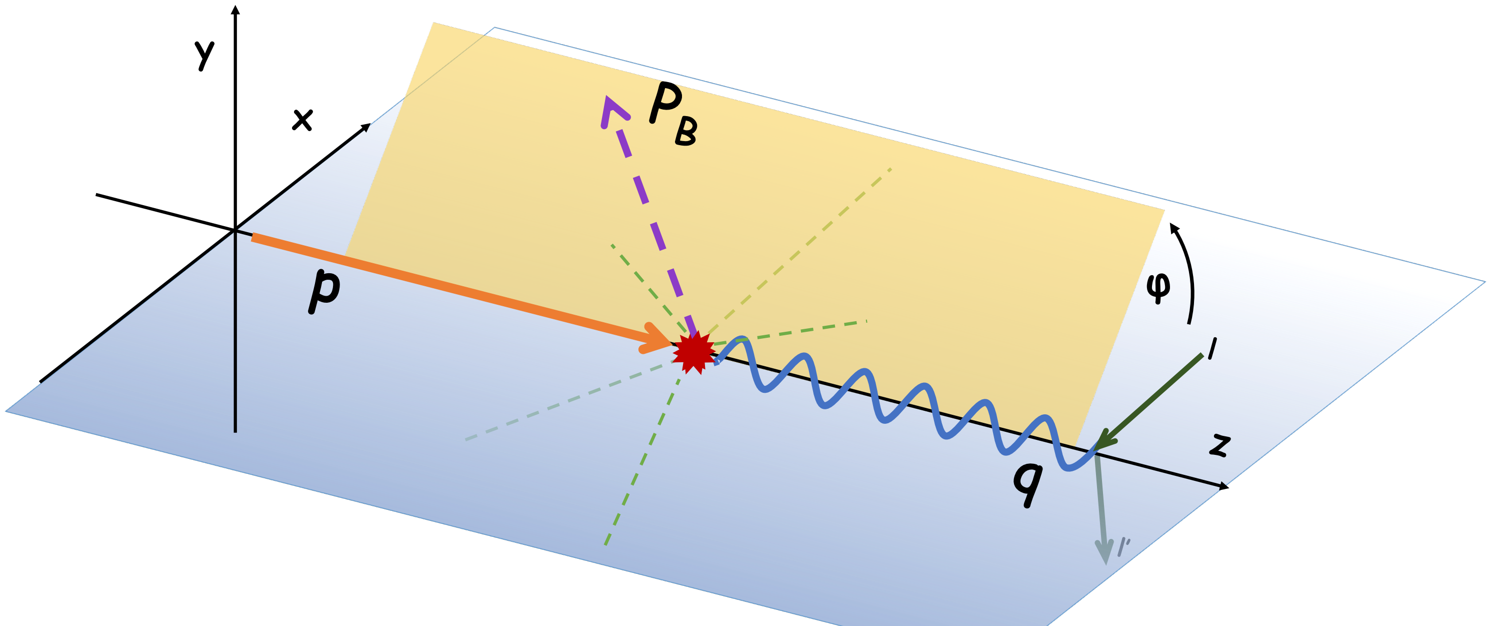
SIDIS

$$\frac{\mathrm{d}\sigma}{\mathrm{d}x\mathrm{d}y\mathrm{d}z\mathrm{d}q_{\mathrm{T}}^2} = \underbrace{W_{\mathrm{SIDIS}}}_{q_{\mathrm{T}} \ll Q} + \underbrace{Y_{\mathrm{SIDIS}}}_{q_{\mathrm{T}} \sim Q} + \mathcal{O}\left(\frac{m^2}{Q^2}\right)$$

FO collinear factorization

$$\mathrm{FO}_{\mathrm{SIDIS}} - \mathrm{ASY}_{\mathrm{SIDIS}}$$

$$H \int \mathrm{d}^2\boldsymbol{k}_{1\mathrm{T}}\mathrm{d}^2\boldsymbol{k}_{2\mathrm{T}} f_{j/p}\left(x, \boldsymbol{k}_{1\mathrm{T}}; \mu, \sqrt{\zeta}\right) D_{h/j}\left(z, z\boldsymbol{k}_{2\mathrm{T}}; \mu, \sqrt{\zeta}\right) \delta^{(2)}\left(\boldsymbol{q}_{\mathrm{T}} + \boldsymbol{k}_{1\mathrm{T}} - \boldsymbol{k}_{2\mathrm{T}}\right)$$



Conventional approach :

①

$$H \int \mathrm{d}^2\boldsymbol{k}_{1\mathrm{T}} \mathrm{d}^2\boldsymbol{k}_{2\mathrm{T}} f_{j/p}\left(x, \boldsymbol{k}_{1\mathrm{T}}; \mu, \sqrt{\zeta}\right) D_{h/j}\left(z, z\boldsymbol{k}_{2\mathrm{T}}; \mu, \sqrt{\zeta}\right) \delta^{(2)}\left(\boldsymbol{q}_\mathrm{T} + \boldsymbol{k}_{1\mathrm{T}} - \boldsymbol{k}_{2\mathrm{T}}\right)$$

Fourier Transform

$$H \int \frac{\mathrm{d}^2\boldsymbol{b}_\mathrm{T}}{(2\pi)^2} e^{-i\boldsymbol{b}_\mathrm{T}\cdot\boldsymbol{q}_\mathrm{T}} \tilde{f}_{j/p}\left(x, \boldsymbol{b}_\mathrm{T}; \mu, \sqrt{\zeta}\right) \tilde{D}_{h/j}\left(z, \boldsymbol{b}_\mathrm{T}; \mu, \sqrt{\zeta}\right)$$

Solve evolution equations relating input scale with SIDIS scale

2

$$\frac{\partial \ln \tilde{f}_{j/p}\left(x, \boldsymbol{b}_\mathrm{T}; \mu, \sqrt{\zeta}\right)}{\partial \ln \sqrt{\zeta}} = \tilde{K}\left(\boldsymbol{b}_\mathrm{T}; \mu\right)$$

$$\frac{\mathrm{d} \ln \tilde{f}_{j/p}\left(x, \boldsymbol{b}_\mathrm{T}; \mu, \sqrt{\zeta}\right)}{\mathrm{d} \ln \mu} = \gamma\left(\alpha_S(\mu); \mu/\sqrt{\zeta}\right)$$

$$\frac{\mathrm{d}\tilde{K}\left(\boldsymbol{b}_\mathrm{T}; \mu\right)}{\mathrm{d} \ln \mu} = -\gamma_K\left(\alpha_S(\mu)\right)$$

Same for FF

$\mu = \sqrt{\zeta}$

$\mu_0 = \sqrt{\zeta_0}$

$$\tilde{f}_{j/p}\left(x, \boldsymbol{b}_\mathrm{T}; \mu, \sqrt{\zeta}\right) = \tilde{f}_{j/p}\left(x, \boldsymbol{b}_\mathrm{T}; \mu_0, \sqrt{\zeta_0}\right) \times$$
$$\times \exp\left\{\int_{\mu_0}^{\mu} \frac{\mathrm{d}\mu'}{\mu'}\left[\gamma\left(\alpha_S(\mu'); 1\right) - \ln\left(\frac{\sqrt{\zeta}}{\mu'}\right)\gamma_K\left(\alpha_S(\mu')\right)\right] + \ln\left(\frac{\sqrt{\zeta}}{\sqrt{\zeta_0}}\tilde{K}\left(\boldsymbol{b}_\mathrm{T}; \mu_0\right)\right)\right\}$$

Separate $b_T < b_{max}$ & $b_T > b_{max}$ regions with a b_* prescription

3

$$\tilde{f}_{j/p}\left(x;\boldsymbol{b}_\mathrm{T};\mu_Q,Q\right) = \tilde{f}_{j/p}\left(x;\boldsymbol{b}_*;\mu_Q,Q\right) \underbrace{\frac{\tilde{f}_{j/p}\left(x;\boldsymbol{b}_\mathrm{T};\mu_Q,Q\right)}{\tilde{f}_{j/p}\left(x;\boldsymbol{b}_*;\mu_Q,Q\right)}}_{\exp\left\{-g_{j/p}(x,\boldsymbol{b}_\mathrm{T})\right\}}$$

Same for FF

Perturbatively calculable with fixed order collinear factorization

Nonperturbative

$$g_K(\boldsymbol{b}_\mathrm{T}) \equiv \tilde{K}\left(\boldsymbol{b}_*;\mu\right) - \tilde{K}\left(\boldsymbol{b}_\mathrm{T};\mu\right)$$

Choose ansatzes for g functions

4

$$g_{j/p}\left(x, \boldsymbol{b}_{\mathrm{T}}\right) = \frac{1}{4} M_F^2 b_{\mathrm{T}}^2$$

$$g_{h/j}\left(z, \boldsymbol{b}_{\mathrm{T}}\right) = \frac{1}{4z^2} M_D^2 b_{\mathrm{T}}^2$$

$$g_K\left(\boldsymbol{b}_{\mathrm{T}}\right) = \frac{g_2}{2M_K^2} \ln\left(1 + M_K^2 b_{\mathrm{T}}^2\right)$$

$$g_K\left(\boldsymbol{b}_{\mathrm{T}}\right) = \frac{1}{2} M_K^2 b_{\mathrm{T}}^2$$

Relate μ_{b_*} with input scale Q_0 and get OPE expansion

5

$$\mu_{b_*} \equiv \frac{2e^{-\gamma_E}}{b_T}$$

$$\tilde{f}_{j/p}\left(x; \boldsymbol{b}_T; \mu_Q, Q\right) = \tilde{f}^{\rm OPE}_{j/p}\left(x; \boldsymbol{b}_*; \mu_{b_*}, \mu_{b_*}\right) \times$$

$$\times \exp\left\{ \int_{\mu_{b_*}}^{\mu_Q} \frac{d\mu'}{\mu'} \left[\gamma\left(\alpha_S(\mu'); 1\right) - \ln\left(\frac{Q}{\mu'}\right) \gamma_K\left(\alpha_S(\mu')\right) \right] + \ln\left(\frac{Q}{\mu_{b_*}}\right) \tilde{K}\left(\boldsymbol{b}_*; \mu_{b_*}\right) \right\}$$

$$\times \exp\left\{ -g_{j/p}\left(x, \boldsymbol{b}_T\right) - g_K\left(\boldsymbol{b}_T\right) \ln\left(\frac{Q}{Q_0}\right) \right\}$$

Nonperturbative

Perturbatively calculable

$$\tilde{f}^{\rm OPE}_{j/p}\left(x, \boldsymbol{b}_*; \mu_{b_*}, \mu_{b_*}\right) = \widetilde{C}_{j/j'}\left(x/\xi, \boldsymbol{b}_*; \mu_{b_*}, \mu_{b_*}\right) \otimes \tilde{f}_{j'/p}\left(\xi; \mu_{b_*}\right) + \mathcal{O}\left(m^2 b_{max}^2\right)$$

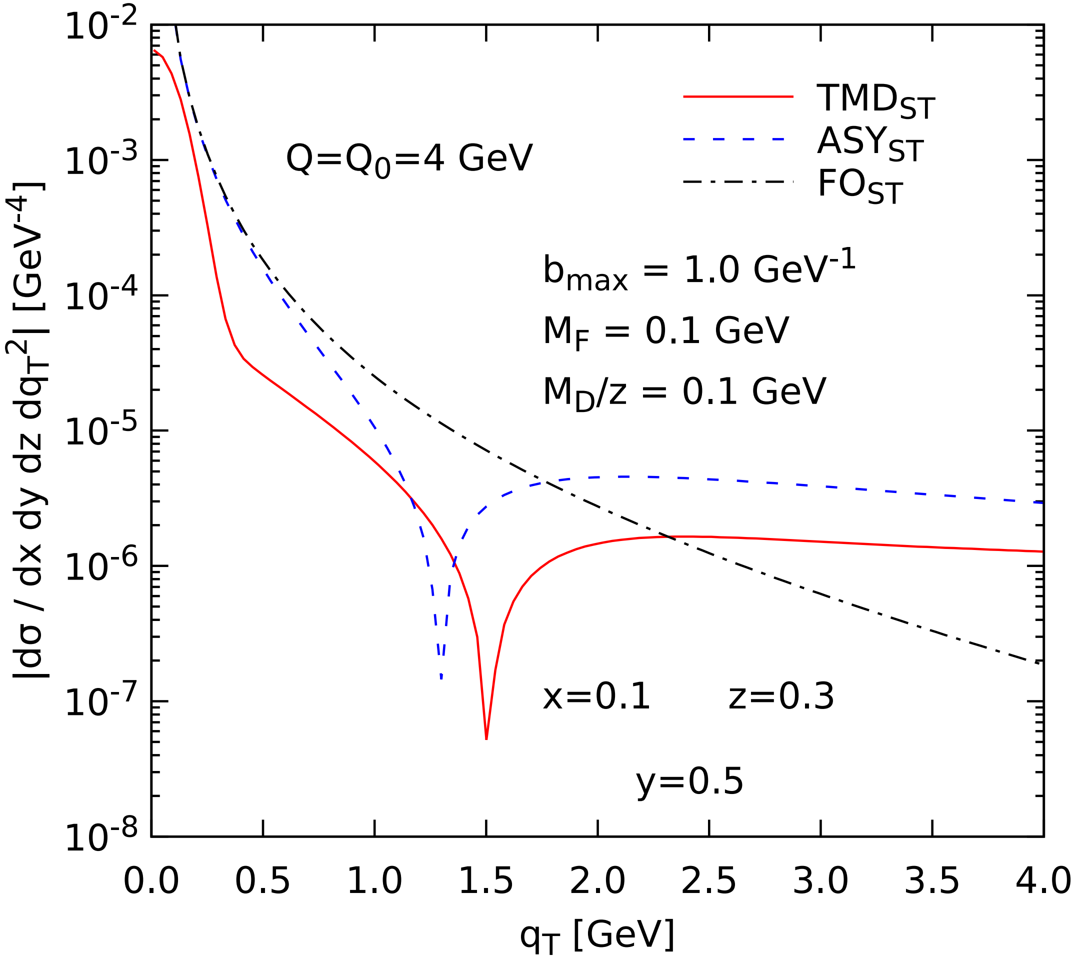
Drop this

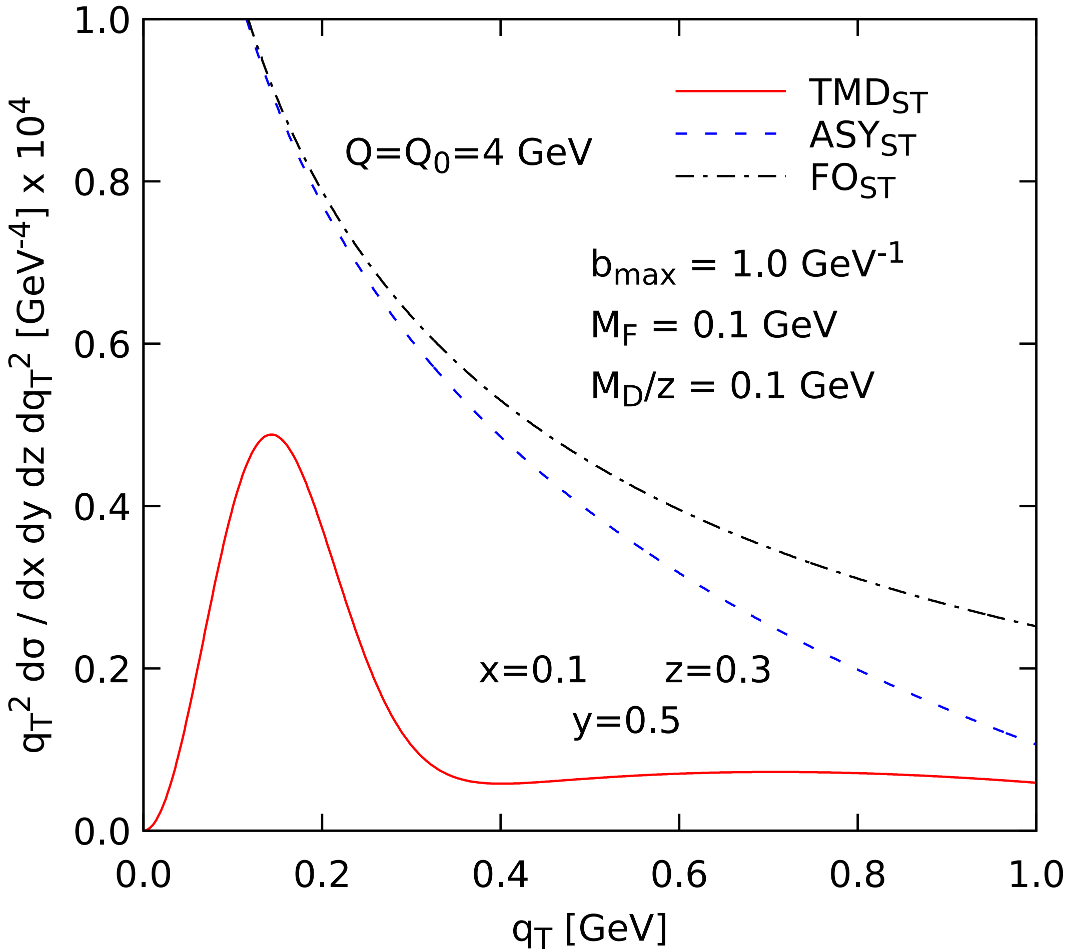
Same for FF

Fixed order collinear factorization

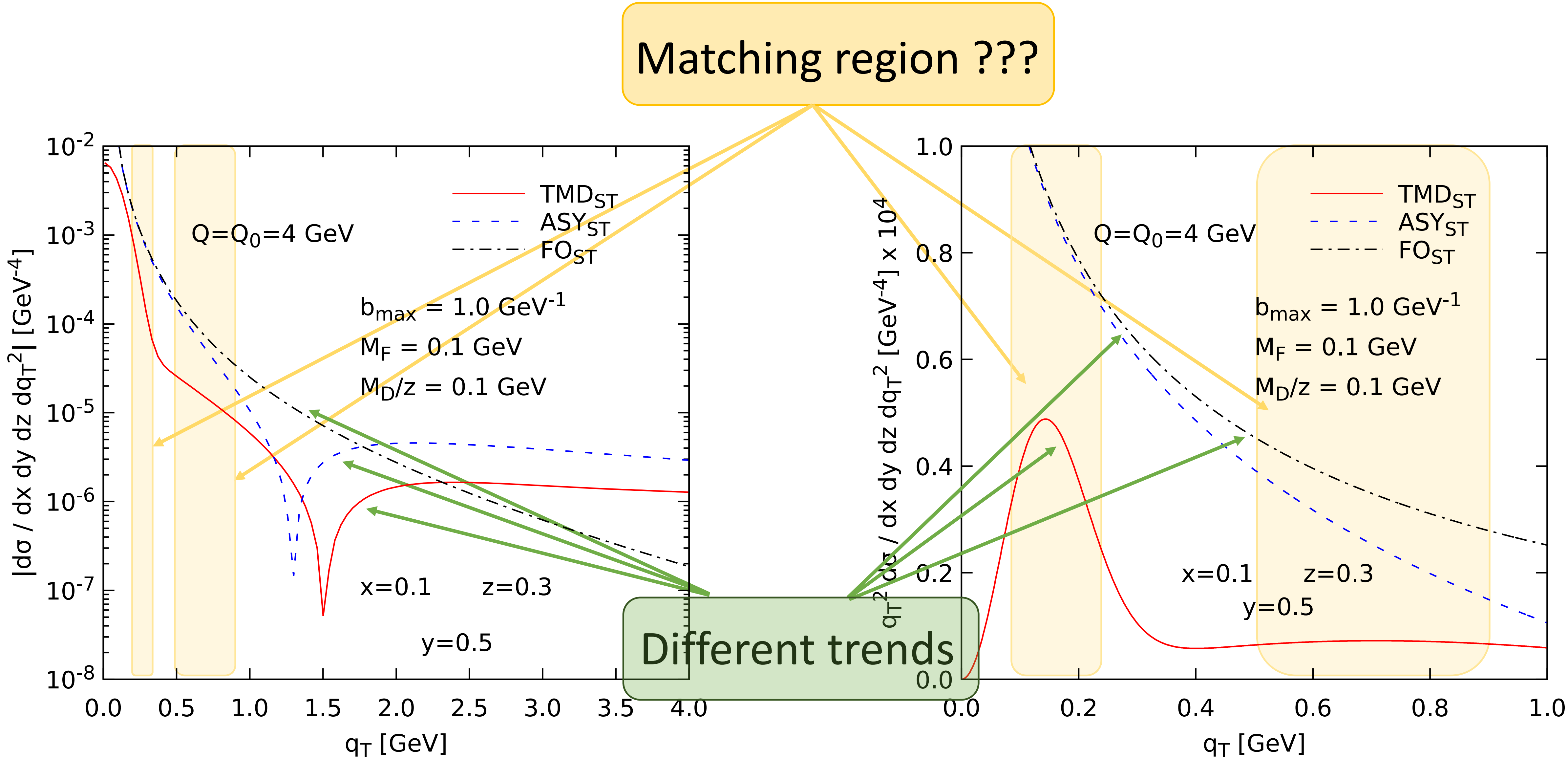
Conventional approach results for SIDIS

$$\Lambda_{\mathrm{QCD}} \ll q_{\mathrm{T}} \ll Q_0$$



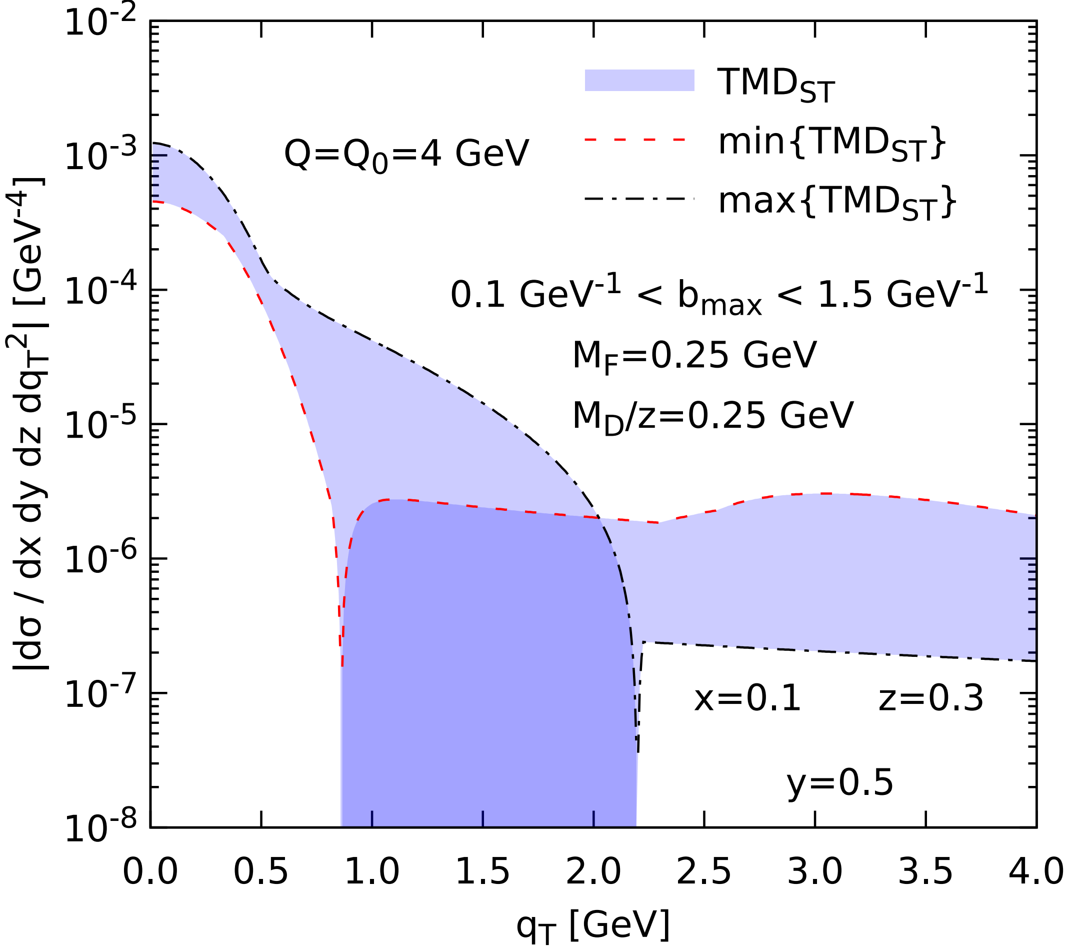


Conventional approach results for SIDIS

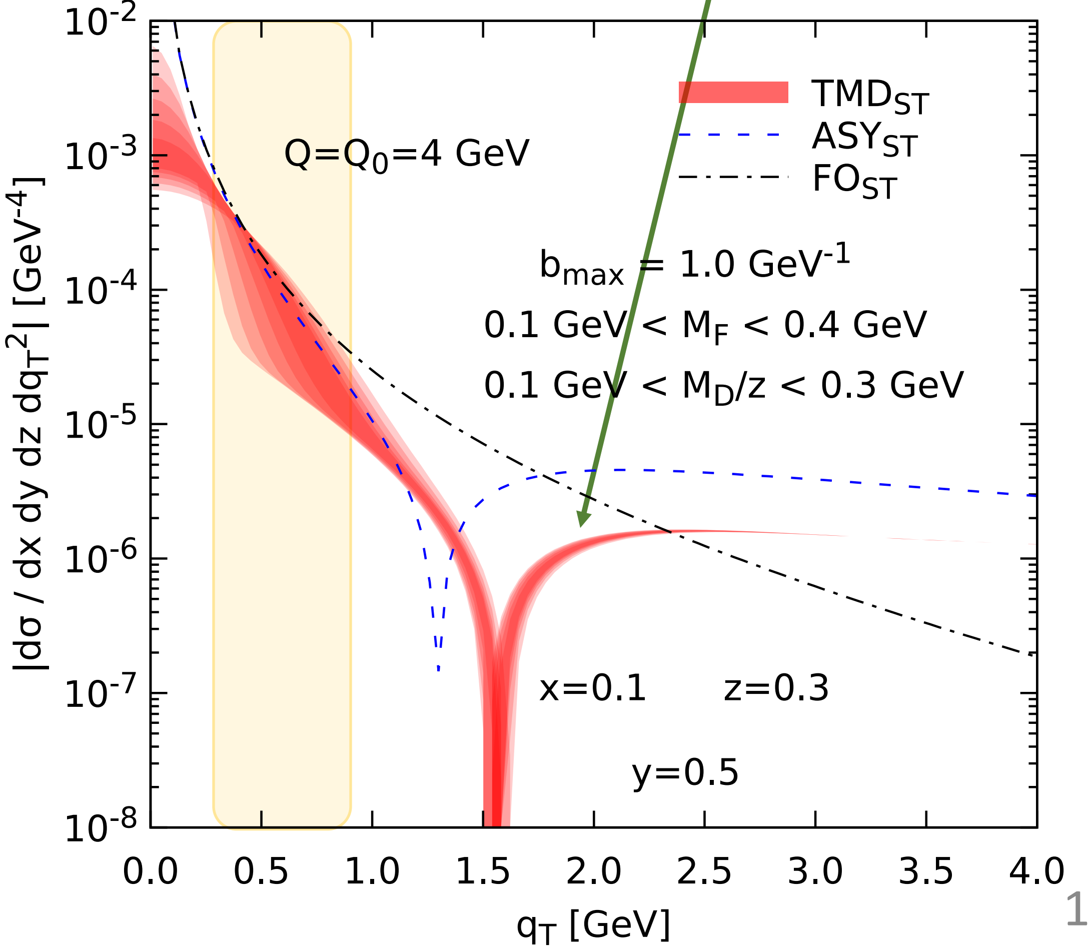


(Other) Issues with conventional approach

Large b_{max} dependence



Model dependence even for large q_T



Hadron Structure Oriented approach (HSO)

- Fixes TMDs parametrization at **input scale** Q_0
- Uses uniquely determined TMDs for **all transverse momenta**
- Interpolates perturbative (large k_T) and nonperturbative (small k_T) TM regions
- Can **swap NP models** easily
- Explicit (approximate) **probability interpretation**

No need for b_* prescription !

TMD PDF & FF HSO parametrization at input scale

Fixed order collinear factorization

$\mathcal{O}(\alpha_S)$

$$f_{j/p}\left(x, \boldsymbol{k}_{\mathrm{T}}; \mu_{Q_0}, Q_0^2\right) = \frac{1}{2\pi}\frac{1}{k_{\mathrm{T}}^2 + m_{f_{j/p}}^2}\left[A_{j/p}^f(x;\mu_Q) + B_{j/p}^f(x;\mu_Q)\ln\left(\frac{Q_0^2}{k_{\mathrm{T}}^2 + m_{f_{j/p}}^2}\right)\right] + \frac{1}{2\pi}\frac{A_{j/p}^{f,g}(x;\mu_Q)}{k_{\mathrm{T}}^2 + m_{f_{g/p}}^2}$$
$$+ C_{j/p}^f f_{\mathrm{core},j,p}\left(x, \boldsymbol{k}_{\mathrm{T}}; Q_0^2\right)$$

Small k_T model

NP parameters

Such that

$$f_{j/p}^c(x;\mu_Q) \equiv 2\pi\int_0^{k_c} \mathrm{d}k_{\mathrm{T}}\, k_{\mathrm{T}}\, f_{j/p}\left(x, \boldsymbol{k}_{\mathrm{T}}; \mu_Q, \sqrt{\zeta}\right)$$
$$= f_{j/p}(x;\mu_Q) + \Delta_{j/p}(x;\mu_Q, k_c) + \mathrm{p.s.}$$

TMD PDF & FF HSO parametrization at input scale

Fixed order collinear factorization

$\mathcal{O}(\alpha_S)$

$$D_{h/j}\left(z, z\boldsymbol{k}_\mathrm{T}; \mu_{Q_0}, Q_0^2\right) = \frac{1}{2\pi z^2}\frac{1}{k_\mathrm{T}^2 + m_{D_{h/j}}^2}\left[A_{h/j}^D(z;\mu_Q) + B_{h/j}^D(z;\mu_Q)\ln\left(\frac{Q_0^2}{k_\mathrm{T}^2 + m_{D_{h/j}}^2}\right)\right] + \frac{1}{2\pi z^2}\frac{A_{h/g}^{D,g}(z;\mu_Q)}{k_\mathrm{T}^2 + m_{D_{h/g}}^2}$$
$$+ C_{h/j}^D D_{\mathrm{core},h/j}\left(z, z\boldsymbol{k}_\mathrm{T}; Q_0^2\right)$$

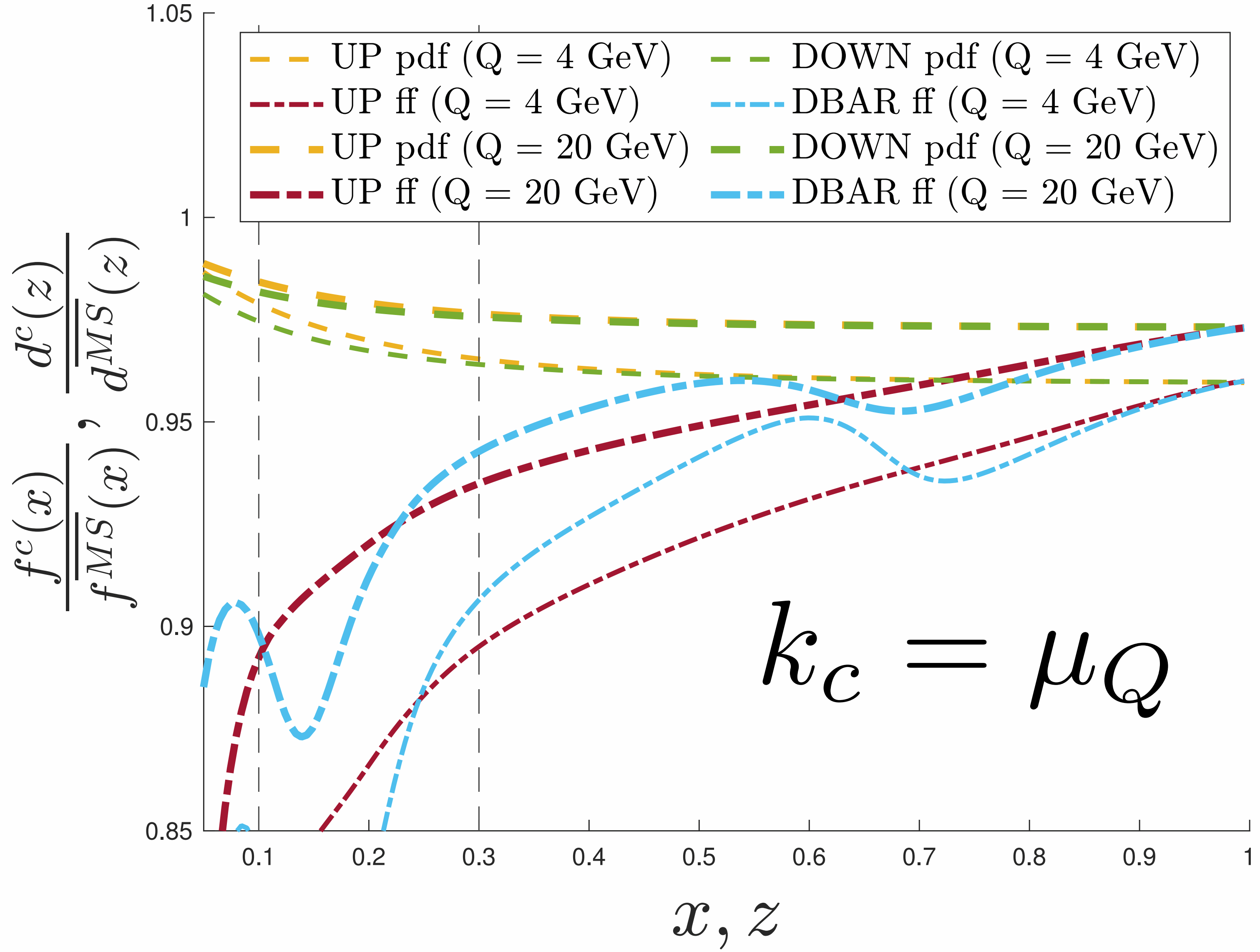
Small k_T model

NP parameters

Such that

$$d_{h/j}^c(z;\mu_Q) \equiv 2\pi z^2 \int_0^{k_c} \mathrm{d}k_\mathrm{T}\, k_\mathrm{T} D_{h/j}\left(z, z\boldsymbol{k}_\mathrm{T}; \mu_Q, \sqrt{\zeta}\right)$$
$$= d_{h/j}(z;\mu_Q) + \Delta_{h/j}(z;\mu_Q, k_c) + \mathrm{p.s.}$$

From $\overline{\mathrm{MS}}$ to Cutoff schemes



Choose "core" models (examples)

$$f^{\text{Gauss}}_{\text{core},i/p}\left(x, \boldsymbol{k}_{\text{T}}; Q_0^2\right) = \frac{e^{-k_{\text{T}}^2/M_F^2}}{\pi M_F^2}$$

$$D^{\text{Gauss}}_{\text{core},h/j}\left(z, z\boldsymbol{k}_{\text{T}}; Q_0^2\right) = \frac{e^{-z^2k_{\text{T}}^2/M_D^2}}{\pi M_D^2}$$

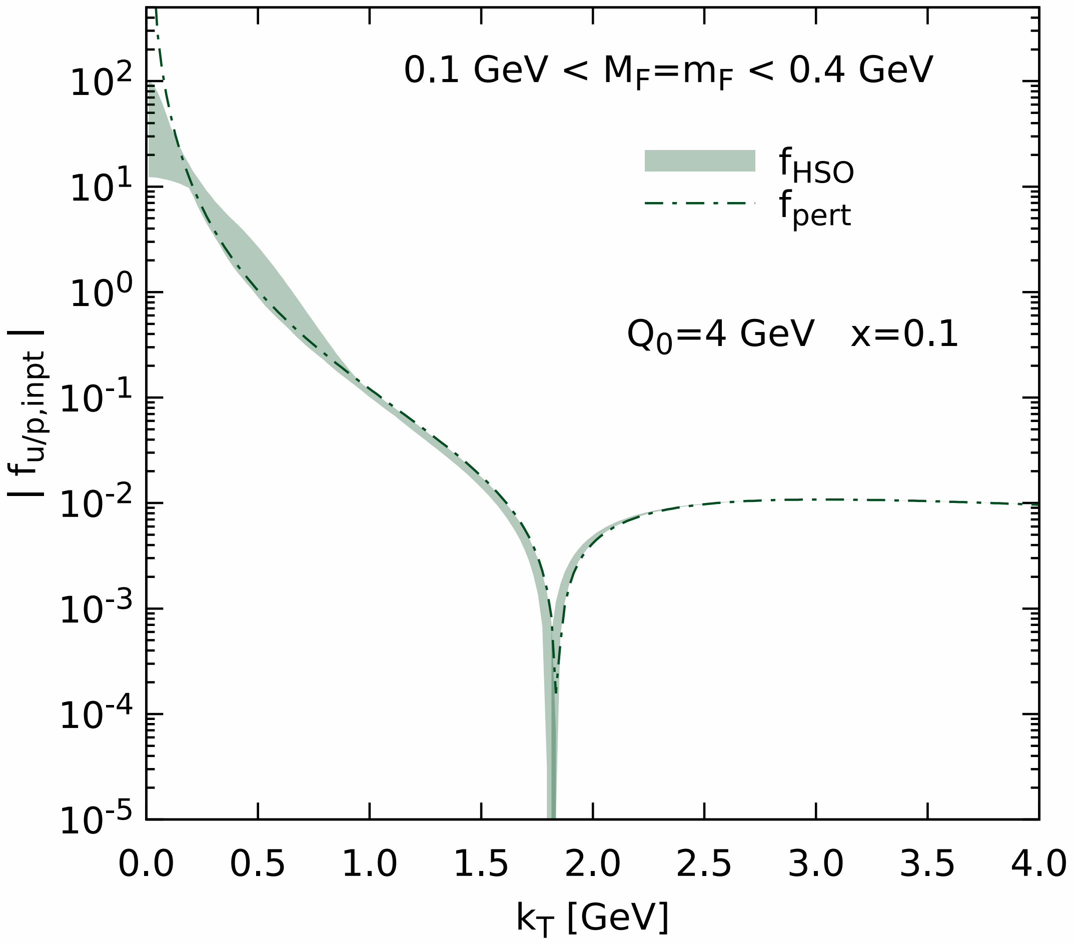
Gaussian "core" models

Spectator-like "core" models

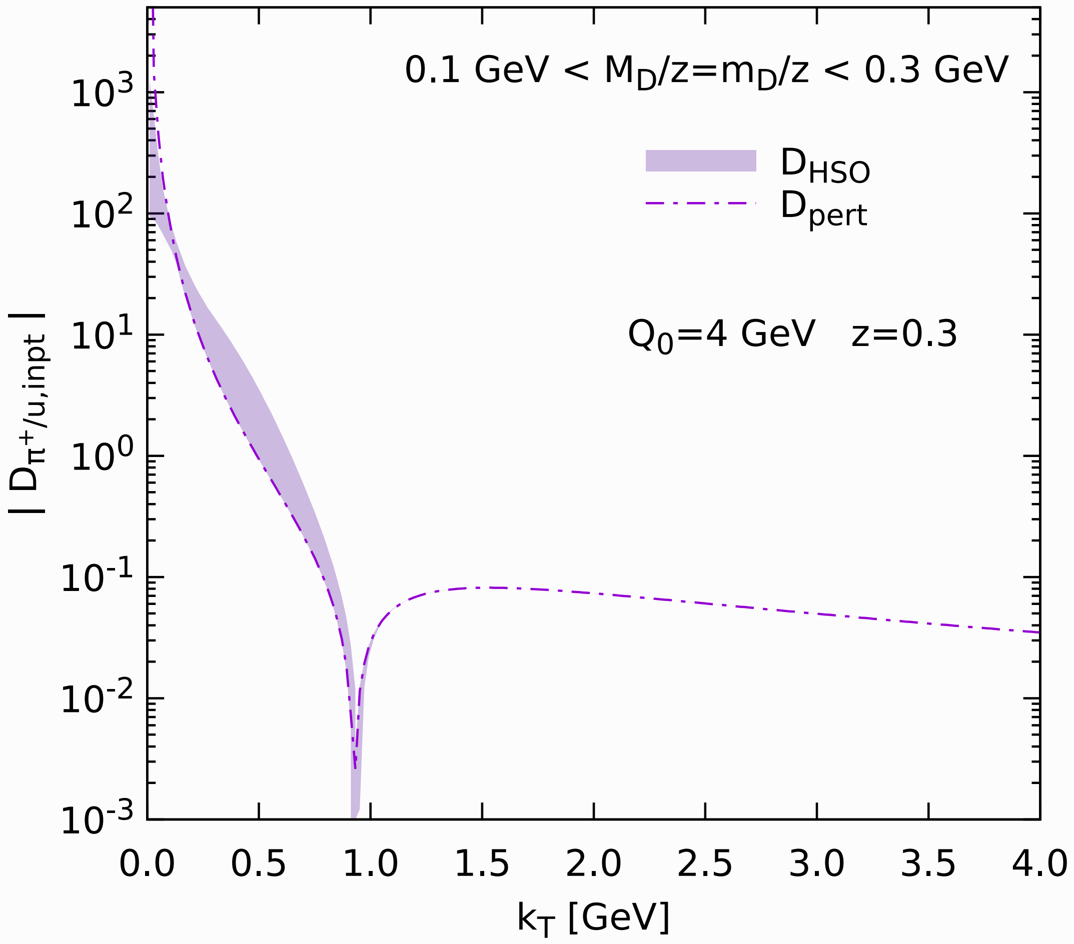
$$f^{\text{Spect}}_{\text{core},j/p}\left(x, \boldsymbol{k}_{\text{T}}; Q_0^2\right) = \frac{6M_{0F}^6}{\pi\left(2M_F^2 + M_{0F}^2\right)} \frac{M_F^2 + k_{\text{T}}^2}{\left(M_{0F}^2 + k_{\text{T}}^2\right)^4}$$

$$D^{\text{Spect}}_{\text{core},h/j}\left(z, z\boldsymbol{k}_{\text{T}}; Q_0^2\right) = \frac{2M_{0D}^4}{\pi\left(M_D^2 + M_{0D}^2\right)} \frac{M_D^2 + z^2k_{\text{T}}^2}{\left(M_{0D}^2 + z^2k_{\text{T}}^2\right)^3}$$

Up-quark from Proton TMD pdf



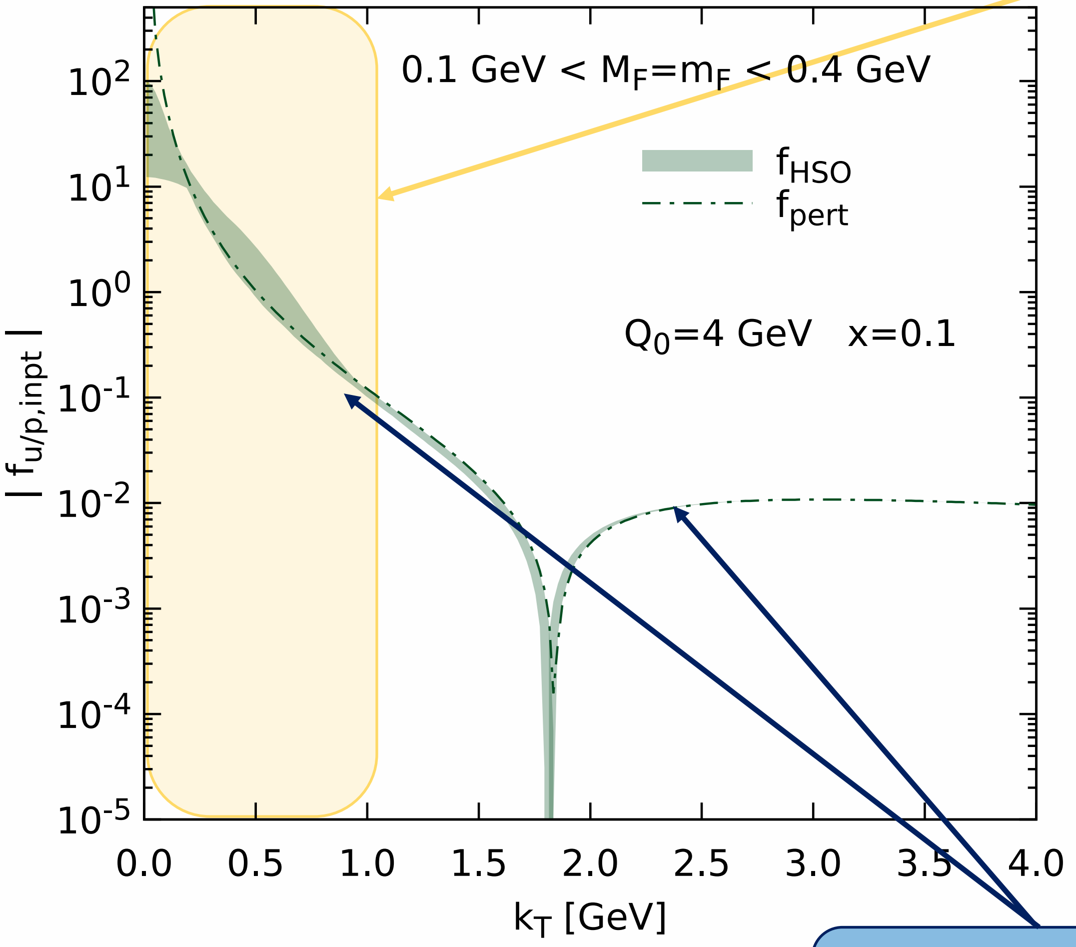
π^+ from Up-quark TMD ff

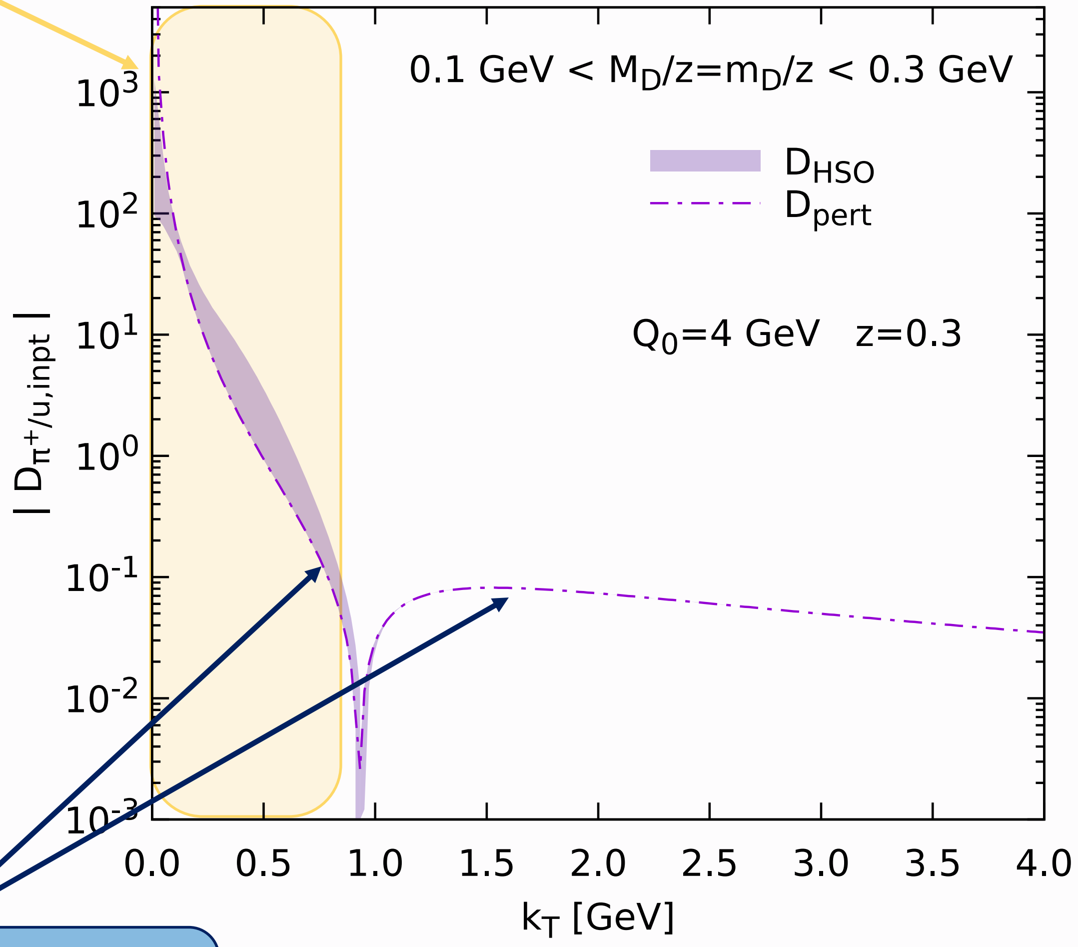


Up-quark from Proton TMD pdf

π^+ from Up-quark TMD ff

Model dependence





Smooth transition

Asymptotic term

HSO

$$\mathrm{ASY_{HSO}} = \lim_{\frac{q_T}{Q}\to\sim 1,\ \frac{m^2}{Q^2}\to 0} \mathrm{W_{HSO}}$$

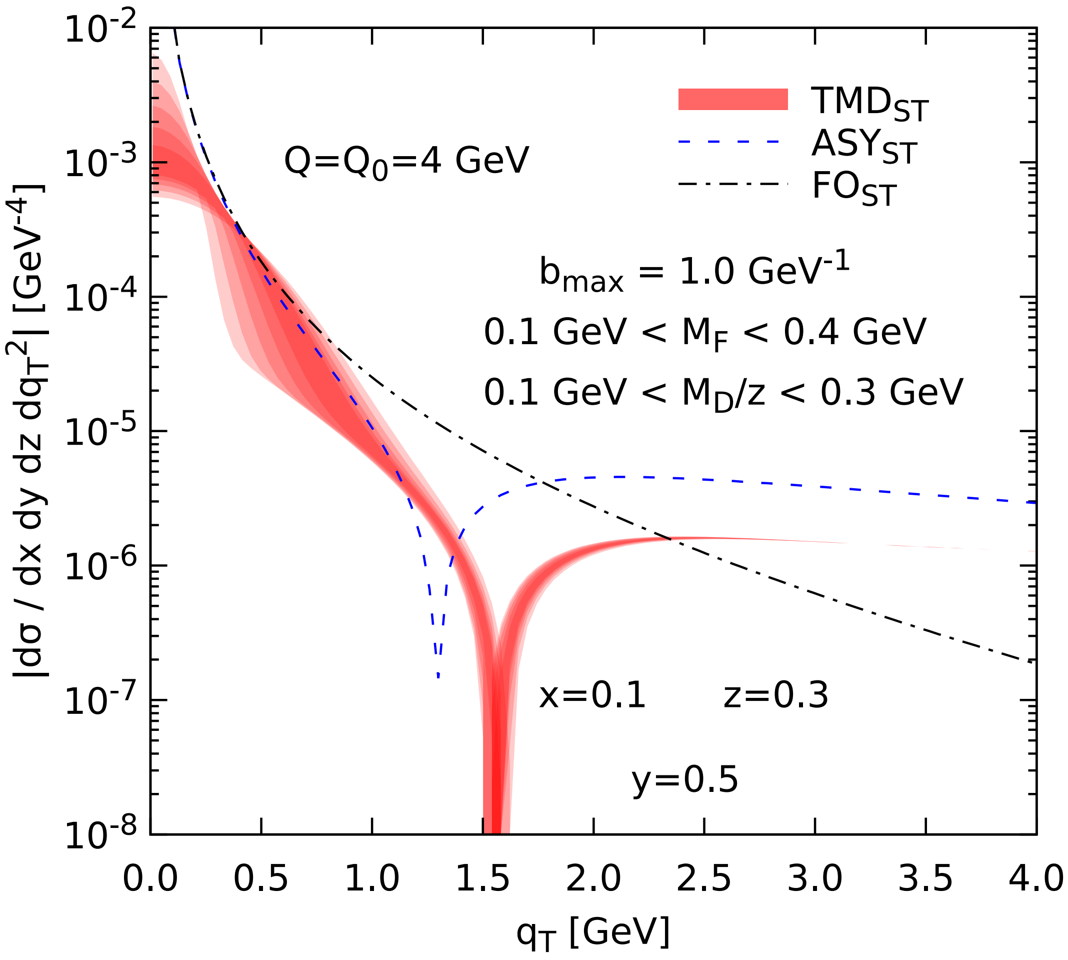
Conventional

$$\mathrm{ASY_{ST}} = \lim_{\frac{q_T}{Q}\to 0,\ \frac{m^2}{Q^2}\to 0} \mathrm{FO_{ST}}$$

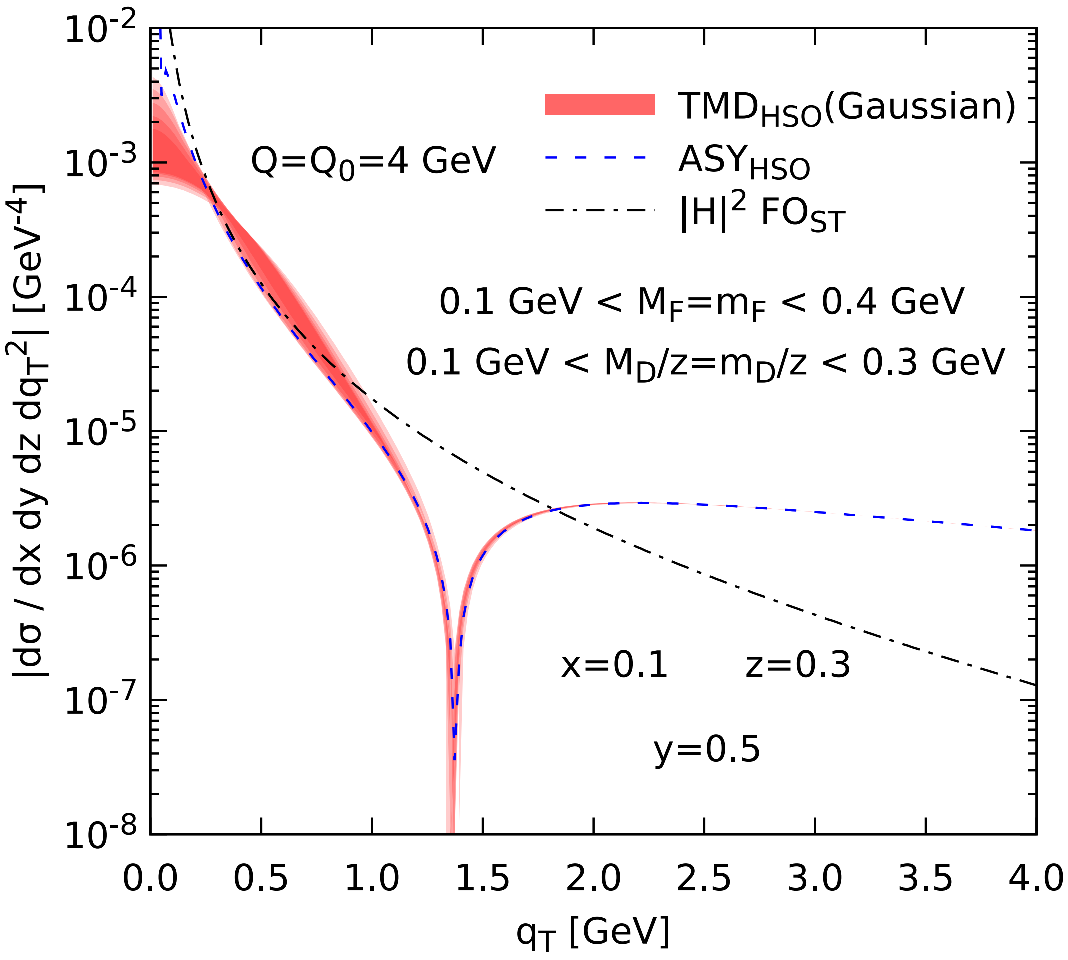
$$\begin{aligned}
[f, D]_{\mathrm{ASY}} = & D^{\mathrm{pert}}\left(z, z\boldsymbol{q}_{\mathrm{T}}; \mu_Q, Q^2\right) f^c\left(x; \mu_Q\right) + \frac{1}{z^2} f^{\mathrm{pert}}\left(x, -\boldsymbol{q}_{\mathrm{T}}; \mu_Q, Q^2\right) d^c\left(z; \mu_Q\right) \\
& + \int \mathrm{d}^2\boldsymbol{k}_{\mathrm{T}} \Big\{ f^{\mathrm{pert}}\left(x, \boldsymbol{k}_{\mathrm{T}} - \boldsymbol{q}_{\mathrm{T}}/2; \mu_Q, Q^2\right) D^{\mathrm{pert}}\left(z, z\left(\boldsymbol{k}_{\mathrm{T}} + \boldsymbol{q}_{\mathrm{T}}/2\right); \mu_Q, Q^2\right) \\
& - D^{\mathrm{pert}}\left(z, z\boldsymbol{q}_{\mathrm{T}}; \mu_Q, Q^2\right) f^{\mathrm{pert}}\left(x, \boldsymbol{k}_{\mathrm{T}} - \boldsymbol{q}_{\mathrm{T}}/2; \mu_Q, Q^2\right) \Theta\left(\mu_Q - |\boldsymbol{k}_{\mathrm{T}} - \boldsymbol{q}_{\mathrm{T}}/2|\right) \\
& - D^{\mathrm{pert}}\left(z, z\left(\boldsymbol{k}_{\mathrm{T}} + \boldsymbol{q}_{\mathrm{T}}/2\right); \mu_Q, Q^2\right) f^{\mathrm{pert}}\left(x, -\boldsymbol{q}_{\mathrm{T}}; \mu_Q, Q^2\right) \Theta\left(\mu_Q - |\boldsymbol{k}_{\mathrm{T}} + \boldsymbol{q}_{\mathrm{T}}/2|\right) \Big\}
\end{aligned}$$

Conventional vs HSO - SIDIS cross section

Conventional

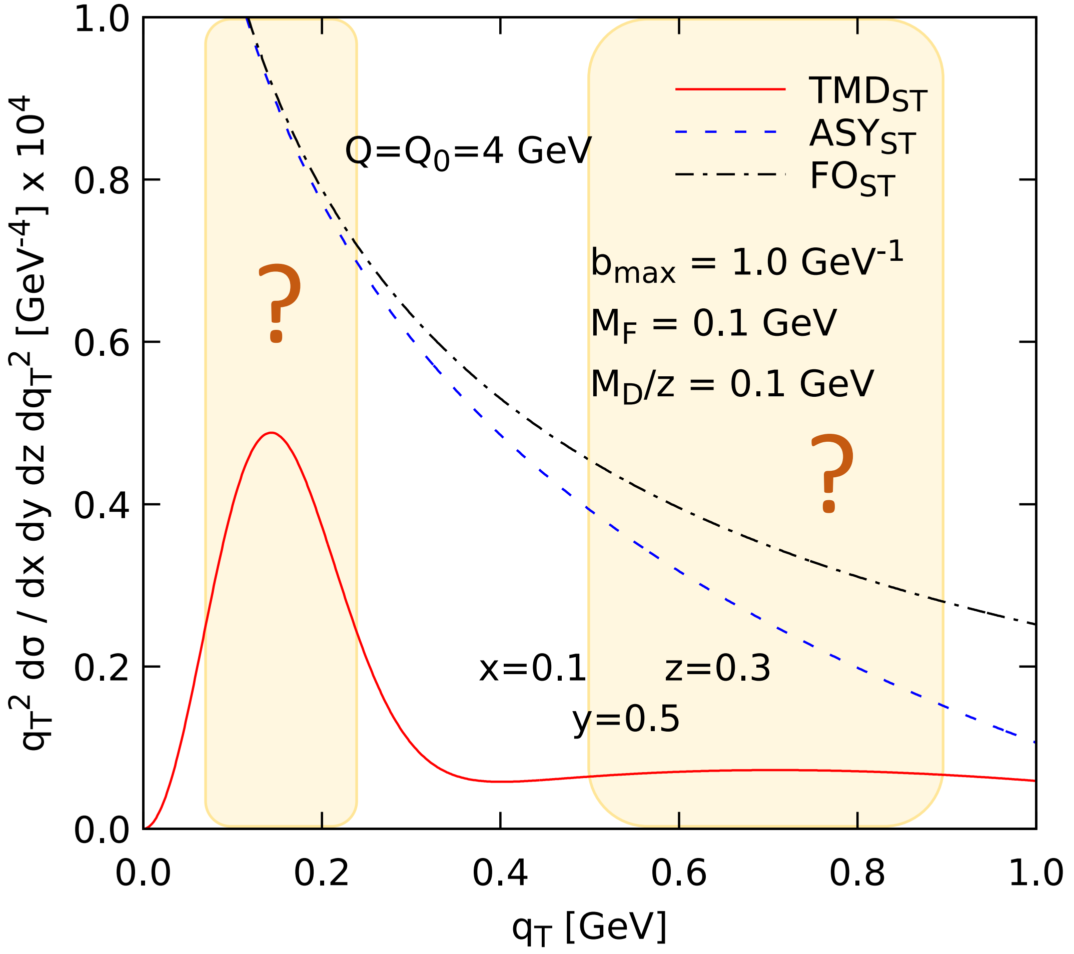


HSO (Gaussian)

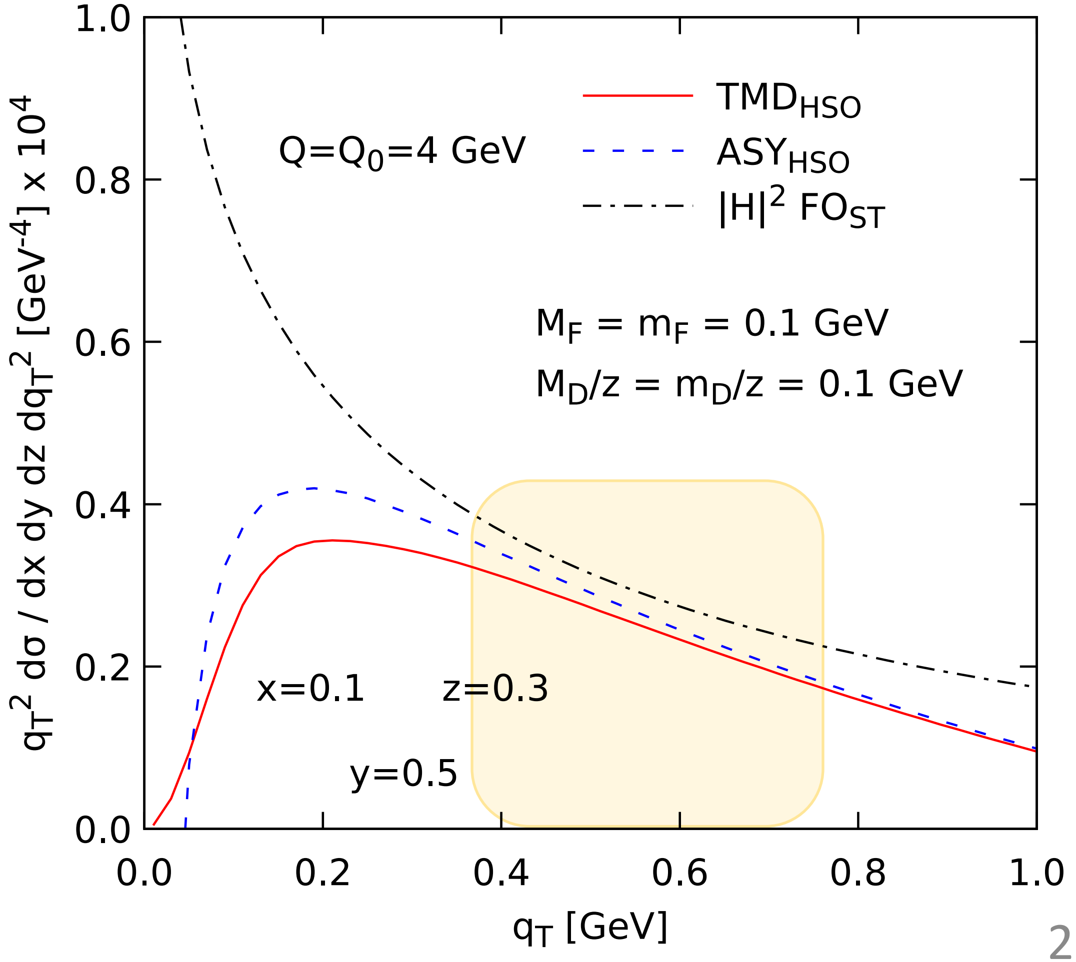


Conventional vs HSO - SIDIS cross section

Conventional



HSO (Gaussian)



Summary

- Consistent TMD parametrization for large TM at input scale
- No need of b_{max}
- Improved TM behavior in matching region

NEXT:

- Check with data (SIDIS, DY, DIA, …)
- Add higher orders
- Incorporate NP calculations (lattice, EFT, …)

Many thanks for the support of
The Gordon and Betty Moore Foundation and the American Physical Society

Thank you

Backup slides

Coefficients (TMD pdf)

$$A^{f}_{i/p}(x;\mu_{Q_0}) \equiv \sum_{ii'} \delta_{i'i} \frac{\alpha_s(\mu_{Q_0})}{\pi} \left\{ [(P_{i'i} \otimes f_{i'/p})(x;\mu_{Q_0})] - \frac{3C_F}{2} f_{i'/p}(x;\mu_{Q_0}) \right\} ,$$

$$B^{f}_{i/p}(x;\mu_{Q_0}) \equiv \sum_{i'i} \delta_{i'i} \frac{\alpha_s(\mu_{Q_0}) C_F}{\pi} f_{i'/p}(x;\mu_{Q_0}) ,$$

$$A^{f,g}_{i/p}(x;\mu_{Q_0}) \equiv \frac{\alpha_s(\mu_{Q_0})}{\pi} [(P_{ig} \otimes f_{g/p})(x;\mu_{Q_0})] ,$$

$$C^{f}_{i/p} \equiv \frac{1}{N^{f}_{i/p}} \left[f_{i/p}(x;\mu_{Q_0}) - A^{f}_{i/p}(x;\mu_{Q_0}) \ln\left(\frac{\mu_{Q_0}}{m_{f_{i,p}}}\right) - B^{f}_{i/p}(x;\mu_{Q_0}) \ln\left(\frac{\mu_{Q_0}}{m_{f_{i,p}}}\right) \ln\left(\frac{Q_0^2}{\mu_{Q_0} m_{f_{i,p}}}\right) , \right.$$

$$\left. - A^{f,g}_{i/p}(x;\mu_{Q_0}) \ln\left(\frac{\mu_{Q_0}}{m_{f_{g,p}}}\right) + \frac{\alpha_s(\mu_{Q_0})}{2\pi} \left\{ \sum_{ii'} \delta_{i'i} [\mathcal{C}^{i/i'}_{\Delta} \otimes f_{i'/p}](x;\mu_{Q_0}) + [\mathcal{C}^{i/g}_{\Delta} \otimes f_{g/p}](x;\mu_{Q_0}) \right\} \right] .$$

Coefficients (TMD ff)

$$A^D_{h/j}(z;\mu_{Q_0}) \equiv \sum_{jj'} \delta_{j'j} \frac{\alpha_s(\mu_{Q_0})}{\pi} \left\{ [(P_{jj'} \otimes d_{h/j'})(z;\mu_{Q_0})] - \frac{3C_F}{2} d_{h/j'}(z;\mu_{Q_0}) \right\} ,$$

$$B^D_{h/j}(z;\mu_{Q_0}) \equiv \sum_{jj'} \delta_{j'j} \frac{\alpha_s(\mu_{Q_0}) C_F}{\pi} d_{h/j'}(z;\mu_{Q_0}) ,$$

$$A^{D,g}_{h/j}(z;\mu_{Q_0}) \equiv \frac{\alpha_s(\mu_{Q_0})}{\pi} [(P_{gj} \otimes d_{h/g})(z;\mu_{Q_0})] ,$$

$$C^D_{h/j} \equiv \frac{1}{N^D_{h/j}} \Bigg[d_{h/j}(z;\mu_{Q_0}) - A^D_{h/j}(z;\mu_{Q_0}) \ln\left(\frac{\mu_{Q_0}}{m_{D_{h,j}}}\right) - B^D_{h/j}(z;\mu_{Q_0}) \ln\left(\frac{\mu_{Q_0}}{m_{D_{h,j}}}\right) \ln\left(\frac{Q_0^2}{\mu_{Q_0} m_{D_{h,j}}}\right) ,$$

$$- A^{D,g}_{h/j}(z;\mu_{Q_0}) \ln\left(\frac{\mu_{Q_0}}{m_{D_{h,g}}}\right) + \frac{\alpha_s(\mu_{Q_0})}{2\pi} \left\{ \sum_{jj'} \delta_{j'j} [\mathcal{C}^{j'/j}_{\Delta} \otimes d_{h/j'}](z;\mu_{Q_0}) + [\mathcal{C}^{g/j}_{\Delta} \otimes d_{h/g}](z;\mu_{Q_0}) \right\} \Bigg] .$$

b_T space TMD distributions at input scale

$$z^2 \tilde{D}_{\mathrm{inpt},h/j}(z, \boldsymbol{b}_\mathrm{T}; \mu_{Q_0}, Q_0^2) = K_0\left(b_\mathrm{T} m_{D_{h,j}}\right) \left[A^D_{h/j}(z; \mu_{Q_0}) + B^D_{h/j}(z; \mu_{Q_0}) \ln\left(\frac{b_\mathrm{T} Q_0^2 e^{\gamma_E}}{2 m_{D_{h,j}}} \right) \right]$$
$$+ K_0\left(b_\mathrm{T} m_{D_{h,g}}\right) A^{D,g}_{h/j}(z; \mu_{Q_0})$$
$$+ C^D_{h/j}\, z^2 \tilde{D}_{\mathrm{core},h/j}(z, \boldsymbol{b}_\mathrm{T}; Q_0^2)\,,$$

$$\tilde{f}_{\mathrm{inpt},i/p}(x, \boldsymbol{b}_\mathrm{T}; \mu_{Q_0}, Q_0^2) = K_0\left(b_\mathrm{T} m_{f_{i,p}}\right) \left[A^f_{i/p}(x; \mu_{Q_0}) + B^f_{i/p}(x; \mu_{Q_0}) \ln\left(\frac{b_\mathrm{T} Q_0^2 e^{\gamma_E}}{2 m_{f_{i,p}}} \right) \right]$$
$$+ K_0\left(b_\mathrm{T} m_{f_{g,p}}\right) A^f_{g/p}(x; \mu_{Q_0})$$
$$+ C^f_{i/p}\, \tilde{f}_{\mathrm{core},i/p}(x, \boldsymbol{b}_\mathrm{T}; Q_0^2)\,,$$

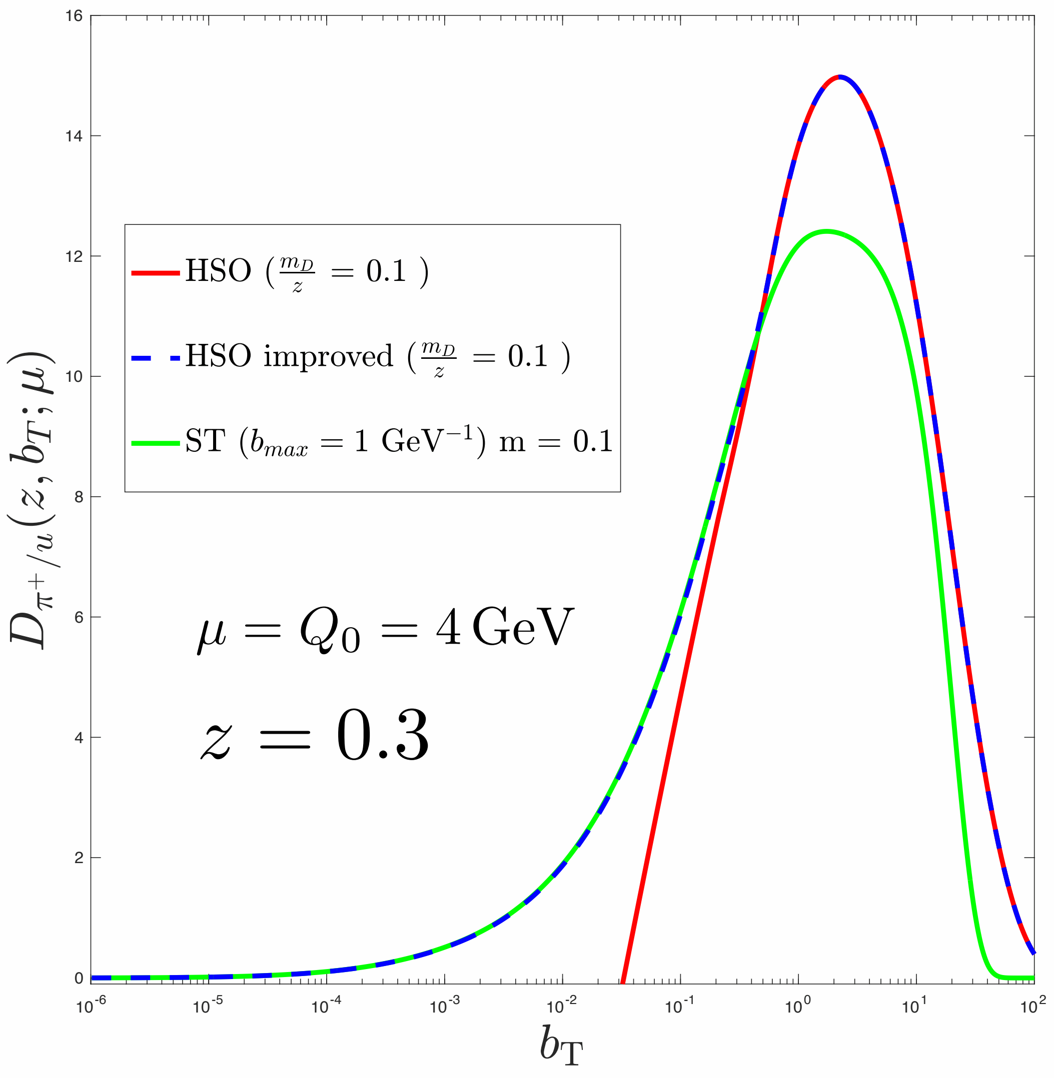
b_T – space RG improvement

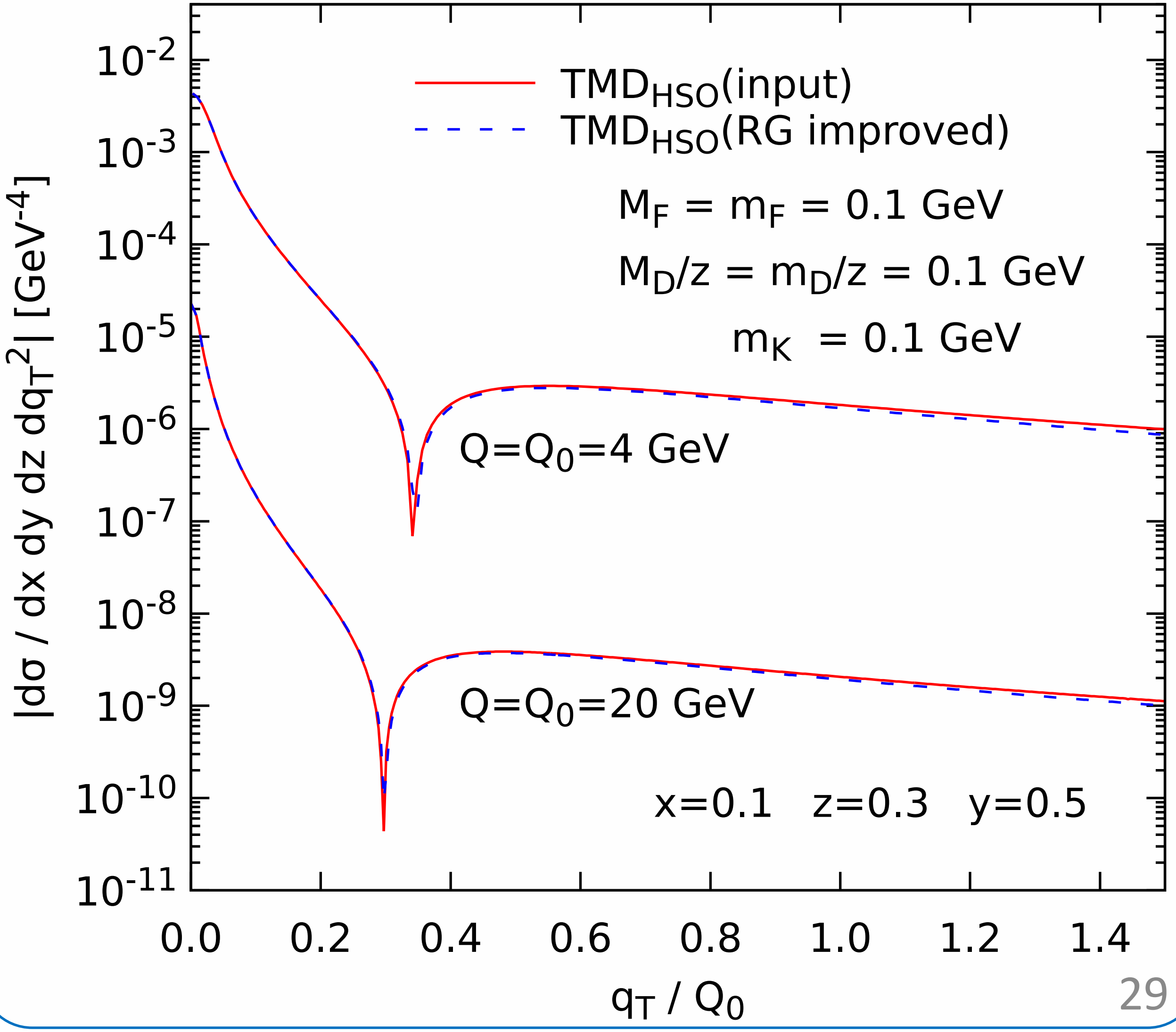
$$\tilde{D}_{h/j}(z, \boldsymbol{b}_{\mathrm{T}}; \mu_{Q_0}, Q_0^2) = \tilde{D}_{\mathrm{inpt},h/j}(z, \boldsymbol{b}_{\mathrm{T}}; \mu_{\overline{Q}_0}, \overline{Q}_0^2) E(\overline{Q}_0/Q_0, b_{\mathrm{T}})$$

$$\tilde{f}_{i/p}(x, \boldsymbol{b}_{\mathrm{T}}; \mu_{Q_0}, Q_0^2) = \tilde{f}_{\mathrm{inpt},i/p}(x, \boldsymbol{b}_{\mathrm{T}}; \mu_{\overline{Q}_0}, \overline{Q}_0^2) E(\overline{Q}_0/Q_0, b_{\mathrm{T}}) \,.$$

$$E(\overline{Q}_0/Q_0, b_{\mathrm{T}}) \equiv \exp\left\{ \int_{\mu_{\overline{Q}_0}}^{\mu_{Q_0}} \frac{d\mu'}{\mu'} \left[\gamma(\alpha_s(\mu'); 1) - \ln\frac{Q_0}{\mu'} \gamma_K(\alpha_s(\mu')) \right] + \ln\frac{Q_0}{\overline{Q}_0} \tilde{K}_{\mathrm{inpt}}(b_{\mathrm{T}}; \mu_{\overline{Q}_0}) \right\}$$

b_T – space RG improvement (an example)





Collins-Soper kernel

$$K^{(1)}_{\rm inpt}\left(k_{\rm T};\mu_{Q_0}\right)=\frac{\alpha_s(\mu_{\overline{Q}_0})C_F}{\pi^2}\frac{1}{k_{\rm T}^2+m_K^2}+C_K\delta^{(2)}(\boldsymbol{k}_{\rm T})$$

$$K^{(1)}_{\rm inpt}\left(b_{\rm T};\mu_{Q_0}\right)=\frac{2\alpha_s(\mu_{\overline{Q}_0})C_F}{\pi}K_0\left(b_{\rm T}m_K\right)+C_K$$

$$\widetilde{K}\left(b_{\rm T};\mu_{Q_0}\right)=\frac{2\alpha_s(\mu_{\overline{Q}_0})C_F}{\pi}\left[K_0\left(b_{\rm T}m_K\right)+\ln\left(\frac{m_K}{\mu_{\overline{Q}_0}}\right)\right]-\int_{\mu_{\overline{Q}_0}}^{\mu_{Q_0}}\frac{d\mu'}{\mu'}\gamma_K\left(\alpha_s(\mu')\right)$$