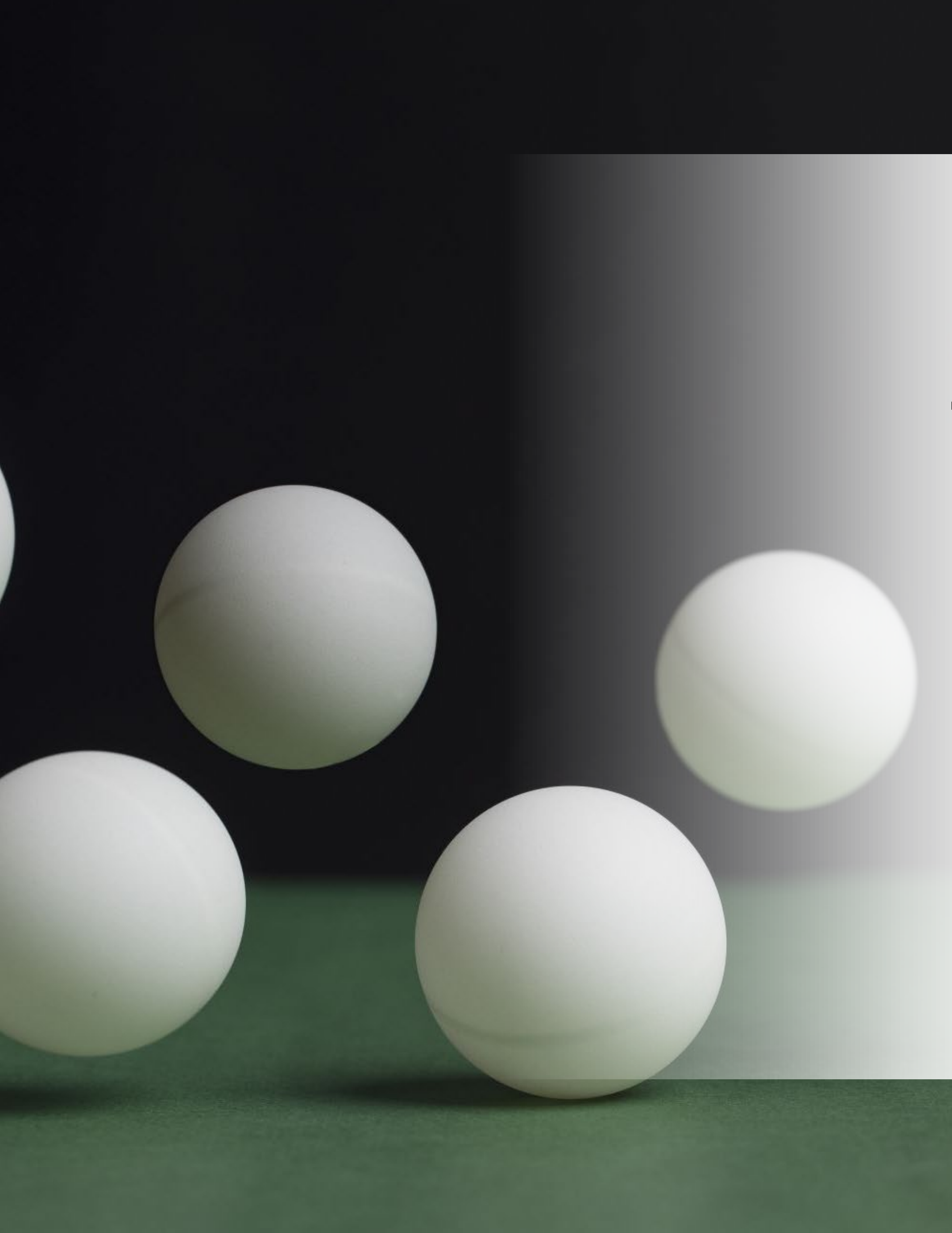

# Three-body dynamics of resonances from lattice QCD

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Review 2B-lattice: [Briceno]
Reviews 3B-lattice: [Hansen] [Mai]
Review hadron resonances: [Mai]

Key publications Finite-Volume Unitary (FVU) approach:
- Three-body unitarity [Mai/JPAC]
- Three-body unitarity finite volume [Mai]
- $a_1$ in finite volume & results from lQCD [Mai]

**Talk outline**:
- 3-body unitarity
- $a_1$ in infinite volume
- $a_1$ in finite volume

Work supported by: National Science Foundation
Grant No. PHY 2012289

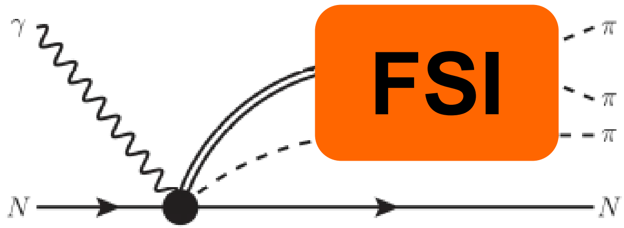
# Progress in last three years alone (narrowly defined for 3B)

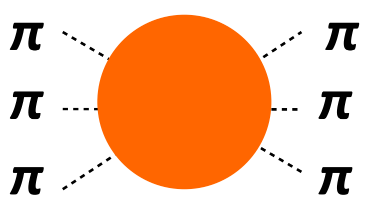
- **Whitepapers**: Snowmass whitepaper amplitude analysis: [1], Snowmass whitepaper lattice: [2]
- **FVU papers**: $a_1$ pole phenomenological: [3], $a_1 \to \pi\sigma$ inf. volume: [4], $a_1$ lQCD/PRL: [5], Review 3B lattice: [6], 3B force: [7], $3K^+$: [8], $a_1$ Dalitz: [9], $3\pi^+$ GWQCD data: [10] $3\pi^+$ interpretation Hanlon Data: [11], cross channel $\pi\pi$: [12], Resonance review (preprint): [13], ($\rho$ with ETMC [14], $\varphi^4$ equivalence FVU/RFT [15])
- **RFT papers**: $3\pi^+$ HadSpec "Dalitz"/inf. vol. amplitude: [16], Decay amplitude to 3 hadrons: [17], 3 pions all isospins: [18], Review 3B fin vol Hansen: [19], QC $\pi^+\pi^+K^+$: [20], Higher-spin isobars: [21], Non-degenerate scalars 3B: [22] Alternative derivation 3B QC [23], ETMC/Bonn $3\pi^+$: [24]. $3\pi^+$ PRL analysis [25] of Hanlon/Hoerz data: [26]
- **(N)REFT**: Resonance form factor from corr functions [27], Spurious poles [28], EFT Book [29], Rel.-inv. formulation [30], $\phi^4$ test scattering [31], Lüscher-Lellouch analog 3-body [32], Analytic energy shift 3B ground state [33], N-particle energy shift [34], Rusetsky Mini-review 3-body [35] Latest (schematic) effort for Roper fin vol [36].
- **Peng/Pang/Koenig, others:** Fin-vol extrapolation eigenvector continuation [37]. 3B resonances pionless EFT [38], Few-body bound states Fin Vol [39], Few-body resonances fin-vol [40], $DDK$ system finite volume [41], Finite volume magnetic field [42, 43], Different fin vol geometries [44], Few-body resonances finite volume [45], Visualization three-body resonances (analytic cont. of L-dependence) [46], Multi-$\pi^+$ and analysis of lattice data [47], Threshold expansion $N$-particle Fin Vol [48], Propagation particle torus [49]
- **inf. vol./Equivalence 3B formalisms:** Equivalence different 3B QC [50], Jackura 3B unitarity PW [51], JPAC hadron physics review [52], 3B unitarity in RFT: [53].

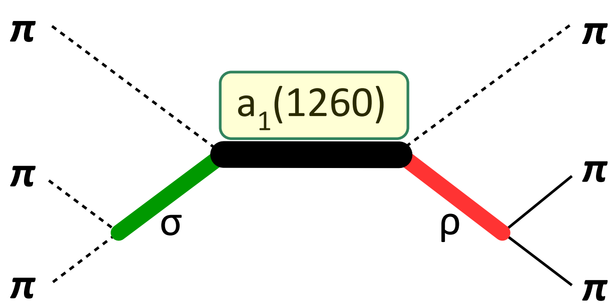
# Three-body aspects: $\pi\pi N$ vs. $\pi\pi\pi$

## Light mesons

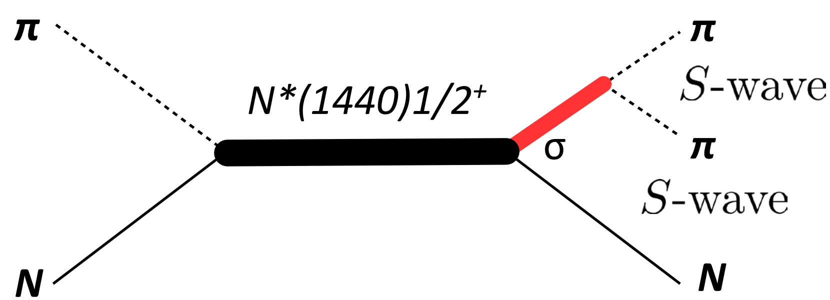
- COMPASS @ CERN: $\pi_1(1600)$ discovery
- GlueX @ Jlab in search of hybrids and exotics,
- Finite volume spectrum from lattice QCD:
  **Lang (2014), Woss [HadronSpectrum] (2018)**
  **Hörz (2019), Culver (2020, 21,…), Fischer (2020),**
  **Hansen/HadSpec (2020)**




## Light baryons




- Roper resonance is debated for ~50 years in experiment.
- 1st calculation w. meson-baryon operators on the lattice: **Lang et al. (2017)**

## Three-body unitarity with isobars * [Mai 2017]

$$\langle q_1,q_2,q_3|(\hat{T}-\hat{T}^\dagger)|p_1,p_2,p_3\rangle = i\int_P \langle q_1,q_2,q_3|\hat{T}^\dagger|k_1,k_2,k_3\rangle\langle k_1,k_2,k_3|\hat{T}|p_1,p_2,p_3\rangle$$

$$\times\prod_{\ell=1}^{3}\left[\frac{\mathrm{d}^4k_\ell}{(2\pi)^4}(2\pi)\boxed{\delta^+(k_\ell^2-m^2)}\right](2\pi)^4\delta^4\left(P-\sum_{\ell=1}^{3}k_\ell\right)$$

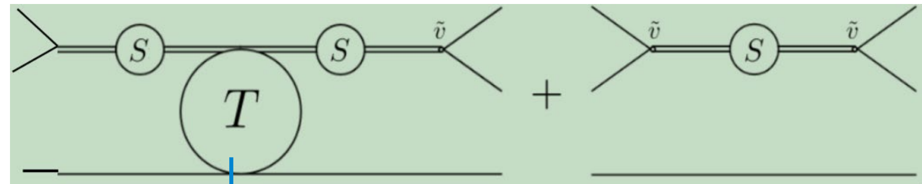
delta function sets all intermediate particles on-shell

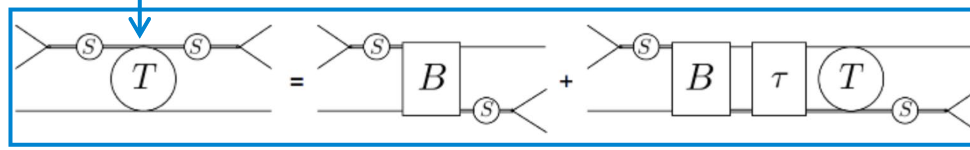
**Idea**: To construct a 3B amplitude, start directly from unitarity (based on ideas of 60's); match a general amplitude to it

* "Isobar" stands for two-body sub-amplitude which can be resonant or not; can be matched to CHPT expansion to one loop if desired. Isobars are re-parametrizations of full 2-body amplitudes [Bedaque] [Hammer]

# Three-body unitarity

$$\langle q_1, q_2, q_3|(\hat{T} - \hat{T}^\dagger)|p_1, p_2, p_3\rangle \quad = \quad i\int_P \langle q_1, q_2, q_3|\hat{T}^\dagger|k_1, k_2, k_3\rangle\langle k_1, k_2, k_3|\hat{T}|p_1, p_2, p_3\rangle$$



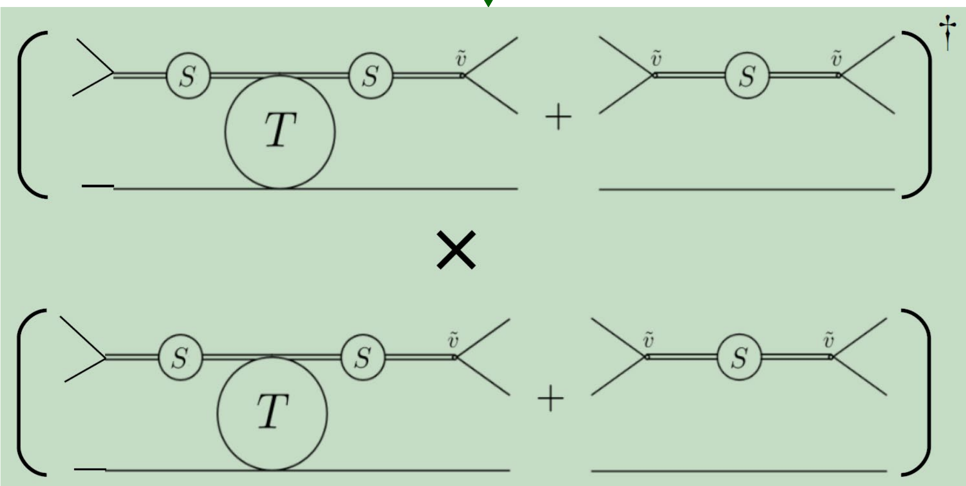




**General Ansatz for the isobar-spectator interaction**

→ **B & τ** are **new** unknown functions

## Three-body unitarity

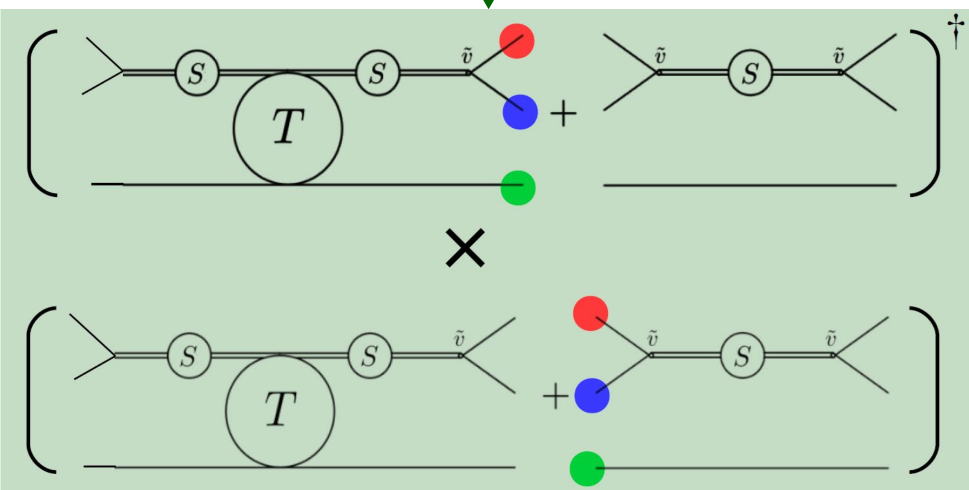
$$\langle q_1, q_2, q_3|(\hat{T}-\hat{T}^\dagger)|p_1, p_2, p_3\rangle \quad = \quad i\int_P \langle q_1, q_2, q_3|\hat{T}^\dagger|k_1, k_2, k_3\rangle\langle k_1, k_2, k_3|\hat{T}|p_1, p_2, p_3\rangle$$

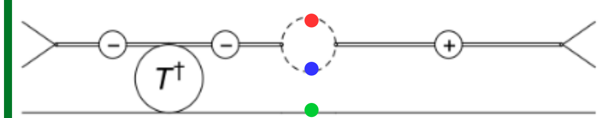



General connected-disconnected structure

## Three-body unitarity

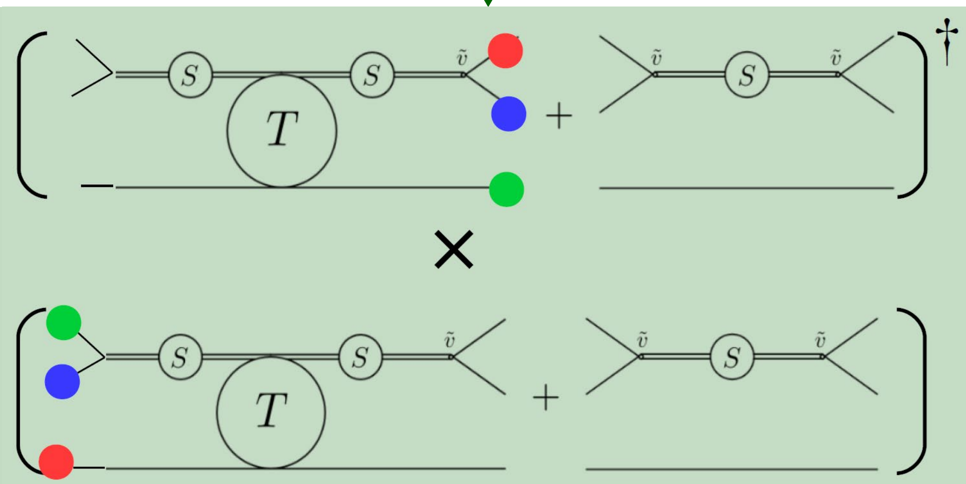
$$\langle q_1, q_2, q_3|(\hat{T} - \hat{T}^\dagger)|p_1, p_2, p_3\rangle \;=\; i\int_P \langle q_1, q_2, q_3|\hat{T}^\dagger|k_1, k_2, k_3\rangle\langle k_1, k_2, k_3|\hat{T}|p_1, p_2, p_3\rangle$$

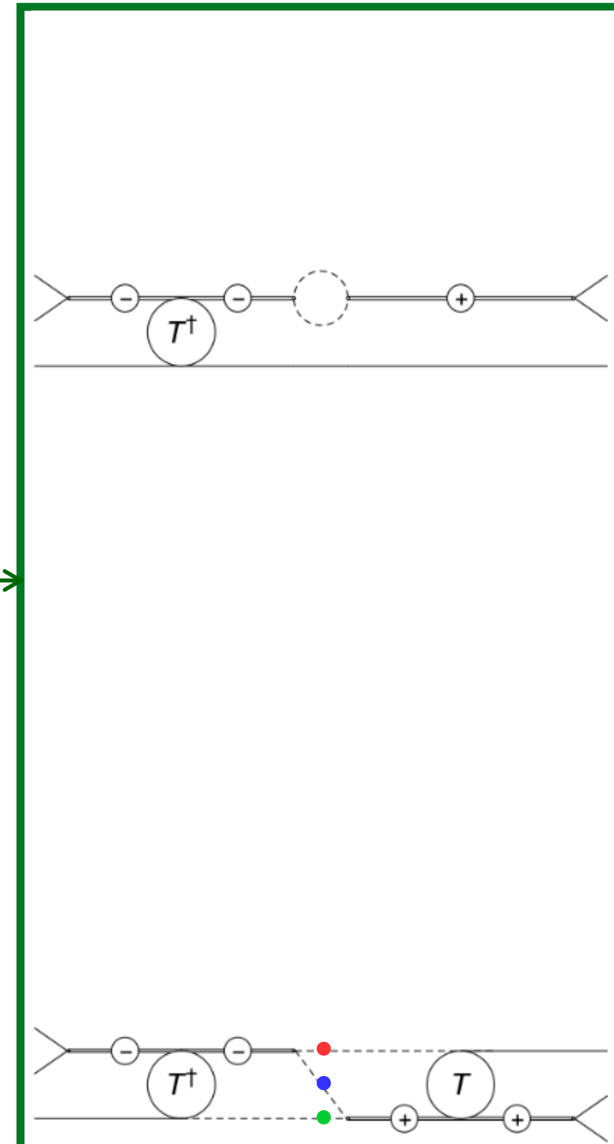


## Three-body unitarity

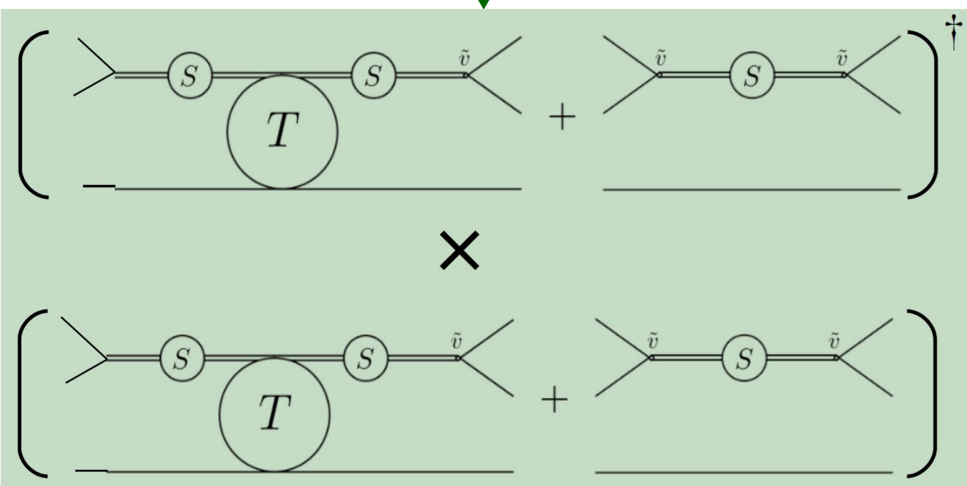
$$\langle q_1, q_2, q_3|(\hat{T}-\hat{T}^\dagger)|p_1, p_2, p_3\rangle \quad = \quad i\int_P \langle q_1, q_2, q_3|\hat{T}^\dagger|k_1, k_2, k_3\rangle\langle k_1, k_2, k_3|\hat{T}|p_1, p_2, p_3\rangle$$



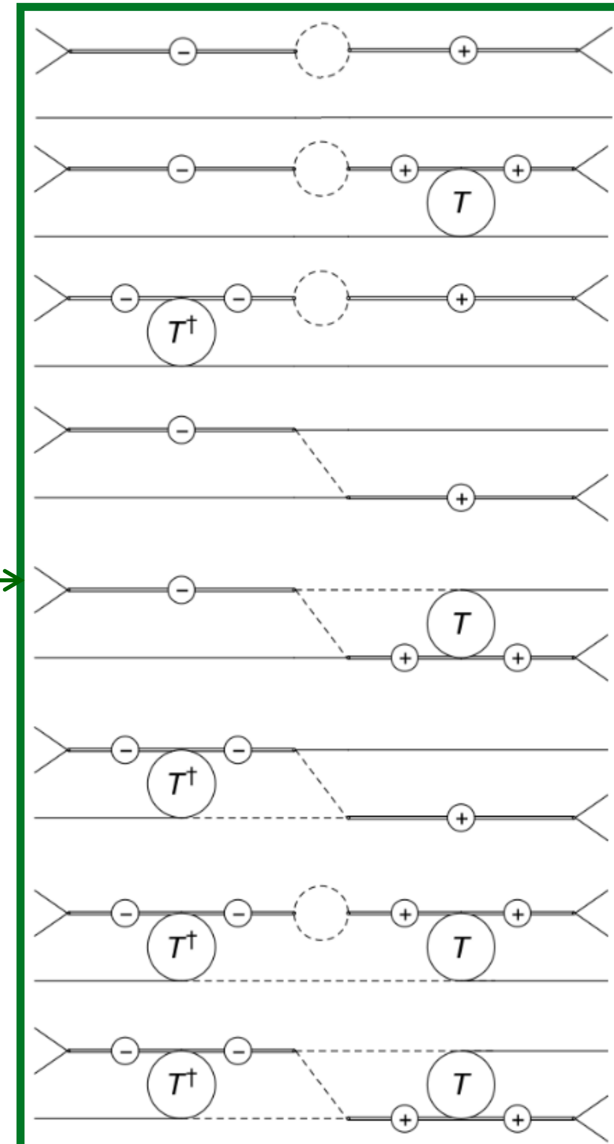



## Three-body unitarity

$$\langle q_1, q_2, q_3|(\hat{T}-\hat{T}^\dagger)|p_1, p_2, p_3\rangle \quad = \quad i\int_P \langle q_1, q_2, q_3|\hat{T}^\dagger|k_1, k_2, k_3\rangle\langle k_1, k_2, k_3|\hat{T}|p_1, p_2, p_3\rangle$$

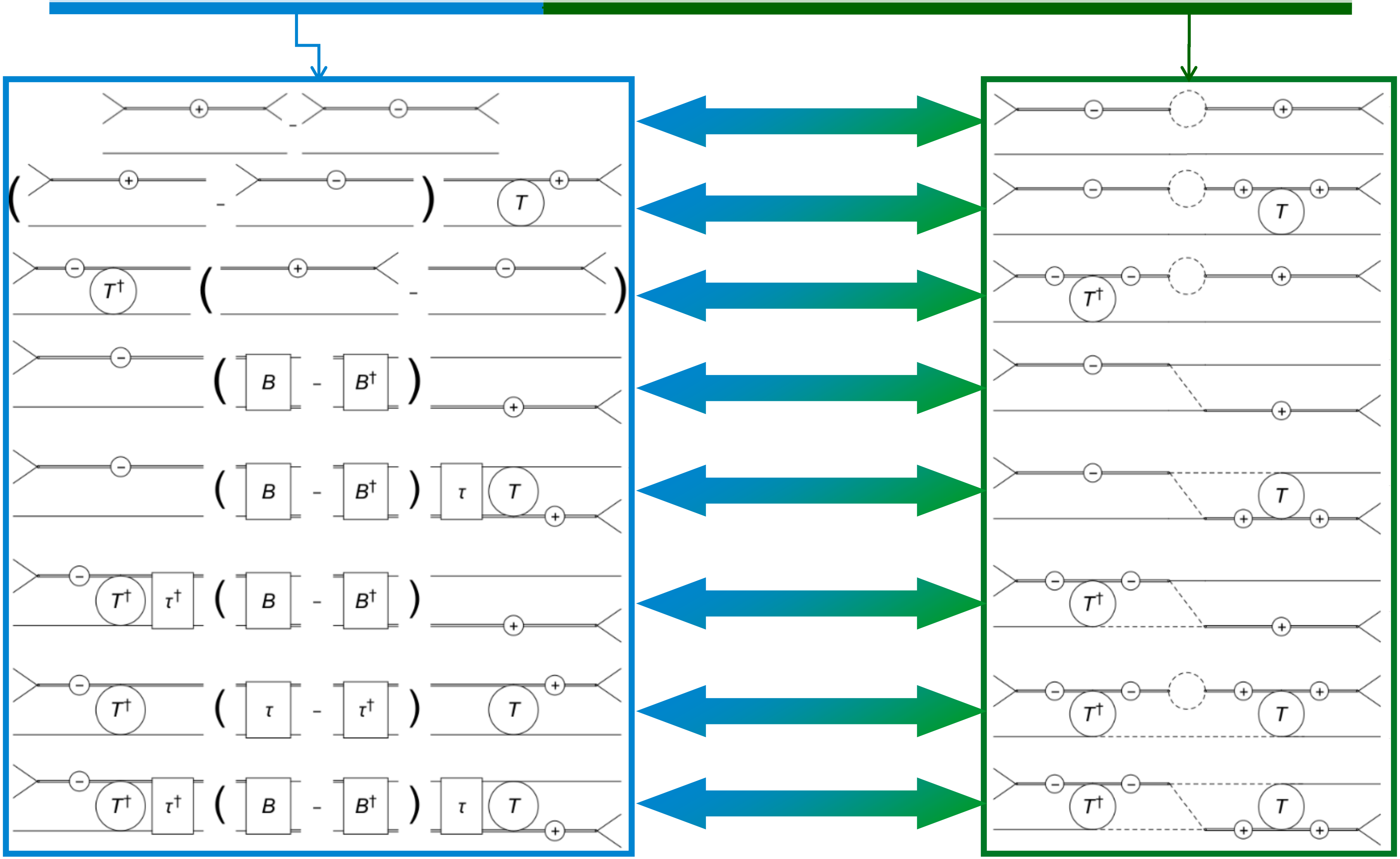



8 topologies



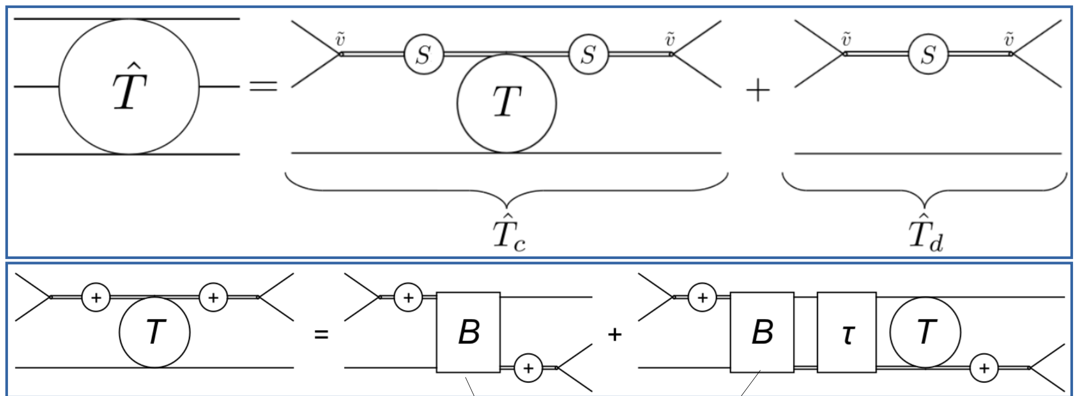
## Three-body unitarity

$$\langle q_1, q_2, q_3 | (\hat{T} - \hat{T}^\dagger) | p_1, p_2, p_3 \rangle \;=\; i \int_P \langle q_1, q_2, q_3 | \hat{T}^\dagger | k_1, k_2, k_3 \rangle \langle k_1, k_2, k_3 | \hat{T} | p_1, p_2, p_3 \rangle$$



# Scattering amplitude

$3 \to 3$ scattering amplitude is a 3-dimensional integral equation




LS-type

- Imaginary parts of ***B, S*** are fixed by **unitarity/matching**
- B, S are determined **consistently** through 8 different relations

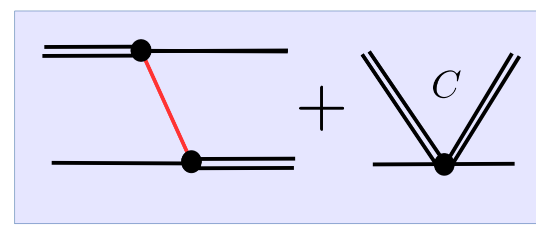
Matching → $\text{Disc}\, B(u) = 2\pi i \lambda^2 \frac{\delta\left(E_Q - \sqrt{m^2 + \mathbf{Q}^2}\right)}{2\sqrt{m^2 + \mathbf{Q}^2}}$

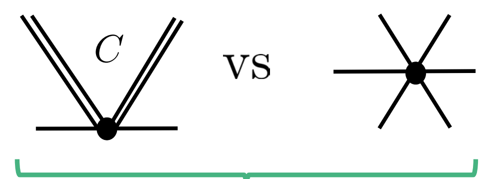
- un-subtracted dispersion relation

$$\langle q|B(s)|p\rangle = -\frac{\lambda^2}{2\sqrt{m^2+\mathbf{Q}^2}\left(E_Q-\sqrt{m^2+\mathbf{Q}^2}+i\epsilon\right)} + C$$

- one-**π** exchange in TOPT → *RESULT, NOT INPUT !*

- One <u>can</u> map to field theory but does not have to.
  Result is a-priori dispersive.



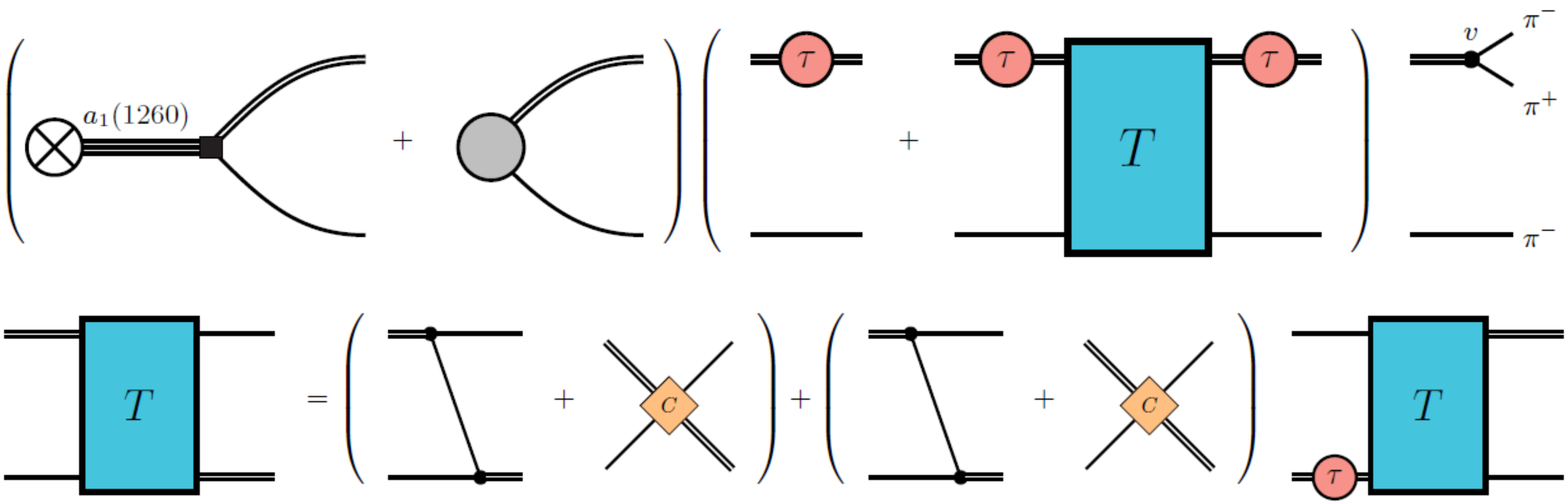




Add. Steps to map to theory might be needed [Brett (2021)]

# The $a_1(1260)$ and its Dalitz plots

[Sadasivan 2020]

- Disconnected and connected decays for three-body untarity

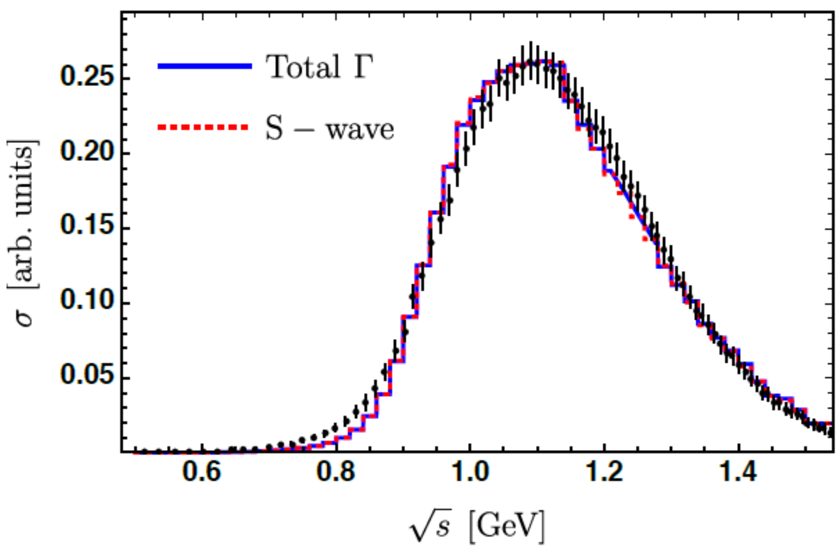


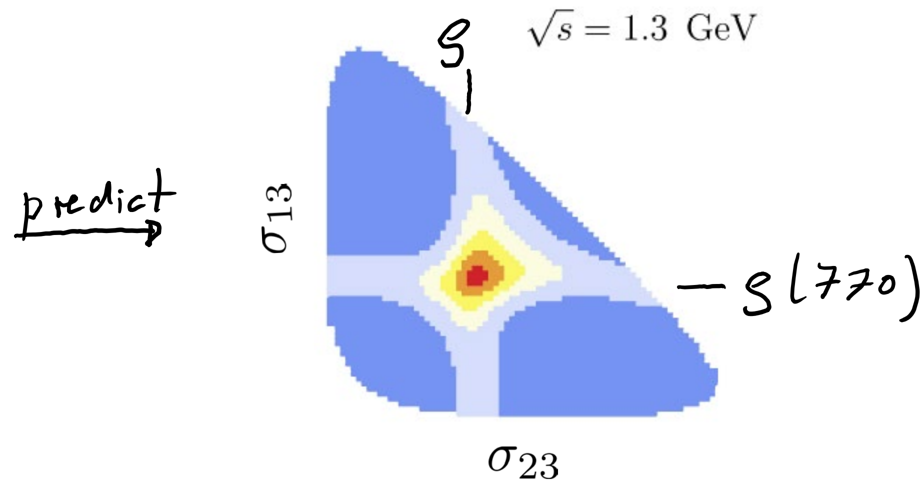
# Fitting the lineshape & predicting Dalitz plots

[Sadasivan 2020]

- One can have $\pi\rho$ in S- and D-wave coupled channels
- Fit contact terms to the lineshape from Experiment (ALEPH)

$a_1 \to \pi^-\pi^-\pi^+$ (symmetrize $\pi^-$'s!)

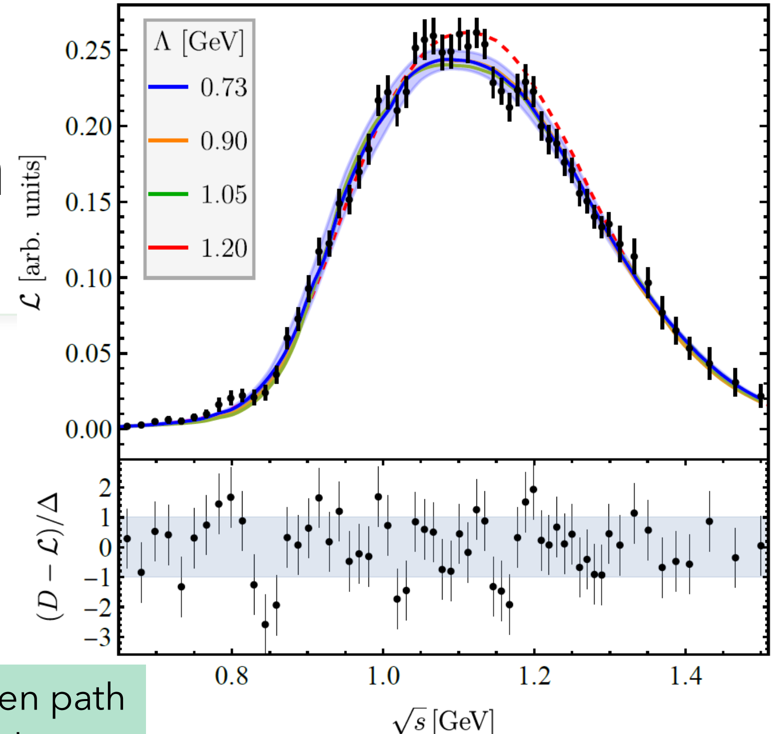


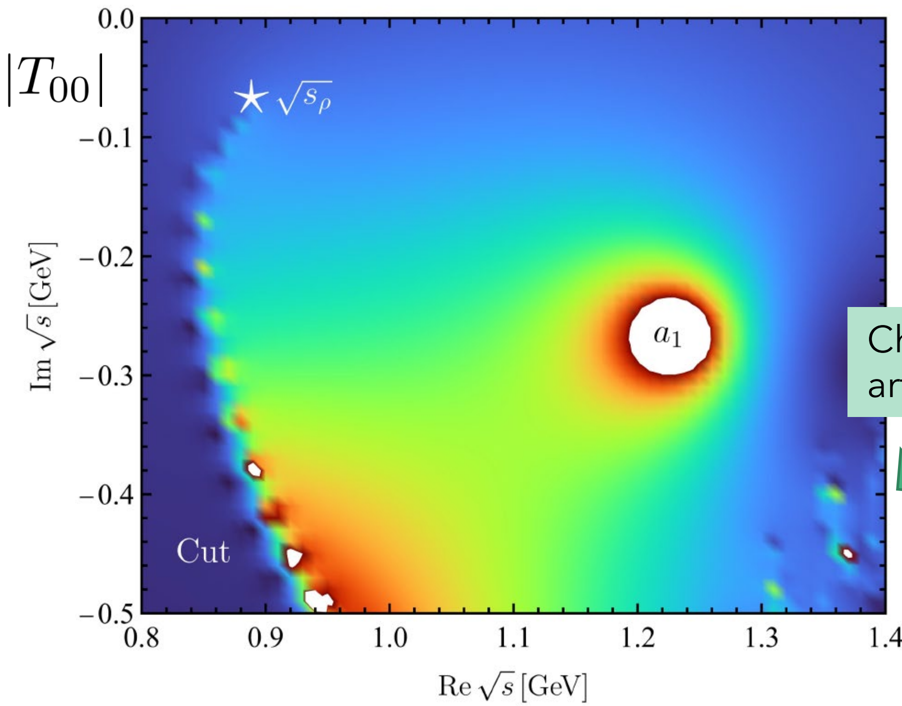




Where is the resonance pole in s?

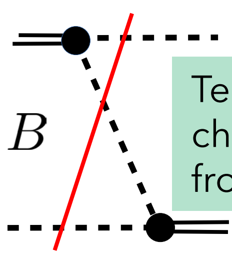
# Result: Pole position

Technicalities analytic cont.: contour deformation } [Sadasivan (2021)] [Doering (2009)]



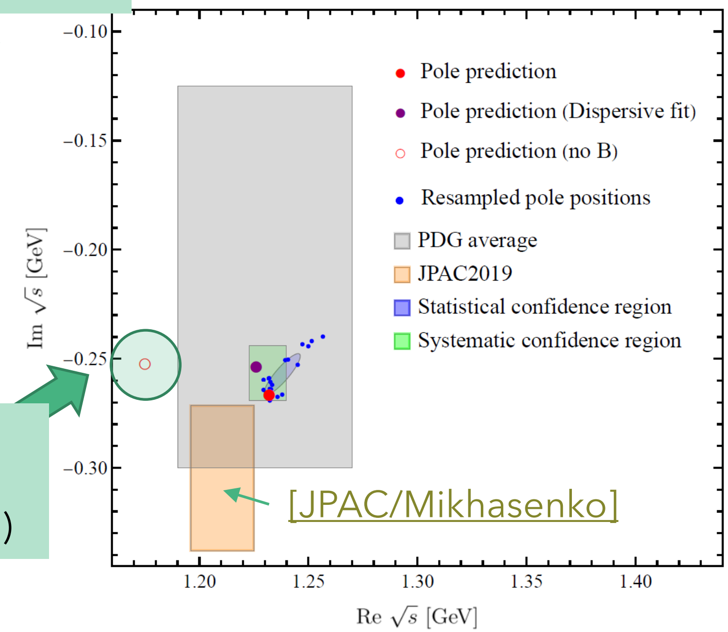




Chosen path artifacts




Technical challenges from 3B cuts

If the B-term is neglected + refit (unitarity violated)

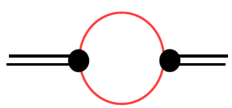



[JPAC/Mikhasenko]

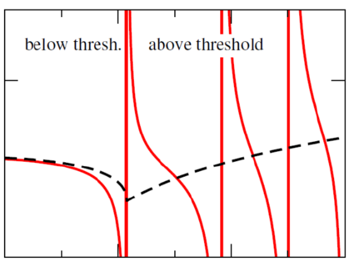
See also: S. M. Dawid [ 2303.04394 [nucl-th]]

# Lattice QCD: Finite-volume unitarity (FVU)

**Two-body unitarity** → **On-shell condition**



→ **Imaginary parts**




**Infinite → Fin. Vol**

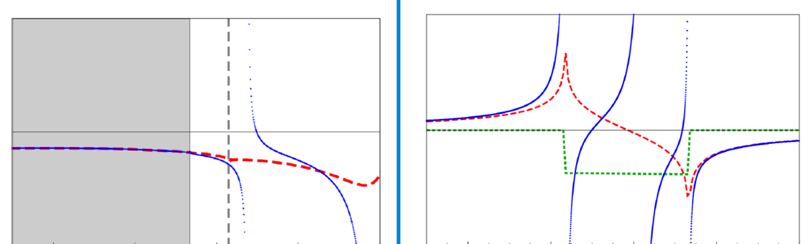
→ **Power-law fin-vol. effects** → **Lüscher**

$$p\cot\delta(p) = -8\pi\sqrt{s}\left(\tilde{G}(E) - \mathrm{Re}\, G(E)\right)$$

.

**Three-body unitarity** → **On-shell condition**

→ **Imaginary parts**



→ **Power-law fin-vol. effects** → **Quantization Condition**

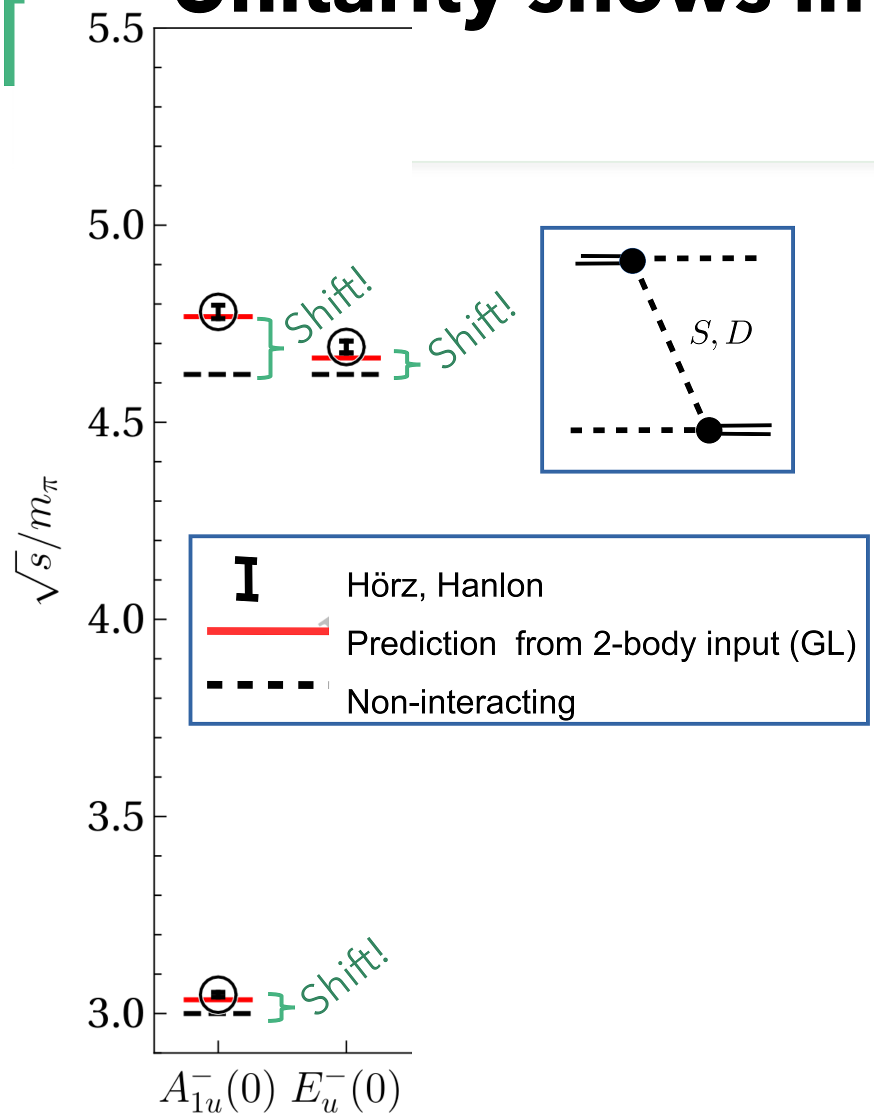
$$\mathrm{Det}\left(\mathbf{B}^{\Gamma ss'}_{\mathbf{uu'}}(\mathbf{W^2}) + \frac{\mathbf{2E_s\, L^3}}{\vartheta(\mathbf{s})}\tau_{\mathbf{s}}(\mathbf{W^2})^{-1}\delta_{\mathbf{ss'}}\delta_{\mathbf{uu'}}\right) = \mathbf{0}$$

.

[Review Mai et al.]

# Unitarity shows in FV spectrum: $3\pi^+$

[GWQCD/Culver 2019]




**S** **D** (lowest participating wave)

- D-wave prediction qualitatively good

→ Relative/absolute strength between S- and D-wave matched

→ Consequence that 3-body interaction dominated by exchange

→ Consequence of 3-body Unitarity

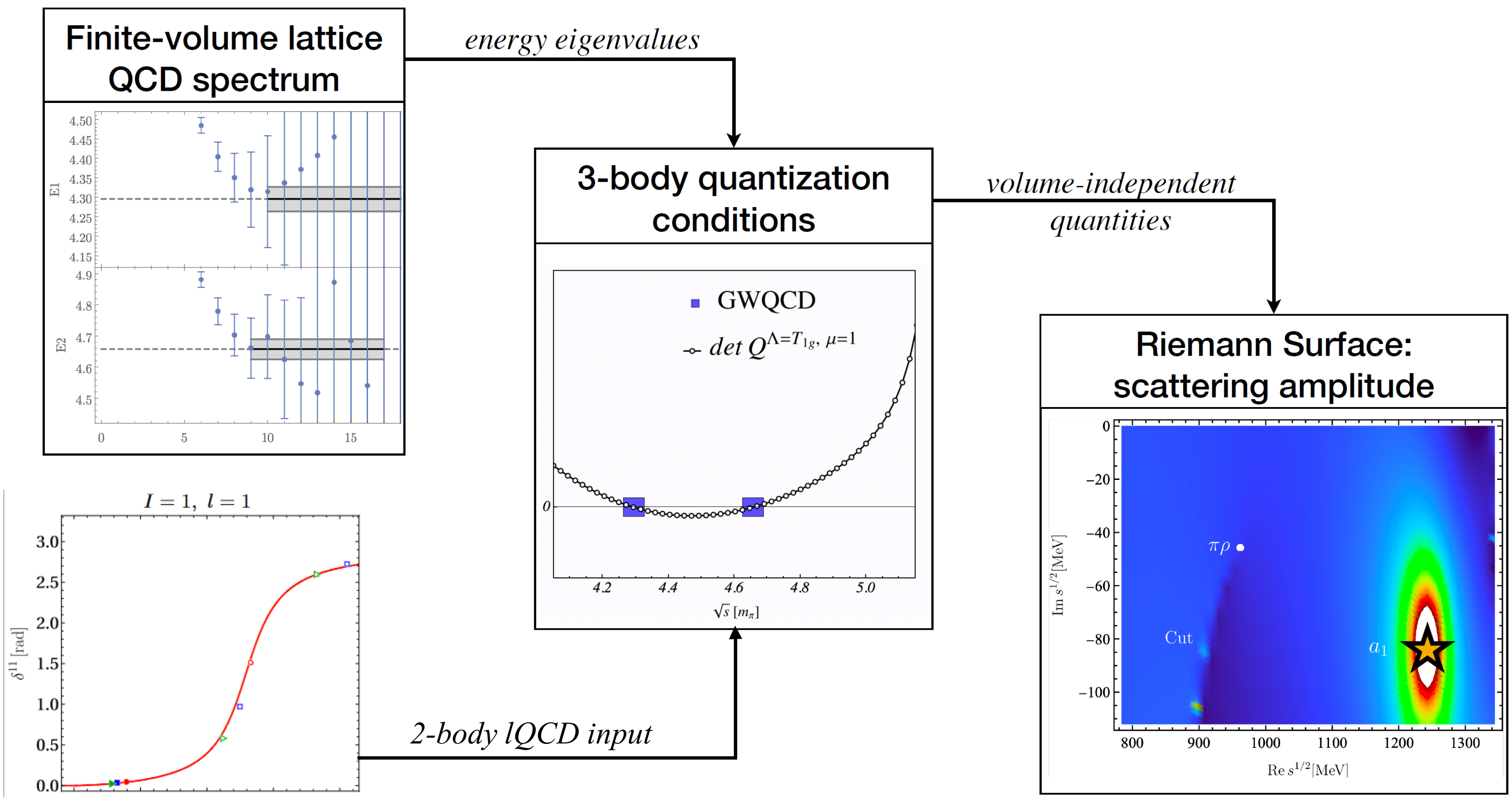
- Three-body unitarity directly visible in the eigenvalue spectrum of lattice QCD

- Many additional levels, including boosts (not shown)

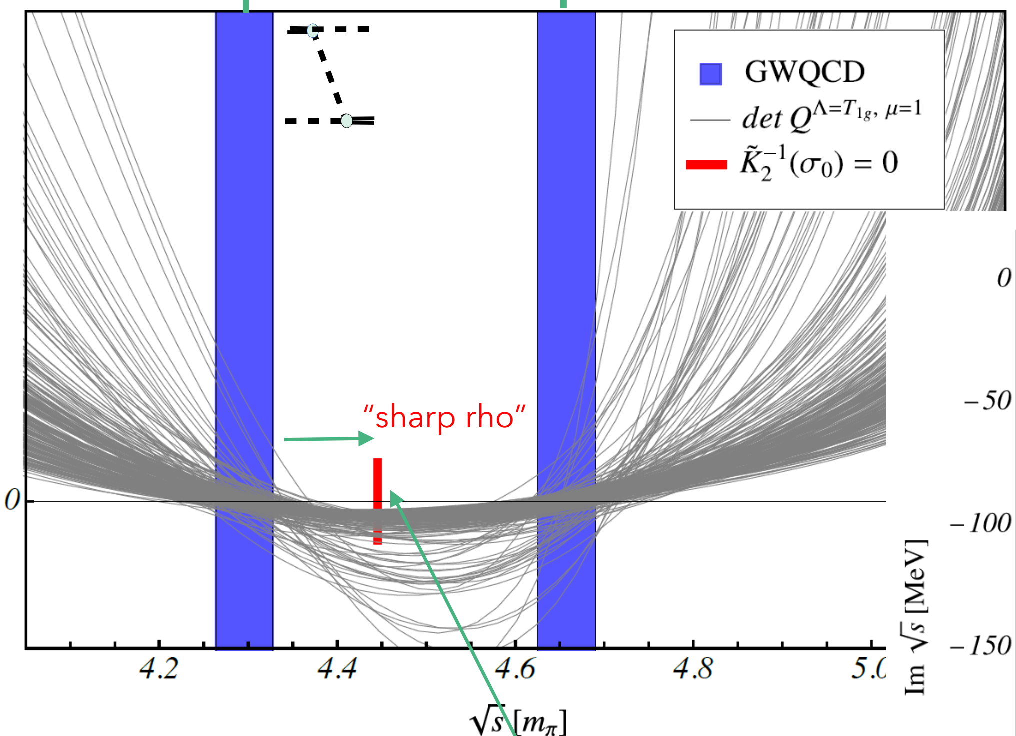
# Extraction of $a_1(1260)$ from lQCD

[Mai/GWQCD, PRL 2021]

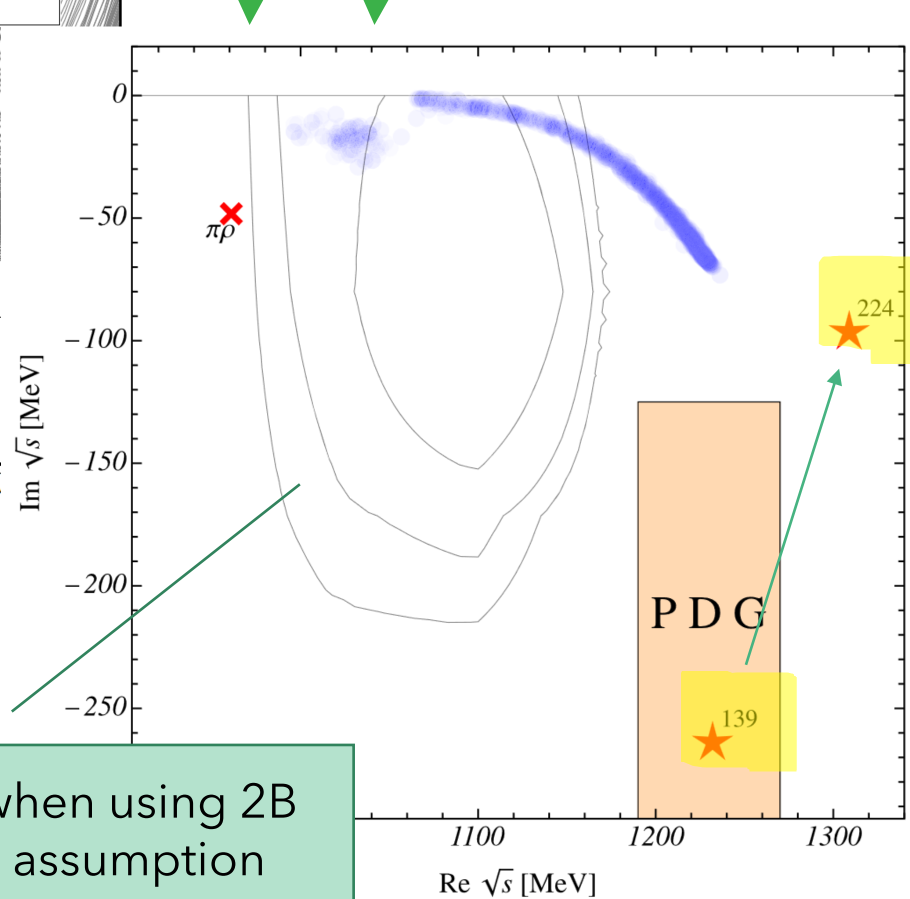
- First-ever three-body resonance from 1st principles (with explicit three-body dynamics).



# Results – overview




$$0 = \det\left[B(s) + C(s) - E_L\left(\tilde{K}_2^{-1}(s) - \Sigma_2^L(s)\right)\right]_{(\lambda'\lambda)\,(\boldsymbol{p}'\boldsymbol{p})}$$




PDF for pole when using 2B Breit-Wigner assumption

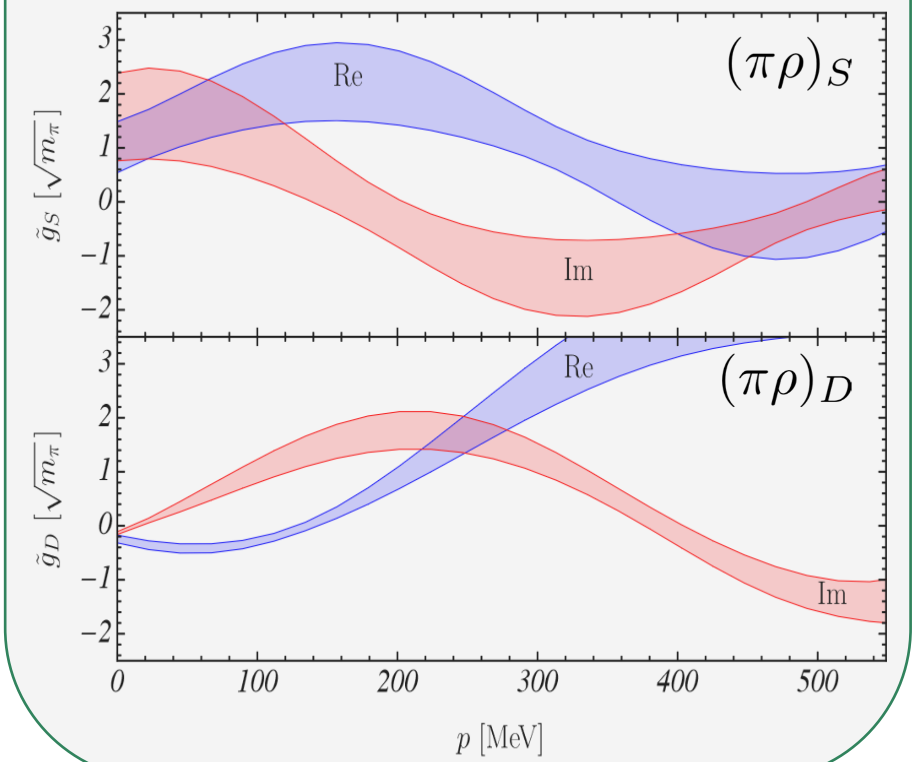
# Branching ratios

- Calculate the residue at the pole:

$$\mathrm{Res}(T^{c}_{\ell'\ell}(\sqrt{s})) = \tilde{g}_{\ell'}\tilde{g}_{\ell}$$

- This result is not as reliable as pole position/existence of $a_1$
- More energy eigenvalues needed to better pin down the decay channels

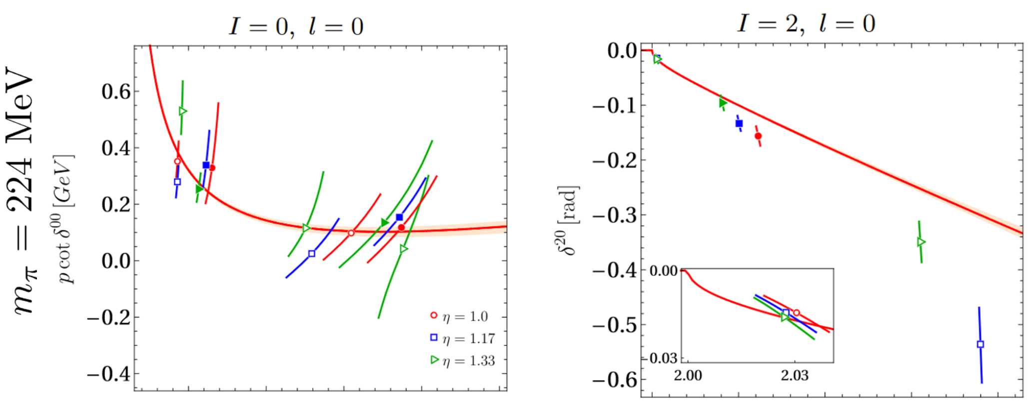
"**Branching ratios**" in 3B decays are momentum -dependent, complex pole residues



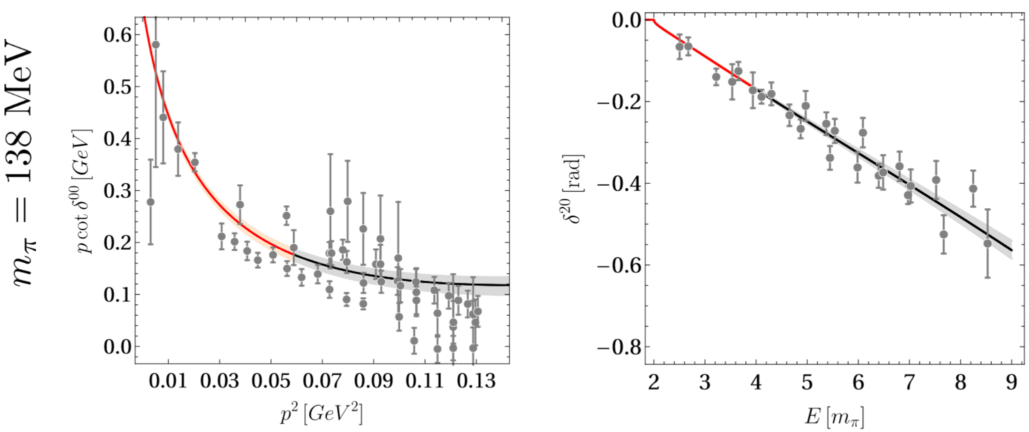
# Outlook: 4+ coupled channels

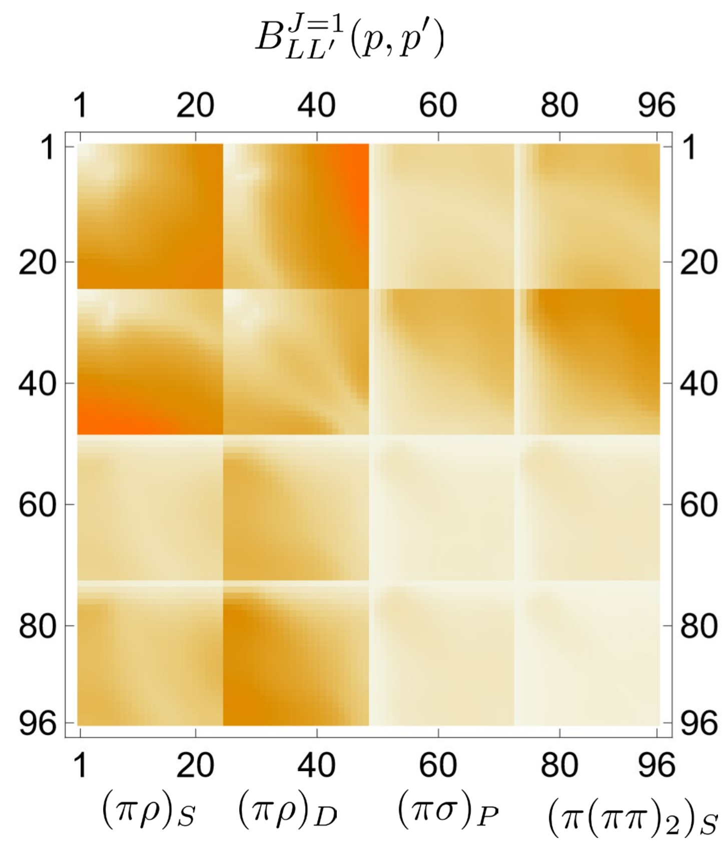
$$a_1 \leftrightarrow (\pi\rho)_S \leftrightarrow (\pi\rho)_D \leftrightarrow (\pi\sigma)_P \leftrightarrow (\pi(\pi\pi)_{S,I=2})$$

- Inclusion of all S- and P-wave isobars (from 2B lQCD input)
- Current status: Solved in infinite volume/awaiting FVU implement.

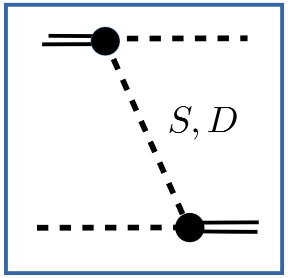



[Mai 2019]



# Summary




- Lattice QCD progress in determining the explicit dynamics of three-body systems:
  - Three pions at maximal isospin well understood (FVU, RFT, Peng,…)
  - First determination of existence and properties of a three-body resonance – the $a_1(1260)$ - in coupled channels
- **Outlook:** More (isospin) channels; other physical systems
  - Lattice: more energy eigenvalues to assess uncertainties and put limits on decay properties. More pion masses to map out chiral trajectory
  - Phenomenology: Fit Dalitz plots instead of predicting them. Coupled-channel, unitary final-state interaction for data analysis (potentially GlueX)

# Spare slides

# Partial-wave decomposition

- Plane-wave basis

$$T_{\lambda'\lambda}(\boldsymbol{p},\boldsymbol{q}_1) = (B_{\lambda'\lambda}(\boldsymbol{p},\boldsymbol{q}_1) + C) + \sum_{\lambda''}\int \frac{d^3\boldsymbol{l}}{(2\pi)^3 2E_l}(B_{\lambda'\lambda''}(\boldsymbol{p},\boldsymbol{l}) + C)\,\tau(\sigma(l))T_{\lambda''\lambda}(\boldsymbol{l},\mathbf{q}_1)$$

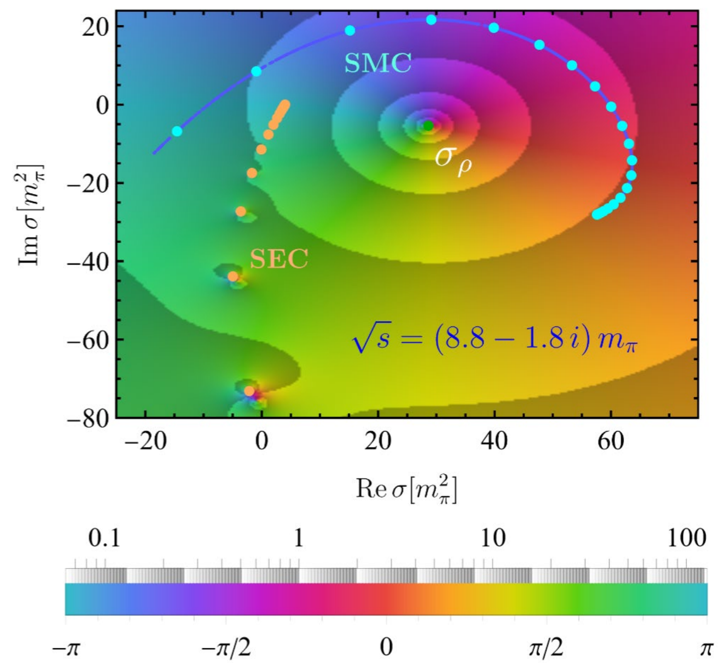
$$B^J_{\lambda\lambda'}(q_1,p) = 2\pi\int_{-1}^{+1}\mathrm{d}x\, d^J_{\lambda\lambda'}(x)B_{\lambda\lambda'}(\boldsymbol{q}_1,\boldsymbol{p}) \qquad B^J_{LL'}(q_1,p) = U_{L\lambda}B^J_{\lambda\lambda'}(q_1,p)U_{\lambda'L'}$$

- JLS basis:

$$T^J_{LL'}(q_1,p) = \left(B^J_{LL'}(q_1,p) + C_{LL'}(q_1,p)\right) + \int_0^\Lambda \frac{\mathrm{d}l\, l^2}{(2\pi)^3 2E_l}\left(B^J_{LL''}(q_1,l) + C_{LL''}(q_1,l)\right)\tau(\sigma(l))T^J_{L''L'}(l,p)$$

# Analytic cont. 3-body

[Sadasivan (2021)]
[Doering (2009)]




SMC

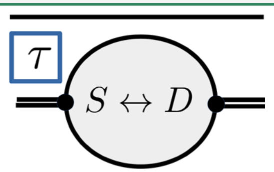
$$T^J_{LL'}(q_1,p) = \left(B^J_{LL'}(q_1,p) + C_{LL'}(q_1,p)\right) + \int_0^\Lambda \frac{\mathrm{d}l\, l^2}{(2\pi)^3 2E_l} \left(B^J_{LL''}(q_1,l) + C_{LL''}(q_1,l)\right) \tau(\sigma(l)) T^J_{L''L'}(l,p)$$

$$\tau^{-1}(\sigma) = K^{-1} - \Sigma\,,$$

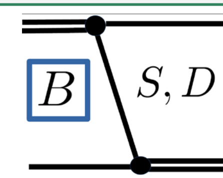
$$\Sigma = \int_0^\infty \frac{\mathrm{d}k\, k^2}{(2\pi)^3} \frac{1}{2E_k} \frac{\sigma^2}{\sigma'^2} \frac{\tilde{v}(k)^*\tilde{v}(k)}{\sigma - 4E_k^2 + i\epsilon}$$

$$B_{\lambda\lambda'}(\boldsymbol{p},\boldsymbol{p}') = \frac{v^*_\lambda(P-p-p',p)\, v_{\lambda'}(P-p-p',p')}{2E_{p'+p}\left(\sqrt{s} - E_p - E_{p'} - E_{p'+p} + i\epsilon\right)}$$

SEC




Singularities

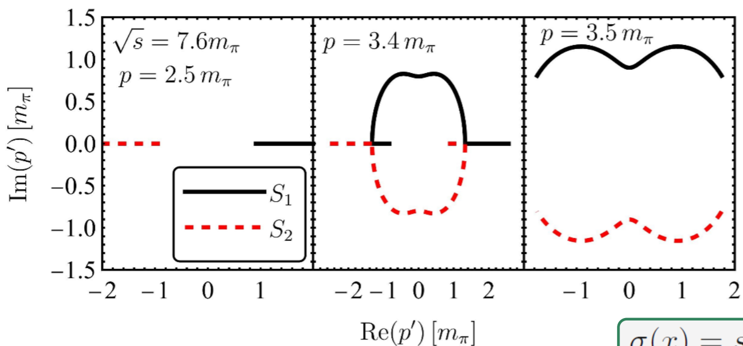


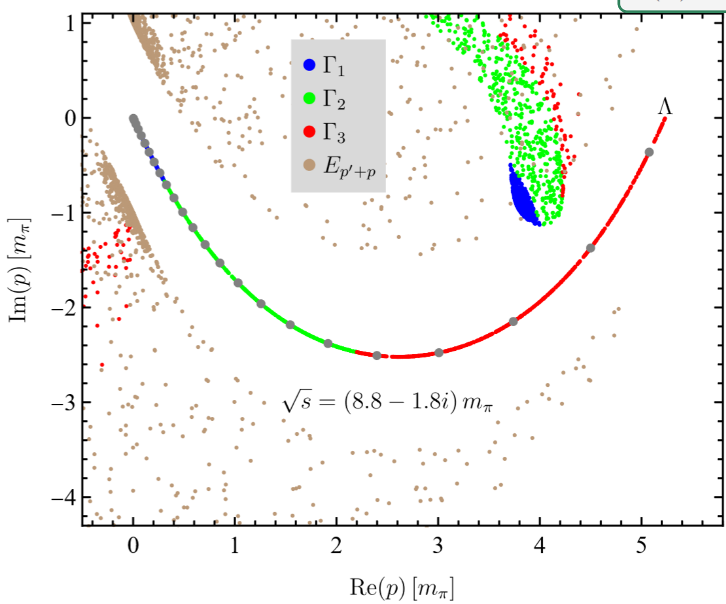

- Two contours (SMC and SEC)
- Deform both “adiabatically” to go to complex s
- Set of rules:
    - Contours cannot intersect with each others
    - Contours cannot intersect with (3-body) cuts
- Passing singularities left or right determines sheet

# Analytic continuation 3-body (contd.)

- Three-body cuts

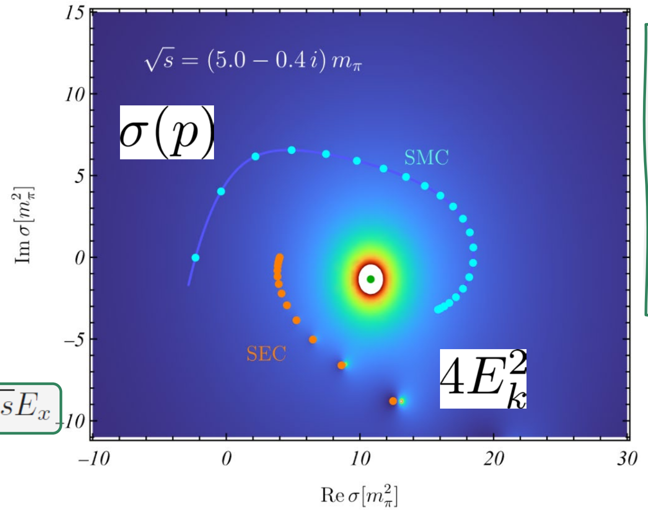
$$\sqrt{s} - E_p - E_{p'} - E_{p+p'} + i\epsilon = 0$$



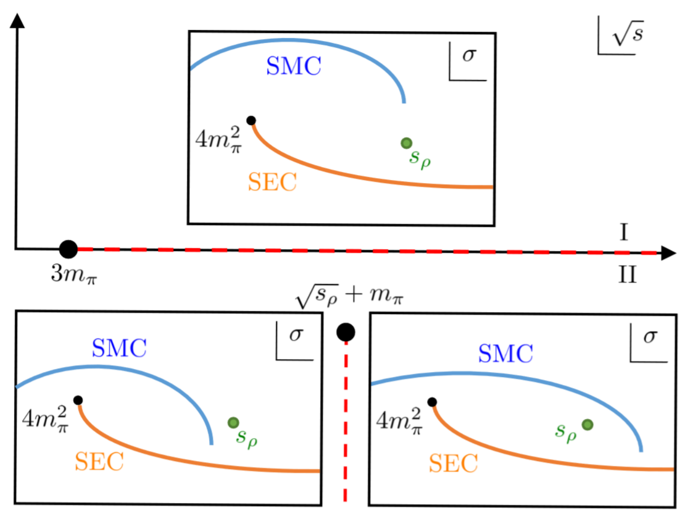




$\sigma(x) = s + m_\pi^2 - 2\sqrt{s}E_x$

- Complex branch points



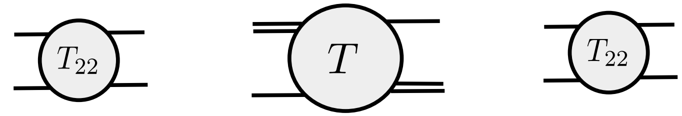

Integration limits at poles induce branch points

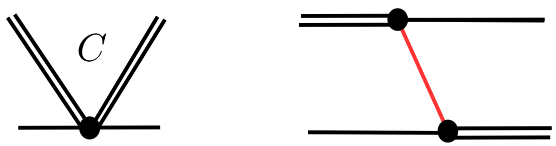


## Scattering amplitude (Details)

Here: Version in which isobar rewritten in on-shell 2 → 2 scattering amplitude $T_{22}$

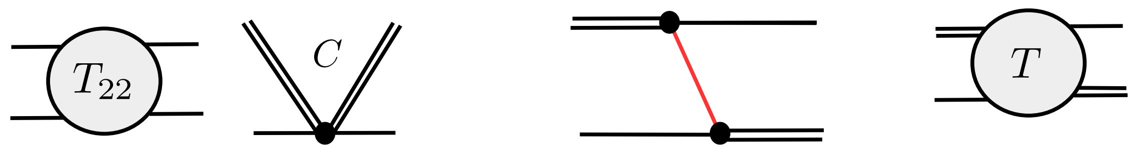
$$\langle q_1, q_2, q_3|\hat{T}_c(s)|p_1, p_2, p_3\rangle = \frac{1}{3!}\sum_{n=1}^{3}\sum_{m=1}^{3} T_{22}(\sigma(q_n))\langle q_n|T(s)|p_m\rangle T_{22}(\sigma(p_m))$$



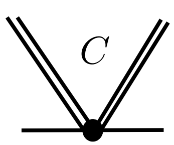



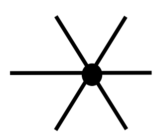

$$\langle q|T(s)|p\rangle = \langle q|C(s)|p\rangle + \frac{1}{m^2 - (P-p-q)^2 - i\epsilon}$$

$$-\int \frac{d^3\ell}{(2\pi)^3}\frac{1}{2E_\ell}T_{22}(\sigma(\ell))\left(\langle p|C(s)|\ell\rangle + \frac{1}{m^2 - (P-p-\ell)^2 - i\epsilon}\right)\langle \ell|T(s)|p\rangle$$




(S-wave)

Technical Detail: 

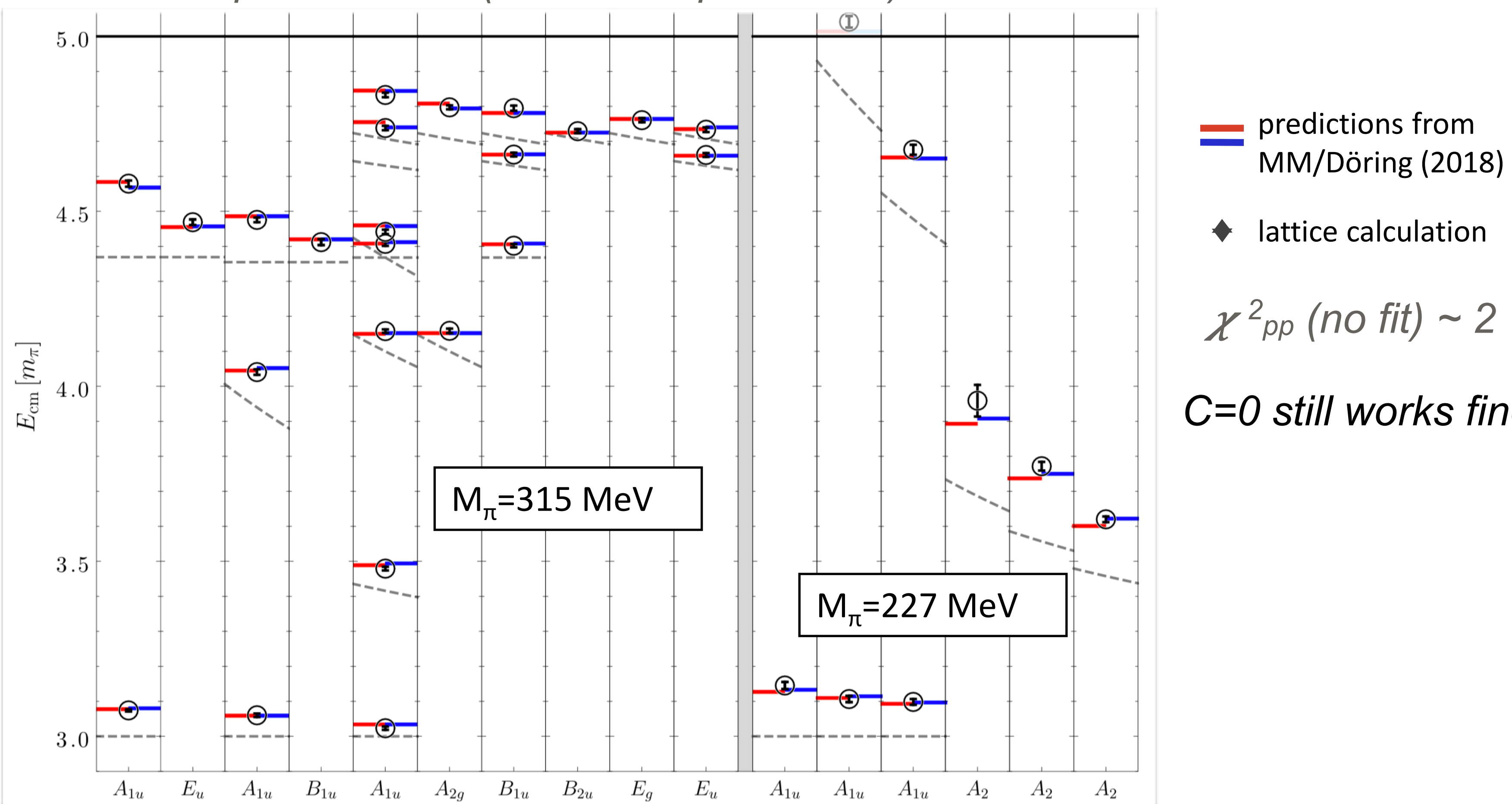
vs 

Scheme-dependent 3-body force requires a mapping [Brett (2021)]

# GWUQCD data

Culver, MM, Brett, Alexandru, Döring (2019) PRD

- ***More recent*** *data is available*
  - *very dense spectrum from elongated boxes*
  - *different pion masses (chiral extrapolations?)*




*$\chi^2_{pp}$ (no fit) ~ 2*

*C=0 still works fine*

# Plane-wave implementation of the C-term

- **Step 1**: JM-basis → Helicity basis
- **Step 2**: partial-wave basis → Plane-wave basis
- **Step 3**: C (and B, and 3B propagator) from plane-wave basis to irreps by suitable rotations

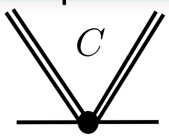
$$\mathcal{A}_{\lambda'\lambda}(s,\boldsymbol{p}',\boldsymbol{p}) := \sum_{M=-J}^{J} \frac{2J+1}{4\pi}\, \mathfrak{D}^{J*}_{M\lambda'}(\phi_{\boldsymbol{p}'},\theta_{\boldsymbol{p}'},0)\, \mathcal{A}^{J}_{\lambda'\lambda}(s,p',p)\, \mathfrak{D}^{J}_{M\lambda}(\phi_{\boldsymbol{p}},\theta_{\boldsymbol{p}},0)\,, \qquad \text{Step 2}$$

$$\mathcal{A}^{J}_{\lambda'\lambda}(s,p',p) = U_{\lambda'\ell'}\mathcal{A}_{\ell'\ell}(s,p',p)U_{\ell\lambda}\,,$$

$$U_{\ell\lambda} := \sqrt{\frac{2\ell+1}{2J+1}}(\ell 0 1\lambda|J\lambda)(1\lambda 00|1\lambda)) = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}, \\ \frac{1}{\sqrt{6}} & -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{6}}, \end{pmatrix}, \qquad \text{Step 1}$$

# 4 different fits to 2 energy eigenvalues

- Fitted isobar-spectator interaction (case 1, 2) for $|\boldsymbol{p}| \leq 2\pi/L|(1,1,0)| \approx 2.69\ m_\pi$.




$$C_{\ell'\ell}(s,\boldsymbol{p}',\boldsymbol{p}) = \sum_{i=-1}^{\infty} c^{(i)}_{\ell'\ell}(\boldsymbol{p}',\boldsymbol{p})(s-m_{a_1}^2)^i$$

- $a_1$ can be generated as pole even though no built-in singularity

| Non-zero coefficients | No of fit parameters | $\chi^2$ |
|---|---|---|
| $c_{00}^{0}$ (no built-in pole) | 1 | 9 |
| $c_{00}^{0}$, $c_{00}^{1}$ (no built-in pole) | 2 | 0.15 |
| | | |
| $g_0$, $g_2$, $m_{a1}$, c | 4 | $10^{-7}$ |

$$C_{\ell'\ell}(s,\boldsymbol{p}',\boldsymbol{p}) = g_{\ell'}\left(\frac{|\boldsymbol{p}'|}{m_\pi}\right)^{\ell'} \frac{m_\pi^2}{s-m_{a_1}^2} g_\ell \left(\frac{|\boldsymbol{p}|}{m_\pi}\right)^{\ell} + c\,\delta_{\ell'0}\delta_{\ell 0}$$

- In these cases, there is a built-in singularity, leading to resonance poles

# Three kaons at maximal isospin

[Alexandru 2020]

- First study of three kaons from lattice QCD with chiral amplitudes
- Other groups have improved on this in the meantime:
  - Max. isospin, non-identical masses ( $\pi^+\pi^+K^+$, $\pi^+K^+K^+$) [Blanton 2021]
  - Pions and kaons at maximal isospin with unprecedented accuracy and no. of levels ($\pi^+\pi^+\pi^+$, $K^+K^+K^+$) [Blanton 2021]

- Two mass-degenerate light quarks (u,d); valence strange quark
- nHYP-smeared clover action
- quark propagation is treated using the LapH method with optimized inverters
- Lattice spacing determined from Wilson flow parameter $w_0$