Rapidity-only TMD factorization at one loop

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TMD factorization

TMD factorization formula for particle production in hadron-hadron scattering looks like

$$\frac{d\sigma}{d\eta d^2q_\perp} = \sum_{\text{flavors}} e_f^2 \int d^2k_\perp \mathcal{D}_{f/A}(x_A, k_\perp)\mathcal{D}_{f/B}(x_B, q_\perp - k_\perp)C(q, k_\perp)$$
$$+ \text{ power corrections } + \text{ ``Y} - \text{terms}''$$

- $\mathcal{D}_{f/A}(x_A, k_\perp)$ is the TMD density of a parton f in hadron A with fraction of momentum x_A and transverse momentum $k_\perp$,
- $\mathcal{D}_{f/B}(x, q_\perp - k_\perp)$ is a similar quantity for hadron B,
- $C_i(q, k)$ are determined by the cross section $\sigma(f\!f \to \mu^+\mu^-)$ of production of DY pair of invariant mass q^2 in the scattering of two partons.

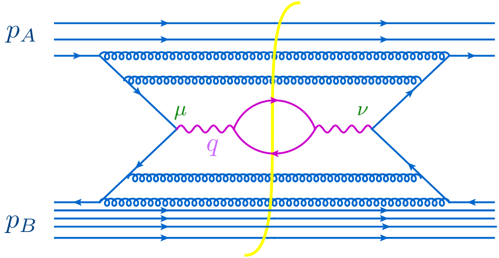
Examples: Drell-Yan process with Q being the mass of DY pair of Higgs production by gluon-gluon fusion

TMD approach is relevant when the transverse momentum $q_\perp \ll Q$

Classical example: DY hadronic tensor

DY cross section is given by the product of leptonic tensor and hadronic tensor. The hadronic tensor $W_{\mu\nu}$ is defined as

$$W_{\mu\nu}(p_A, p_B, q) \;=\; \frac{1}{(2\pi)^4}\int d^4x\, e^{-iqx}\langle p_A, p_B|J_\mu(x)J_\nu(0)|p_A, p_B\rangle$$



p_A, p_B = hadron momenta, q = the momentum of DY pair, and J_μ is the electromagnetic or Z-boson current.

Long-term project: TMD factorization valid at small and moderate x

$$\frac{d\sigma}{d\eta d^2q_\perp} = \sum_{\text{flavors}} e_f^2 \int d^2k_\perp \mathcal{D}_{f/A}(x_A, k_\perp)\mathcal{D}_{f/B}(x_B, q_\perp - k_\perp)C(q, k_\perp)$$
$$+ \text{ power corrections } + \text{ ``Y} - \text{terms''}$$

The quantities $\mathcal{D}_{f/A}(x_A, k_\perp)$, $\mathcal{D}_{f/B}(x_B, q_\perp - k_\perp)$, and $C(q, k_\perp)$ are defined with cutoffs. The dependence on the cutoffs cancels in their product order by order in α_s.

At moderate x_A, x_B: CSS approach. The TMDs $\mathcal{D}_{f/A}(x_A, k_\perp)$ are defined with a combination of UV and rapidity cutoffs.

At $x_A, x_B \ll 1$: k_T-factorization approach. The TMDs are defined with rapidity-only cutoffs.

It is impossible to extend CSS approach to small x ($\Leftrightarrow$ nobody tried)

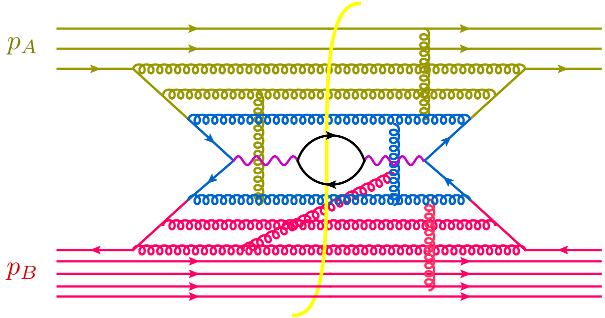
It *is* possible to study TMD factorization at moderate x using small-x methods (rapidity-only factorization etc.) (A. Tarasov, G. Chirilli, I.B, 2015-2023)

Visible success: power corrections $\sim \frac{1}{Q^2}$ for small-x DY hadronic tensor $\Rightarrow$ EM gauge invariance of DY tensor. It is not obtained (yet?) using CSS or SCET

TMD factorization from rapidity factorization (A. Tarasov and I.B.)

Sudakov variables:

$$p = \alpha p_1 + \beta p_2 + p_\perp, \qquad p_1 \simeq p_A, \;\; p_2 \simeq p_B, \;\; p_1^2 = p_2^2 = 0$$



"Projectile″ fields : $|\beta| < \sigma_p$

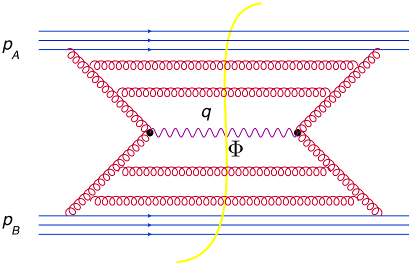
"Central" fields

"Target" fields : $|\alpha| < \sigma_t$

The result of the integration over "central" fields in the background of projectile and target fields is a series of TMD operators made from projectile (or target) fields multiplied by powers of $\frac{1}{Q^2} \Rightarrow$ power corrections

Coefficient function for TMD factorization at one loop

Particle production by gluon-gluon fusion (point $gg\Phi$ vertex is a $\frac{m_H}{m_t} \ll 1$ approximation for Higgs production.)

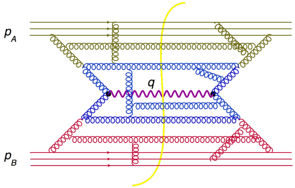


$$s \gtrsim Q^2 \quad \gg \quad Q_\perp^2 \gtrsim m^2$$
$$q^2 \equiv Q^2 = M_\Phi^2, \quad Q_\perp^2 \equiv q_\perp^2$$

Goal: one-loop TMD factorizaton formula for hadronic tensor.

Result of calculations:

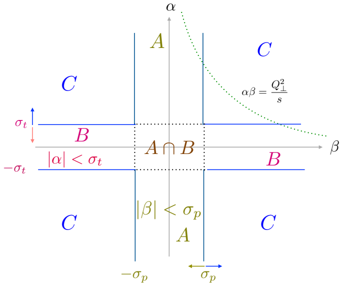
$$\begin{aligned} W(p_A, p_B; q) \; &= \int db_\perp \; e^{i(q,b)_\perp} \mathcal{D}_{g/A}(x_A, b_\perp; \sigma_a) \mathcal{D}_{g/B}(x_B, b_\perp; \sigma_b) \\ &\times \; \exp\Big\{ \frac{\alpha_s N_c}{2\pi} \Big[\ln^2 \frac{b_\perp^2 s \sigma_p \sigma_t}{4} - 2\Big(\ln \frac{\alpha_q}{\sigma_t} + \gamma\Big)\Big(\ln \frac{\beta_q}{\sigma_p} + \gamma\Big) + \frac{\pi^2}{2} \Big] \Big\} \\ &\qquad + \text{NLO terms} \sim O\big(\alpha_s^2\big) \; + \; \text{power corrections} \end{aligned}$$



"Projectile" fields: $|\beta| < \sigma_a$

"Central" fields

"Target" fields: $|\alpha| < \sigma_b$



Matching: $\ln \sigma_p$ in the projectile TMDs and $\ln \sigma_t$ in the target TMDs should cancel with $\ln \sigma_p$ and $\ln \sigma_t$ in the coefficient functions.

$A \cap B$, $k_\perp \sim m_\perp$: Glauber gluons

$A \cap B$, $k_\perp \ll m_\perp$: soft gluons

$A \cap B$ gluons $\equiv$ soft/Glauber (sG) gluons

sG gluons cancel out

Formal rescaling: $s = \zeta s_0$, $\zeta \to \infty$, $Q_\perp^2$-fixed

$$\text{Rapidity cutoffs :}\quad \alpha_a \gg \sigma_t \gg \frac{Q_\perp^2}{\beta_b s} \sim \zeta^{-1},\quad \beta_b \gg \sigma_p \gg \frac{Q_\perp^2}{\alpha_a s} \sim \zeta^{-1},\quad \frac{\sigma_p \sigma_t s}{Q_\perp^2} \sim \zeta^{-1/2}$$

Coefficient function in the functional-integral language

After integration over central fields

$$\frac{1}{16}(N_c^2-1)\langle p'_A,p'_B|g^2F^a_{\mu\nu}F^{a\mu\nu}(x_2)g^2F^b_{\lambda\rho}F^{b\lambda\rho}(x_1)|p_A,p_B\rangle$$
$$=\int\mathscr{D}\Phi_{\mathscr{A}}\ \Psi^*_{p'_A}(t_i)\Psi_{p_A}(t_i)\Psi^*_{p'_B}(t_i)\Psi_{p_B}(t_i)\Big[\mathcal{O}^{\sigma_p}_{ij}(x_2^-,x_{2_\perp};z_1^-,x_{1_\perp})\mathcal{O}^{ij;\sigma_t}(x_2^+,x_{2_\perp};x_1^+,x_{1_\perp})$$
$$+\int dz_1^-dz_{1_\perp}dz_2^-dz_{2_\perp}dw_1^+dw_{1_\perp}dw_2^+dw_{2_\perp}\frac{\alpha_sN_c}{2\pi}\mathfrak{C}_1(x_1,x_2;z_i^-,z_{i_\perp},w_i^+,w_{i_\perp};\sigma_p,\sigma_t)$$
$$\times\ \mathcal{O}^{\sigma_p}_{ij}(z_2^-,z_{2_\perp};z_1^-,z_{1_\perp})\mathcal{O}^{ij;\sigma_t}(z_2^+,z_{2_\perp};z_1^+,z_{1_\perp})\ +\ldots\Big]$$

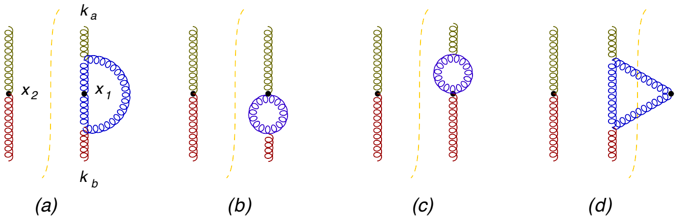
where $\mathscr{A}=A+B+sG$

Calculation of coefficient function $\mathfrak{C}_1$ in the background field $\mathbb{A}=\bar{A}+\bar{B}+\bar{C}$

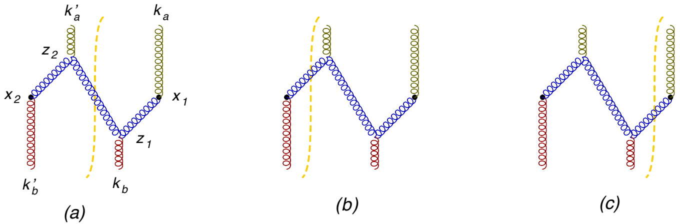
$$\int dz_2^-dz_{2_\perp}dz_1^-dz_{1_\perp}dw_1^+dw_{1_\perp}dw_2^+dw_{2_\perp}\frac{\alpha_sN_c}{2\pi}\mathfrak{C}_1(x_1,x_2;z_i^-,z_{i_\perp},w_i^+,w_{i_\perp};\sigma_p,\sigma_t)$$
$$\times\ U^{-i,a}(z_2^+,z_{2_\perp})U^{-j,a}(z_1^+,z_{1_\perp})V^{+i,a}(z_2^-,z_{2_\perp})V^{+j,a}(z_1^-,z_{1_\perp})$$
$$=\ \frac{N_c^2-1}{16}g^4\langle\tilde{F}^a_{\mu\nu}\tilde{F}^{a\mu\nu}(x_2)F^b_{\lambda\rho}F^{b\lambda\rho}(x_1)\rangle_{\mathbb{A}}$$
$$-\ \langle\hat{\mathcal{O}}^{ij,\sigma_p}(x_2^-,x_{2_\perp};x_1^-,x_{1_\perp})\hat{\mathcal{O}}^{ij;\sigma_t}(x_2^+,x_{2_\perp};x_1^+,x_{1_\perp})\rangle_{\mathbb{A}}$$

Diagrams for $\langle \tilde{F}^a_{\mu\nu}\tilde{F}^{a\mu\nu}(x_2)F^b_{\lambda\rho}F^{b\lambda\rho}(x_1)\rangle_{\mathbb{A}}$ in background fields

"Virtual" diagrams



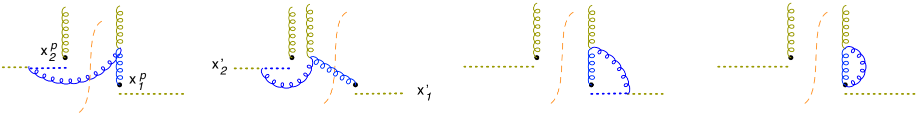
"Real" diagrams



Diagrams for subtracted TMD matrix elements

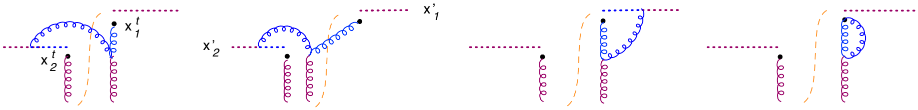
"Projectile" TMD matrix elements.

The rapidity-only $e^{-i\frac{\beta}{\sigma_p}}$ regularization is depicted by point splitting:
F^{+k} shown by dots stand at $x_1^p = x_{1_\perp} + x_1^-$ and $x_2^p = x_{2_\perp} + x_2^-$
Wilson lines start from $x_1' = x_2 + \delta^+$ and $x_2' = x_1 + \delta^+$ where $\delta^+ = \frac{1}{\varrho\sigma_p}$



(a) (b) (c) (d)

"Target" TMD matrix elements. The rapidity-only $e^{-i\frac{\alpha}{\sigma_t}}$ regularization is depicted by point splitting.



(a) (b) (c) (d)

Typical divergent integral ($\varepsilon = \frac{d}{2} - 2$, $\bar{d}^n p \equiv \frac{d^n p}{(2\pi)^n}$)

$$-i\mu^{-2\varepsilon}\int \bar{d}\alpha \bar{d}\beta \bar{d}p_\perp \frac{1}{\beta - i\epsilon}\frac{1}{\alpha\beta s - p_\perp^2 + i\epsilon}\frac{s(\beta-\beta_B)}{\alpha(\beta-\beta_B)s - p_\perp^2 + i\epsilon}\big(1 - e^{i(p,x)_\perp}\big)$$

$$= \mu^{-2\varepsilon}\int \frac{\bar{d}p_\perp}{p_\perp^2}\big(1 - e^{i(p,x)_\perp}\big)\int_0^{\beta_B}\frac{\bar{d}\beta}{\beta_B}\frac{\beta_B - \beta}{\beta - i\epsilon} = -\frac{1}{8\pi^2}\frac{\Gamma(\varepsilon)}{(x_\perp^2\mu^2)^\varepsilon}\int_0^{\beta_B}\frac{d\beta}{\beta_B}\frac{\beta_B-\beta}{\beta - i\epsilon}$$

δ-regularization with $A^-(z^+) \to A^-(z^+)e^{\pm\delta z^+}$

$$-\frac{1}{8\pi^2}\frac{\Gamma(\varepsilon)}{(x_\perp^2\mu^2)^\varepsilon}\int_0^{\beta_B}\frac{d\beta}{\beta_B}\frac{\beta_B-\beta}{\beta - i\delta} \simeq \frac{1}{8\pi^2}\Big(-\frac{1}{\varepsilon} + \ln\mu^2\frac{x_\perp^2}{4} + \gamma_E\Big)\Big(\ln\frac{\beta_B}{-i\delta} - 1\Big)$$

Rapidity-only cutoff

$$-i\int \bar{d}\alpha \bar{d}\beta \bar{d}p_\perp \frac{1}{\beta - i\epsilon}\frac{e^{-i\frac{\alpha}{\sigma}}}{\alpha\beta s - p_\perp^2 + i\epsilon}\frac{s(\beta-\beta_B)}{\alpha(\beta-\beta_B)s - p_\perp^2 + i\epsilon}\big(1 - e^{i(p,x)_\perp}\big)$$

$$= \int \frac{\bar{d}p_\perp}{p_\perp^2}\big(1 - e^{i(p,x)_\perp}\big)\int_0^{\infty}\bar{d}\alpha\frac{\beta_B s}{\alpha\beta_B s + p_\perp^2}e^{-i\frac{\alpha}{\sigma}} = \frac{1}{16\pi^2}\ln^2\Big(-i\beta_B\sigma s\frac{x_\perp^2}{4}e^{\gamma_E}\Big)$$

$$\mathcal{W}(x_1,x_2)\;-\;\mathcal{W}^{\rm tmd}(x_1,x_2)$$
$$= \int d\!\!\!{}^-\alpha'_a d\!\!\!{}^- k'_{a_\perp} d\!\!\!{}^-\beta'_b d\!\!\!{}^- k_{b_\perp} d\!\!\!{}^-\alpha_a d\!\!\!{}^- k'_{a_\perp} d\!\!\!{}^-\beta_b d\!\!\!{}^- k'_{b_\perp}\, e^{-i\alpha'_a\varrho x_2^- - i\alpha_a\varrho x_1^-} e^{-i\beta'_b\varrho x_2^+ - i\beta_b\varrho x_1^+}$$
$$\times\, e^{-i(k_a+k_b,x_1)_\perp - i(k'_a+k'_b,x_2)_\perp} U^{+,b}_i(\alpha'_a,k'_{a_\perp}) V^{-i,a}(\beta'_b,k'_{b_\perp}) U^{+,b}_j(\alpha_a,k_{a_\perp}) V^{-j,a}(\beta_b,k_{b_\perp})$$
$$\times\, g^2[I - I^{\sigma_p,\sigma_t}_{\rm tmd}](\alpha_a,\alpha'_a,\beta_b,\beta'_b,k_{a_\perp},k'_{a_\perp},k_{b_\perp},k'_{b_\perp},x_1,x_2)$$

with

$$[I - I^{\sigma_p,\sigma_t}_{\rm tmd}](\alpha'_a,\alpha_a,\beta'_b,\beta_b,k'_{a_\perp},k'_{a_\perp},k_{b_\perp},k'_{b_\perp},x_2,x_1)$$
$$= -\ln\frac{(-i\alpha'_a)k'^2_{a_\perp}}{(-i\alpha_a)k'^2_{a_\perp}}\ln\frac{(-i\beta'_b)k'^2_{b_\perp}}{(-i\beta_b)k^2_{b_\perp}} + \ln^2\frac{x^2_{12_\perp}s\sigma_p\sigma_t}{4}$$
$$-\;\ln\frac{(-i\alpha'_a)e^\gamma}{\sigma_t}\ln\frac{(-i\beta'_b)e^\gamma}{\sigma_p} - \ln\frac{(-i\alpha_a)e^\gamma}{\sigma_t}\ln\frac{(-i\beta_b)e^\gamma}{\sigma_p} + \pi^2$$

where $(-i\alpha_a)\equiv -i(\alpha_a+i\epsilon)$ etc. Power corrections $\sim\zeta^{-1}$ and $\sim\zeta^{-1/2}$ are neglected.

(Intermediate) Result

$$\begin{aligned}
&\mathcal{W}(x_1,x_2)\ -\ \mathcal{W}^{\rm tmd}(x_1,x_2)\\
&= \int d\!\!{}^-\alpha'_a d\!\!{}^-k'_{a_\perp} d\!\!{}^-\beta'_b d\!\!{}^-k_{b_\perp} d\!\!{}^-\alpha_a d\!\!{}^-k'_{a_\perp} d\!\!{}^-\beta_b d\!\!{}^-k'_{b_\perp}\, e^{-i\alpha'_a\varrho x_2^- - i\alpha_a\varrho x_1^-} e^{-i\beta'_b\varrho x_2^+ - i\beta_b\varrho x_1^+}\\
&\quad\times e^{-i(k_a+k_b,x_1)_\perp - i(k'_a+k'_b,x_2)_\perp} U^{+,b}_i(\alpha'_a,k'_{a_\perp}) V^{-i,a}(\beta'_b,k'_{b_\perp}) U^{+,b}_j(\alpha_a,k_{a_\perp}) V^{-j,a}(\beta_b,k_{b_\perp})\\
&\quad\times g^2[I - I^{\sigma_p,\sigma_t}_{\rm tmd}](\alpha_a,\alpha'_a,\beta_b,\beta'_b,k_{a_\perp},k'_{a_\perp},k_{b_\perp},k'_{b_\perp},x_1,x_2)
\end{aligned}$$

with

$$\begin{aligned}
&[I - I^{\sigma_p,\sigma_t}_{\rm tmd}](\alpha'_a,\alpha_a,\beta'_b,\beta_b,k'_{a_\perp},k'_{a_\perp},k_{b_\perp},k'_{b_\perp},x_2,x_1)\\
&\quad = -\ln\frac{(-i\alpha'_a)k'^2_{a_\perp}}{(-i\alpha_a)k'^2_{a_\perp}}\ln\frac{(-i\beta'_b)k'^2_{b_\perp}}{(-i\beta_b)k^2_{b_\perp}} + \ln^2\frac{x^2_{12_\perp}s\sigma_p\sigma_t}{4}\\
&\qquad - \ln\frac{(-i\alpha'_a)e^\gamma}{\sigma_t}\ln\frac{(-i\beta'_b)e^\gamma}{\sigma_p} - \ln\frac{(-i\alpha_a)e^\gamma}{\sigma_t}\ln\frac{(-i\beta_b)e^\gamma}{\sigma_p} + \pi^2
\end{aligned}$$

where $(-i\alpha_a) \equiv -i(\alpha_a + i\epsilon)$ etc. Power corrections $\sim \zeta^{-1}$ and $\sim \zeta^{-1/2}$ are neglected.

This formula is not yet the final result for the coefficient function. The coefficient function was defined as a result of integration over C-fields with $\alpha > \sigma_t$ and $\beta > \sigma_p$. Since we did not impose these restrictions while calculating the loop integrals, we need to subtract sG contributions (with $\alpha < \sigma_t, \beta < \sigma_p$) to these integrals.

Result for the coefficient function

Result of sG subtraction:

term $-\ln\frac{(-i\alpha'_a)k'^2_{a_\perp}}{(-i\alpha_a)k'^2_{a_\perp}}\ln\frac{(-i\beta'_b)k'^2_{b_\perp}}{(-i\beta_b)k^2_{b_\perp}}$ disappears $\Rightarrow$ no dynamics in the transverse plane

$$\begin{aligned}
&\mathcal{W}(x_1,x_2)\ -\ \mathcal{W}^{\rm tmd}(x_1,x_2)\ -\ \mathcal{W}^{\rm sG}(x_1,x_2)\\
&=\int đ\alpha'_a đ\beta'_b đ\alpha_a đ\beta_b e^{-i\alpha'_a\varrho x_2^- - i\alpha_a\varrho x_1^-}e^{-i\beta'_b\varrho x_2^+ - i\beta_b\varrho x_1^+}\\
&\quad\times U_i^{+,b}(\alpha'_a,x_{2_\perp})V^{-i,a}(\beta'_b,x_{2_\perp})U_j^{+,b}(\alpha_a,x_{1_\perp})V^{-j,a}(\beta_b,x_{1_\perp})\\
&\quad\times g^2\mathfrak{C}_1(\alpha'_a,\alpha_a,\beta'_b,\beta_b;x_1,x_2)
\end{aligned}$$

where

$$\begin{aligned}
&\mathfrak{C}_1(\alpha'_a,\alpha_a,\beta'_b,\beta_b;x_2,x_1)\ =\ I - I^{\sigma_p,\sigma_t}_{\rm tmd} - I^{\sigma_p,\sigma_t}_{\rm sG}\\
&=\ \ln^2\frac{x^2_{12_\perp}s\sigma_p\sigma_t}{4} - \ln\frac{(-i\alpha'_a)e^\gamma}{\sigma_t}\ln\frac{(-i\beta'_b)e^\gamma}{\sigma_p} - \ln\frac{(-i\alpha_a)e^\gamma}{\sigma_t}\ln\frac{(-i\beta_b)e^\gamma}{\sigma_p}+\pi^2
\end{aligned}$$

The coefficient function in the coordinate space is made of $(+)$ - prescriptions since

$$\int d\alpha\ e^{i\alpha z}\Big[\ln\big(-i\frac{\alpha}{\sigma}+\epsilon\big)\ =\ \frac{\theta(-z)}{z}+\delta(z)\int_0^{1/\sigma}\frac{dz'}{z'}$$

Result for the coefficient function

Our formula

$$\frac{1}{16}(N_c^2-1)\langle p_A',p_B'|g^2F^a_{\mu\nu}F^{a\mu\nu}(x_2)g^2F^b_{\lambda\rho}F^{b\lambda\rho}(x_1)|p_A,p_B\rangle$$
$$= \int \mathscr{D}\Phi_{\mathscr{A}}\ \Psi^*_{p_A'}(t_i)\Psi_{p_A}(t_i)\Psi^*_{p_B'}(t_i)\Psi_{p_B}(t_i)\Big[\mathcal{O}^{\sigma_p}_{ij}(x_2^-,x_{2_\perp};z_1^-,x_{1_\perp})\mathcal{O}^{ij;\sigma_t}(x_2^+,x_{2_\perp};x_1^+,x_{1_\perp})$$
$$+\int dz_1^-dz_2^-dw_1^+dw_2^+\frac{\alpha_sN_c}{2\pi}\mathfrak{C}_1(x_1,x_2;z_i^-,w_i^+;\sigma_p,\sigma_t)$$
$$\times\ \mathcal{O}^{\sigma_p}_{ij}(z_2^-,x_{2_\perp};z_1^-,x_{1_\perp})\mathcal{O}^{ij;\sigma_t}(z_2^+,x_{2_\perp};z_1^+,x_{1_\perp})\ +\ O(\alpha_s^2)\Big]$$

is not yes TMD formula since $\mathscr{A}=A+B+sG$ and soft/Glauber gluons connect "projectile" and "target" gluons.

It is well known that Glauber gluons cancel and soft gluons form soft factors.

With rapidity-only cutoffs, soft factors are power corrections $\Rightarrow$ TMD formula

$$\frac{1}{16}(N_c^2-1)\langle p_A',p_B'|g^2F^a_{\mu\nu}F^{a\mu\nu}(x_2)g^2F^b_{\lambda\rho}F^{b\lambda\rho}(x_1)|p_A,p_B\rangle$$
$$=\ \langle p_A'|\hat{\mathcal{O}}^{\sigma_p}_{ij}(x_2^-,x_{2_\perp};x_1^-,x_{1_\perp})|p_A\rangle\langle p_B'|\hat{\mathcal{O}}^{ij;\sigma_t}(x_2^+,x_{2_\perp};x_1^+,x_{1_\perp})|p_B\rangle$$
$$+\int dz_1^-dz_2^-dw_1^+dw_2^+\frac{\alpha_sN_c}{2\pi}\mathfrak{C}_1(x_1,x_2;z_i^-,w_i^+;\sigma_p,\sigma_t)$$
$$\times\ \langle p_A'|\hat{\mathcal{O}}^{\sigma_p}_{ij}(z_2^-,x_{2_\perp};z_1^-,x_{1_\perp})|p_A\rangle\langle p_B'|\hat{\mathcal{O}}^{ij;\sigma_t}(z_2^+,x_{2_\perp};z_1^+,x_{1_\perp})|p_B\rangle$$

Matching of coefficient function and TMDs

TMD evolution equations

$$\sigma_p \frac{d}{d\sigma_p} \hat{\mathcal{O}}^{ij;\sigma_t}(\alpha'_a, \alpha_a, x_{2_\perp}, x_{1_\perp})$$
$$= -\frac{\alpha_s N_c}{2\pi}\Big[2\ln\frac{sx^2_{12_\perp}}{4} + \ln(-i\alpha'_a\sigma_p + \epsilon) + \ln(-i\alpha_a\sigma_p + \epsilon) + 2\gamma\Big]\hat{\mathcal{O}}^{ij;\sigma_t}(\alpha'_a, \alpha_a, x_{2_\perp}, x_{1_\perp})$$
$$\sigma_t \frac{d}{d\sigma_t} \hat{\mathcal{O}}^{ij;\sigma_t}(\beta'_b, \beta_b, x_{2_\perp}, x_{1_\perp})$$
$$= -\frac{\alpha_s N_c}{2\pi}\Big[2\ln\frac{sx^2_{12_\perp}}{4} + \ln(-i\beta'_b\sigma_t + \epsilon) + \ln(-i\beta_b\sigma_t + \epsilon) + 2\gamma\Big]\hat{\mathcal{O}}^{ij;\sigma_t}(\beta'_b, \beta_b, x_{2_\perp}, x_{1_\perp})$$

Matching of σ_p and σ_t evolutions $\Rightarrow$

$$\sigma_t \frac{d}{d\sigma_t}\mathfrak{C}(x_{1_\perp}, x_{2_\perp}; \alpha'_a, \alpha_a, \beta'_b, \beta_b; \sigma_p, \sigma_t) = \frac{\alpha_s N_c}{2\pi}\Big[2\ln\frac{sx^2_{12_\perp}}{4}$$
$$+ \ln(-i\beta'_b\sigma_t + \epsilon) + \ln(-i\beta_b\sigma_t + \epsilon) + 2\gamma\Big]\mathfrak{C}(x_1, x_2; \alpha'_a, \alpha_a, \beta'_b, \beta_b; \sigma_p, \sigma_t)$$
$$\sigma_p \frac{d}{d\sigma_p}\mathfrak{C}(x_{1_\perp}, x_{2_\perp}; \alpha'_a, \alpha_a, \beta'_b, \beta_b; \sigma_p, \sigma_t) = \frac{\alpha_s N_c}{2\pi}\Big[2\ln\frac{sx^2_{12_\perp}}{4}$$
$$+ \ln(-i\alpha'_a\sigma_p + \epsilon) + \ln(-i\alpha_a\sigma_p + \epsilon) + 2\gamma\Big]\ \mathfrak{C}(x_{1_\perp}, x_{2_\perp}; \alpha'_a, \alpha_a, \beta'_b, \beta_b; \sigma_p, \sigma_t)$$

Matching of coefficient function and TMDs

The solution of this equations compatible with our first-order result is

$$\mathfrak{C}(x_{1_\perp}, x_{2_\perp}; \alpha'_a, \alpha_a, \beta'_b, \beta_b; \sigma_p, \sigma_t) \ = \ e^{\frac{\alpha_s N_c}{2\pi}\mathfrak{C}_1(x_{12_\perp}, \alpha'_a, \alpha_a, \beta'_b, \beta_b; \sigma_p, \sigma_t)}$$

⇒ hadronic tensor is

$$W(\alpha'_a, \alpha_a, \beta'_b, \beta_b, x_{1_\perp}, x_{2_\perp}) \ = \ \int \text{đ}\alpha'_a \text{đ}\alpha_a \text{đ}\beta'_b \text{đ}\beta_b \ e^{\frac{\alpha_s N_c}{2\pi}\mathfrak{C}_1(x_{12_\perp}, \alpha'_a, \alpha_a, \beta'_b, \beta_b; \sigma_p, \sigma_t)}$$
$$\times \langle p'_A|\hat{\mathcal{O}}^{\sigma_p}_{ij}(\alpha'_a, \alpha_a, x_{2_\perp}, x_{1_\perp})|p_A\rangle\langle p'_B|\hat{\mathcal{O}}^{ij;\sigma_t}(\beta'_b, \beta_b, x_{2_\perp}, x_{1_\perp})|p_B\rangle \ + \ldots$$

Reminder

$$\mathfrak{C}_1(\alpha'_a, \alpha_a, \beta'_b, \beta_b; x_1, x_2; \sigma_p, \sigma_t)$$
$$= \ \ln^2 \frac{x^2_{12_\perp} s\sigma_p\sigma_t}{4} - \ \ln \frac{(-i\alpha'_a)e^\gamma}{\sigma_t} \ln \frac{(-i\beta'_b)e^\gamma}{\sigma_p} - \ln \frac{(-i\alpha_a)e^\gamma}{\sigma_t} \ln \frac{(-i\beta_b)e^\gamma}{\sigma_p} + \pi^2$$

Forward case ($\equiv$ particle production by gluon fusion)

$$W(p_A,p_B;q) \;=\; \int db_\perp\; e^{i(q,b)_\perp} W(p_A,p_B;\alpha_q,\beta_q,b_\perp),$$

$$W(p_A,p_B;\alpha_q,\beta_q,b_\perp) \;=\; \frac{\pi^2}{2} Q^2 \mathcal{G}_{ij}^{\sigma_p}(\alpha_q,b_\perp;p_A)\,\mathcal{G}^{ij;\sigma_t}(\beta_q,b_\perp;p_B)$$
$$\times\;\exp\Big\{\frac{\alpha_s N_c}{2\pi}\Big[\ln^2\frac{b_\perp^2 s\sigma_p\sigma_t}{4} - 2\Big(\ln\frac{\alpha_q}{\sigma_t}+\gamma\Big)\Big(\ln\frac{\beta_q}{\sigma_p}+\gamma\Big)+\frac{\pi^2}{2}\Big]\Big\}$$
$$+\;\text{NLO terms} \sim O\big(\alpha_s^2\big) \;+\; \text{power corrections}$$

where $\mathcal{G}_{ij}^{\sigma_p}$, $\mathcal{G}_{ij}^{\sigma_t}$ are gluon TMDs:

$$\langle p_A|\hat{\mathcal{O}}_{ij}^{\sigma_p}(z^-,0^-,b_\perp)|p_A\rangle \;=\; -g^2\varrho^2\int_0^1 du\; u\,\mathcal{G}_{ij}^{\sigma_p}(u,b_\perp)\cos u\varrho z^-,$$

$$\langle p_B|\hat{\mathcal{O}}_{ij}^{\sigma_t}(z^-,0^-,b_\perp)|p_B\rangle \;=\; -g^2\varrho^2\int_0^1 du\; u\,\mathcal{G}_{ij}^{\sigma_t}(u,b_\perp)\cos u\varrho z^-,$$

Matching of coefficient function and TMDs

For the "forward" case $p'_A = p_A,\ p'_B = p_B$ The r.h.s. of the evolution formula (1) does not depend on cutoffs σ_p and σ_t as long as $\sigma_p \geq \tilde{\sigma}_p = \frac{4b_\perp^{-2}}{\alpha_q s}$ and $\sigma_t \geq \tilde{\sigma}_t \equiv \frac{4b_\perp^{-2}}{\beta_q s}$. Thus, the result of double-log Sudakov evolution reads

$$W(p_A, p_B; \alpha_q, \beta_q, b_\perp) \ = \ \frac{\pi^2}{2} Q^2 \mathcal{G}_{ij}^{\tilde{\sigma}_p}(\alpha_q, b_\perp; p_A) \mathcal{G}^{ij;\tilde{\sigma}_t}(\beta_q, b_\perp; p_B)$$
$$\times \ \exp\Big\{ - \frac{\alpha_s N_c}{2\pi} \Big[\Big(\ln\frac{Q^2 b_\perp^2}{4} + 2\gamma \Big)^2 - 2\gamma^2 - \frac{\pi^2}{2} \Big] \Big\} + \ O\big(\alpha_s^2\big) \text{ terms } + \text{ power corrections}$$

This result is universal for moderate x and small-x hadronic tensor. The difference lies in the continuation of the evolution beyond Sudakov region.

Double-log Sudakov evolution should stop at $\beta_B \sigma_0 s \simeq b_\perp^{-2}$. After that:

- If $\beta_B \equiv x_B \sim 1$ - DGLAP-type evolution from $\sigma_0 = \frac{b_\perp^{-2}}{x_B s}$ to $\sigma_{\text{fin}} = \frac{m_N^2}{s}$: summation of $\big(\alpha_s \ln \frac{b_\perp^{-2}}{m_N^2}\big)^n$
- If $\beta_B \equiv x_B \ll 1$ - BFKL-type evolution from $\sigma_0 = \frac{b_\perp^{-2}}{x_B s}$ to $\sigma_{\text{fin}} = \frac{b_\perp^{-2}}{s}$: summation of $\big(\alpha_s \ln x_B\big)^n$

Conclusions

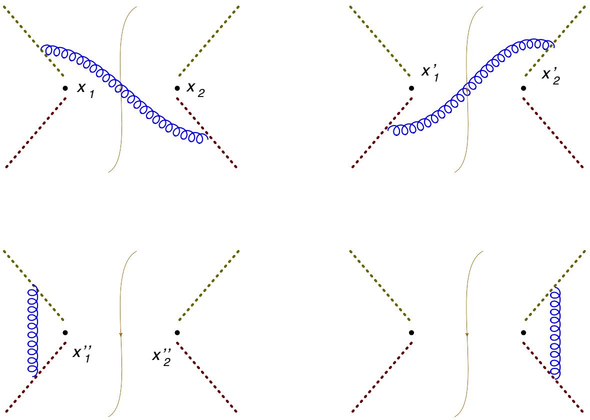
1. Conclusion: rapidity-only TMD factorization works!
   - Rapidity factorization at the one-loop level gives Sudakov-type double logs for both small and intermediate x_B
2. Outlook
   - Matching to DGLAP and BFKL/BK evolutions
   - Conformal invariance of rapidity-only factorization

Thank you for attention!

BACKUP SLIDES

Backup slide: soft factor

Leading-order diagrams



Result of calculation: $\frac{1}{4\pi^2}\mathrm{Li}_2\left(-\frac{x^2_{12\perp}}{2\delta^+\delta^-}\right) \sim O\left(\frac{\Delta^2_\perp}{2\delta^+\delta^-}\right) \sim O\left(\frac{\sigma_p\sigma_t s}{Q^2_\perp}\right) \sim O(\zeta^{-1/2})$

Soft factor with rapidity-only regularization does not have perturbative contributions which can mix with the TMD evolution