# Lattice QCD calculation of the pion generalized parton distributions (GPDs)

**Qi Shi**
**CCNU & BNL**

In collaboration with ANL, Adam Mickiewicz U., Boon U. and Temple U.

25TH INTERNATIONAL
SPIN PHYSICS
SYMPOSIUM

**September 24 - 29, 2023**
Durham Convention Center
*Durham, NC, USA*

# Introduction

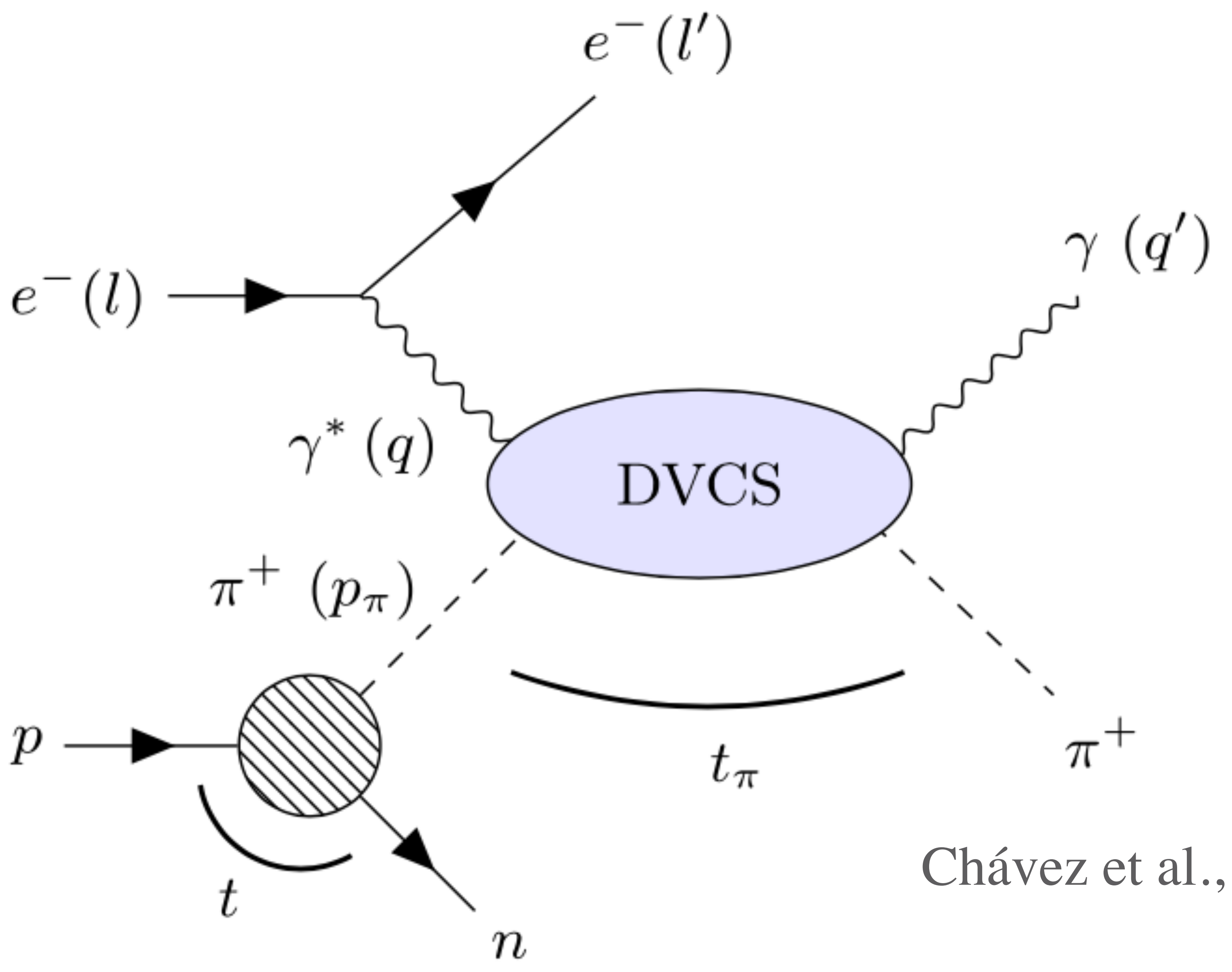
1D

Form Factor (pion-electron scattering)

Parton Distribution Functions (Drell-Yan process)

3D

Generalized Parton Distributions



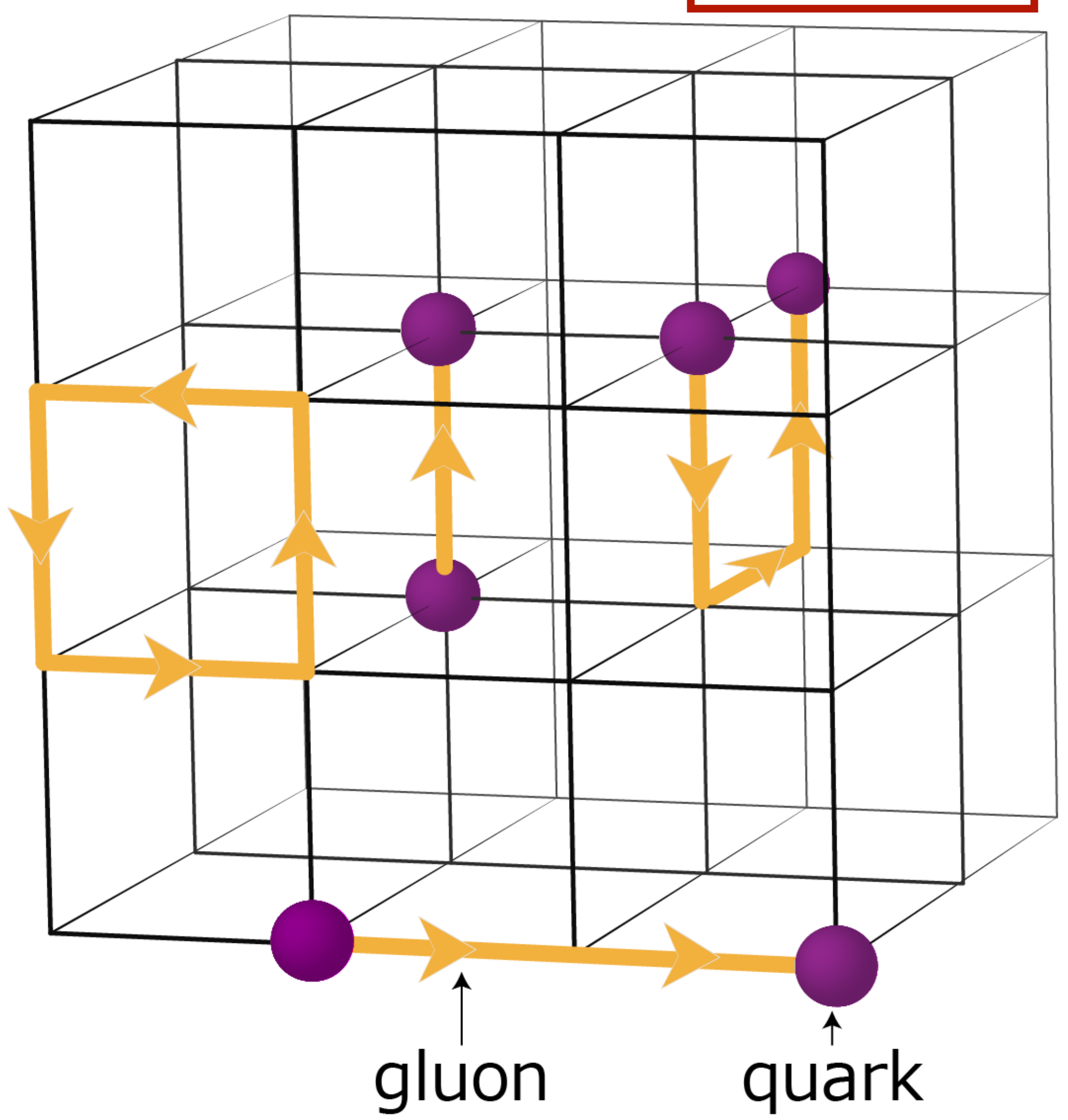

Chávez et al., PRL 128 (2022) 202501

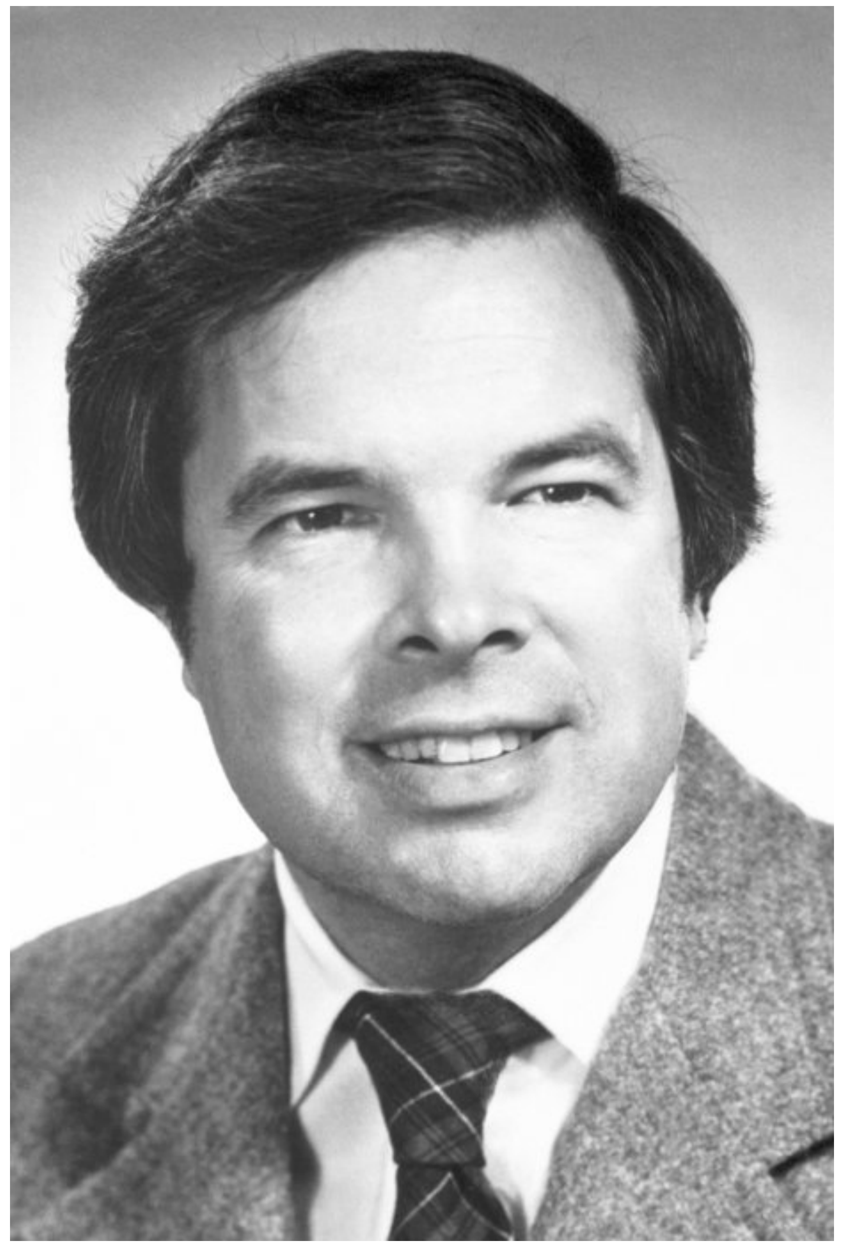
+ Models

**!!! Lattice QCD: from first principle**

# Lattice QCD

**1974**



Kenneth G. Wilson

## From first principle

- Discretizing Spacetime -> $N_s^3 \times N_t$
- QCD Lagrangian $\mathscr{L}$ -> action $\mathscr{S}[\phi, \bar{\phi}, U]$
- Path Integral and Partition Function

$$\langle \mathcal{O} \rangle = \frac{1}{Z} \int \mathscr{D}[U]\, \mathscr{D}[\phi]\, \mathscr{D}[\bar{\phi}]\, \mathcal{O}\, e^{-\mathscr{S}}$$

## Computational cost

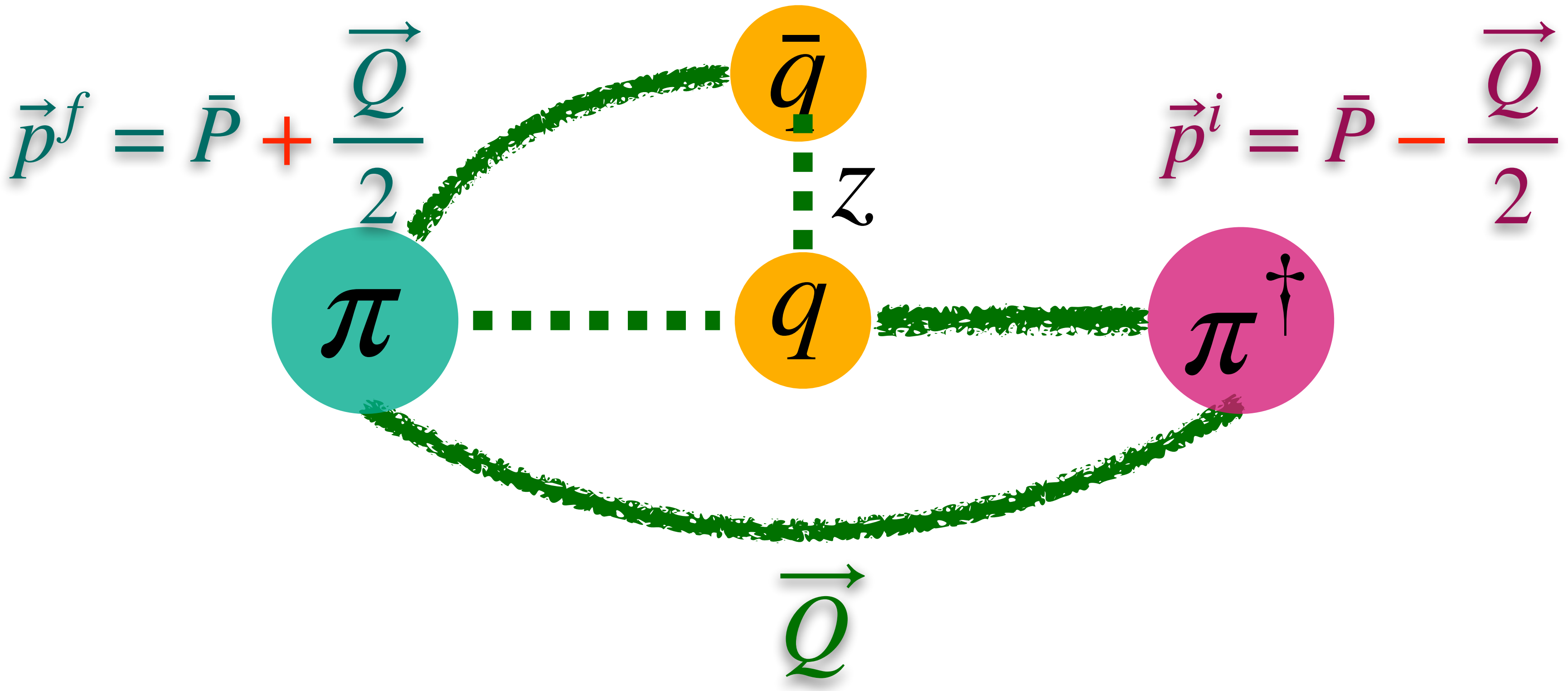
$$N_{\text{color}} \otimes N_{\text{flavor}} \otimes N_{\text{spin}} \otimes N_{\text{space}}^3 \otimes N_{\text{time}} \gtrsim 10^9 \quad + \quad N_{\text{polarization}}, N_{\text{momentum}}, N_{\text{temperature}}, \cdots$$

Several months ~ years

# Frame independent approach

**Lattice: each $\vec{p}^f$ require a seperate calculation**

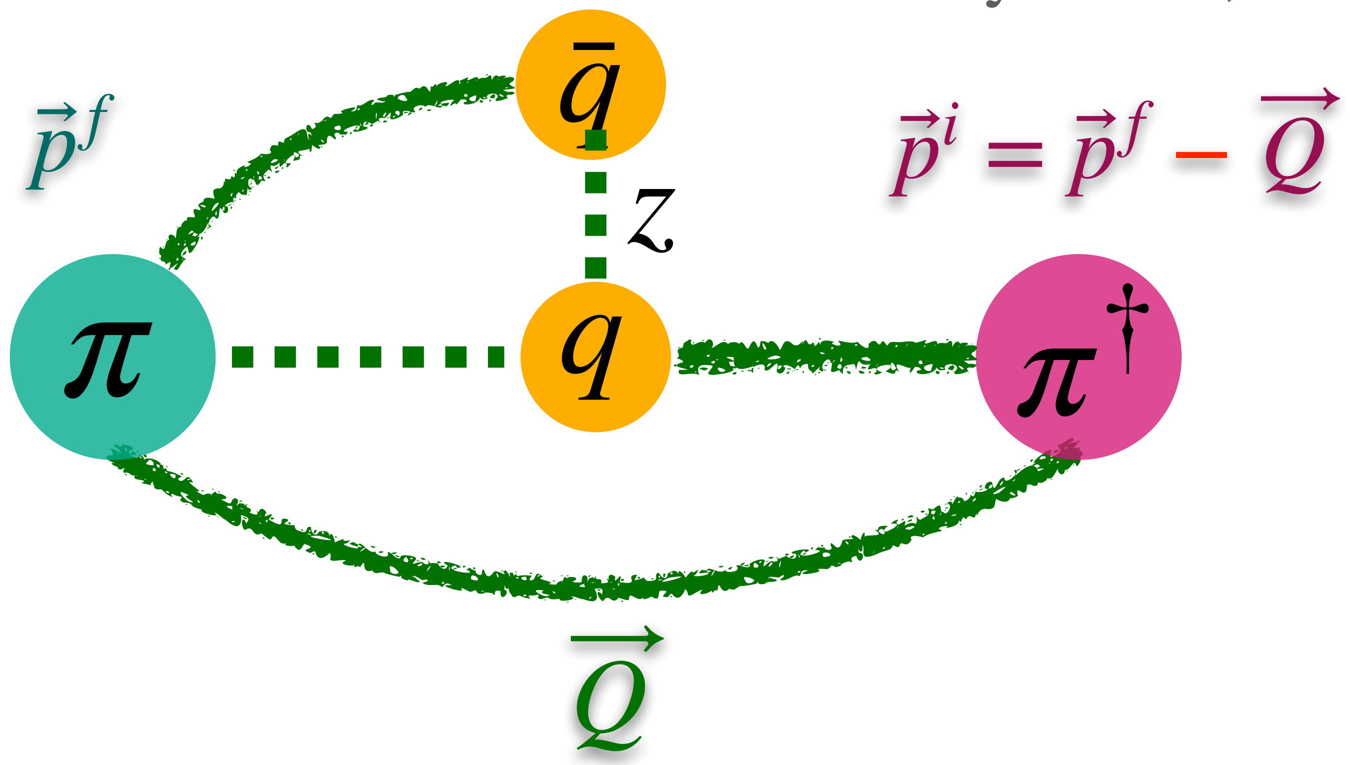
**Traditional: Symmetric**




- One $\vec{p}^f$ — only one $Q^2$ is useful
- Each $Q^2$ requires a seperate calculation

**Newly proposed: Asymmetric**

Bhattacharya et al., PRD 106 (2022) 114512




- One $\vec{p}^f$ — several $Q^2$ are useful


**Computational cost**

# Frame independent approach

Shohini Bhattacharya's plenary talk on Friday

- Lorentz invariant amplitudes $A_i$'s

$$F^{\mu}(P,z,Q)=\frac{1}{\sqrt{E^iE^f}}(P^{\mu}A_1+m^2z^{\mu}A_2+Q^{\mu}A_3),\qquad P^{\mu}=(p_f^{\mu}+p_i^{\mu})/2,\quad Q^{\mu}=p_f^{\mu}-p_i^{\mu}.$$

- Frame independent GPD $H$

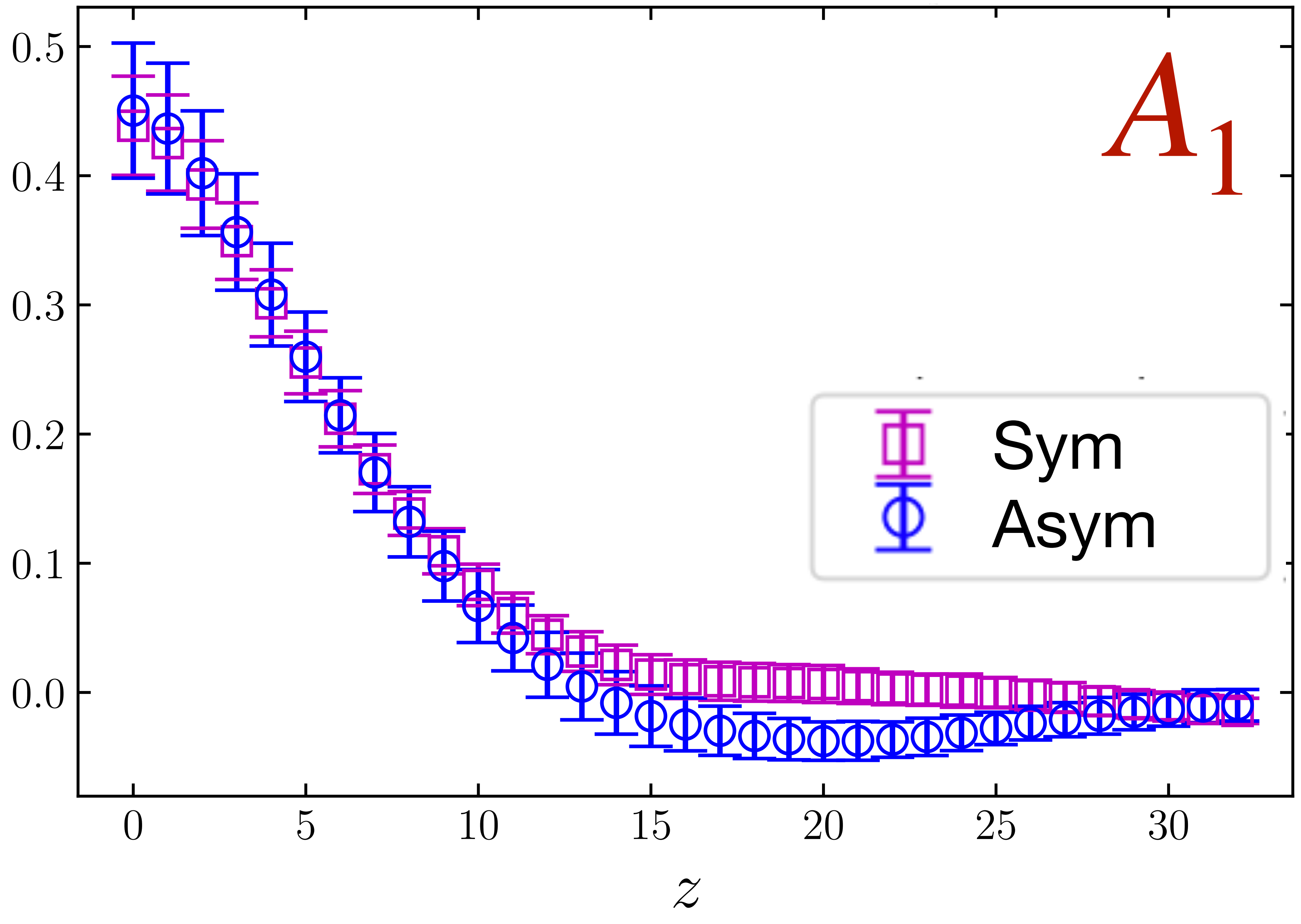
$$H(P,z,\Delta)=A_1+\frac{z\cdot Q}{z\cdot P}A_3$$

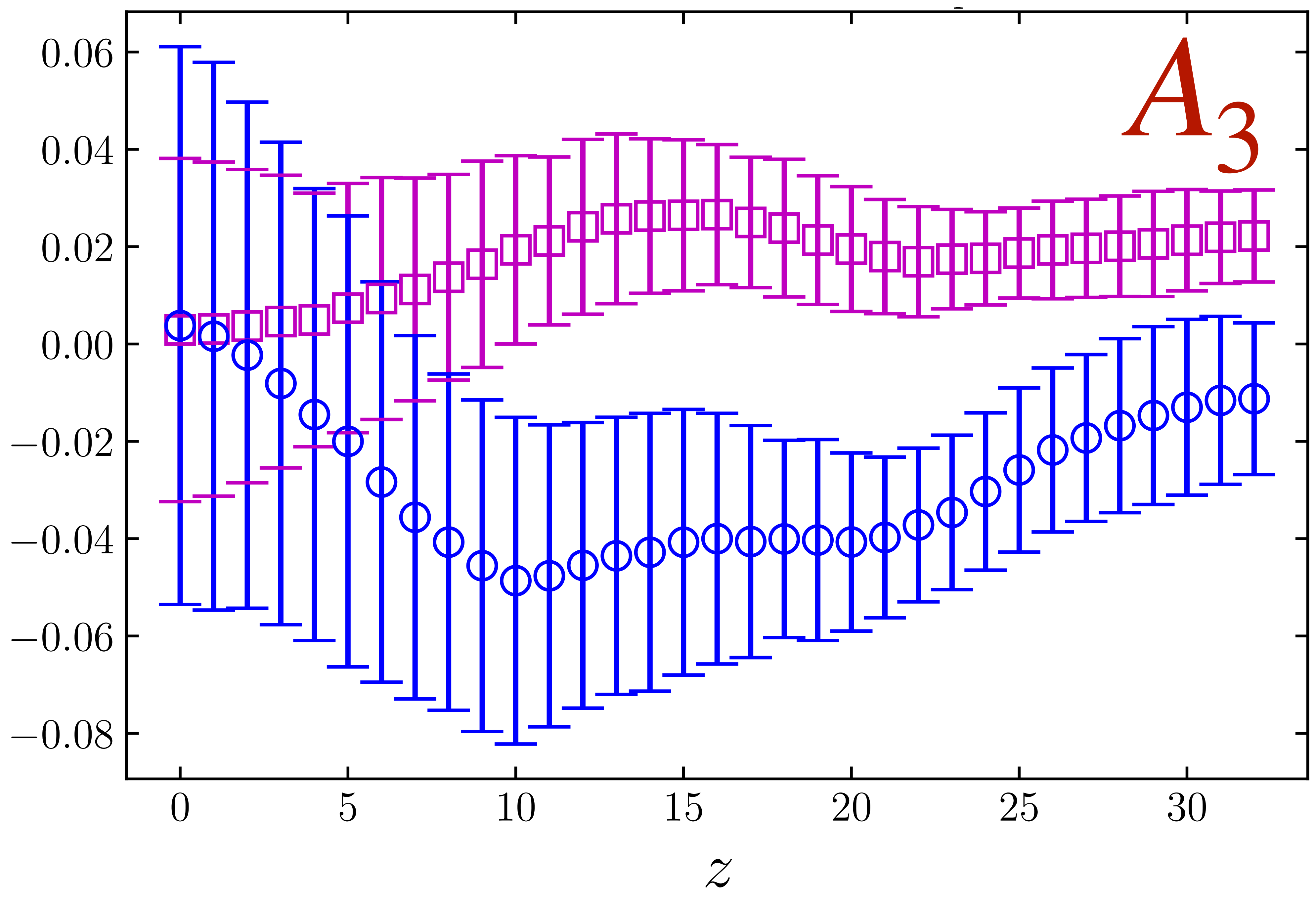
$$A_3(-z\cdot Q)=-A_3(z\cdot Q)\xrightarrow{z=z_3,\ Q_3=0}A_3(z\cdot Q=0)=0$$

$$H(P,z,\Delta)=A_1$$

- $A_1(\mathrm{Sym})\sim A_1(\mathrm{Asym})$
- $A_3(z\cdot Q=0)=0$

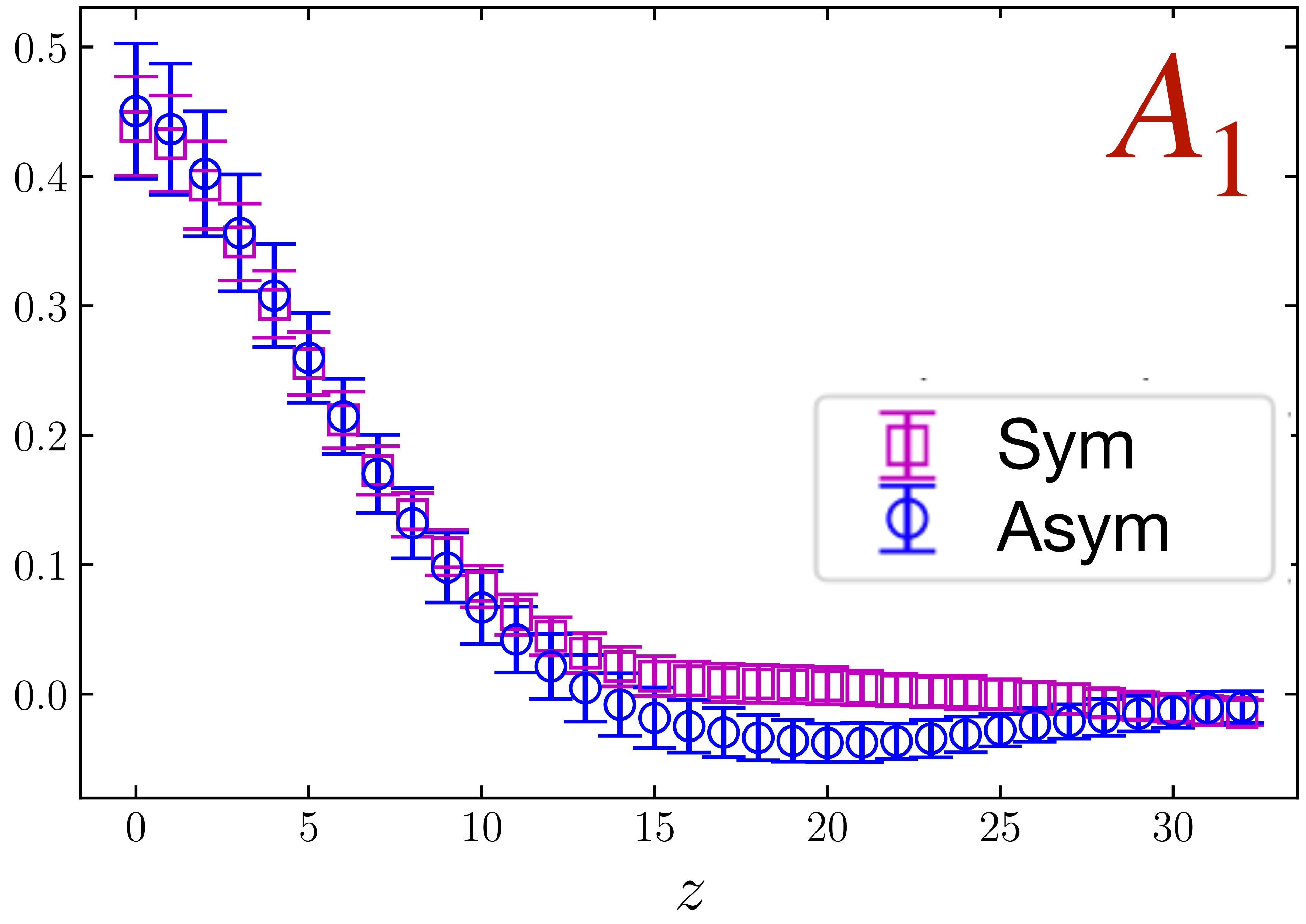
# Comparison of $A_i$ got from both frames

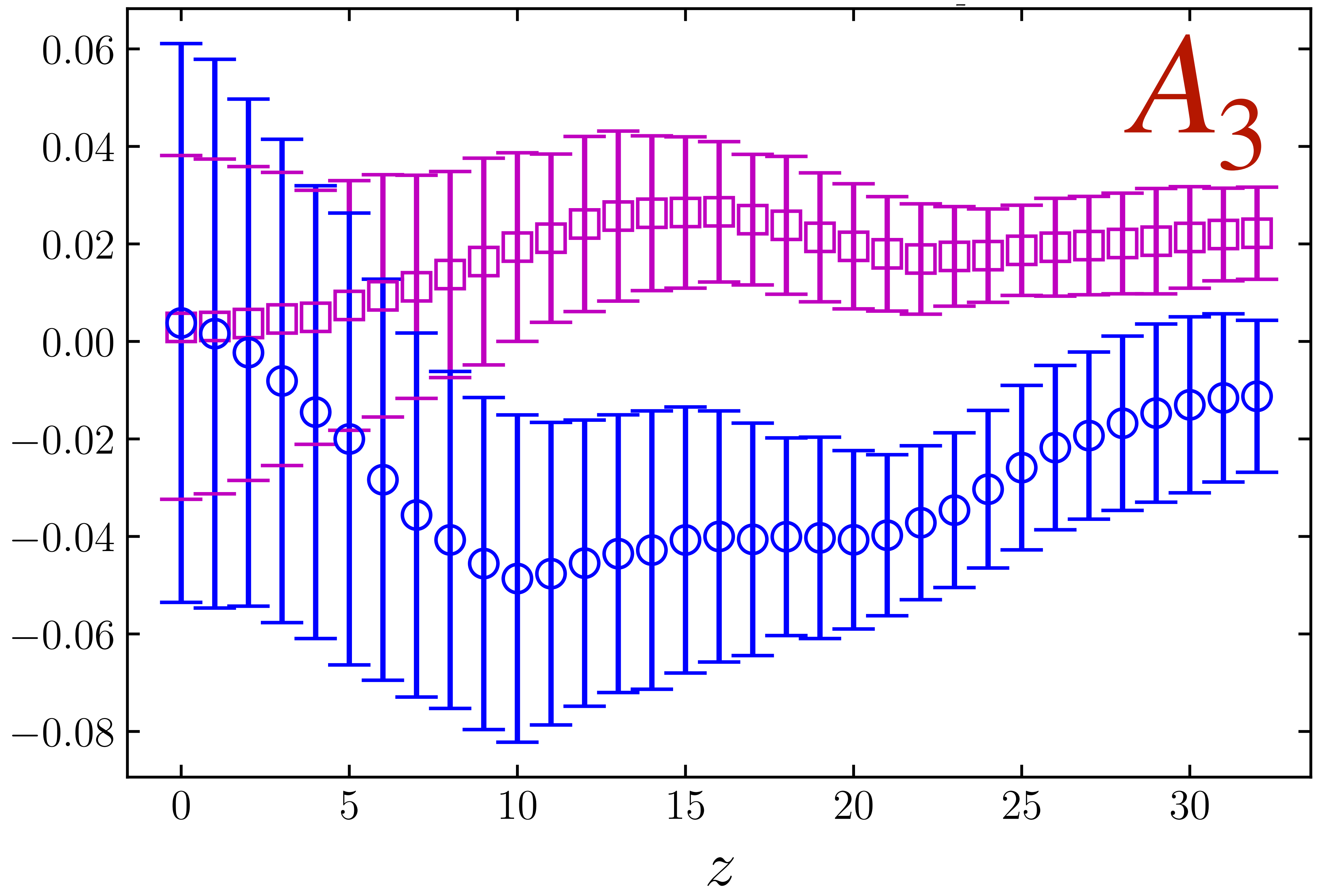



- $A_1(\mathrm{Sym}) \sim A_1(\mathrm{Asym})$
- $A_3(z \cdot Q = 0) = 0$

# Comparison of $A_i$ got from both frames



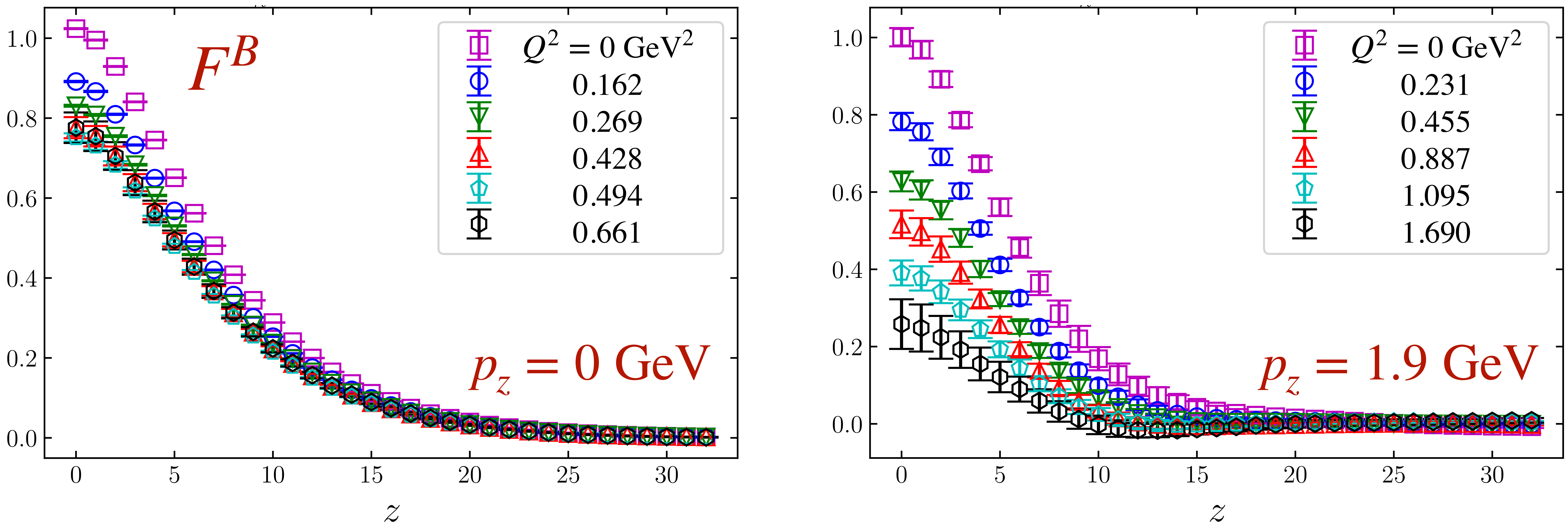

- Work on the asymmetric frame

$$F^{\mu}(P,z,\Delta) = \frac{1}{\sqrt{E^i E^f}}(P^{\mu}A_1 + m^2 z^{\mu} A_2 + Q^{\mu} A_3) \quad -\!>> \quad F^t(P,z,\Delta) = \frac{E^i + E^f}{2\sqrt{E^i E^f}} A_1,$$

$$H(P,z,\Delta) = A_1 + \frac{z \cdot Q}{z \cdot P} A_3 \quad -\!>> \quad H(P,z,\Delta) = A_1 = \frac{2\sqrt{E^i E^f}}{E^i + E^f} F^t$$

# Bare Matrix elements

- Fix $p_z = 0$ and 1.9 GeV, vary $Q^2$




- Two branches with different renormalization methods

# Renormalization

**Ratio-scheme:**

- Short distance
- The twist-2 factorization formula:

$$\mathscr{M}_0(z, p, q) = \sum_{n=0} \frac{(-i\lambda)^n}{n!} C_n(\mu^2 z^2) < x^n > + \mathcal{O}(\Lambda_{\mathrm{QCD}}^2 z^2)$$

- **Mellin Moments**

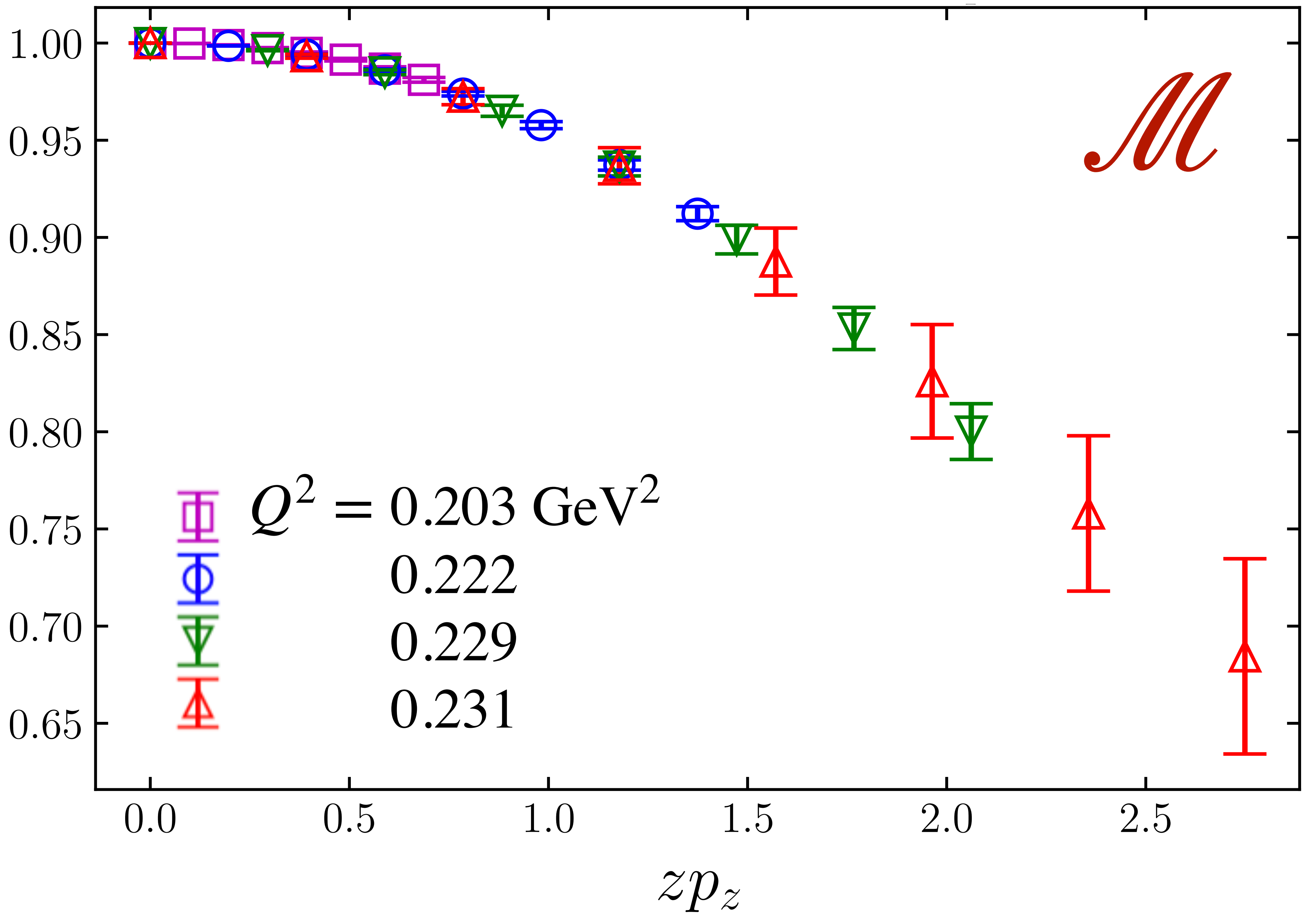
**Hybrid-scheme:**

- Long distance
- **L**arge **M**omentum **E**ffective **T**heory
- **Valence GPD**

# Ratio-scheme

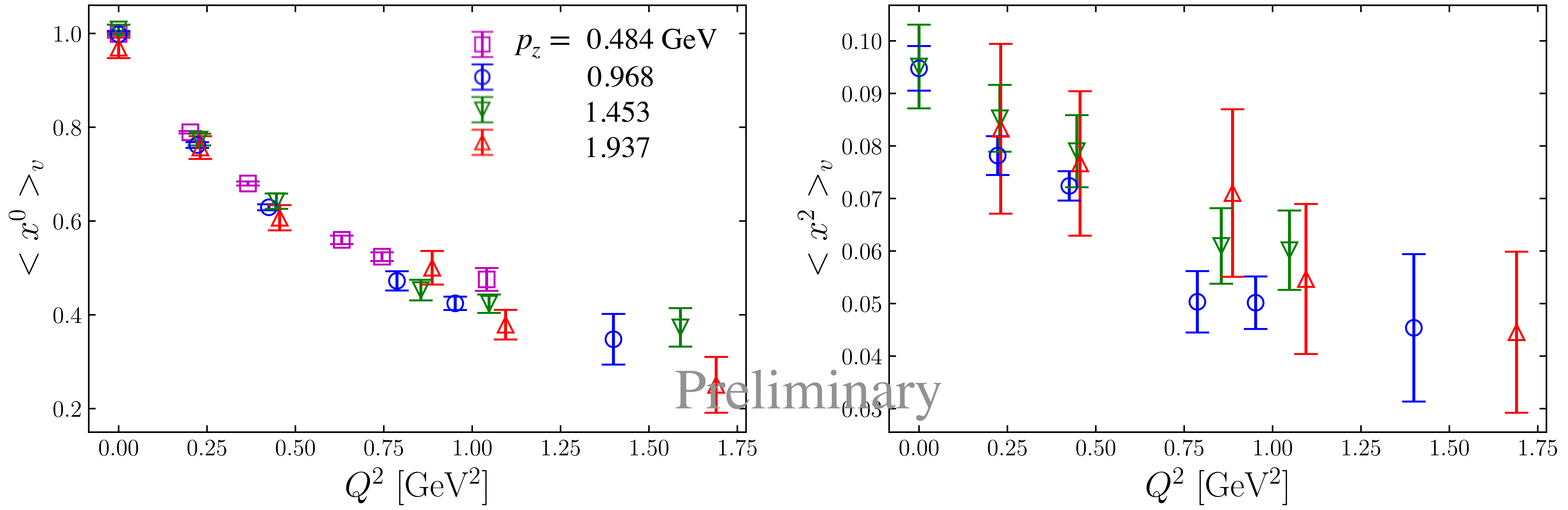
$$F^B(z,a) = e^{-\delta m\, z} Z(a) F^R(z) \quad \begin{cases} \mathcal{M}_0(z;p,q;0,0) = \dfrac{F^B(z,\vec{p},\vec{q})}{F^B(z,0,0)} = \dfrac{F^R(z,\vec{p},\vec{q})}{F^R(z,0,0)} \\[2ex] \mathcal{M}(z,p,q) = \dfrac{\mathcal{M}_0(z,p,q)}{\mathcal{M}_0(0,p,q)} = \dfrac{\mathcal{M}_0(z,p,q)}{F_\pi(Q^2)} \end{cases}$$

$$\mathcal{M}(z,p,Q^2) = \sum_{n=0} \frac{(-i\lambda)^n}{n!} \frac{C_n(\mu^2 z^2)}{C_0(\mu^2 z^2)} \frac{< x^n >}{F_\pi(Q^2)}$$



# $Q^2$ dependence of the Mellin moments

Real part, NLO $C_n(\mu^2 z^2)$




**Electromagnetic Form Factor**

# Hybrid-scheme

$$\begin{cases} z \leq z_S: \quad \text{Ratio-scheme } \mathcal{M}(z,p,q) = \dfrac{\mathcal{M}_0(z,p,q)}{F_\pi(Q^2)} \\ z \geq z_S: \quad \mathcal{M}(z,p,q) = e^{\delta m(a)\, z - z_S} \dfrac{F^B(z,\vec{p},\vec{q})}{F^B(z_S,0,0)} \Big/ F_\pi(Q^2) \end{cases}$$

Linear

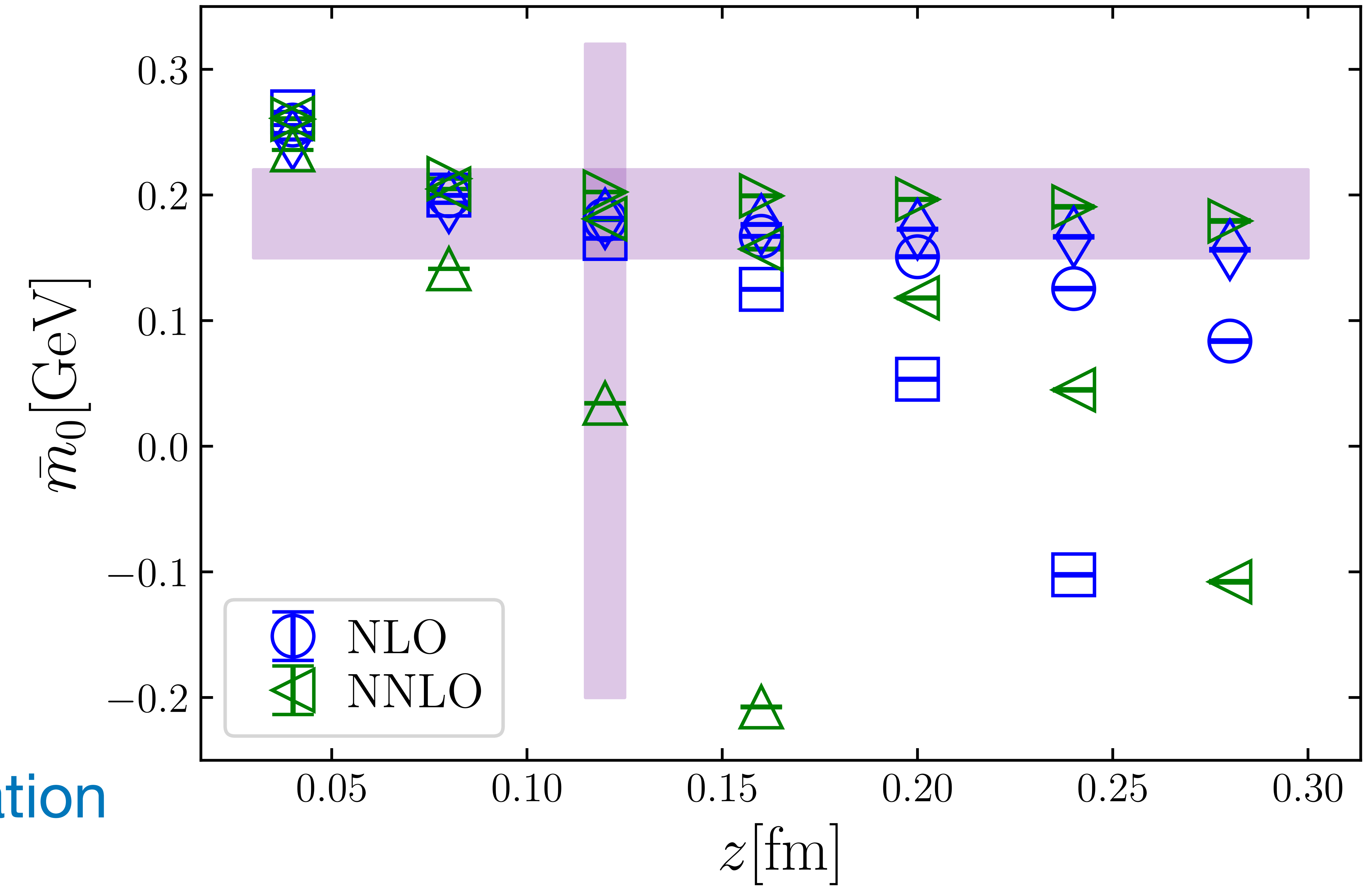
Renormalon ambiguity

$$\delta m = \frac{m_{-1}}{a} + \bar{m}_0$$

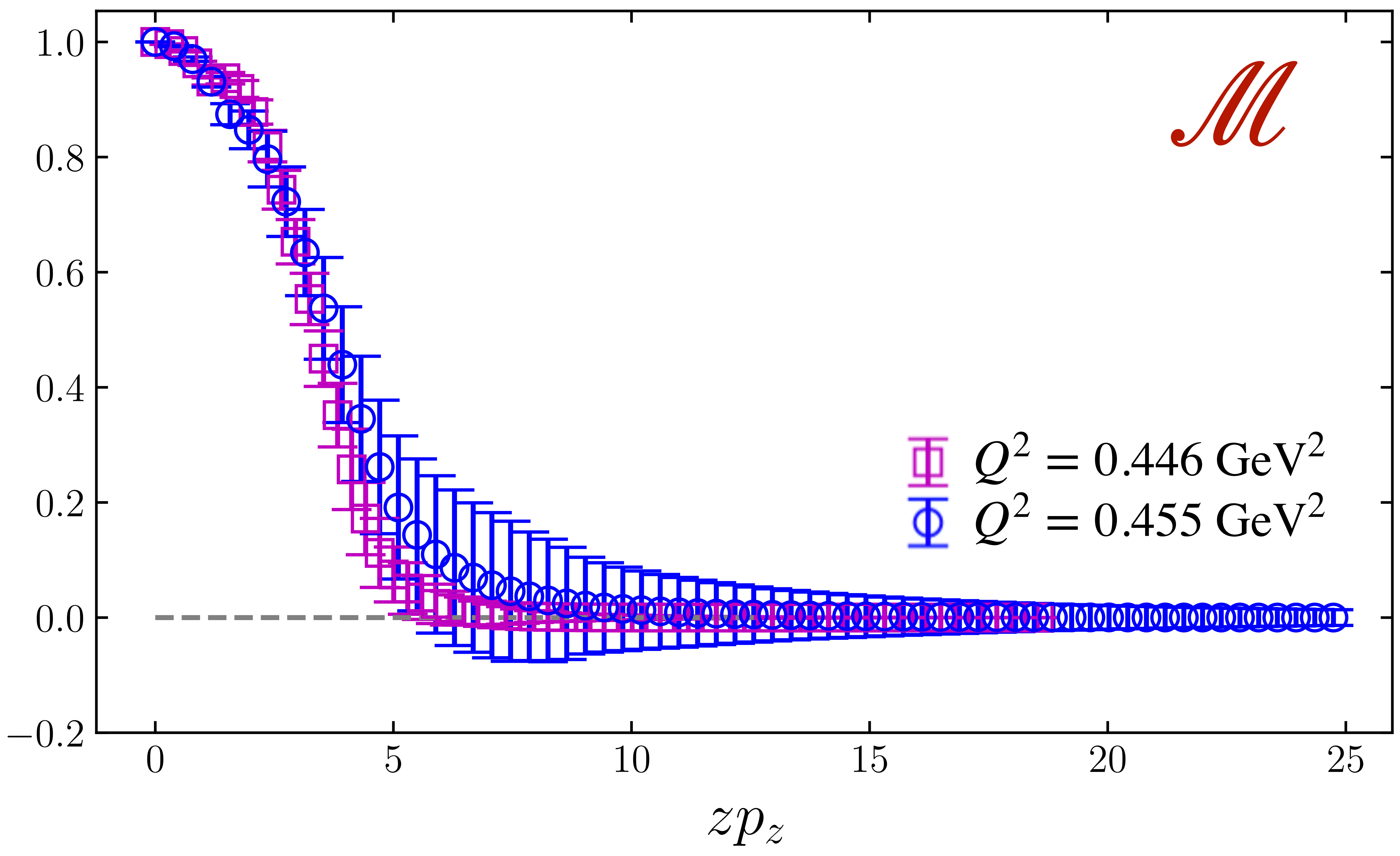
$F^B$ at $p_z = 0, \vec{q} = \vec{0}$ -> extract $\bar{m}_0$

$$e^{(\frac{m_{-1}}{a}+\bar{m}_0)\Delta z}\frac{F^B(z+\Delta z)}{F^B(z)} = \frac{C_0^{\mathrm{LRR}}(\mu^2(z+\Delta z)^2)}{C_0^{\mathrm{LRR}}(\mu^2 z^2)}$$

LRR: leading renormalon resummation



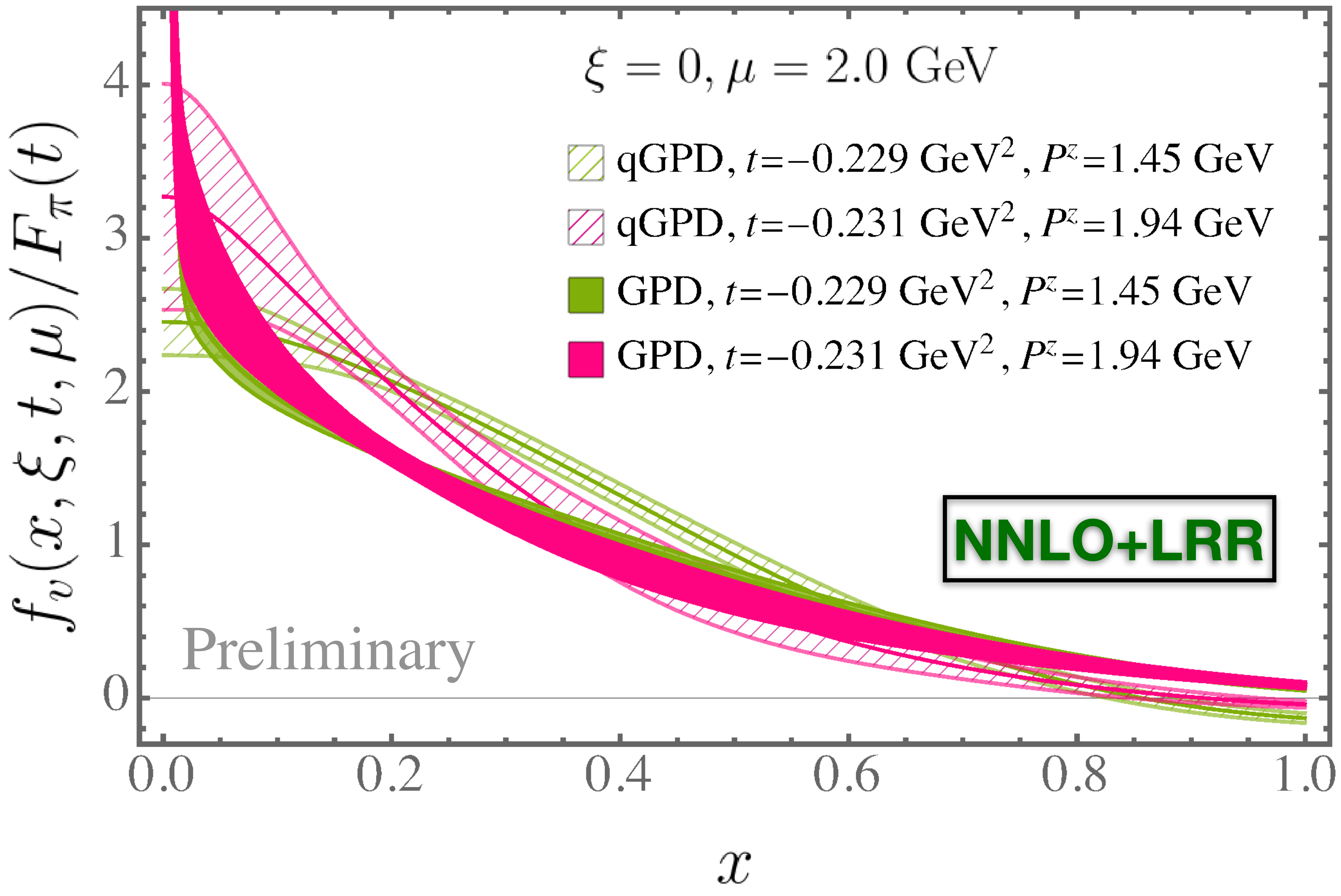
# Fourier Transform & Perturbative Matching




$$\tilde{f}(x, z_S, p_z) = \int \frac{d\lambda}{2\pi} e^{ix\lambda} \mathscr{M}(\lambda, z_S, p_z)$$ —> coordinate space, quasi-GPD

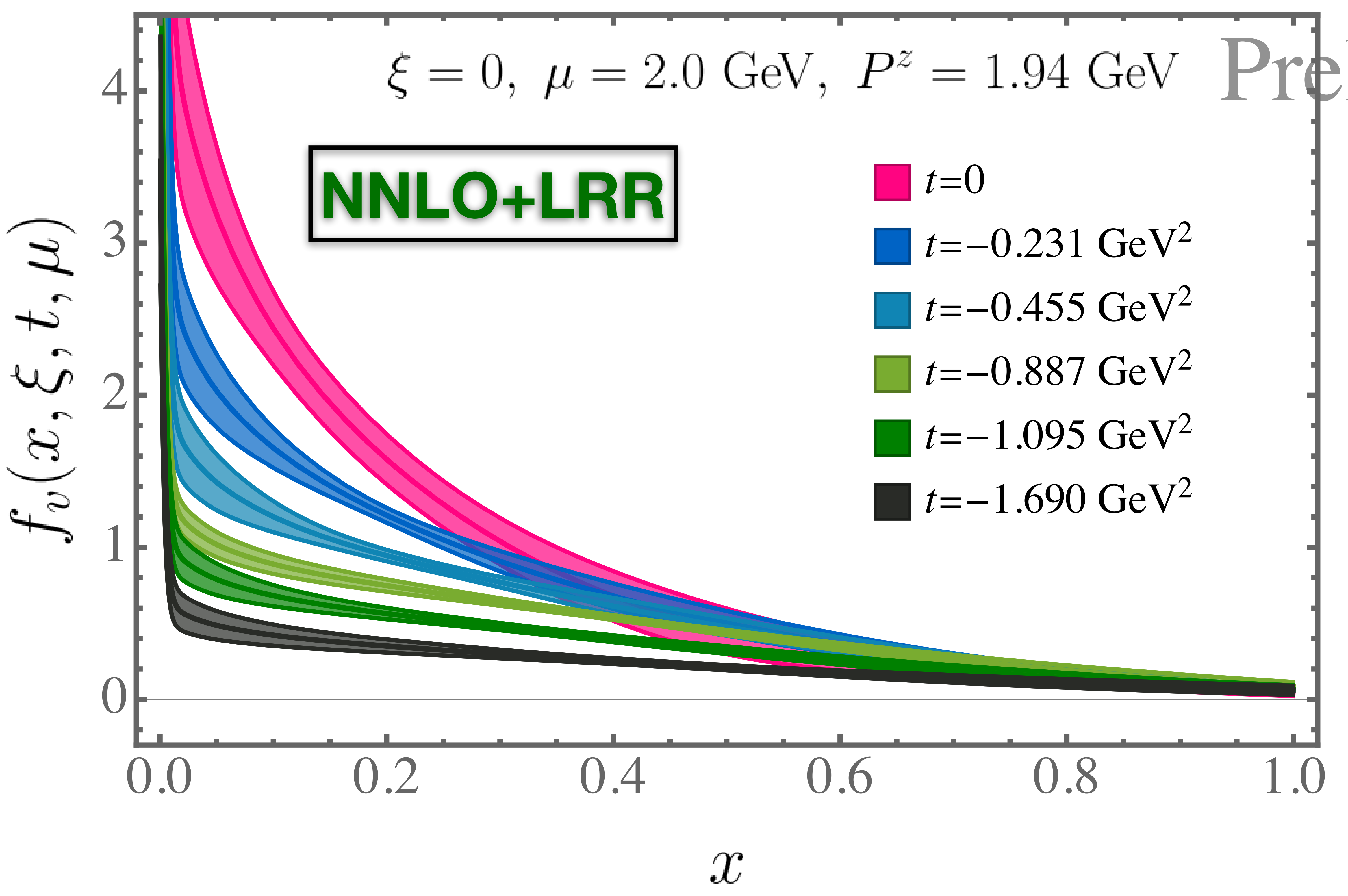
$$\frac{f^v(x, \xi = 0, Q^2, \mu)}{F_\pi(Q^2)} = \int_{-\infty}^{\infty} \frac{dy}{y} C_{\mathrm{LRR}}^{-1}\left(\frac{x}{y}, \frac{\mu}{yp_z}, \ y \ \lambda_S\right) \tilde{f}^v(y, \xi = 0, Q^2, \lambda_S, p_z, \mu) + \mathcal{O}\left(\frac{\Lambda_{\mathrm{QCD}}^2}{(xp_z)^2}, \frac{\Lambda_{\mathrm{QCD}}^2}{((1-x)p_z)^2}\right)$$

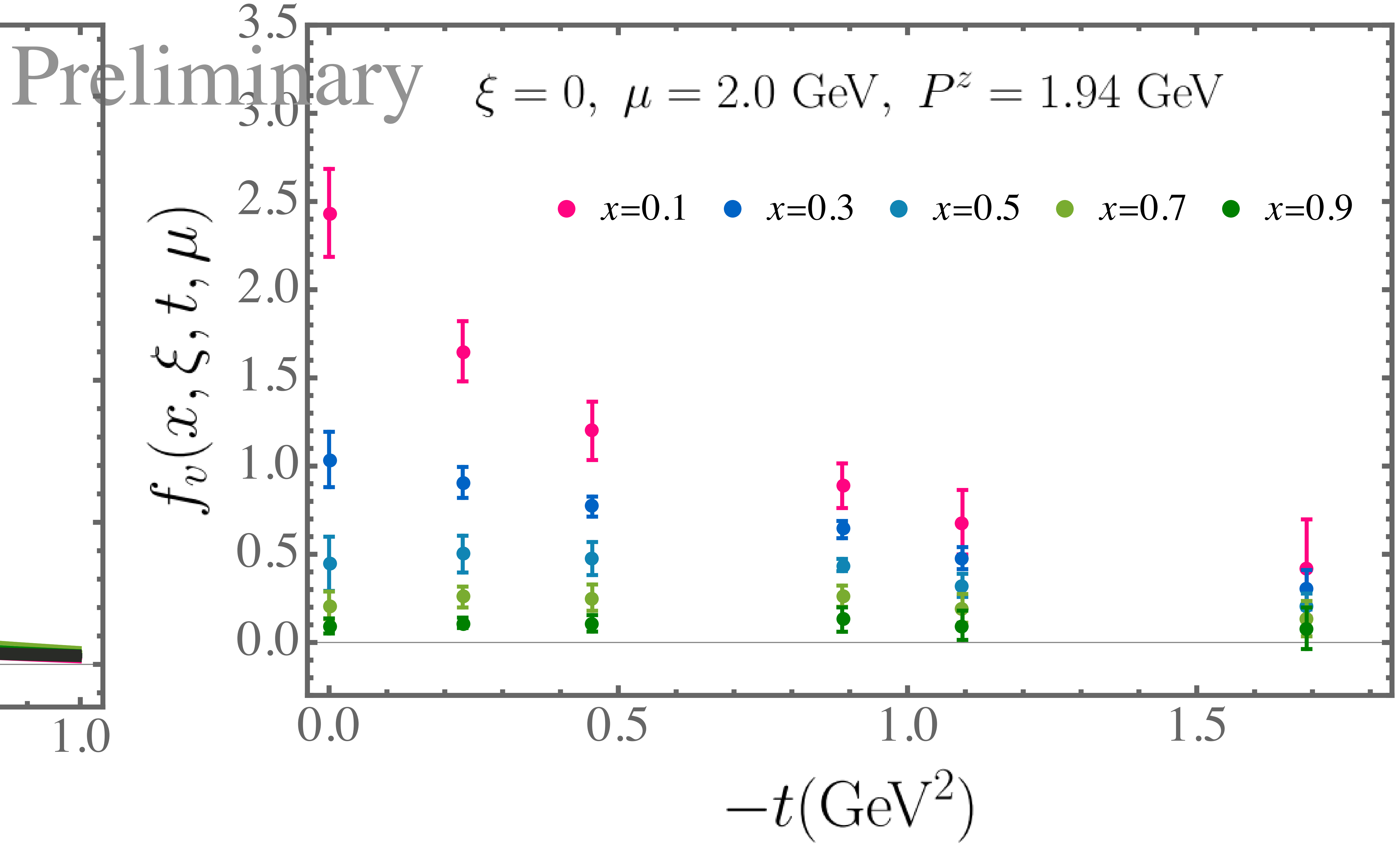
# Valance GPD: $p_z$-dependence




- Convergence in $p_z$ → Effectiveness of LaMET

# Valance GPD: $Q^2$-dependence




- Monotonically decreasing -> flat

# Summary & Outlook

Confirm the frame independence of the amplitudes $A_i$'s

- Get the $Q^2$-dependence of the first two Mellin moments with NLO

  -> More moments

  -> More reliable results, up to NNLO

  -> Imaginary part: **G**ravitational **F**orm **F**actor

- Get the pion valance GPD at $\mu = 2$ GeV up to NNLO+LRR

  -> DGLAP evolution, scale variation

Thanks for your attention!

# Backup

# Amplitude

Non-breit

$$F^t_{\mathrm{M}}(z,p,\Delta) = \frac{1}{\sqrt{E^i E^f}}\left(p^t_{\mathrm{M}} A_1 + \Delta^t_{\mathrm{M}} A_3\right),$$

$$F^x_{\mathrm{M}}(z,p,\Delta) = \frac{1}{\sqrt{E^i E^f}}\left(p^x_{\mathrm{M}} A_1 + \Delta^x_{\mathrm{M}} A_3\right),$$

$$F^y_{\mathrm{M}}(z,p,\Delta) = \frac{1}{\sqrt{E^i E^f}}\left(p^y_{\mathrm{M}} A_1 + \Delta^y_{\mathrm{M}} A_3\right),$$

$$F^z_{\mathrm{M}}(z,p,\Delta) = \frac{1}{\sqrt{E^i E^f}}\left(p^z_{\mathrm{M}} A_1 + m^2 z^z_{\mathrm{M}} A_2 + \Delta^z_{\mathrm{M}} A_3\right).$$

Breit

$$F^t_{\mathrm{M}}(z,p,\Delta) = \frac{E^i + E^f}{2\sqrt{E^i E^f}} A_1 = A_1,$$

$$F^x_{\mathrm{M}}(z,p,\Delta) = \frac{p^f_x - p^i_x}{\sqrt{E^i E^f}} A_3,$$

$$F^y_{\mathrm{M}}(z,p,\Delta) = \frac{p^f_y - p^i_y}{\sqrt{E^i E^f}} A_3,$$

$$F^z_{\mathrm{M}}(z,p,\Delta) = \frac{p^i_z + p^f_z}{2\sqrt{E^i E^f}} A_1 + \frac{m^2 z_z}{\sqrt{E^i E^f}} A_2,$$

# Lattice Setup

- $N_s^3 \times N_t = 64^3 \times 64, \qquad a = 0.04$ fm
- HISQ action + Wilson-Clover action ⇒ $m_\pi^{\rm val} = 0.3$ GeV
- Using boost smearing to enhance the signal
- Momentum transfer $Q^2$:  0 ~ 1.7 GeV$^2$

| Frame | $n_p^z$, $p^z$(GeV) | $\Gamma$ | $n_q$ | $N_{\rm cfg} * N_{\rm src}$ |
|---|---|---|---|---|
| Breit | (0, 1, 2), 1.083 | $\gamma_x, \gamma_y, \gamma_t$ | (0, -2, 0) | 115*32=3680 |
| Non-breit | (0, 0, 0), 0 | $\gamma_x, \gamma_y, \gamma_t$ | (0, 0, 0), (0, -1, 0), (-1, -1, 0), (0, -2, 0), (-1, -2, 0), (-2, -2, 0) | 314*96=30144 |
| | (0, 0, 1), 0.484 | | | |
| | (0, 0, 2), 0.968 | | | |
| | (0, 0, 3), 1.453 | | | |
| | (0, 0, 4), 1.937 | | | 564*96=54144 |

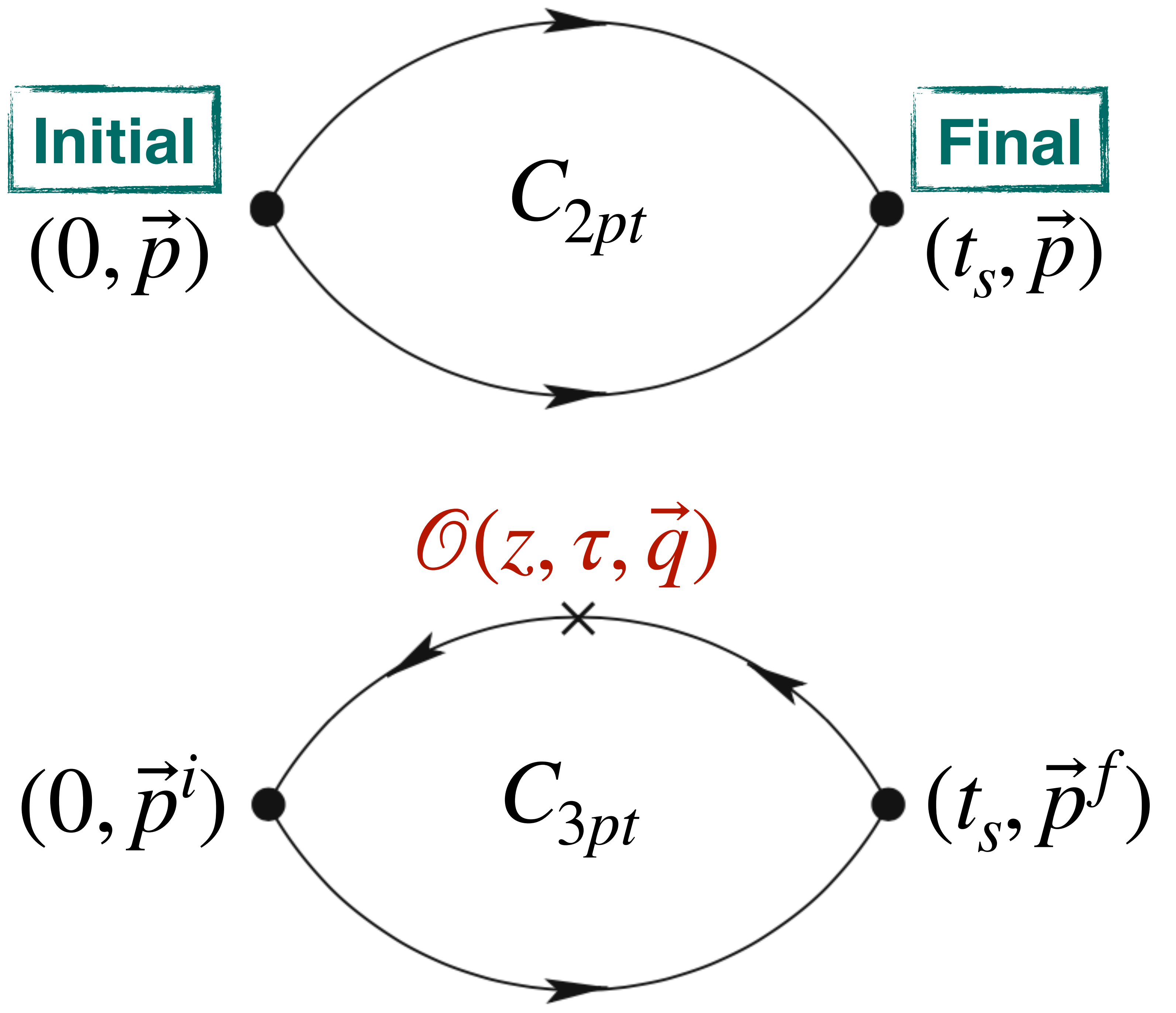
# How to the matrix elements from lattice

$$C_{2pt}(t,\vec{p}) = \left\langle H(t_s,\vec{p})H^\dagger(0,\vec{p})\right\rangle$$

Insert $\mathcal{O}(\tau,\vec{q})$ ↓ $\vec{p}^f = \vec{p}^i + \vec{q}$

$$C_{3pt}(z;\tau,t_s;\vec{p}^i,\vec{p}^f) = \left\langle H(t_s,\vec{p}^f)\hat{\mathcal{O}}_\Gamma(z,\tau,\vec{q})H^\dagger(0,\vec{p}^i)\right\rangle$$

$\Gamma : \hat{1}, \gamma^\mu, \sigma^{\mu\nu}$

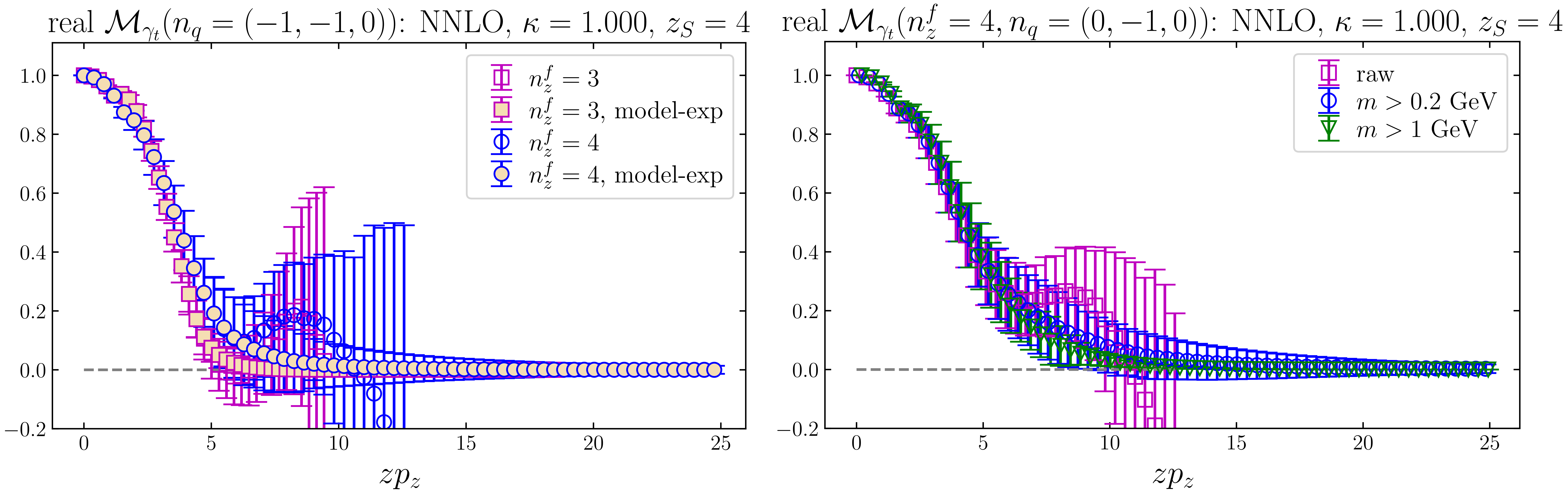



→ $F^B = \left\langle E_0,\vec{p}^f \;\; \hat{\mathcal{O}}_{\gamma^\mu}(z,\tau,\vec{q}) \;\; E_0,\vec{p}^i\right\rangle$ —Renormalization→ $F(Q^2),\ Q^2 = -t: 0 \sim 1.7\ \mathrm{GeV}^2$

from ~ $C_{3pt}$ / $C_{2pt}$

# Extrapolation

Exponential decay model: $M^R = A\frac{e^{-mz}}{\lambda^d}$



# Quasi & Matched GPD