Origin of Spin in Non-relativistic Quantum Physics

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Rotating charged sphere

- Component of electron spin along z axis $S = \hbar/2 = 5.3 \cdot 10^{-35} \mathrm{m}^2 \cdot \mathrm{kg} \cdot \mathrm{s}^{-1}$
- But $S = I\omega$, I=Moment of inertia, ω is angular speed.
- $I = \frac{2}{5} m_e r_{cl}^2$, where m_e is electron mass and r_{cl} is classical electron radius (an upper bound of the size of the electron$\sim 2.8 \cdot 10^{-15}$ m).
- Tangential speed at the equator:

$$v = \omega r_{cl} = \frac{S}{I} r_{cl} = \frac{5h}{4 m_e r_{cl}} \sim 170c$$

- Smaller value of r_{cl} imples a greater tangential velocity.

Quantum spin can't be modeled as a charged sphere spinning around its axis.

Acceptable model of Spin

- The Hamiltonian must commute with total angular momentum $\vec{J} = \vec{L} + \vec{S}$.
- The probability density must be positive definite.

- Dirac equation: 1st order time/space derivatives [Lorentz covariant]

$$\hat{H}_D \psi = (\vec{\alpha} \cdot \vec{p}) \psi = i \partial \psi / \partial t$$

- Wavefunction has new degrees of freedom (4-component spinors).
- $[\hat{H}_D, \vec{L} + \vec{S}] = 0$ and the probability density $\psi^\dagger \psi > 0$.

- Origin of spin in non-relativistic domain (say just a 2-component spinor)?
- Preferably some palpable connection with angular momentum or rotation as in classical mechanics.

Alternate view of angular momentum

Geometrical Optics

- $\nabla \times \vec{k} = \mathbf{0} \implies \oint \vec{k} \cdot d\vec{l} = 0$ and $\vec{k} = \nabla \psi$.
- $\delta \int n\ ds = 0$, $|\vec{k}| = \omega n / c$

- Phase integral (action variable) for the periodic motion: $J = \oint p_i dq_i$, where the integral is over a full cycle of rotation or vibration.
- Momenta $p_i = \frac{\partial \mathcal{S}}{\partial q_i}$, where $\mathcal{S}$ is classical action.

Consider the action variable for angular momentum and angle
$\mathcal{S} = \int L_i d\theta_i \implies L_i = \frac{\partial \mathcal{S}}{\partial \theta_i}$

- Transition to quantum mechanics as phase integral takes only discrete values: $J = \oint p_i dq_i = nh$, where h is Planck's constant. As $h \to 0$, $\oint p_i dq_i = 0$.

Angular momentum as a field

- With the definition $\mathcal{L}_i = \frac{\partial \mathcal{S}}{\partial \theta_i}$, one can easily verify:

$$\nabla_{\vec{\theta}} \times \vec{\mathcal{L}} = \nabla_{\vec{\theta}} \times \nabla_{\vec{\theta}} \mathcal{S} = \nabla_{\vec{\theta}} \times \left(\frac{\partial \mathcal{S}}{\partial \vec{\theta}} \right) = \begin{vmatrix} \hat{\theta}_x & \hat{\theta}_y & \hat{\theta}_z \\ \frac{\partial}{\partial \theta_x} & \frac{\partial}{\partial \theta_y} & \frac{\partial}{\partial \theta_z} \\ \frac{\partial \mathcal{S}}{\partial \theta_x} & \frac{\partial \mathcal{S}}{\partial \theta_y} & \frac{\partial \mathcal{S}}{\partial \theta_z} \end{vmatrix} = \mathbf{0},$$

- $\vec{\theta} = (\theta_x, \theta_y, \theta_z)$ and their conjugate momenta, i.e. $\vec{\mathcal{L}} = (\mathcal{L}_x, \mathcal{L}_y, \mathcal{L}_z)$ constitute a phase space.
- Transition to quantum theory: Action variable $\mathcal{S}^{(x)} = \oint \mathcal{L}_x d\theta_x = nh$, etc.

Eigenvalue equation

$$-i\hbar\frac{\partial \psi_L}{\partial \vec{\theta}} = \vec{\mathcal{L}}\psi_L$$

$$\implies \left(-i\hbar\frac{\partial}{\partial \vec{\theta}}\right)\cdot\left[-i\hbar\frac{\partial \psi_L}{\partial \vec{\theta}}\right] = \left(-i\hbar\frac{\partial}{\partial \vec{\theta}}\right)\cdot\vec{\mathcal{L}}\psi_L$$

$$-\hbar^2\frac{\partial^2 \psi_L}{\partial \vec{\theta}^2} = \left(-i\hbar\frac{\partial}{\partial \vec{\theta}}\cdot\vec{\mathcal{L}}\right)\psi_L + \vec{\mathcal{L}}\cdot\left(-i\hbar\frac{\partial \psi_L}{\partial \vec{\theta}}\right)$$

$$\boxed{\hbar^2\nabla^2_{\vec{\theta}}\psi_L + \mathcal{L}^2\psi_L = i\hbar\left(\nabla_{\vec{\theta}}\cdot\vec{\mathcal{L}}\right)\psi_L = i\hbar(\nabla^2_{\vec{\theta}}S)\psi_L} \tag{1}$$

RHS non-zero: source charge of angular momentum field?

Quantum Theory of Light Rays[1]

Transition to the quantum description of light by treating momenta by appropriate linear differential operators: $p_x \to \hat{p}_x \equiv -i\frac{\lambda}{2\pi}\frac{\partial}{\partial x}$ etc.

$$\hat{p}_x\psi = n_x\psi \quad \Longrightarrow \quad \hat{p}_x^2\psi = n_x^2\psi \quad \Longrightarrow \quad \bar{\lambda}^2\nabla^2\psi + n^2\psi = 0$$

Quantum effect manifests in the limit $\bar{\lambda} \not\to 0$
Geometrical optics emerges in the limit when $\bar{\lambda} \to 0$.

[1] Gloge & Marcuse, Journal of Optical Society of America, 1969

Semi-classical description of electrostatic fields

Wavefunctions of electrostatic fields ψ satisfy:

$$\bar{\gamma}^2\nabla^2\psi + \mathcal{E}^2\psi = 0 \qquad \text{in } \underline{\text{absence}} \text{ of source charge.}^2 \tag{2a}$$

$$\bar{\gamma}^2\nabla^2\psi + \mathcal{E}^2\psi = i\frac{\rho\bar{\gamma}}{\epsilon_0}\psi \qquad \text{in } \underline{\text{presence}} \text{ of source charge.}^3 \tag{2b}$$

Basis solutions of this equation are given by $e^{-i\Phi(\mathbf{r})/\bar{\gamma}}$. For a non-zero electric field, the solution is given by: $\Psi := \int c(\vec{E})\mathbf{e}^{-i\Phi/\gamma}d\vec{E}$

- This model was used to argue that:
  - Aharonov-Bohm effect is actually a local phenomenon
  - electric charge quantization is possible without magnetic monopoles.

[2] Kolahal Bhattacharya, Physica Scripta, vol. 96(8), 2021
[3] Kolahal Bhattacharya, Physica Scripta, vol. 98(8), 2023

Klein-Gordon type anomaly

- Note similarity of Eq.(1) with KG equation.
- Set $W^2 = \mathcal{L}^2 - i\hbar\nabla^2_{\vec{\theta}} S$
- Treat θ_z as a pseudotime variable in Eq.(1):

$$\hbar^2 \frac{\partial^2 \psi}{\partial \theta_z^2} = \left(-W^2 - \hbar^2 \frac{\partial^2}{\partial \theta_x^2} - \hbar^2 \frac{\partial^2}{\partial \theta_y^2} \right) \psi$$

- Thus $\psi(\theta_x, \theta_y, 0)$ and $\partial\psi/\partial\theta_z(\theta_x, \theta_y, 0)|_{\theta_z=0}$ should be independently specified. This arbitrary specification may lead to -ve probability density.[4]

[4] G Eichmann, Journal of the Optical Society of America, Vol. 61(2),1971

Following Eichmann's track...

- First, set

$$H = -i\hbar\frac{\partial}{\partial\theta_z} \implies H^2 = -\hbar^2\frac{\partial^2}{\partial\theta_z^2}$$

- Consider:

$$-H = i\hbar\frac{\partial}{\partial\theta_z} = \sqrt{W^2 + (ip_{\theta x})^2 + (ip_{\theta y})^2}$$

where $p_{\theta x} = -i\hbar\frac{\partial}{\partial\theta_x}$, and: $(ip_{\theta x})^2 = \hbar^2\frac{\partial^2}{\partial\theta_x^2}$, etc.

- Assume $-H = \sum_{i=1}^{3}\alpha_i\tau_i$. This leads to:

$$W^2 - p_{\theta x}^2 - p_{\theta y}^2 = \sum_{i,j}\alpha_i\alpha_j\tau_i\tau_j$$

$$\alpha_1 = ip_{\theta x} \qquad \alpha_2 = ip_{\theta y} \qquad \alpha_3 = W$$

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \tau_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \qquad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Only for constant W

- Note:

$$\begin{aligned} H^2 &= i^2\alpha_1^2 + i^2\alpha_2^2 + \alpha_3^2 + 2(\alpha_1\alpha_2\tau_1\tau_2 + \alpha_2\alpha_1\tau_2\tau_1) \\ &+ 2(\alpha_2\alpha_3\tau_2\tau_3 + \alpha_3\alpha_2\tau_3\tau_2) + 2(\alpha_3\alpha_1\tau_3\tau_1 + \alpha_1\alpha_3\tau_1\tau_3) \\ &= W^2 - p_{\theta x}^2 - p_{\theta y}^2 + 2\cdot 0 - 2i(\tau_2\tau_3 p_{\theta y} W + \tau_3\tau_2 W p_{\theta y}) \\ &- 2i(\tau_1\tau_3 p_{\theta x} W + \tau_3\tau_1 W p_{\theta x}) \\ &= W^2 - p_{\theta x}^2 - p_{\theta y}^2 \qquad \text{Only for constant W} \end{aligned}$$

- The resulting wave equation becomes

$$i\hbar(\partial/\partial z)\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}\begin{pmatrix}u_1\\ u_2\end{pmatrix} = \left[\hbar(\partial/\partial x)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} + \hbar(\partial/\partial y)\begin{pmatrix}0 & i\\ -i & 0\end{pmatrix} + W\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}\right]\begin{pmatrix}u_1\\ u_2\end{pmatrix}$$

- Eichmann shows $u^\dagger u > 0$.
- Commutation of angular momentum & spin with Hamiltonian yet to be done.

Let there be light

Fermat's principle for such media still conceives light rays as lines in space (i.e., no polarisation vectors yet), but the refractive index along the paths of the rays in the medium is allowed to depend on both position and direction[4]

- For wavefront surface denoted by S, $\nabla S = \mathcal{N}\frac{d\mathbf{r}}{ds}$.
- Scalar eikonal equation:

$$|\nabla S|^2 = \left(\frac{d\mathbf{r}}{ds}\right)^T (\mathcal{N}^T\mathcal{N}) \left(\frac{d\mathbf{r}}{ds}\right) = \left(\frac{d\mathbf{r}}{ds}\right)^T \mathcal{G} \left(\frac{d\mathbf{r}}{ds}\right) = n^2\left(\mathbf{r}, \frac{d\mathbf{r}}{ds}\right)$$

- Set $d\sigma = n\left(\mathbf{r}, \frac{d\mathbf{r}}{ds}\right) ds$. Then one can show that the geodesic paths can be found from the Variational principle:

$$\delta A = \delta \int \frac{1}{2}\mathbf{r}'(\sigma)\mathcal{G}\mathbf{r}'(\sigma)d\sigma = 0$$

[4] Geometric Mechanics Part I: Dynamics and Symmetry, Darryl D Holm

Analogy with angular momentum

- Angular momentum:

$$\boxed{\vec{\mathcal{L}} = \frac{1}{2}\nabla_{\vec{\omega}} F}$$

where F represents energy (Poinsot) ellipsoid[9]:

$$2T = I_{xx}\omega_x^2 + I_{yy}\omega_y^2 + I_{zz}\omega_z^2 + 2I_{xy}\omega_x\omega_y + 2I_{yz}\omega_y\omega_z + 2I_{zx}\omega_z\omega_z$$

- Energy expression:

$$T = \frac{1}{2}\vec{\omega}\cdot\vec{\mathcal{L}} = \frac{1}{2}\vec{\omega}\cdot\mathcal{I}\cdot\vec{\omega}$$

Unmistakable similarity with variational principle in anisotropic medium

[9] Classical Mechanics, by W Greiner (Springer), pp.217-219

Summary

- Understanding of quantum spin from a classical mechanics point of view.
- For constant angular momentum, we find a wave equation for states with two components.
- It is possible that 4-component spinors will emerge as a consequence of anisotropy in the system.