Observables for scattering on targets with any spin

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Motivation

Advantages

- Systematic approach for any spin j
- "Basic" algebraic construction su(2) $\rightarrow$ su(N) $\rightarrow$ $sl(2, C)$
- Covariant "multipole" basis physical interpretation
- Parity conserving interactions $\rightarrow$ generalized Dirac algebra
- Any form of dynamics (instant form, light front)
- Exact degrees of freedom (chiral reps), no need for constraints

Introduction: Review of Weinberg's formalism

Weinberg's "Feynman rules for Any Spin" (1964) [Wigner(1939)]

- Algebra for Generators of the Lorentz group

$$[\mathbb{J}_l, \mathbb{J}_m] = i\epsilon_{lmn}\mathbb{J}_n \ , \quad [\mathbb{J}_l, \mathbb{K}_m] = i\epsilon_{lmn}\mathbb{K}_n \ , \quad [\mathbb{K}_l, \mathbb{K}_m] = -i\epsilon_{lmn}\mathbb{J}_n$$

- Two independent su(2) subalgebras $\rightarrow$ irreps (j_A, j_B)

$$\mathbb{A}_m = \frac{1}{2}(\mathbb{J}_m + i\mathbb{K}_m) \quad , \quad \mathbb{B}_m = \frac{1}{2}(\mathbb{J}_m - i\mathbb{K}_m)$$

$$[\mathbb{A}_l, \mathbb{A}_m] = i\epsilon_{lmn}\mathbb{A}_n \ , \quad [\mathbb{B}_l, \mathbb{B}_m] = i\epsilon_{lmn}\mathbb{B}_n \ , \quad [\mathbb{A}_l, \mathbb{B}_m] = 0$$

- Simplest irreps that contain spin-j $\rightarrow$ $(2j+1$ components$)$
  - Right-handed $(j, 0)$: $\mathbb{K}_m \rightarrow -i\mathbb{J}_m$
  - Left-handed $(0, j)$: $\mathbb{K}_m \rightarrow +i\mathbb{J}_m$

Causal chiral fields (massive, left- right-handed)

- Lorentz invariant S-matrix using a Hamiltonian density built up from causal fields

$$U_{[\Lambda,a]}\psi_{\sigma}(x)U^{-1}_{[\Lambda,a]} = \sum_{\sigma'}\left(D^{(j)}_{[\Lambda^{-1}]}\right)_{\sigma\sigma'}\psi_{\sigma'}(\Lambda x+a)$$

- No EoM for chiral fields (only obey KG eq.)

- Spinors appearing in the fields (not invariants, depend on choice boost)

$$\text{Canonical} \quad \rightarrow \quad \begin{aligned} D^{(j)}_{[L(p)]} &= e^{-\hat{p}\cdot\vec{J}^{(j)}\theta} \\ \bar{D}^{(j)}_{[L(p)]} &= e^{+\hat{p}\cdot\vec{J}^{(j)}\theta} \end{aligned}$$

Propagator of chiral fields

- Numerator (invariant)

$$\Pi^{(j)}_{\sigma\sigma'}(\vec{p},\omega) = m^{2j} D^{(j)}_{\sigma\sigma'}[L(\vec{p})] \left(D^{(j)}_{\sigma'\sigma''}[L(\vec{p})]\right)^{\dagger} = m^{2j}\left(e^{-2\hat{p}\cdot\vec{J}^{(j)}\theta}\right)_{\sigma\sigma'}$$

$$\bar{\Pi}^{(j)}_{\sigma\sigma'}(\vec{p},\omega) = m^{2j} \bar{D}^{(j)}_{\sigma\sigma'}[L(\vec{p})] \left(\bar{D}^{(j)}_{\sigma'\sigma''}[L(\vec{p})]\right)^{\dagger} = m^{2j}\left(e^{2\hat{p}\cdot\vec{J}^{(j)}\theta}\right)_{\sigma\sigma'}$$

- Introduction of 2j-rank t-tensors

$$\Pi^{(j)}_{\sigma\sigma'}(\vec{p},\omega) = t^{\mu_1\mu_2\ldots\mu_{2j}}_{\sigma\sigma'} p_{\mu_1}p_{\mu_2}\ldots p_{\mu_{2j}}$$

totally symmetric

$$\bar{\Pi}^{(j)}_{\sigma\sigma'}(\vec{p},\omega) = \bar{t}^{\mu_1\mu_2\ldots\mu_{2j}}_{\sigma\sigma'} p_{\mu_1}p_{\mu_2}\ldots p_{\mu_{2j}}$$

covariantly traceless

$$g_{\mu_k\mu_l} t^{\mu_1\ldots\mu_k\ldots\mu_l\ldots\mu_{2j}}_{\sigma\sigma'} = 0$$

- Central role of t-tensors

$$D^{(j)}_{[L(p)]} = t^{\mu_1\mu_2\ldots\mu_{2j}} \tilde{p}_{\mu_1}\tilde{p}_{\mu_2}\ldots\tilde{p}_{\mu_{2j}}$$

used to construct boosts/spinors

$$\bar{D}^{(j)}_{[L(p)]} = \bar{t}^{\mu_1\mu_2\ldots\mu_{2j}} \tilde{p}_{\mu_1}\tilde{p}_{\mu_2}\ldots\tilde{p}_{\mu_{2j}}$$

($\tilde{p}^{\mu}$ not four-vectors) Canonical:

$$\tilde{p}^{\mu}{}_{\mathrm{C}} = \sqrt{\frac{1}{2m(m+p^0)}}\,(p^0+m,\vec{p}\,)$$

Bi-Spinors (direct sum representation $(j,0)\bigoplus(0,j)$)

- For Parity conserving interactions the direct sum of both chiral representations is used, like the spin 1/2 case
- Boosts and bispinor (Weyl rep.)

$$u^{(j)}_{(p,s)} = \mathcal{D}^{(j)}_{[L_p]} \overset{\circ}{u}{}^{(j)}_s = \begin{pmatrix} D^{(j)}_{[L_p]} & 0 \\ 0 & \bar{D}^{(j)}_{[Lp]} \end{pmatrix} \overset{\circ}{u}{}^{(j)}_s = \begin{pmatrix} \Pi^{(j)}_{(\tilde{p})} & 0 \\ 0 & \bar{\Pi}^{(j)}_{(\tilde{p})} \end{pmatrix} \overset{\circ}{u}{}^{(j)}_s$$

$$\overset{\circ}{u}{}^{(j)}_s = \begin{pmatrix} \overset{\circ}{\phi}{}^{(j)}_s \\ \overset{\circ}{\phi}{}^{(j)}_s \end{pmatrix} \quad , \quad \overset{\circ}{\phi}{}^{(j)}_s = m^j \begin{pmatrix} \vdots \\ 1 \\ \vdots \end{pmatrix} \qquad \text{(1 in the s-th position)}$$

- Adjoint bispinor $(\mathcal{D}^{\dagger}_{[\Lambda]} = \beta \mathcal{D}^{-1}_{[\Lambda]} \beta)$

$$\bar{u}^{(j)}_{(p,s)} = u^{(j)}_{(p,s)}{}^{\dagger} \beta = \overset{\circ}{u}{}^{(j)\dagger}_s \mathcal{D}^{(j)}_{[L_p]}{}^{\dagger} \beta = \overset{\circ}{u}{}^{(j)\dagger}_s \begin{pmatrix} 0 & \Pi^{(j)}_{(\tilde{p})} \\ \bar{\Pi}^{(j)}_{(\tilde{p})} & 0 \end{pmatrix} \quad ; \quad \beta = \begin{pmatrix} 0 & \mathbf{1}^{(j)} \\ \mathbf{1}^{(j)} & 0 \end{pmatrix}$$

Dirac Eq. & Gamma matrices

- The bispinor satisfy the Dirac eq.

$$\left(\gamma^{\mu_1\cdots\mu_{2j}}p_{\mu_1}\cdots p_{\mu_{2j}} - m^{2j}\right)u^{(j)}_{(p,s)} = 0$$

$$\bar{u}^{(j)}_{(p,s)}\left(\gamma^{\mu_1\cdots\mu_{2j}}p_{\mu_1}\cdots p_{\mu_{2j}} - m^{2j}\right) = 0$$

- The gamma matrices

$$\gamma^{\mu_1\cdots\mu_{2j}} = \begin{pmatrix} 0 & t^{\mu_1\cdots\mu_{2j}} \\ \bar{t}^{\mu_1\cdots\mu_{2j}} & 0 \end{pmatrix} \quad ; \quad \beta = \gamma^{0\cdots 0} = \begin{pmatrix} 0 & \mathbf{1}^{(j)} \\ \mathbf{1}^{(j)} & 0 \end{pmatrix} \quad ; \quad \gamma_5 = \begin{pmatrix} -\mathbf{1}^{(j)} & 0 \\ 0 & \mathbf{1}^{(j)} \end{pmatrix}$$

Algorithm for construction of t-tensors

Insightful construction for the t-tensors

- The 0-th degree polynomial in the J's is always $\quad t^{0\ldots0} = \mathbf{1}$

- The linear polynomials are the Rotation Group Generators $\quad t^{0\ldots i\ldots0} = \frac{2}{2j} J_i = \frac{1}{j} J_i$

- From pairwise symmetrizations of the rotation generators

$$t^{0\ldots m\ldots0\ldots n\ldots0} = t^{mn0\ldots0} = \frac{1}{\frac{(2j)!}{2!(2j-2)!}} \left(\{J_m, J_n\} - \frac{1}{3}\delta_{mn} \sum_{r=1}^{3} \{J_r, J_r\} \right) + \frac{1}{3} t^{0\ldots0} \delta_{mn}$$

$$= \frac{j}{(2j-1)} \left(\left\{ t^{m0\ldots0}, t^{n0\ldots0} \right\} - \frac{1}{j} \delta_{mn} t^{0\ldots0} \right)$$

Algorithm for construction of t-tensors

- Continues for higher orders
  - Matrices have more and more off-diagonal elements

$$t^{lmn0\ldots0} = t^{0\ldots0l0\ldots0m0\ldots0n0\ldots0} = \frac{j}{(2j-2)}\frac{1}{3}\Big(\{t^{l0\ldots0}, t^{mn0\ldots0}\} + \{t^{m0\ldots0}, t^{nl0\ldots0}\} + \{t^{n0\ldots0}, t^{lm0\ldots0}\}$$
$$-\frac{2}{j}\{\delta_{lm}t^{n0\ldots0} + \delta_{ln}t^{m0\ldots0} + \delta_{mn}t^{l0\ldots0}\}\Big)$$

- Stops after j steps (eigenvalue eq.) $\quad (J-s)(J-s-1)\ldots(J+s) = 0$

- t-tensors contain a basis for su(N=2j+1)
  (Universal Enveloping Algebra)

t^μ-tensor for Spin 1/2

0-th order terms in $J_i^{(1/2)}$: $t^0 = \mathbf{1}$

Linear terms in $J_i^{(1/2)}$: $t^i = \frac{1}{1/2} J_i^{(1/2)} = \sigma_i$
(Pauli matrices)

$$J_1^{(1/2)} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \ , J_2^{(1/2)} = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \ , J_3^{(1/2)} = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

Quadratic terms in $J_i^{(1/2)}$

$$(J^{(1/2)} - \tfrac{1}{2}\mathbf{1})(J^{(1/2)} + \tfrac{1}{2}\mathbf{1}) = 0 \implies (J^{(1/2)})^2 = c_0\mathbf{1} + c_2 J^{(1/2)}$$

$t^{\mu\nu}$-tensor for Spin 1

0-th order terms in $J_i^{(1)}$: $t^{00} = \mathbf{1}$

Linear terms in $J_i^{(1)}$: $t^{0i} = J_i^{(1)}$

$$t^{01} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} \ , \ t^{02} = \frac{i}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix} \ , \ t^{03} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

Quadratic terms in $J_i^{(1)}$: $t^{ij} = \{J_i^{(1)}, J_j^{(1)}\} - \mathbf{1}\delta_{ij}$

$$t^{11} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \ , \ t^{22} = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ -1 & 0 & 0 \end{pmatrix} \ , \ t^{33} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$t^{12} = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix} \ , \ t^{13} = \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & -1 \\ 0 & -1 & 0 \end{pmatrix} \ , \ t^{23} = \frac{i}{\sqrt{2}}\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}$$

Cubic terms in $J_i^{(1)}$: $(J^{(1)} - 1)(J^{(1)})(J^{(1)} + 1) = 0 \implies (J^{(1)})^3 = c_0\mathbf{1} + c_2 J^{(1)} + c_3 (J^{(1)})^2$

Reduction for Cubic Monomials

- Central role of the covariant t-tensors (spinors, boosts, propagators, gamma matrices)
- Bilinear calculus involve products with alternating "barring" pattern: $t\bar{t}t\cdots$
- Matrices in t-tensors form a basis of $su(2j+1) \to$ Products can be linearized
- Cubic products are reduced with an Invariant Tensor

$$t^{\mu_1\cdots\mu_{2j}}\bar{t}^{\rho_1\cdots\rho_{2j}}t^{\sigma_1\cdots\sigma_{2j}} = \frac{1}{[(2j)!]^2}\underset{\{\rho_1\ldots\rho_{2j}\}}{\mathcal{S}}\underset{\{\sigma_1\ldots\sigma_{2j}\}}{\mathcal{S}}\left(\prod_{l=1}^{2j}\mathcal{C}^{\mu_l\rho_l\sigma_l\alpha_l}\right)t_{\alpha_1\cdots\alpha_{2j}}$$

$$\bar{t}^{\mu_1\cdots\mu_{2j}}t^{\rho_1\cdots\rho_{2j}}\bar{t}^{\sigma_1\cdots\sigma_{2j}} = \frac{1}{[(2j)!]^2}\underset{\{\rho_1\ldots\rho_{2j}\}}{\mathcal{S}}\underset{\{\sigma_1\ldots\sigma_{2j}\}}{\mathcal{S}}\left(\prod_{l=1}^{2j}\bar{\mathcal{C}}^{\mu_l\rho_l\sigma_l\alpha_l}\right)\bar{t}_{\alpha_1\cdots\alpha_{2j}}$$

$$\mathcal{C}^{\mu\rho\alpha\beta} = g^{\mu\rho}g^{\alpha\beta} - g^{\mu\alpha}g^{\rho\beta} + g^{\mu\beta}g^{\rho\alpha} + i\epsilon^{\mu\rho\alpha\beta}$$

$$\bar{\mathcal{C}}^{\mu\rho\alpha\beta} = g^{\mu\rho}g^{\alpha\beta} - g^{\mu\alpha}g^{\rho\beta} + g^{\mu\beta}g^{\rho\alpha} - i\epsilon^{\mu\rho\alpha\beta}$$

(Lorentz Invariants)

- Trade matrix multiplication by number multiplication

Reduction for Quadratic Monomials

- Central role of the covariant t-tensors (spinors, boosts, propagators, gamma matrices)
- Since, $t^{0\cdots 0} = \bar{t}^{0\cdots 0} = 1 \quad \rightarrow \quad t^{\mu_1\cdots\mu_{2j}}\bar{t}^{\nu_1\cdots\nu_{2j}} = t^{\mu_1\cdots\mu_{2j}}\bar{t}^{\nu_1\cdots\nu_{2j}}\left(t^{\rho_1\cdots\rho_{2j}}\eta_{\rho_1}\cdots\eta_{\rho_{2j}}\right)$

$$t^{\mu_1\cdots\mu_{2j}}\bar{t}^{\rho_1\cdots\rho_{2j}} = \frac{1}{(2j)!}\underset{\{\rho_1\ldots\rho_{2j}\}}{\mathcal{S}}\left(\prod_{l=1}^{2j}\mathcal{C}^{\mu_l\rho_l\sigma_l\alpha_l}\eta_{\sigma_l}\right)t_{\alpha_1\cdots\alpha_{2j}}$$

$\eta^\mu = (1,0,0,0)$

$\mathcal{C}^{\mu\rho\sigma\alpha}\eta_\sigma = g^{\mu\rho}\eta^\alpha - g^{\rho\alpha}\eta^\mu + g^{\mu\alpha}\eta^\rho + i\epsilon^{\mu\rho\sigma\alpha}\eta_\sigma \qquad$ (Rotational Invariant)

- General result $\left(\mathcal{Q}_{\text{red}}^{\mu\rho\alpha} \equiv \mathcal{C}^{\mu\rho\sigma\alpha}\eta_\sigma - g^{\mu\rho}\eta^\alpha\right)$

$$t^{\mu_1\cdots\mu_{2j}}\bar{t}^{\rho_1\cdots\rho_{2j}} = \sum_{m=0}^{2j}\frac{1}{(2j)!}\underset{\{\rho_1\ldots\rho_{2j}\}}{\mathcal{S}}\left[\sum_{n=1}^{B_m^{2j}}\left(\prod_{l\in\pi_{m,n}}\mathcal{Q}_{\text{red}}^{\mu_l\rho_l\alpha_l}\ \prod_{k\in\pi_{m,n}^{\complement}}g^{\mu_k\rho_k}\eta^{\alpha_k}\right)\right]t_{\alpha_1\cdots\alpha_{2j}}$$

each $0 \le m \le 2j$ corresponds to a Lorentz independent tensor

Algebra of t-tensors

Covariant sl(2,C) Multipole expansion

[S. Cotogno, C. Lorcé, P. Lowdon, M. Morales (2020)]

- $m = 0$ → Identity $$\prod_{r=1}^{0} \mathcal{Q}_{\text{red}}^{\mu_r\rho_r\alpha_r} \left(\prod_{s=1}^{2j} \eta^{\alpha_s} \right) t_{\alpha_1\cdots\alpha_{2j}} = t_{0\cdots0} = \mathbf{1}$$

- $m = 1$ → Gen of Lorentz transf $\left(i\,[\mathbb{M}^{\mu\rho}, \mathbb{M}^{\nu\lambda}] = g^{\rho\lambda}\mathbb{M}^{\mu\nu} - g^{\mu\nu}\mathbb{M}^{\rho\lambda} + g^{\rho\nu}\mathbb{M}^{\mu\lambda} - g^{\mu\lambda}\mathbb{M}^{\rho\nu} \right)$

$$\prod_{r=1}^{1} \mathcal{Q}_{\text{red}}^{\mu_r\rho_r\alpha_r} \left(\prod_{s=2}^{2j} \eta^{\alpha_s} \right) (j) t_{\alpha_1\cdots\alpha_{2j}} = \mathcal{Q}_{\text{red}}^{\mu\rho\alpha}(j) t_{\alpha0\cdots0} = -i\mathbb{M}^{\mu\rho}$$

- In general

$$\prod_{r=1}^{m} \mathcal{Q}_{\text{red}}^{\mu_r\rho_r\alpha_r}(j) t_{\alpha_1\cdots\alpha_m0\cdots0} = \frac{(-i)^m}{m!} \underset{\{\mu_1\rho_1,\cdots,\mu_m\rho_m\}}{\mathcal{S}} \prod_{r=1}^{m} \mathbb{M}^{\mu_r\rho_r} - (\text{Lower Multipoles})$$

Decompose operators with physical interpretation for each term

Multipole expansion → mono-, di-, quadrupole, ...

Algebra of γ-tensors

Generalized Dirac basis (Weyl rep)

- $2j$-rank symmetric tensors: $\gamma^{\mu_1\cdots\mu_{2j}} = \begin{pmatrix} 0 & t^{\mu_1\cdots\mu_{2j}} \\ \bar{t}^{\mu_1\cdots\mu_{2j}} & 0 \end{pmatrix}$
  (2j+1) independent matrices
- $2j$-rank symmetric psuedo-tensors: $\gamma^{\mu_1\cdots\mu_{2j}}\gamma_5 = \begin{pmatrix} 0 & t^{\mu_1\cdots\mu_{2j}} \\ -\bar{t}^{\mu_1\cdots\mu_{2j}} & 0 \end{pmatrix}$
  (2j+1) independent matrices
- $4j$-rank bi-tensors: $\gamma^{\mu_1\cdots\mu_{2j}}\gamma^{\rho_1\cdots\rho_{2j}} = \begin{pmatrix} t^{\mu_1\cdots\mu_{2j}}\bar{t}^{\rho_1\cdots\rho_{2j}} & 0 \\ 0 & \bar{t}^{\mu\cdots}t^{\rho_1\cdots\rho_{2j}} \end{pmatrix}$
  2(2j+1) independent matrices

$$\gamma^{\mu_1\cdots\mu_{2j}}\gamma^{\rho_1\cdots\rho_{2j}} = \sum_{m=0}^{2j}\frac{1}{(2j)!}\underset{\{\rho_1\ldots\rho_{2j}\}}{\mathcal{S}}\sum_{n=1}^{B_m^{2j}}\left(\mathrm{Re}\left\{\prod_{l\in\pi_{m,n}}\mathcal{Q}_{\mathrm{red}}^{\mu_l\rho_l\alpha_l}\right\}\gamma_{\alpha_1\cdots\alpha_{2j}}\gamma^{0\cdots 0}\right.$$
$$\left.+i\,\mathrm{Im}\left\{\prod_{l\in\pi_{m,n}}\mathcal{Q}_{\mathrm{red}}^{\mu_l\rho_l\alpha_l}\right\}\gamma_{\alpha_1\cdots\alpha_{2j}}\gamma_5\gamma^{0\cdots 0}\right)\prod_{k\in\pi_{m,n}^{\complement}} g^{\mu_k\rho_k}\eta^{\alpha_k}$$

Generalized Bilinears

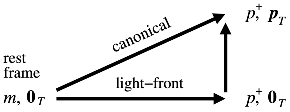
- $\bar{u}^{(j)}_{(p_f,s_f)}\Gamma u^{(j)}_{(p_i,s_i)} = \mathring{u}^{(j)\dagger}_{s_f} \begin{pmatrix} 0 & t^{\beta_1\cdots}\tilde{p}^f_{\beta_1\cdots} \\ \bar{t}^{\beta_1\cdots}(\tilde{p}^f_{\beta_1\cdots})^* & 0 \end{pmatrix} \Gamma \begin{pmatrix} t^{\alpha_1\cdots}\tilde{p}^i_{\beta_1\cdots} & 0 \\ 0 & \bar{t}^{\bar{\alpha}_1\cdots}(\tilde{p}^i_{\alpha_1\cdots})^* \end{pmatrix} \mathring{u}^{(j)}_{s_i}$

Canonical: $\tilde{p}^\mu_{\mathrm{C}} = \sqrt{\frac{1}{2m(m+p^0)}}(p^0+m,\vec{p}\,)$

Light-Front: $\tilde{p}^\mu_{\mathrm{LF}} = \sqrt{\frac{1}{4mp^+}}(p^+ + m, p_\ell, ip_\ell, p^+ - m)$

$p^+ = p^0 + p_z$

$p_\ell = p_x - ip_y$



- $2j$-rank Tensor bilinear $\widetilde{P} = \frac{1}{2}(\tilde{p}_f + \tilde{p}_i)$, $\widetilde{\Delta} = \tilde{p}_f - \tilde{p}_i$

$$\bar{u}_f\,\gamma^{\mu_1\cdots\mu_{2j}}\,u_f = m^{2j}\prod_{l=1}^{2j}\left[2\left(\widetilde{P}^{\mu_l}\widetilde{P}^{\tau_l} - \tfrac{1}{4}\widetilde{\Delta}^{\mu_l}\widetilde{\Delta}^{\tau_l}\right) - \left(\widetilde{P}^2 - \tfrac{1}{4}\widetilde{\Delta}^2\right)g^{\mu_l\tau_l} + i\varepsilon^{\mu_l\tau_l\widetilde{P}\widetilde{\Delta}}\right]\langle\lambda_f|t_{\tau_1\cdots\tau_{2j}}|\lambda_i\rangle$$

$$+m^{2j}\prod_{l=1}^{2j}\left[2\left(\widetilde{P}^{\mu_l}\widetilde{P}^{\tau_l} - \tfrac{1}{4}\widetilde{\Delta}^{\mu_l}\widetilde{\Delta}^{\tau_l}\right) - \left(\widetilde{P}^2 - \tfrac{1}{4}\widetilde{\Delta}^2\right)g^{\mu_l\tau_l} + i\varepsilon^{\mu_l\tau_l\widetilde{P}\widetilde{\Delta}}\right]^*\langle\lambda_f|\bar{t}_{\tau_1\cdots\tau_{2j}}|\lambda_i\rangle$$

Generalized Bilinears

Canonical: $\tilde{p}^{\mu}_{\mathrm{C}} = \sqrt{\frac{1}{2m(m+p^0)}}(p^0 + m, \vec{p}\,)$ Light-Front: $\tilde{p}^{\mu}_{\mathrm{LF}} = \sqrt{\frac{1}{4mp^+}}(p^+ + m, p_\ell, ip_\ell, p^+ - m)$

- $4j$-rank bi-Tensor bilinear $\widetilde{P} = \frac{1}{2}\,(\tilde{p}_f + \tilde{p}_i)$, $\widetilde{\Delta} = \tilde{p}_f - \tilde{p}_i$

$$\bar{u}(p_f,\lambda_f)\gamma^{\mu_1\cdots\mu_{2j}}\gamma^{\rho_1\cdots\rho_{2j}}u(p_i,\lambda_i) = \sum_{m=0}^{2j} \mathring{u}_f^\dagger \left[\frac{1}{2}\frac{1}{(2j)!}\underset{\{\rho_1\ldots\rho_{2j}\}}{\mathcal{S}}\sum_{n=1}^{B_m^{2j}}\right.$$

$$\mathrm{Re}\left\{\prod_{r\in\pi_{m,n}}\tilde{p}^f_{\beta_r}\tilde{p}^i_{\alpha_r}\left(\mathcal{C}^{\beta_r\mu_r\rho_r}{}_{\sigma_r}\mathcal{Q}^{\sigma_r\alpha_r\xi_r}_{\mathrm{red}} - \mathcal{C}^{\alpha_r\rho_r\mu_r}{}_{\sigma_r}\mathcal{Q}^{\sigma_r\beta_r\xi_r}_{\mathrm{red}}\right)\prod_{k\in\pi^{\complement}_{m,n}}\tilde{p}^f_{\beta_r}\tilde{p}^i_{\alpha_r}\mathcal{C}^{\beta_k\mu_k\rho_k\alpha_k}\eta^{\xi_k}\right\}\gamma_{\xi_1\cdots\xi_{2j}}$$

$$\left.+i\,\mathrm{Im}\left\{\prod_{r\in\pi_{m,n}}\tilde{p}^f_{\beta_r}\tilde{p}^i_{\alpha_r}\left(\mathcal{C}^{\beta_r\mu_r\rho_r}{}_{\sigma_r}\mathcal{Q}^{\sigma_r\alpha_r\xi_r}_{\mathrm{red}} - \mathcal{C}^{\alpha_r\rho_r\mu_r}{}_{\sigma_r}\mathcal{Q}^{\sigma_r\beta_r\xi_r}_{\mathrm{red}}\right)\prod_{k\in\pi^{\complement}_{m,n}}\tilde{p}^f_{\beta_r}\tilde{p}^i_{\alpha_r}\mathcal{C}^{\beta_k\mu_k\rho_k\alpha_k}\eta^{\xi_k}\right\}\gamma_{\xi_1\cdots\xi_{2j}}\gamma_5\right]\mathring{u}_i$$

Generalized Gordon Identities

- Dirac Equation $\left(\gamma^{\mu_1\cdots\mu_{2j}} p_{\mu_1}\ldots p_{\mu_{2j}} - m^{2j}\right) u_p^s = 0$ leads to On-Shell (Gordon) identities

$$\bar{u}_{p'}^{s'}\left(\Gamma\right) u_p^s = \frac{1}{2\bar{m}^{2j}} u_{p'}^{s'} \left(\left\{\not{P}^{(j)}, \Gamma\right\} + \frac{1}{2}\left[\not{\Delta}^{(j)}, \Gamma\right]\right) u_p^s$$

$$0 = \bar{u}_{p'}^{s'} \left(\frac{1}{2}\left\{\not{\Delta}^{(j)}, \Gamma\right\} + \left[\not{P}^{(j)}, \Gamma\right]\right) u_p^s$$

$$P_{\mu_1\ldots\mu_{2j}} = \tfrac{1}{2}\left(p'_{\mu_1}\ldots p'_{\mu_{2j}} + p_{\mu_1}\ldots p_{\mu_{2j}}\right)$$

$$\Delta_{\mu_1\ldots\mu_{2j}} = p'_{\mu_1}\ldots p'_{\mu_{2j}} - p_{\mu_1}\ldots p_{\mu_{2j}}$$

$$P^{\mu_1\cdots\mu_{2j}}_{(p',p)} = P^{\mu_1\cdots\mu_{2j}}_{(p,p')}$$

$$\Delta^{\mu_1\cdots\mu_{2j}}\left(p',p\right) = -\Delta^{\mu_1\cdots\mu_{2j}}\left(p,p'\right)$$

$$P^{\mu_1\cdots\mu_{2j}}\Delta_{\mu_1\ldots\mu_{2j}} = 0$$

- Useful to reduce independent Dirac structures

Applications

- Using basis of bilinears and Gordon identities we can identify minimal set of independent bilinears
- These will form the basis in decompositions of matrix elements of QCD operators (currents/correlators)
- Each basis element comes with FF/distribution
- Has multipole interpretation, construction is identical for all spin cases
- Unified framework to discuss spin in hadronic physics

Many applications and extensions possible

- Local operators parameterizations: Generalized Form Factors
  (two independent four-vectors)
- Bilocal operators parameterizations
  (more than two independent four-vectors)
- Transition matrix elements
- Use in χEFT's for high energy processes

Summary

- Construction allows for an efficient and manifestly covariant calculations
- Central role of the covariant t-tensors (spinors, boosts, propagators, gamma matrices)
- Very simple. Only need to know the matrices for the Generators of Rotations
- Covariant $sl(2,\mathrm{C})$-multipole basis for operators, transparent interpretation
- Unique framework for any spin.
  Built intuition on spin 1/2 can be carried over to higher spin
- Avoid calculations with (Dirac) matrices.
  Everything reduces to number multiplication ($\mathcal{C}^{\mu\rho\sigma\alpha}$, $\mathcal{Q}^{\mu\rho\alpha}$)
- $\vdots$

Thank You For Your Time!

Questions?

Backup Slides

Properties of the t-tensors

- Each $t^{\mu_1 \ldots \mu_{2j}}$ is a 2j-rank tensor
- Symmetric and (covariantly) traceless $\quad g_{\mu_k \mu_l} t_{\sigma\sigma'}^{\mu_1 \ldots \mu_k \ldots \mu_l \ldots \mu_{2j}} = 0$
- Transform covariantly $\quad \left(D^{(j)}_{[\Lambda]}\right)_{\sigma\delta} t_{\delta\delta'}^{\mu_1 \ldots \mu_{2j}} \left({D^{(j)}}^{\dagger}_{[\Lambda]}\right)_{\delta'\sigma'} = \Lambda_{\nu_1}{}^{\mu_1} \ldots \Lambda_{\nu_{2j}}{}^{\mu_{2j}} \; t_{\sigma\sigma'}^{\nu_1 \ldots \nu_{2j}}$
- Right chiral (t) and left chiral ($\bar{t}$) are related by charge conjugation $\quad \bar{t}_{\sigma\sigma'}^{\mu_1 \mu_2 \ldots \mu_{2j}} = (\pm) t_{\sigma\sigma'}^{\mu'_1 \mu'_2 \ldots \mu'_{2j}}$

(+ for even (− for odd) spacelike indices)

Spin 1/2 Example: Spinors

Right Chiral Rep

- $t^0 = \mathbf{1} \quad , \quad t^i = \sigma_i$
- t^μ Transform Covariantly: $\quad D^{(1/2)}_{[\Lambda]} t^\mu D^{(1/2)\,\dagger}_{[\Lambda]} = \Lambda_\rho{}^\mu t^\rho$
- Propagator (Lorentz invariant): $\Pi^{(1/2)}(p) = t^\mu p_\mu = \begin{pmatrix} E - p_z & -(p_x - ip_y) \\ -(p_x + ip_y) & E + p_z \end{pmatrix}$
  $p_\mu = (E_p, \vec{p})$
- Boost/spinors (Canonical): $D^{(1/2)}_{\mathrm{IF}} = t^\mu \tilde{p}^{\mathrm{C}}_\mu = \dfrac{1}{\sqrt{2m\,(m + p_0)}} \begin{pmatrix} m + p^- & -p_\ell \\ -p_r & m + p^+ \end{pmatrix}$
  $\tilde{p}^\mu_{\mathrm{C}} = \sqrt{\frac{m}{2(m+p^0)}}(p^0 + m, \vec{p}\,)$

Similarly for the Left Chiral Rep, only change is: $J_i^{(1/2)} \to \bar{J}^\mu = (1, -\vec{J}^{\,(1/2)})$

Right Chiral Rep

- $t^{00} = \mathbf{1} \quad , \quad t^{0i} = t^{i0} = J_i^{(1)} \quad , \quad t^{ij} = \{J_1^{(1)}, J_1^{(1)}\} - \mathbf{1}\delta_{ij}$
- $t^{\mu\nu}$ Transform covariantly $\quad D_{[\Lambda]}^{(1)} t^{\mu\nu} D^{(1)}{}_{[\Lambda]}^{\dagger} = \Lambda_\rho{}^\mu \Lambda_\sigma{}^\nu t^{\rho\sigma}$
- Propagator ($p_\mu = (E_p, \vec{p})$): $\quad \Pi^{(1)}(p) = t^{\mu\nu} p_\mu p_\nu = \begin{pmatrix} (p^-)^2 & -\sqrt{2} p_\ell p^- & p_\ell^2 \\ \sqrt{2} p_r p^+ & p^+ p^- + p_\mathrm{T}^2 & \sqrt{2} p_\ell p^- \\ p_r^2 & \sqrt{2} p_r p^- & (p^+)^2 \end{pmatrix}$
- Boost/spinors ($t^{\mu\nu}\tilde{p}_\mu \tilde{p}_\nu$)

  Canonical: $D_{\mathrm{IF}}^{(1)} = \dfrac{1}{2m\,(m+p_0)} \begin{pmatrix} (m+p^-)^2 & -\sqrt{2} p_\ell (m+p^-) & p_\ell^2 \\ -\sqrt{2} p_r (m+p^-) & 2(m^2 + m p_0 + p_\mathrm{T}^2) & -\sqrt{2} p_\ell (m+p^+) \\ p_r^2 & -\sqrt{2} p_r (m+p^+) & (m+p^+)^2 \end{pmatrix}$

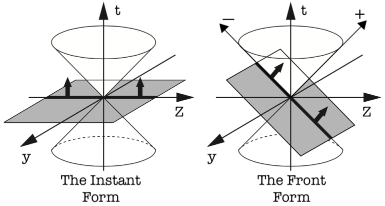
  $$\tilde{p}_\mathrm{C}^\mu = \sqrt{\frac{m}{2(m+p^0)}}\,(p^0 + m, \vec{p}\,)$$

Similarly for the Left Chiral Rep, only change is: $J_i^{(1)} \to \bar{J}^\mu = (1, -\vec{J}^{(1)})$

Canonical Space-Time Parameterization

Parameterizations (Foliations) of space-time $\rightarrow$ Specify equal time surfaces

Canonical or Instant time: $x^0 = t$ [Wigner(1939)]



- Defined by rotationless boosts from rest: $\overset{\circ}{p}{}^{\mu} = (m, 0, 0, 0)$

  to final momentum: $p^\mu = (E_p, \vec{p}) = (\sqrt{m^2 + \vec{p}^{\,2}}, \vec{p})$

$$\Lambda^{\rm IF} = \exp\left(i\vec{\mathbb{K}}\cdot\vec{\phi}\right) = \exp\left(i\phi\vec{\mathbb{K}}\cdot\hat{\phi}\right)$$

- Then, $p^\mu = (E, \vec{p}) = (\Lambda^{\rm IF})^\mu{}_\nu \overset{\circ}{p}{}^{\nu}$

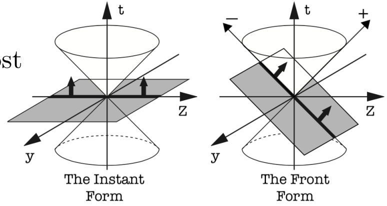
  implies, $\cosh(\phi) = \frac{E}{m}$, $\hat{\phi}_j \sinh(\phi) = \frac{p_j}{m}$

Leading to the well known result: $(\Lambda^{\rm IF})^\mu{}_\nu = \begin{pmatrix} \frac{E}{m} & \frac{\vec{p}}{m} \\ \frac{\vec{p}}{m} & \delta_{ij} + \frac{p_i p_j}{(E+m)m} \end{pmatrix}$

Light-Front Space-Time Parameterization

Light Front time: $x^+ = t + z$ Dirac(1949)

$p^+ = E_p + p_z \ , \ \ p^- = E_p - p_z$



- Defined by a longitudinal boost followed by a transverse boost

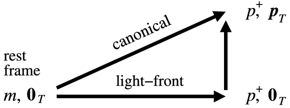
$$\Lambda^{\rm LF}_{\rm def.} = \exp\left[i\vec{\mathbb{G}}\cdot\vec{\rm v}_{\rm T}\right]\cdot\exp\left[i\mathbb{K}_3\eta\right]$$

- LF Boost Generators (light front along $z-$axis),

$$\mathbb{G}_1 = \mathbb{G}_x = \mathbb{K}_x - \mathbb{J}_y \ , \quad \mathbb{G}_2 = \mathbb{G}_y = \mathbb{K}_y + \mathbb{J}_x \ , \quad \mathbb{K}_3 = \mathbb{K}_z$$

- Comparing the action of both boosts on the same rest momentum one finds the LF boost parameters

$$e^{\eta} = \frac{p^+}{m} \ , \quad \vec{\rm v}_T = \frac{\vec{p}_T}{p^+} \to \Lambda^{\rm LF} = \exp\left[i\frac{\eta}{p^+-m}\vec{p}_T\cdot\vec{\mathbb{G}} + i\eta\mathbb{K}_3\right]$$



The boosts/spinors for the most used forms of dynamics

- In general

$$D^{(j)}_{[L(p)]} = t^{\mu_1\mu_2\ldots\mu_{2j}}\tilde{p}_{\mu_1}\tilde{p}_{\mu_2}\ldots\tilde{p}_{\mu_{2j}}$$

$$\bar{D}^{(j)}_{[L(p)]} = \bar{t}^{\mu_1\mu_2\ldots\mu_{2j}}\tilde{p}_{\mu_1}\tilde{p}_{\mu_2}\ldots\tilde{p}_{\mu_{2j}}$$

Instant form dynamics
(Canonical)

$$\tilde{p}^{\mu}_{\mathrm{C}} = \sqrt{\frac{1}{2m(m+p^0)}}(p^0 + m, \vec{p}\,)$$

Light-Front dynamics
(Light Cone time)

$$\tilde{p}^{\mu}_{\mathrm{LF}} = \sqrt{\frac{1}{4mp^+}}(p^+ + m, p_\ell, ip_\ell, p^+ - m)$$

$\tilde{p}^{\mu}$ not four-vectors, but same for any spin. Left/right related by complex conjugation.
(Helicity spinor also recovered with specific parameters)

Generalized Bilinears

- $\bar{u}^{(j)}_{(p_f,s_f)}\Gamma u^{(j)}_{(p_i,s_i)} = \frac{1}{(m_f m_i)^{2j}} \overset{\circ}{u}^{(j)\dagger}_{s_f} \begin{pmatrix} 0 & t^{\beta_1\cdots}\tilde{p}^f_{\beta_1\cdots} \\ \bar{t}^{\beta_1\cdots}(\tilde{p}^f_{\beta_1\cdots})^* & 0 \end{pmatrix} \Gamma \begin{pmatrix} t^{\alpha_1\cdots}\tilde{p}^i_{\beta_1\cdots} & 0 \\ 0 & \bar{t}^{\bar{\alpha}_1\cdots}(\tilde{p}^i_{\alpha_1\cdots})^* \end{pmatrix} \overset{\circ}{u}^{(j)}_{s_i}$

- Dirac basis: $4(2j+1)^2$ ind. elements

$\Gamma = \mathbf{1}\ (1)\ ,\quad \gamma^{\mu_1\cdots\mu_{2j}}\ (2j+1)^2\ ,\quad \gamma_5\gamma^{\mu_1\cdots\mu_{2j}}\ (2j+1)^2\ ,\quad \gamma_5\ (1)$

$[\gamma^{\mu_1\cdots\mu_{2j}},\gamma^{\nu_1\cdots\nu_{2j}}]\quad 2\sum_{n=1,3,\cdots}^{2j}(2n+1)$

$\{\gamma^{\mu_1\cdots\mu_{2j}},\gamma^{\nu_1\cdots\nu_{2j}}\}_{\text{traceless}}\quad 2\sum_{n=0,2,\cdots}^{2j}(2n+1)$

Examples

  - Spin-1/2 (16): $\mathbf{1}(1),\ \gamma^\mu(4),\ [\gamma^\mu,\gamma^\nu](6),\ (\{\gamma^\mu,\gamma^\nu\}-2g^{\mu\nu})\ (0),\ \gamma^\mu\gamma_5(4),\ \gamma_5(1)$
  - Spin-1 (36): $\mathbf{1}(1),\ \gamma^{\mu\nu}(9),\ [\gamma^{\mu_1\mu_2},\gamma^{\mu_3\mu_4}](6),\ \{\gamma^{\mu_1\mu_2},\gamma^{\mu_3\mu_4}\}_{\text{trless}}(10),\ \gamma^{\mu\nu}\gamma_5(9),\ \gamma_5(1)$

- Matrix elements of Operators, covariant Density matrices, Amplitudes, $\cdots$

Spin 1/2 Bilinears

Final evaluations recover the results of [Lorcé(2017)] (Canonical)

- Scalar $\bar{u}^{(1/2)}_{(p_f,s_f)} u^{(1/2)}_{(p_i,s_i)} = \widetilde{N}\phi^{\dagger}_{s_f}\left[4P^2\mathbf{1}_2 + 4mP_\lambda\left(\sigma^\lambda + \bar{\sigma}^\lambda\right) - \frac{i}{2}\left(\sigma_\lambda + \bar{\sigma}_\lambda\right)\varepsilon^{\lambda\beta\alpha\rho}\Delta_\beta P_\alpha\left(\sigma_\rho - \bar{\sigma}_\rho\right)\right]\phi_{s_i}$

$$= \widetilde{N}\phi^{\dagger}_{s_f}\left[4\left(P^2 + mP^0\right) + 2i\varepsilon^{0\beta\alpha\rho}\Delta_\beta P_\alpha \sigma_\rho\right]\phi_{s_i}$$

- Pseudoscalar

$$\bar{u}^{(1/2)}_{(p_f,s_f)}\gamma_5 u^{(1/2)}_{(p_i,s_i)} = \widetilde{N}\phi^{\dagger}_{s_f}\left[m\Delta_\lambda\left(\sigma^\lambda - \bar{\sigma}^\lambda\right) + \left(P_\mu\left(\sigma^\mu + \bar{\sigma}^\mu\right)\right)\left(\Delta_\nu\left(\sigma^\nu - \bar{\sigma}^\nu\right)\right) - \left(\Delta_\mu\left(\sigma^\mu + \bar{\sigma}^\mu\right)\right)\left(P_\nu\left(\sigma^\nu - \bar{\sigma}^\nu\right)\right)\right]\phi_{s_i}$$

$$= \widetilde{N}\phi^{\dagger}_{s_f}\left[2\Delta^0\vec{P}\cdot\vec{\sigma} - 2\left(P^0 + m\right)\vec{\Delta}\cdot\vec{\sigma}\right]\phi_{s_i}$$

$$\widetilde{N} = \widetilde{N}_f\widetilde{N}_i = \frac{1}{2m}\left[\left(p^0 + m\right)^2 - \left(\frac{1}{2}\Delta\right)^2\right]^{-1}$$

Bilinears

- Vector

$$\bar{u}^{(j)}_{(p_f,s_f)}\gamma^\mu u^{(j)}_{(p_i,s_i)} = \tilde{N}\phi_{s_f} + \Big[\frac{1}{2}\Delta^2\left(\sigma^\mu + \bar{\sigma}^\mu\right) - \frac{1}{2}\Delta_\lambda\left(\sigma^\lambda + \bar{\sigma}^\lambda\right)\Delta^\mu$$
$$+ 4mP^\mu \mathbf{1}_2 + 2P_\lambda\left(\sigma^\lambda + \bar{\sigma}^\lambda\right)P^\mu$$
$$i\varepsilon^{\mu\beta\alpha\rho}\Delta_\beta\left(m\left(\sigma_\alpha + \bar{\sigma}_\alpha\right) + P_\alpha\right)\left(\sigma_\rho - \bar{\sigma}_\rho\right)\Big]\phi_{s_i}$$
$$= \tilde{N}\phi^\dagger_{s_f}\Big[\left(4\left(P^0 + m\right)P^\mu + \Delta^2 g^{0\mu} - \Delta^0\Delta^\mu\right)\mathbf{1}_2$$
$$+2i\varepsilon^{0\mu\beta\rho}\Delta_\beta\sigma_\rho + i\varepsilon^{\mu\beta\alpha\rho}\Delta_\beta P_\alpha\left(\sigma_\rho + \bar{\sigma}_\rho\right)\Big]\phi_{s_i}$$

- Pseudovector

$$\bar{u}^{(1/2)}_{(p_f,s_f)}\gamma^\mu\gamma_5 u^{(1/2)}_{(p_i,s_i)} = \tilde{N}\phi^\dagger_{s_f}\Big[-\left(4P^\mu P_\alpha - \Delta^\mu\Delta_\alpha\right)\left(\sigma^\alpha - \bar{\sigma}^\alpha\right)$$
$$+\left(P^2 - \frac{1}{4}\Delta^2\right)\left(\sigma^\mu - \bar{\sigma}^\mu\right) - i\varepsilon^{\mu\alpha\beta\rho}\Delta_\alpha P_\beta\left(\sigma_\rho + \bar{\sigma}_\rho\right)\Big]\phi_{s_i}$$

Spin 1 Example: EM Current

Using spinor representation: $\langle p', s' | j^\mu(0) | p, s\rangle = \overset{\circ}{\phi}{}^{(1)}_{s'} \Gamma^\mu_{(p',p)} \overset{\circ}{\phi}{}^{(1)}_{s}$

$P = \frac{1}{2}(p' + p)$

$\Delta = p' - p \quad (\Delta^2 = -Q^2)$

$n_t^\nu = (1, 0, 0, 0)$

$$m^2 \Gamma^\mu_{(p',p)} = 2P^\mu \left[P^2 \mathbf{1} G_C\left(Q^2\right) - \Delta^\rho \Delta^\sigma \left(t_{\rho\sigma} - \frac{1}{3} g_{\rho\sigma} \mathbf{1} \right) G_Q\left(Q^2\right) \right]$$
$$- i\epsilon^{\mu\rho\sigma\lambda} \left[\Delta_\rho P_\sigma \left(t_{\lambda\nu} - \frac{1}{3} g_{\lambda\nu} \mathbf{1} \right) n_t^\nu G_M\left(Q^2\right) \right]$$

Using polarization vectors: $\langle p', s' | j^\mu(0) | p, s\rangle = \varepsilon^{*\,\alpha}_{s'}(p')\, \Gamma^\mu_{\alpha\beta}(P, \Delta) \varepsilon^\beta_s(p)$

[Wang & Lorcé (2022)]

$$\Gamma^{\mu\alpha\beta} = 2P^\mu \left(\Pi^{\alpha\beta} G_C\left(Q^2\right) - \frac{\Delta^\rho \Delta^\sigma \left(\Sigma_{\rho\sigma}\right)^{\alpha\beta}}{2m^2} \frac{P^2}{m^2} G_Q\left(Q^2\right) \right)$$
$$- i\epsilon^{\mu\rho\sigma\lambda} \left(\frac{\Delta_\rho P_\sigma \left(\Sigma_\lambda\right)^{\alpha\beta}}{\sqrt{P^2}} G_M\left(Q^2\right) \right)$$