Exact Polarization in Relativistic Fluids at Global Equilibrium

Based on Eur.Phys.J.Plus 138 (2023) 6
In collaboration with F. Becattini

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Motivation

The global polarization of the Lambda particle is well described by the thermal vorticity
Becattini, Chandra, Del Zanna, Grossi, Annals Phys. 338 (2013) 32-49

$$S^\mu(p) = -\frac{1}{8m}\epsilon^{\mu\rho\sigma\tau}p_\tau \frac{\int \mathrm{d}\Sigma\cdot p\, n_F(1-n_F)\varpi_{\rho\sigma}(x)}{\int \mathrm{d}\Sigma\cdot p\, n_F}$$

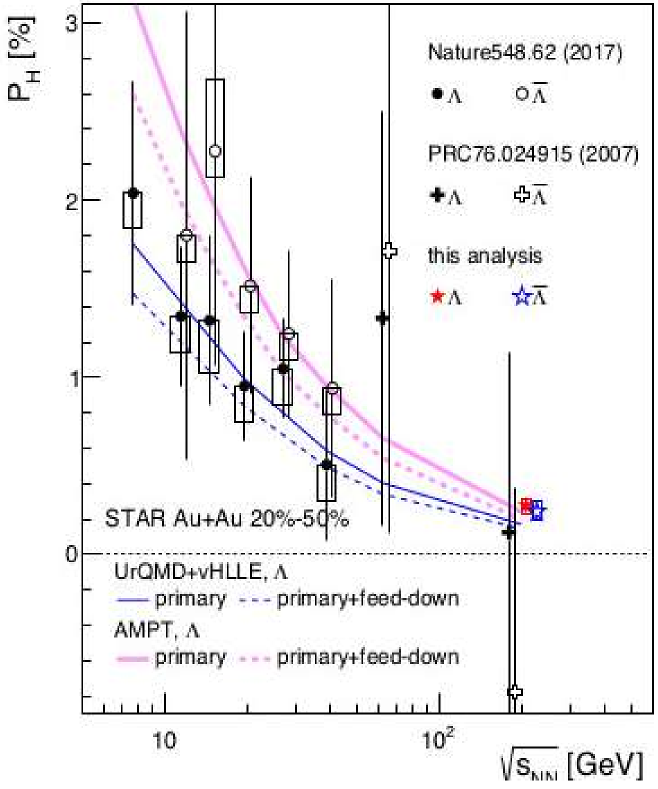
$$\varpi_{\mu\nu} = -\frac{1}{2}\left(\partial_\mu\beta_\nu - \partial_\nu\beta_\mu\right)$$

Thermal vorticity

$$\beta^\mu = \frac{u^\mu}{T}$$

Four-temperature

The formula is only a **first order approximation.** Higher order corrections in vorticity are unknown, even in thermal equilibrium.



T. Niida, Nucl.phys.A,2019

Main results

Exact spin density matrix for spin-S particles at general global equilibrium:

$$\Theta(p) = \frac{\left[(-1)^{2S}\mathbb{I} - e^{-b\cdot p + \boldsymbol{\theta_0}\cdot D^S(\mathbf{J})}\right]^{-1} - (-1)^{2S}\mathbb{I}}{\sum_{k=-S}^{S}\left(e^{b\cdot p - k\sqrt{-\theta^2}} - (-1)^{2S}\right)^{-1}}$$

Exact spin vector for Dirac field at global equilibrium

$$\theta^\mu = -\frac{1}{2m}\epsilon^{\mu\nu\rho\sigma}\varpi_{\nu\rho}p_\sigma \qquad S^\mu(p) = \frac{1}{2}\,\frac{\theta^\mu}{\sqrt{-\theta^2}}\,\frac{\sinh\left(\frac{\sqrt{-\theta^2}}{2}\right)}{\cosh\left(\frac{\sqrt{-\theta^2}}{2}\right) + e^{-b\cdot p+\zeta}}$$

Including **all quantum corrections** in vorticity

Expectation values

To compute mean values we need the spin density matrix:

$$\Theta_{sr}(p) = \frac{\mathrm{Tr}\left(\widehat{\rho}\,\widehat{a}_r^\dagger(p)\widehat{a}_s(p)\right)}{\sum_l \mathrm{Tr}\left(\widehat{\rho}\,\widehat{a}_l^\dagger(p)\widehat{a}_l(p)\right)} = \frac{\langle\widehat{a}_r^\dagger(p)\widehat{a}_s(p)\rangle}{\sum_l\langle\widehat{a}_l^\dagger(p)\widehat{a}_l(p)\rangle}$$

Once we are given a spin density matrix, the expectation value of the Pauli-Lubanski operator is:

[F. Becattini, Lect.Notes Phys. 987 (2021) 15-52, AP, F. Becattini Eur.Phys.J.Plus 138 (2023) 6, 547]

$$S^\mu(p) = \sum_{i=1}^{3} [p]^\mu_{\ i}\,\mathrm{tr}\left(\Theta(p) D^S(\mathrm{J}^i)\right), \qquad m \neq 0$$

Other methods to compute polarization rely on the Wigner function.

Global equilibrium

Density operator at global equilibrium:

$$\widehat{\rho} = \frac{1}{Z} \exp\left[-b_\mu \widehat{P}^\mu + \frac{1}{2} \varpi_{\mu\nu} \widehat{J}^{\mu\nu} \right] \qquad \langle \widehat{O} \rangle = \mathrm{Tr}\left(\widehat{\rho} \widehat{O} \right)$$

The vector b is constant and the thermal vorticity ϖ is a constant antisymmetric tensor. The four-temperature β vector is a Killing vector:

$$\beta^\mu(x) = b^\mu + \varpi^{\mu\nu} x_\nu \equiv \frac{u^\mu}{T}$$

At global equilibrium:

$$\frac{A^\mu}{T} = \varpi^{\mu\nu} u_\nu$$

Acceleration

$$\frac{\omega^\mu}{T} = -\frac{1}{2} \epsilon^{\mu\nu\rho\sigma} \varpi_{\nu\rho} u_\sigma$$

Angular velocity

The generators of the Poincaré group appear in the density operator.

Analytic continuation of the thermal vorticity: $\varpi \mapsto -i\phi$

$$\widehat{\rho} = \frac{1}{Z} \exp\left[-b_\mu \widehat{P}^\mu - \frac{i}{2}\phi_{\mu\nu}\widehat{J}^{\mu\nu}\right]$$

P $\mapsto$ translations
J$\mapsto$ Lorentz transformations

Factorization of the density operator:

$$\widehat{\rho} = \frac{1}{Z} \exp\left[-\widetilde{b}_\mu(\phi)\widehat{P}^\mu\right] \exp\left[-i\frac{\phi_{\mu\nu}}{2}\widehat{J}^{\mu\nu}\right] \equiv \frac{1}{Z}\exp\left[-\widetilde{b}_\mu(\phi)\widehat{P}^\mu\right]\widehat{\Lambda}$$

$$\widetilde{b}^\mu(\phi) = \sum_{k=0}^{\infty} \frac{1}{(k+1)!} \underbrace{(\phi^\mu{}_{\alpha_1}\phi^{\alpha_1}{}_{\alpha_2}\ldots\phi^{\alpha_{k-1}}{}_{\alpha_k})}_{k \text{ times}} b^{\alpha_k} \qquad \widehat{\Lambda} \equiv \mathrm{e}^{-i\frac{\phi_{\mu\nu}}{2}\widehat{J}^{\mu\nu}}$$

We can use **group theory** to calculate **thermal expectation values.**

[AP, F. Becattini, M. Buzzegoli JHEP 10 (2021) 077]

The number operator at (imaginary) global equilibrium:

$$\langle \widehat{a}_s^\dagger(p)\widehat{a}_t(p')\rangle = 2\varepsilon' \sum_{n=1}^{\infty}(-1)^{2S(n+1)}\delta^3(\Lambda^n \boldsymbol{p} - \boldsymbol{p}')D^S(W(\Lambda^n, p))_{ts}\mathrm{e}^{-\widetilde{b}\cdot\sum_{k=1}^{n}\Lambda^k p}$$

$D(W) = [\Lambda p]^{-1}\Lambda[p]$ is the Wigner rotation in the spin-S representation of the rotation group.

For vanishing vorticity (i.e. Λ=I) we recover Bose and Fermi statistics:

$$\langle \widehat{a}_s^\dagger(p)\widehat{a}_t(p')\rangle = 2\varepsilon' \sum_{n=1}^{\infty}(-1)^{2S(n+1)}\delta^3(\boldsymbol{p} - \boldsymbol{p}')\delta_{ts}\,\mathrm{e}^{-nb\cdot p} = \frac{2\varepsilon\,\delta^3(\boldsymbol{p} - \boldsymbol{p}')\delta_{ts}}{\mathrm{e}^{b\cdot p} + (-1)^{2S+1}}$$

We want to use this result to compute the spin density matrix:

$$\Theta(p)_{rs} = \frac{\langle \widehat{a}_s^\dagger(p)\widehat{a}_r(p)\rangle}{\sum_t \langle \widehat{a}_t^\dagger(p)\widehat{a}_t(p)\rangle}$$

From the **analytic continuation** of the density operator:

$$\langle \widehat{a}_s^\dagger(p)\widehat{a}_t(p)\rangle = 2\varepsilon \sum_{n=1}^{\infty} (-1)^{2S(n+1)} \delta^3(\Lambda^n \boldsymbol{p} - \boldsymbol{p}) D^S(W(\Lambda^n, p))_{ts} \mathrm{e}^{-\widetilde{b}\cdot\sum_{k=1}^{n}\Lambda^k p}$$

The spin density matrix is singular unless $\Lambda \mathrm{p} = \mathrm{p}$. The analytic continuation of the density operator is forced to be in the little group of p.

Exact spin physics

The constraint equation:

$$\Lambda p = p \implies \phi^{\mu\nu} p_\nu = 0$$

$$\phi^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma} \xi_\rho \frac{p_\sigma}{m} \qquad \xi^\rho = -\frac{1}{2m} \epsilon^{\rho\mu\nu\sigma} \phi_{\mu\nu} p_\sigma$$

Thanks to the constraint, it is possible to compute the spin density matrix for any spin

$$\xi^\mu \mapsto \theta^\mu = -\frac{1}{2m} \epsilon^{\mu\nu\rho\sigma} \varpi_{\nu\rho} p_\sigma \qquad \theta_0^\mu = [p]^{-1\,\mu}{}_\nu \theta^\nu$$

$$\Theta(p) = \frac{\left[(-1)^{2S}\mathbb{I} - e^{-b\cdot p + \boldsymbol{\theta_0}\cdot D^S(\mathbf{J})}\right]^{-1} - (-1)^{2S}\mathbb{I}}{\sum_{k=-S}^{S} \left(e^{b\cdot p - k\sqrt{-\theta^2}} - (-1)^{2S}\right)^{-1}}$$

We can compute the spin vector for Dirac fermions:

$$S^\mu(p) = \frac{1}{2}\,\frac{\theta^\mu}{\sqrt{-\theta^2}}\,\frac{\sinh\left(\frac{\sqrt{-\theta^2}}{2}\right)}{\cosh\left(\frac{\sqrt{-\theta^2}}{2}\right) + e^{-b\cdot p+\zeta}}$$

And for generic spin-S particles:

$$S^\mu(p) = \frac{\theta^\mu}{\sqrt{-\theta^2}}\,\frac{\sum_{k=-S}^{S} k\left[\mathrm{e}^{b\cdot p-\zeta-k\sqrt{-\theta^2}} - (-1)^{2S}\right]^{-1}}{\sum_{k=-S}^{S}\left[\mathrm{e}^{b\cdot p-\zeta-k\sqrt{-\theta^2}} - (-1)^{2S}\right]^{-1}}$$

These formulae account for corrections to **all orders in vorticity**

The formula:

- Reproduces linear results

$$S^{\mu}(p)_{\theta\to 0} \sim \theta^{\mu}\frac{S(S+1)}{3}(1+(-1)^{2S}n_{F/B}(b\cdot p-\zeta))$$

- Exact calculation with Boltzmann statistics

$$S^{\mu}_{B} = \frac{\theta^{\mu}}{\sqrt{-\theta^{2}}}\frac{\chi'(\sqrt{-\theta^{2}})}{\chi(\sqrt{-\theta^{2}})} \qquad \chi\left(\sqrt{-\theta^{2}}\right) = \sum_{k=-S}^{S} e^{k\sqrt{-\theta^{2}}}$$

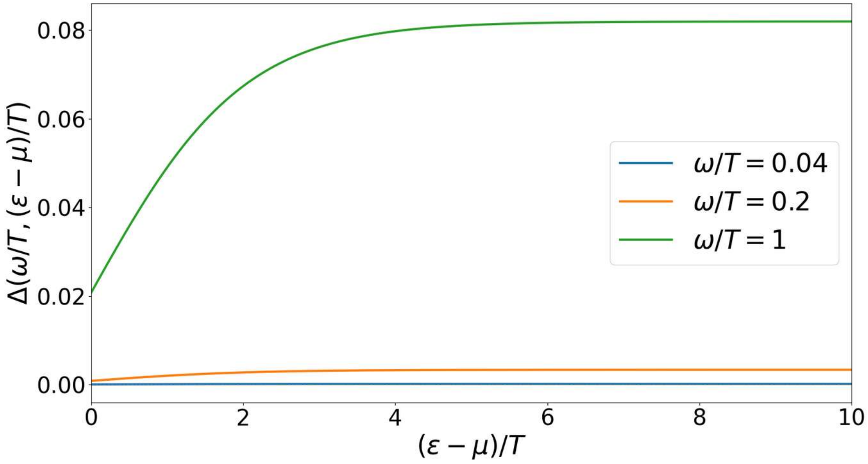
- Is unitary

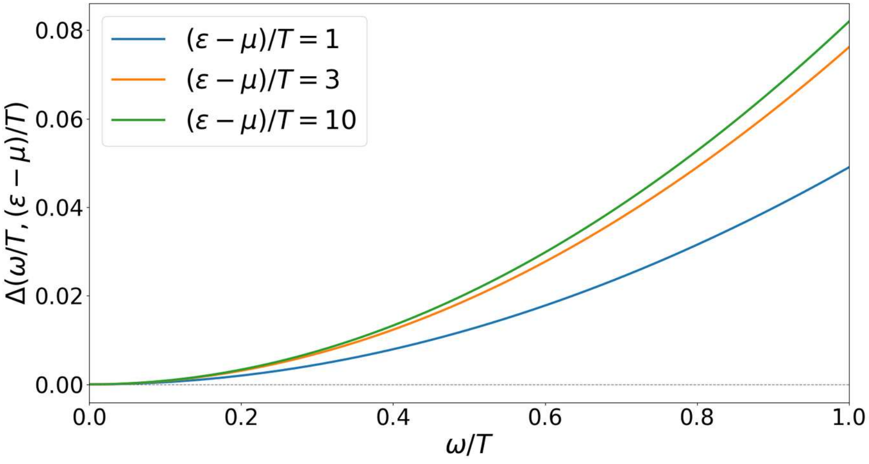
$$S^{\mu}(p)_{\theta\to\infty} \sim S\frac{\theta^{\mu}}{\sqrt{-\theta^{2}}}$$

Exact polarization in heavy-ion collisions

Consider the Λ polarization in relativistic heavy ion collisions $\omega \sim 10^{22}$ s^{-1} and $\omega/T \sim 0.04$.

$$\Delta = \left| \frac{S_E - S_L}{S_E} \right|$$

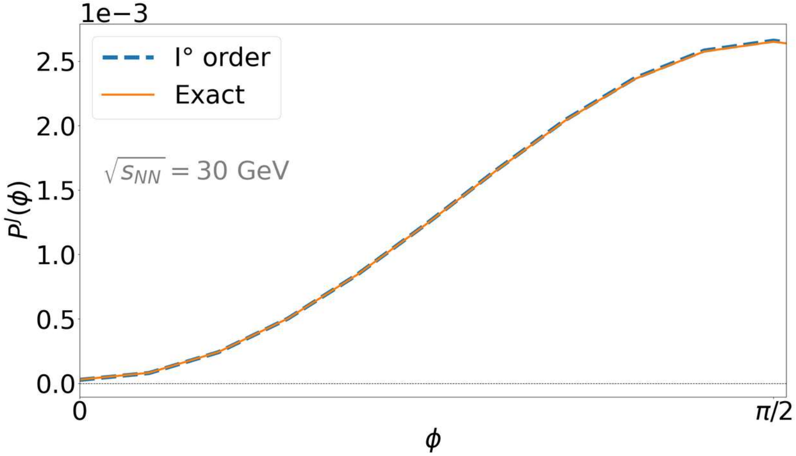


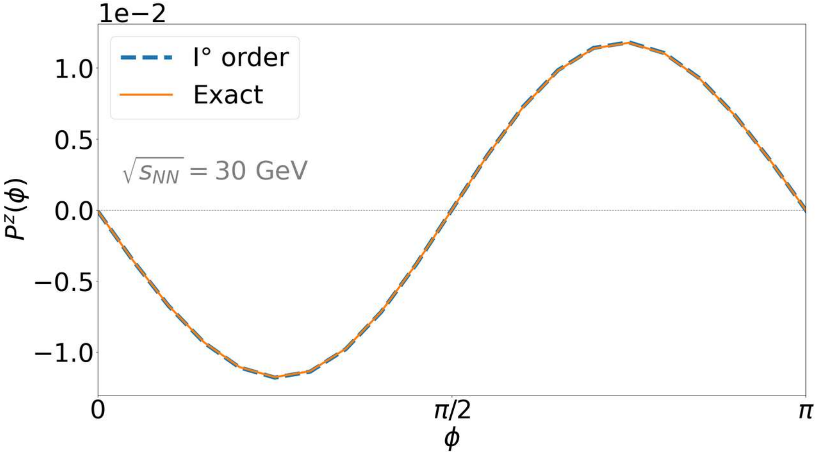


The difference is very small in most physical cases.

Extending the formula to local equilibrium with $\varpi(x)$

$$S^\mu(p) = -\frac{1}{4m}\epsilon^{\mu\nu\rho\sigma}p_\sigma \frac{\int \mathrm{d}\Sigma\cdot p\, n_F \frac{\varpi_{\nu\rho}}{\sqrt{-\theta^2}} \frac{\sinh\left(\sqrt{-\theta^2}/2\right)}{\cosh\left(\sqrt{-\theta^2}/2\right)+\mathrm{e}^{-b\cdot p+\zeta}}}{\int \mathrm{d}\Sigma\cdot p\, n_F}$$





Conclusions

Exact expectation values can be computed using the **analytic continuation of the density operator.**

Exact spin polarization vector and **spin density matrix** for particles at global equilibrium.

Higher order corrections in vorticity are negligible, at least in high energy collisions...

Thank you for the attention!

Backup

Pauli-Lubanski Vector

The Hilbert space of relativistic particles is built using the four-momentum and the Pauli-Lubanski vector

$$\widehat{\Pi}^\mu = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\widehat{J}_{\nu\rho}\widehat{P}_\sigma \qquad [\widehat{\Pi}^\mu, \widehat{P}^\nu] = 0$$

For states with definite momentum p

$$\widehat{\Pi}^\mu|p\rangle = \widehat{\Pi}^\mu(p)|p\rangle \qquad \widehat{\Pi}^\mu(p) = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\widehat{J}_{\nu\rho}p_\sigma$$

One has

$$p_\mu\widehat{\Pi}^\mu(p) = 0$$

If we consider **massive** particles the Pauli-Lubanski vector is connected to the generators of rotations

$$\widehat{S}^\mu(p) = \frac{\widehat{\Pi}^\mu(p)}{m} = -\frac{1}{2m}\epsilon^{\mu\nu\rho\sigma}\widehat{J}_{\nu\rho}p_\sigma$$

$$\widehat{S}^\mu(p) = \sum_{i=1}^{3}\widehat{S}_i(p)n_i^\mu(p) \qquad [\widehat{S}_i(p), \widehat{S}_j(p)] = i\epsilon_{ijk}\widehat{S}_k(p)$$

In the **massless** case:

$$\widehat{\Pi}^\mu(p) = \widehat{h}(p)p^\mu + \widehat{\Pi}_1(p)n_1^\mu(p) + \widehat{\Pi}_2(p)n_2^\mu(p) \qquad \widehat{\Pi}_{1,2}(p)|p, h\rangle = 0$$

$$\widehat{h}(p) = \frac{\widehat{\Pi}(p)\cdot q}{p\cdot q} = -\frac{1}{2p\cdot q}\epsilon^{\mu\nu\rho\sigma}\widehat{J}_{\nu\rho}p_\sigma q_\mu$$

$$q\cdot p \neq 0, \qquad q^2 = 0, \qquad q\cdot n_i(p) = 0$$

Helicity is the only physical degree of freedom

Polarization and the Wigner function

To compute expectation values it is useful to use the Wigner function.

$$W(x,k)_{ab} = -\frac{1}{(2\pi)^4}\int \mathrm{d}^4y e^{-ik\cdot y}\langle : \bar{\Psi}_b(x+y/2)\Psi_a(x-y/2) : \rangle$$

Massive Dirac fermions [F. Becattini, Lect.Notes Phys. 987 (2021) 15-52]:

$$S^\mu(p) = \frac{1}{2}\frac{\int \mathrm{d}\Sigma\cdot p\ \mathrm{tr}(\gamma^\mu\gamma_5 W_+(x,p))}{\int \mathrm{d}\Sigma\cdot p\ \mathrm{tr}(W_+(x,p))}$$

Massless Dirac fermions

$$\Pi^\mu(p) = p^\mu \sum_{h=\pm S} h\Theta_{hh}(p)$$

$$\Pi^\mu(p) = \frac{p^\mu}{2}\frac{\int \mathrm{d}\Sigma\cdot p\ \mathrm{tr}(\not{a}\gamma^5 W_+(x,p))}{\int \mathrm{d}\Sigma\cdot p\ \mathrm{tr}(W_+(x,p)\not{a})}$$

In the massless case the mean spin always points in the direction of momentum.
See also *[Y.-C. Liu, K. Mameda, X.-G. Huang, Chin. Phys. C 44(9), 094101]*

Exact Wigner function for free fermions at global equilibrium:

$$W(x,k) = \frac{1}{(2\pi)^3}\int \frac{\mathrm{d}^3 p}{2\varepsilon}\sum_{n=1}^{\infty}(-1)^{n+1}\mathrm{e}^{-n\tilde{\beta}(in\phi)\cdot p}\times$$
$$\left[\mathrm{e}^{-in\frac{\phi:\Sigma}{2}}(m+\not{p})\delta^4\left(k-(\Lambda^n p+p)/2\right)+(m-\not{p})\mathrm{e}^{in\frac{\phi:\Sigma}{2}}\delta^4\left(k+(\Lambda^n p+p)/2\right)\right]$$

Where $\Lambda = \mathrm{e}^{-i\frac{\phi}{2}:J}$ is in the four-vector representation.
Solves the Wigner equation! Full summation of the "$\hbar$ expansion".

Can be used to compute expectation values:

$$\langle : \{\bar{\Psi}\Psi, j^\mu, T^{\mu\nu}\} : \rangle = \int \mathrm{d}^4 k \left\{\mathrm{tr}\left(W\right), \mathrm{tr}\left(\gamma^\mu W\right), k^\mu \mathrm{tr}\left(\gamma^\nu W\right)\right\}$$

The Dirac delta is integrated out and results are expressed as series of functions that can be regularized using the **analytic distillation**.

Spin vector

Spin vector of massive Dirac fermions:

$$S^\mu(p) = \frac{1}{2}\frac{\int \mathrm{d}\Sigma\cdot p\ \mathrm{tr}\left(\gamma^\mu\gamma_5 W_+(x,p)\right)}{\int \mathrm{d}\Sigma\cdot p\ \mathrm{tr}\left(W_+(x,p)\right)}$$

Exact spin vector at global equilibrium:

$$S^\mu(p) = \frac{1}{2m}\frac{\sum_{n=1}^\infty (-1)^{n+1}\mathrm{e}^{-n\widetilde{b}(in\phi)\cdot p}\mathrm{tr}\left(\gamma^\mu\gamma_5\mathrm{e}^{-in\frac{\phi:\Sigma}{2}}\not{p}\right)\delta^3(\Lambda^n p - p)}{\sum_{n=1}^\infty (-1)^{n+1}\mathrm{e}^{-n\widetilde{b}(in\phi)\cdot p}\mathrm{tr}\left(\mathrm{e}^{-in\frac{\phi:\Sigma}{2}}\right)\delta^3(\Lambda^n p - p)}$$

How to handle a ratio of series of δ-functions? Where does it come from?

Any thermal expectation value in a free quantum field theory is obtained from:

$$\langle \widehat{a}_s^\dagger(p)\widehat{a}_t(p')\rangle = \frac{1}{Z}\mathrm{Tr}\left(\exp[-\tilde{b}_\mu(\phi)\widehat{P}^\mu]\widehat{\Lambda}\ \widehat{a}_s^\dagger(p)\widehat{a}_t(p')\right)$$

$$[\widehat{a}_s^\dagger(p),\widehat{a}_t(p')]_\pm = 2\varepsilon\delta^3(\boldsymbol{p}-\boldsymbol{p'})\delta_{st}$$

Using Poincaré transformation rules and (anti)commutation relations (particle with spin S):

$$\begin{aligned}\langle \widehat{a}_s^\dagger(p)\widehat{a}_t(p')\rangle =&(-1)^{2S}\sum_r D^S(W(\Lambda,p))_{rs}\mathrm{e}^{-\widetilde{b}\cdot\Lambda p}\langle \widehat{a}_r^\dagger(\Lambda p)\widehat{a}_t(p')\rangle + \\ &+ 2\varepsilon\,\mathrm{e}^{-\widetilde{b}\cdot\Lambda p}D^S(W(\Lambda,p))_{ts}\delta^3(\Lambda\boldsymbol{p}-\boldsymbol{p}')\end{aligned}$$

$D(W)=[\Lambda p]^{-1}\Lambda[p]$ is the “Wigner rotation” in the S-spin representation.

We find a solution by iteration:

I

$$\langle \widehat{a}^\dagger_s(p)\widehat{a}_t(p')\rangle \sim 2\varepsilon\, \mathrm{e}^{-\widetilde{b}\cdot\Lambda p} D^S(W(\Lambda,p))_{ts}\delta^3(\Lambda\boldsymbol{p}-\boldsymbol{p}')$$

II

$$\langle \widehat{a}^\dagger_s(p)\widehat{a}_t(p')\rangle \sim 2\varepsilon\,(-1)^{2S} D^S(W(\Lambda^2,p))_{ts}\mathrm{e}^{-\widetilde{b}\cdot(\Lambda p+\Lambda^2 p)}\delta^3(\Lambda^2\boldsymbol{p}-\boldsymbol{p}')+ \\ + 2\varepsilon\, \mathrm{e}^{-\widetilde{b}\cdot\Lambda p} D^S(W(\Lambda,p))_{ts}\delta^3(\Lambda\boldsymbol{p}-\boldsymbol{p}')$$

∞

$$\langle \widehat{a}^\dagger_s(p)\widehat{a}_t(p')\rangle = 2\varepsilon' \sum_{n=1}^{\infty}(-1)^{2S(n+1)}\delta^3(\Lambda^n\boldsymbol{p}-\boldsymbol{p}')D^S(W(\Lambda^n,p))_{ts}\mathrm{e}^{-\widetilde{b}\cdot\sum_{k=1}^{n}\Lambda^k p}$$

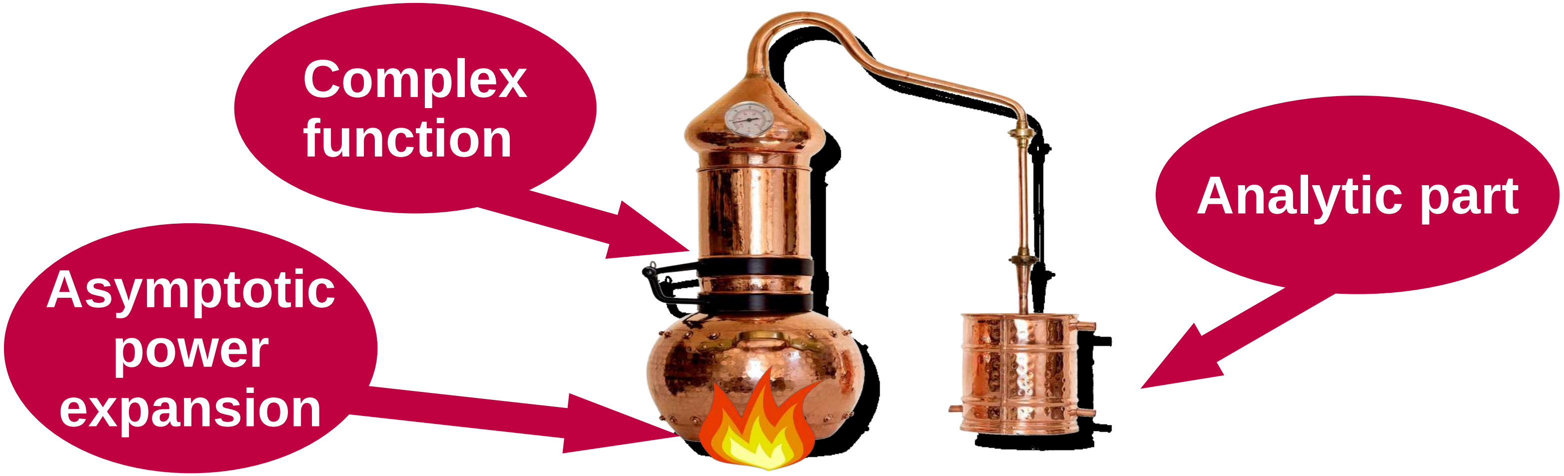
Energy density for massless fermions, equilibrium with acceleration ($\phi = ia/T$)

$$\rho = \frac{3T^4}{8\pi^2} \sum_{n=1}^{\infty} (-1)^{n+1} \phi^4 \frac{\sinh n\phi}{\sinh^5(n\phi/2)}$$

The series is finite as long as ϕ is real. For real thermal vorticity it diverges!

The series includes terms which are non analytic at $\phi = 0$.

Analytic distillation:



The series boils down to polynomials: $\alpha^\mu = \frac{A^\mu}{T}$ $w^\mu = \frac{\omega^\mu}{T}$

$$\rho = \frac{7\pi^2}{60\beta^4} - \frac{\alpha^2}{24\beta^4} - \frac{17\alpha^4}{960\pi^2\beta^4}$$

Expectation values vanish at the Unruh temperature $T_U = \sqrt{-A\cdot A}/2\pi$
[G.Prokhorov, O. Teryaev, V. Zakharov, JHEP03(2020)137]

Axial current under rotation: [V. Ambrus, E. Winstanley, 1908.10244]

$$j_A^\mu = T^2\left(\frac{1}{6} - \frac{w^2}{24\pi^2} - \frac{\alpha^2}{8\pi^2}\right)\frac{w^\mu}{\sqrt{\beta^2}}$$

First exact results at equilibrium with **both rotation and acceleration.**
[V. Ambrus, E. Winstanley Symmetry 2021, 13(11)]

$$\rho = T^4\left(\frac{7\pi^2}{60} - \frac{\alpha^2}{24} - \frac{w^2}{8} - \frac{17\alpha^4}{960\pi^2} + \frac{w^4}{64\pi^2} + \frac{23\alpha^2 w^2}{1440\pi^2} + \frac{11(\alpha\cdot w)^2}{720\pi^2}\right)$$

Massless Dirac field

$$\Pi^\mu = \frac{p^\mu}{2} \frac{\sum_{n=1}^{\infty} (-1)^{n+1} \mathrm{e}^{-\tilde{b}(\phi)\cdot\sum_{k=1}^{n}\Lambda^k p} \mathrm{tr}\left(\not{a}\gamma_5 \exp[-in\phi : \Sigma/2]\not{p}\right) \delta^3(\Lambda^n p - p)}{\sum_{n=1}^{\infty} (-1)^{n+1} \mathrm{e}^{-\tilde{b}(\phi)\cdot\sum_{k=1}^{n}\Lambda^k p} \mathrm{tr}\left(\not{a} \exp[-in\phi : \Sigma/2]\not{p}\right) \delta^3(\Lambda^n p - p)}$$

We can deal with the series as we did before

$$\phi^{\mu\nu} = \epsilon^{\mu\nu\rho\sigma} \frac{h_\rho p_\sigma}{p \cdot q} \qquad\qquad h^\mu = -\frac{1}{2}\epsilon^{\mu\nu\rho\sigma}\phi_{\nu\rho} q_\sigma$$

The series reduces to

$$\eta = \frac{h \cdot p}{q \cdot p} = \frac{1}{2(p \cdot q)} \epsilon^{\mu\nu\alpha\beta} \phi_{\alpha\beta} p_\nu q_\mu$$

$$\Pi^\mu(p) = \frac{p^\mu}{2} \frac{i \sin(\eta/2)}{\cos(\eta/2) + e^{-b\cdot p}}$$

Dirac fermions:

$$\Pi^{\mu}(p) = -\frac{p^{\mu}}{2} \frac{\sinh\left(H/2\right)}{\cosh\left(H/2\right) + \mathrm{e}^{-b\cdot p}}$$

$$H = \frac{1}{2(p\cdot q)} \epsilon^{\mu\nu\alpha\beta} \varpi_{\alpha\beta} p_{\nu} q_{\mu} = -\frac{1}{2\varepsilon} \epsilon^{0\nu\alpha\beta} \varpi_{\alpha\beta} p_{\nu}$$

The mean Pauli-Lubanski vector depends on the orientation of the momentum

$$\varphi^{\nu} = -\frac{1}{2} \epsilon^{\nu\alpha\beta 0} \varpi_{\alpha\beta} \qquad \hat{\mathbf{p}} = \frac{\mathbf{p}}{\varepsilon}$$

Massless particles

For massless particles of arbitrary helicity S:

$$\Theta_{hk}(p) = \frac{\sum_{n=1}^{\infty}(-1)^{2S(n+1)}\mathrm{e}^{-nb\cdot p}\mathrm{e}^{n\eta h}\delta_{hk}}{\sum_{n=1}^{\infty}(-1)^{2S(n+1)}\mathrm{e}^{-nb\cdot p}2\cos n\eta}$$

$$\Pi^{\mu}(p) = -p^{\mu} S \frac{\sinh(S\,H)}{\cosh(S\,H) - (-1)^{2S}\mathrm{e}^{-b\cdot p}}$$

$$H = \frac{1}{2(p\cdot q)}\epsilon^{\mu\nu\alpha\beta}\varpi_{\alpha\beta}p_{\nu}q_{\mu} = -\frac{1}{2\varepsilon}\epsilon^{0\nu\alpha\beta}\varpi_{\alpha\beta}p_{\nu}$$

The reason for this simplified structure is that massless particles only have +S and -S helicity and that

$$D^{S}(W(\Lambda^{n},p))_{ts} = \mathrm{e}^{i\eta r}\delta_{hk} \qquad \widehat{\Pi}_{1,2}(p)|p,h\rangle = 0$$