Analytic Solution for the Revised Helicity Evolution at Small x and Large N_c:

New Resummed Gluon-Gluon Polarized Anomalous Dimension and Intercept

arXiv:2304.06161

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Motivation

Proton spin sum rule: $S_q + L_q + S_G + L_G = \frac{1}{2}$ (Jaffe, Manohar) 10.1016/0550

$$S_q(Q^2) = \frac{1}{2}\int_0^1 \mathrm{d}x \Delta\Sigma(x, Q^2)$$

$$S_q(Q^2 = 10\,\mathrm{GeV}^2) \approx 0.15 \div 0.20$$

for

$$x \in [0.001, 0.7]$$

$$S_G(Q^2) = \int_0^1 \mathrm{d}x \Delta G(x, Q^2)$$

$$S_G(Q^2 = 10\,\mathrm{GeV}^2) \approx 0.13 \div 0.26$$

for

$$x \in [0.05, 0.7]$$

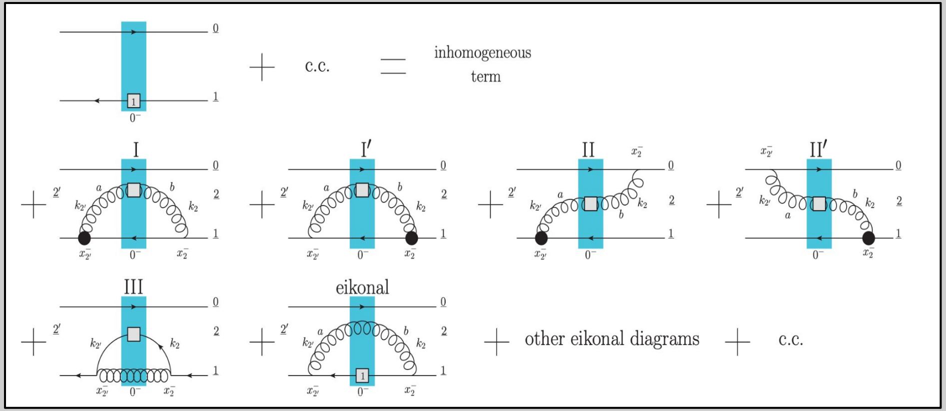
(see e.g. arXiv:1212.1701v3)

Still short of ½

How much spin at small-*x*?

Small-*x* Helicity Evolution

Cougoulic, Kovchegov, Tarasov, Tawabutr arXiv:2204.11898v3
{Kovchegov, Pitonyak, Sievert} arXiv:1511.06737v3, arXiv:1808.09010v1, arXiv:1610.06197v1, arXiv:1706.04236v3



Novel small-*x* helicity evolution equations (KPS-CTT)

Already solved numerically (at large-N_c), giving numerical agreement with existing results (BER)

Bartels, Ermolaev, Ryskin arXiv:hep-ph/9603204v1

Intercepts: BER $3.66\sqrt{\frac{\alpha_s N_c}{2\pi}}$

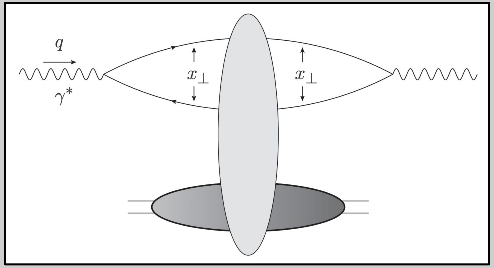
KPS-CTT $3.66\sqrt{\frac{\alpha_s N_c}{2\pi}}$

What about an analytic solution?

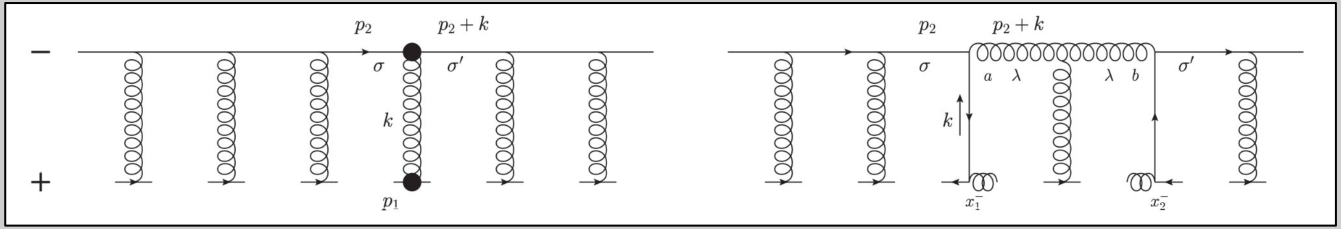
→ Cross check numerical results? Anything new to learn?

Quark and gluon helicity evolution at small-x

Dipole picture of DIS



Helicity evolution enters at the sub-eikonal level



g_1 structure function expressed in terms of the 'polarized dipole amplitudes'

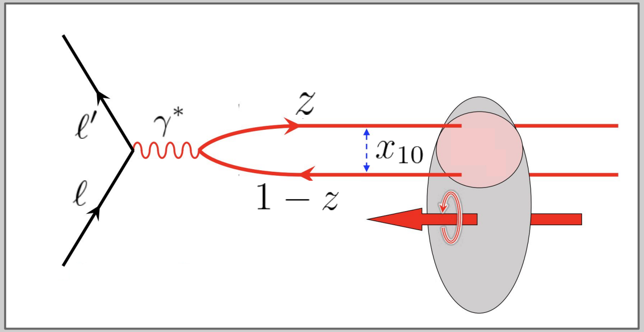
$$G(x_{10}^2, zs),\ G_2(x_{10}^2, zs)$$

$$g_1(x, Q^2) = -\sum_f \frac{N_c Z_f^2}{4\pi^3} \int\limits_{\Lambda^2/s}^{1} \frac{\mathrm{d}z}{z} \int\limits_{\frac{1}{zs}}^{\min\{\frac{1}{zQ^2}, \frac{1}{\Lambda^2}\}} \frac{\mathrm{d}x_{10}^2}{x_{10}^2} \left[G\left(x_{10}^2, zs\right) + 2G_2\left(x_{10}^2, zs\right)\right]$$

Polarized Dipole Amplitudes

$$G_{10}(zs) = \frac{1}{2N_c}\mathrm{Re}\left\langle\left\langle \mathrm{T}\,\mathrm{tr}\left[V_{\underline{0}} V_{\underline{1}}^{G[1]\dagger}\right] + \mathrm{T}\,\mathrm{tr}\left[V_{\underline{1}}^{G[1]} V_{\underline{0}}^{\dagger}\right]\right\rangle\right\rangle(zs)$$

$$G_{10}^{i}(zs) = \frac{1}{2N_c}\left\langle\left\langle \mathrm{tr}\left[V_{\underline{0}}^{\dagger} V_{\underline{1}}^{iG[2]} + \left(V_{\underline{1}}^{iG[2]}\right)^{\dagger} V_{\underline{0}}\right]\right\rangle\right\rangle(zs)$$



$$\int \mathrm{d}^2\left(\frac{x_0+x_1}{2}\right) G_{10}(zs) = \underline{G(x_{10}^2, zs)}$$

$$\int \mathrm{d}^2\left(\frac{x_0+x_1}{2}\right) G_{10}^{i}(zs) = (x_{10})_\perp^i\, G_1(x_{10}^2, zs) + \epsilon^{ij}(x_{10})_\perp^j\, \underline{G_2(x_{10}^2, zs)}$$

$V_{\underline{0}}$ is ordinary (unpolarized) fundamental Wilson line

$$V_{\underline{x}} = \mathcal{P}\exp\left[ig\int_{-\infty}^{\infty} \mathrm{d}x^- A^+\left(0^+, x^-, \underline{x}\right)\right]$$

$V_{\underline{1}}^{G[1]}$, $V_{\underline{1}}^{iG[2]}$ are polarized Wilson line operators

$\Downarrow$

polarization-dependent interactions sandwiched between ordinary Wilson lines

Polarized (Fundamental) Wilson Line Operators

$$V_{\underline{x}}^{\mathrm{pol}[1]} = V_{\underline{x}}^{\mathrm{G}[1]} + V_{\underline{x}}^{\mathrm{q}[1]}, \quad V_{\underline{x},\underline{y}}^{\mathrm{pol}[2]} = V_{\underline{x},\underline{y}}^{\mathrm{G}[2]} + V_{\underline{x}}^{\mathrm{q}[2]}\,\delta^2(\underline{x}-\underline{y}),$$

$$V_{\underline{x}}^{\mathrm{G}[1]} = \frac{i\,g\,P^+}{s}\int\limits_{-\infty}^{\infty} dx^- V_{\underline{x}}[\infty, x^-]\, F^{12}(x^-, \underline{x})\; V_{\underline{x}}[x^-, -\infty],$$

$$V_{\underline{x}}^{\mathrm{q}[1]} = \frac{g^2 P^+}{2\,s}\int\limits_{-\infty}^{\infty} dx_1^- \int\limits_{x_1^-}^{\infty} dx_2^- V_{\underline{x}}[\infty, x_2^-]\, t^b\, \psi_\beta(x_2^-, \underline{x})\, U_{\underline{x}}^{ba}[x_2^-, x_1^-]\; \left[\gamma^+\gamma^5\right]_{\alpha\beta}\, \bar{\psi}_\alpha(x_1^-, \underline{x})\, t^a\, V_{\underline{x}}[x_1^-, -\infty],$$

$$V_{\underline{x},\underline{y}}^{\mathrm{G}[2]} = -\frac{i\,P^+}{s}\int\limits_{-\infty}^{\infty} dz^- d^2z\; V_{\underline{x}}[\infty, z^-]\, \delta^2(\underline{x}-\underline{z})\, \overleftarrow{D}^i(z^-, \underline{z})\, D^i(z^-, \underline{z})\, V_{\underline{y}}[z^-, -\infty]\, \delta^2(\underline{y}-\underline{z}),$$

$$V_{\underline{x}}^{\mathrm{q}[2]} = -\frac{g^2 P^+}{2\,s}\int\limits_{-\infty}^{\infty} dx_1^- \int\limits_{x_1^-}^{\infty} dx_2^- V_{\underline{x}}[\infty, x_2^-]\, t^b\, \psi_\beta(x_2^-, \underline{x})\, U_{\underline{x}}^{ba}[x_2^-, x_1^-]\; \left[\gamma^+\right]_{\alpha\beta}\, \bar{\psi}_\alpha(x_1^-, \underline{x})\, t^a\, V_{\underline{x}}[x_1^-, -\infty].$$

$$V_{\underline{z}}^{i\,\mathrm{G}[2]} \equiv \frac{P^+}{2s}\int\limits_{-\infty}^{\infty} dz^-\, V_{\underline{z}}[\infty, z^-]\, \left[D^i(z^-, \underline{z}) - \overleftarrow{D}^i(z^-, \underline{z})\right]\, V_{\underline{z}}[z^-, -\infty].$$

Dipole Amplitudes Also Give helicity TMDs, PDFs

Gluon helicity TMD
$$g_{1L}^{G\,dip}(x,k_T^2) = \frac{N_c}{\alpha_s 2\pi^4}\int d^2x_{10}\, e^{-i\underline{k}\cdot\underline{x}_{10}}\left[1 + x_{10}^2\frac{\partial}{\partial x_{10}^2}\right] G_2\left(x_{10}^2, zs = \frac{Q^2}{x}\right)$$

Flavor Singlet quark helicity TMD
$$g_{1L}^{S}(x,k_T^2) = \frac{8iN_cN_f}{(2\pi)^5}\int\limits_{\Lambda^2/s}^{1}\frac{dz}{z}\int d^2x_{10}\, e^{i\underline{k}\cdot\underline{x}_{10}}\frac{\underline{x}_{10}}{x_{10}^2}\cdot\frac{\underline{k}}{\underline{k}^2}\left[G(x_{10}^2,zs) + 2G_2(x_{10}^2,zs)\right]$$

Gluon helicity PDF
$$\Delta G(x,Q^2) = \frac{2N_c}{\alpha_s\pi^2}\left[\left(1 + x_{10}^2\frac{\partial}{\partial x_{10}^2}\right)G_2\left(x_{10}^2, zs = \frac{Q^2}{x}\right)\right]_{x_{10}^2 = 1/Q^2}$$

Flavor Singlet quark helicity PDF
$$\Delta\Sigma(x,Q^2) = -\frac{N_cN_f}{2\pi^3}\int\limits_{\Lambda^2/s}^{1}\frac{dz}{z}\int\limits_{\frac{1}{zs}}^{\min\{\frac{1}{zQ^2},\frac{1}{\Lambda^2}\}}\frac{dx_{10}^2}{x_{10}^2}\left[G(x_{10}^2,zs) + 2G_2(x_{10}^2,zs)\right]$$

g_1 structure function
$$g_1(x,Q^2) = -\sum_f\frac{N_cZ_f^2}{4\pi^3}\int\limits_{\Lambda^2/s}^{1}\frac{dz}{z}\int\limits_{\frac{1}{zs}}^{\min\{\frac{1}{zQ^2},\frac{1}{\Lambda^2}\}}\frac{dx_{10}^2}{x_{10}^2}\left[G(x_{10}^2,zs) + 2G_2(x_{10}^2,zs)\right]$$

Small-*x* evolution of the dipole amplitudes

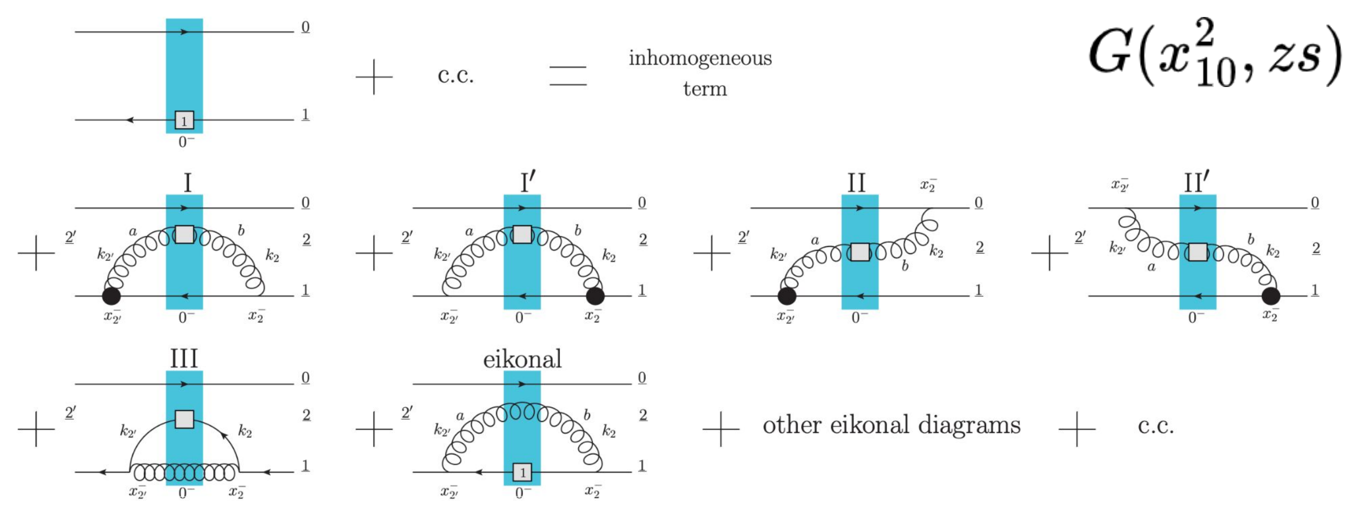
Cougoulic, Kovchegov, Tarasov, Tawabutr
arXiv:2204.11898v3
{Kovchegov, Pitonyak, Sievert}
arXiv:1511.06737v3, arXiv:1808.09010v1, arXiv:1610.06197v1, arXiv:1706.04236v3

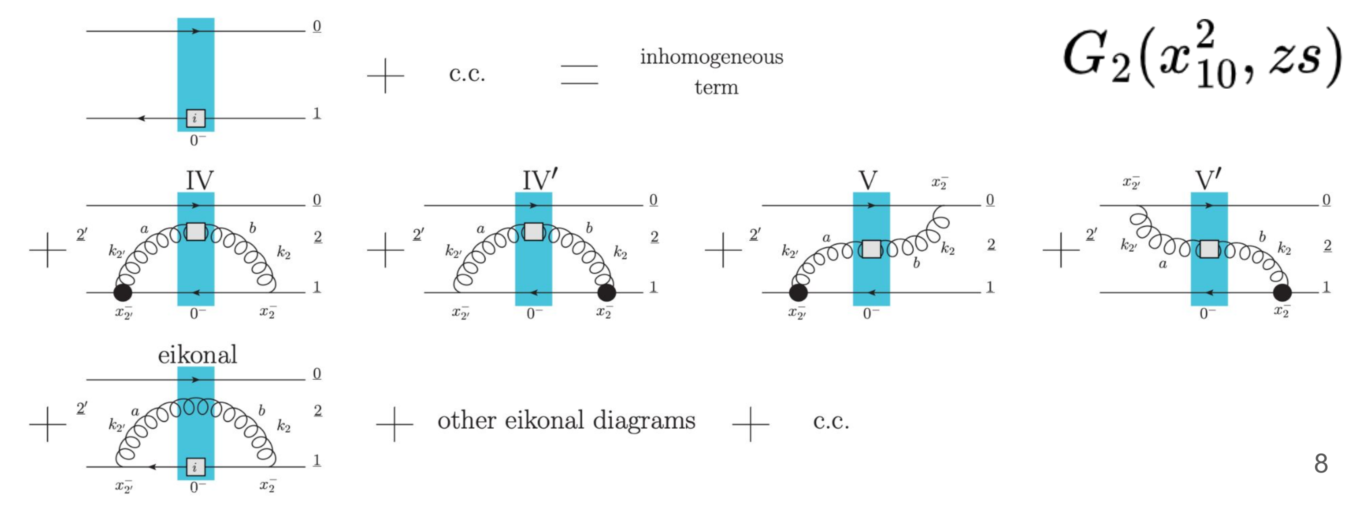
Double-logarithmic - resumming powers of

$$\alpha_s \ln^2(1/x)$$

Full evolution equations don’t close
(like Balitsky hierarchy)

See Balitsky arXiv:hep-ph/9509348v1, arXiv:hep-ph/9812311v1





Equations do close in the large-$\mathrm{N_c}$ limit

Cougoulic, Kovchegov, Tarasov, Tawabutr
arXiv:2204.11898v3

$$G(x_{10}^2, zs) = G^{(0)}(x_{10}^2, zs) + \frac{\alpha_s N_c}{2\pi} \int\limits_{1/sx_{10}^2}^{z} \frac{\mathrm{d}z'}{z'} \int\limits_{1/z's}^{x_{10}^2} \frac{\mathrm{d}x_{21}^2}{x_{21}^2} \left[\Gamma(x_{10}^2, x_{21}^2, z's) + 3G(x_{21}^2, z's) + 2G_2(x_{21}^2, z's) + 2\Gamma_2(x_{10}^2, x_{21}^2, z's)\right]$$

$$\Gamma(x_{10}^2, x_{21}^2, z's) = G^{(0)}(x_{10}^2, z's) + \frac{\alpha_s N_c}{2\pi} \int\limits_{1/sx_{10}^2}^{z'} \frac{\mathrm{d}z''}{z''} \int\limits_{1/z''s}^{\min\{x_{10}^2, x_{21}^2 \frac{z'}{z''}\}} \frac{\mathrm{d}x_{32}^2}{x_{32}^2} \left[\Gamma(x_{10}^2, x_{32}^2, z''s) + 3G(x_{32}^2, z''s) + 2G_2(x_{32}^2, z''s) + 2\Gamma_2(x_{10}^2, x_{32}^2, z''s)\right]$$

$$G_2(x_{10}^2, zs) = G_2^{(0)}(x_{10}^2, zs) + \frac{\alpha_s N_c}{\pi} \int\limits_{\Lambda^2/s}^{z} \frac{\mathrm{d}z'}{z'} \int\limits_{\max\{x_{10}^2, \frac{1}{z's}\}}^{\min\{x_{10}^2 \frac{z}{z'}, \frac{1}{\Lambda^2}\}} \frac{\mathrm{d}x_{21}^2}{x_{21}^2} \left[G(x_{21}^2, z's) + 2G_2(x_{21}^2, z's)\right]$$

$$\Gamma_2(x_{10}^2, x_{21}^2, z's) = G_2^{(0)}(x_{10}^2, z's) + \frac{\alpha_s N_c}{\pi} \int\limits_{\Lambda^2/s}^{z' \frac{x_{21}^2}{x_{10}^2}} \frac{\mathrm{d}z''}{z''} \int\limits_{\max\{x_{10}^2, \frac{1}{z''s}\}}^{\min\{x_{21}^2 \frac{z'}{z''}, \frac{1}{\Lambda^2}\}} \frac{\mathrm{d}x_{32}^2}{x_{32}^2} \left[G(x_{32}^2, z''s) + 2G_2(x_{32}^2, z''s)\right]$$

$\boldsymbol{\Gamma}$ and $\boldsymbol{\Gamma}_2$ are auxiliary functions (‘neighbor dipole amplitudes’)

Would like to solve these equations analytically

Solution

$$G_2(x_{10}^2, zs) = \int \frac{\mathrm{d}\omega}{2\pi i} \int \frac{\mathrm{d}\gamma}{2\pi i} e^{\omega \ln(zsx_{10}^2) + \gamma \ln\left(\frac{1}{x_{10}^2\Lambda^2}\right)} G_{2\omega\gamma}$$

$$G(x_{10}^2, zs) = \int \frac{\mathrm{d}\omega}{2\pi i} \int \frac{\mathrm{d}\gamma}{2\pi i} e^{\omega \ln(zsx_{10}^2) + \gamma \ln\left(\frac{1}{x_{10}^2\Lambda^2}\right)} G_{\omega\gamma}$$

Starting point - double inverse Laplace transforms for dipole amplitudes G_2 and G (along with corresponding transforms for the initial conditions of the evolution)

Can then manipulate the large-N_c equations to find expressions for the neighbor dipole amplitudes and constrain the double-Laplace images $G_{2\omega\gamma}$, $G_{\omega\gamma}$

After some work, the results are…

Solution

$$G_2(x_{10}^2, zs) = \int \frac{d\omega}{2\pi i} \int \frac{d\gamma}{2\pi i} e^{\omega \ln(zsx_{10}^2) + \gamma \ln\left(\frac{1}{x_{10}^2 \Lambda^2}\right)} G_{2\omega\gamma}$$

$$\overline{\alpha}_s = \frac{\alpha_s N_c}{2\pi}$$

$$G(x_{10}^2, zs) = \int \frac{d\omega}{2\pi i} \int \frac{d\gamma}{2\pi i} e^{\omega \ln(zsx_{10}^2) + \gamma \ln\left(\frac{1}{x_{10}^2 \Lambda^2}\right)} \left[\frac{\omega\gamma}{2\overline{\alpha}_s}\left(G_{2\omega\gamma} - G_{2\omega\gamma}^{(0)}\right) - 2G_{2\omega\gamma}\right]$$

$$G_{2\omega\gamma} = G_{2\omega\gamma}^{(0)} + \frac{\overline{\alpha}_s}{\omega\left(\gamma - \gamma_\omega^-\right)\left(\gamma - \gamma_\omega^+\right)}\left[2\left(\gamma - \delta_\omega^+\right)\left(G_{\delta_\omega^+\gamma}^{(0)} + 2G_{2\delta_\omega^+\gamma}^{(0)}\right) - 2\left(\gamma_\omega^+ - \delta_\omega^+\right)\left(G_{\delta_\omega^+\gamma_\omega^+}^{(0)} + 2G_{2\delta_\omega^+\gamma_\omega^+}^{(0)}\right) + 8\delta_\omega^-\left(G_{2\omega\gamma}^{(0)} - G_{2\omega\gamma_\omega^+}^{(0)}\right)\right]$$

$$\delta_\omega^\pm = \frac{\omega}{2}\left[1 \pm \sqrt{1 - \frac{4\overline{\alpha}_s}{\omega^2}}\right] \qquad \gamma_\omega^\pm = \frac{\omega}{2}\left[1 \pm \sqrt{1 - \frac{16\overline{\alpha}_s}{\omega^2}\sqrt{1 - \frac{4\overline{\alpha}_s}{\omega^2}}}\right]$$

Note $G_{2\omega\gamma}^{(0)}$, $G_{\omega\gamma}^{(0)}$ are the double-Laplace images of the initial conditions $G_2^{(0)}(x_{10}^2, zs)$, $G^{(0)}(x_{10}^2, zs)$

Using the Dipole Amplitudes

Can write down small-*x* large-N_c expressions for hTMDs and hPDFs (for arbitrary initial conditions)

$\mathit{\Gamma}$ functions, not neighbor dipole amplitude

$$g_{1L}^{G\,dip}(x,k_T^2) = \frac{2N_c}{\alpha_s \pi^3}\frac{1}{k_T^2}\int\frac{\mathrm{d}\omega}{2\pi i}\int\frac{\mathrm{d}\gamma}{2\pi i}e^{\omega\ln\left(\frac{Q^2}{xk_T^2}\right)+\gamma\ln\left(\frac{k_T^2}{\Lambda^2}\right)}2^{2\omega-2\gamma}\frac{\Gamma(\omega-\gamma+1)}{\Gamma(\gamma-\omega)}G_{2\omega\gamma}$$

$$g_{1L}^{S}(x,k_T^2) = -\frac{N_f}{\alpha_s 2\pi^3}\frac{1}{k_T^2}\int\frac{\mathrm{d}\omega}{2\pi i}\int\frac{\mathrm{d}\gamma}{2\pi i}\left[e^{\omega\ln\left(\frac{Q^2}{xk_T^2}\right)+\gamma\ln\left(\frac{k_T^2}{\Lambda^2}\right)}-e^{(\gamma-\omega)\ln\left(\frac{k_T^2}{\Lambda^2}\right)}\right]2^{2\omega-2\gamma}\frac{\Gamma(\omega-\gamma+1)}{\Gamma(\gamma-\omega+1)}\gamma\left(G_{2\omega\gamma}-G^{(0)}_{2\omega\gamma}\right)$$

$$\Delta G(x,Q^2) = \frac{2N_c}{\alpha_s\pi^2}\int\frac{\mathrm{d}\omega}{2\pi i}\int\frac{\mathrm{d}\gamma}{2\pi i}e^{\omega\ln\left(\frac{1}{x}\right)+\gamma\ln\left(\frac{Q^2}{\Lambda^2}\right)}G_{2\omega\gamma}$$

$$\Delta\Sigma(x,Q^2) = -\frac{N_f}{\alpha_s 2\pi^2}\int\frac{\mathrm{d}\omega}{2\pi i}\int\frac{\mathrm{d}\gamma}{2\pi i}\frac{\omega}{\omega-\gamma}\left(G_{2\omega\gamma}-G^{(0)}_{2\omega\gamma}\right)e^{\omega\ln\left(\frac{1}{x}\right)+\gamma\ln\left(\frac{Q^2}{\Lambda^2}\right)}$$

$$g_1(x,Q^2) = -\frac{1}{2}\sum_f Z_f^2\frac{1}{\alpha_s 2\pi^2}\int\frac{\mathrm{d}\omega}{2\pi i}\int\frac{\mathrm{d}\gamma}{2\pi i}\frac{\omega}{\omega-\gamma}\left(G_{2\omega\gamma}-G^{(0)}_{2\omega\gamma}\right)e^{\omega\ln\left(\frac{1}{x}\right)+\gamma\ln\left(\frac{Q^2}{\Lambda^2}\right)}$$

Resummed Anomalous Dimension

Now fix the initial conditions of the evolution to be simply

$$G_2^{(0)}(x_{10}^2, zs) = 1$$
$$G^{(0)}(x_{10}^2, zs) = 0$$

Gluon helicity PDF becomes:

$$\Delta G(x, Q^2) = \frac{2N_c}{\alpha_s \pi^2} \int \frac{d\omega}{2\pi i} e^{\omega \ln\left(\frac{1}{x}\right) + \gamma_\omega^- \ln\left(\frac{Q^2}{\Lambda^2}\right)} \frac{1}{\omega}$$

Pure-glue polarized anomalous dimension

$$\overline{\alpha}_s = \frac{\alpha_s N_c}{2\pi}$$

$$\Delta\gamma_{GG}(\omega) = \gamma_\omega^- = \frac{\omega}{2}\left[1 - \sqrt{1 - \frac{16\overline{\alpha}_s}{\omega^2}\sqrt{1 - \frac{4\overline{\alpha}_s}{\omega^2}}}\right] = \underline{\frac{4\overline{\alpha}_s}{\omega} + \frac{8\overline{\alpha}_s^2}{\omega^3} + \frac{56\overline{\alpha}_s^3}{\omega^5}} + \frac{496\overline{\alpha}_s^4}{\omega^7} + \mathcal{O}(\alpha_s^5)$$

Agrees with fixed-order calculations up to $\mathcal{O}\left(\alpha_s^3\right)$

Altarelli, Parisi 10.1016/0550-3213(77)90384-4
Mertig & van Neerven arXiv:hep-ph/9506451v3
Moch, Vermaseren, & Vogt arXiv:1409.5131v1
Blümlein, Marquard, Schneider, & Schönwald arXiv:2111.12401v2

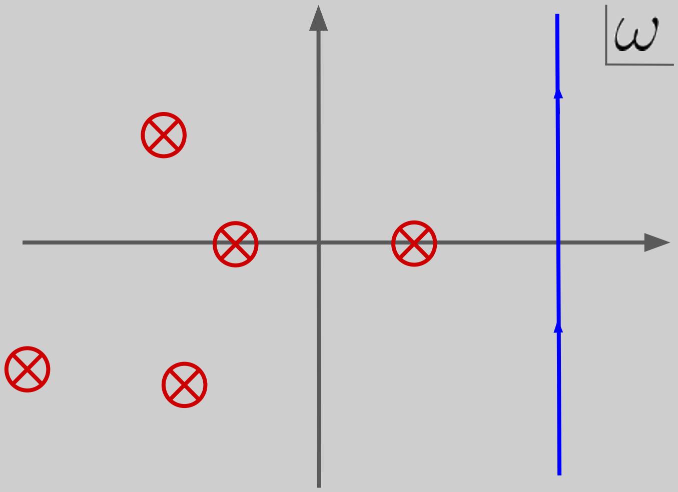
Small-x Asymptotics

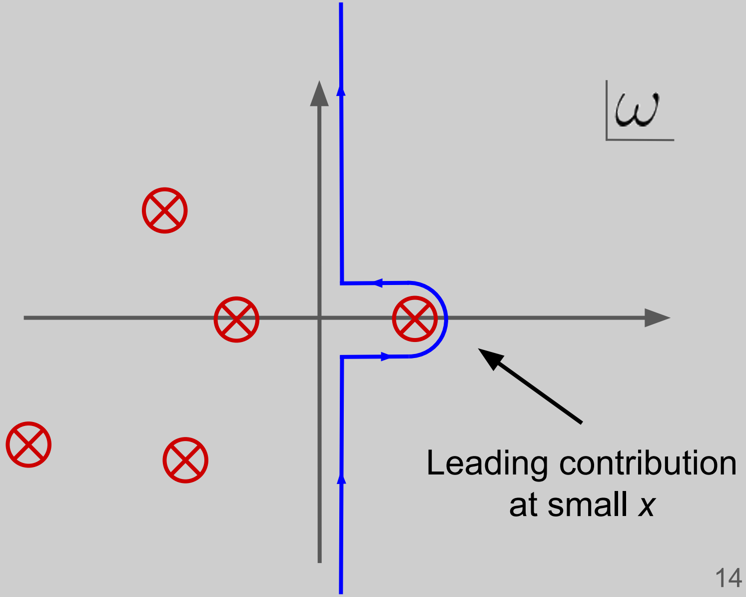
Asymptotics governed by the intercept α_h $\Longrightarrow$ $\Delta\Sigma(x,Q^2) \sim \Delta G(x,Q^2) \sim g_1(x,Q^2) \sim \left(\frac{1}{x}\right)^{\alpha_h}$

corresponds to the rightmost singularity in the ω-plane

$$F(t) = \int \frac{\mathrm{d}\omega}{2\pi i} e^{\omega t} f_\omega$$

Contour for inverse Laplace - parallel to imaginary axis, right of all singularities





Small-x Asymptotics

Rightmost singularity here comes from the polarized anomalous dimension $\Delta\gamma_{GG}(\omega) = \gamma_\omega^-$

See e.g. gluon helicity PDF $\Delta G(x, Q^2) = \frac{2N_c}{\alpha_s \pi^2} \int \frac{d\omega}{2\pi i} e^{\omega \ln\left(\frac{1}{x}\right) + \gamma_\omega^- \ln\left(\frac{Q^2}{\Lambda^2}\right)} \frac{1}{\omega}$

$$\gamma_\omega^- = \frac{\omega}{2}\left[1 - \underbrace{\sqrt{1 - \frac{16\bar{\alpha}_s}{\omega^2}\sqrt{1 - \frac{4\bar{\alpha}_s}{\omega^2}}}}_{}\right]$$

Branch point from the large square root

$$\boxed{\alpha_h = \frac{4}{3^{1/3}}\sqrt{\mathrm{Re}\left[\left(-9 + i\sqrt{111}\right)^{1/3}\right]}\sqrt{\frac{\alpha_s N_c}{2\pi}} \approx 3.66074\sqrt{\frac{\alpha_s N_c}{2\pi}}}$$

Comparison to BER

Bartels, Ermolaev, and Ryskin (BER) IR evolution

Bartels, Ermolaev, Ryskin 9603204v1

$$\overline{\alpha}_s = \frac{\alpha_s N_c}{2\pi}$$

Polarized GG anomalous dimension

$$\Delta\gamma_{GG}^{\rm BER}(\omega) = \frac{\omega}{2}\left[1 - \sqrt{1 - \frac{16\overline{\alpha}_s}{\omega^2}\,\frac{1-\frac{3\overline{\alpha}_s}{\omega^2}}{1-\frac{\overline{\alpha}_s}{\omega^2}}}\right] = \underline{\frac{4\overline{\alpha}_s}{\omega} + \frac{8\overline{\alpha}_s^2}{\omega^3} + \frac{56\overline{\alpha}_s^3}{\omega^5}} + \underline{\frac{504\overline{\alpha}_s^4}{\omega^7}} + \mathcal{O}(\alpha_s^5)$$

Compare to us

$$\Delta\gamma_{GG}(\omega) = \gamma_\omega^- = \frac{\omega}{2}\left[1 - \sqrt{1 - \frac{16\overline{\alpha}_s}{\omega^2}\sqrt{1 - \frac{4\overline{\alpha}_s}{\omega^2}}}\right] = \underline{\frac{4\overline{\alpha}_s}{\omega} + \frac{8\overline{\alpha}_s^2}{\omega^3} + \frac{56\overline{\alpha}_s^3}{\omega^5}} + \underline{\frac{496\overline{\alpha}_s^4}{\omega^7}} + \mathcal{O}(\alpha_s^5)$$

Comparison to BER

Bartels, Ermolaev, and Ryskin (BER) IR evolution

Bartels, Ermolaev, Ryskin 9603204v1

Small-*x* (pure-glue) intercept

$$\alpha_h^{\mathrm{BER}} = \sqrt{\frac{17+\sqrt{97}}{2}}\sqrt{\frac{\alpha_s N_c}{2\pi}} \approx \underline{3.66394}\sqrt{\frac{\alpha_s N_c}{2\pi}}$$

Compare to us

$$\alpha_h = \frac{4}{3^{1/3}}\sqrt{\mathrm{Re}\left[\left(-9+i\sqrt{111}\right)^{1/3}\right]}\sqrt{\frac{\alpha_s N_c}{2\pi}} \approx \underline{3.66074}\sqrt{\frac{\alpha_s N_c}{2\pi}}$$

Why the (*very small*) disagreements with BER?

No hard non-ladder gluons in IREE (?)

Kovchegov, Pitonyak, & Sievert 1610.06197v1

See also Boussarie, Hatta, Yuan arXiv:1904.02693v2

Takeaways

- Analytic solution at small-*x* and large-N_c for the dipole amplitudes
  - → Analytic expressions in the same regime for gluon and flavor-singlet quark helicity TMDs and PDFs, along with g_1
- Small-*x* asymptotics $\Delta\Sigma(x, Q^2) \sim \Delta G(x, Q^2) \sim g_1(x, Q^2) \sim \left(\frac{1}{x}\right)^{\alpha_h}$ $\qquad \alpha_h \approx \underline{3.66074}\sqrt{\frac{\alpha_s N_c}{2\pi}}$
  - A *very* small discrepancy compared to the prediction of BER: $\alpha_h^{\mathrm{BER}} \approx \underline{3.66394}\sqrt{\frac{\alpha_s N_c}{2\pi}}$
- Resummed small-*x* anomalous dimension

$$\Delta\gamma_{GG}(\omega) = \gamma_\omega^- = \frac{\omega}{2}\left[1 - \sqrt{1 - \frac{16\bar{\alpha}_s}{\omega^2}\sqrt{1 - \frac{4\bar{\alpha}_s}{\omega^2}}}\right] = \underline{\frac{4\bar{\alpha}_s}{\omega} + \frac{8\bar{\alpha}_s^2}{\omega^3} + \frac{56\bar{\alpha}_s^3}{\omega^5}} + \underline{\frac{496\bar{\alpha}_s^4}{\omega^7}} + \mathcal{O}(\alpha_s^5)$$

  - Comparison with BER again yields a *very* small discrepancy, only at $\mathcal{O}(\alpha_s^4)$

$$\Delta\gamma_{GG}^{\mathrm{BER}}(\omega) = \frac{\omega}{2}\left[1 - \sqrt{1 - \frac{16\bar{\alpha}_s}{\omega^2}\frac{1 - \frac{3\bar{\alpha}_s}{\omega^2}}{1 - \frac{\bar{\alpha}_s}{\omega^2}}}\right] = \underline{\frac{4\bar{\alpha}_s}{\omega} + \frac{8\bar{\alpha}_s^2}{\omega^3} + \frac{56\bar{\alpha}_s^3}{\omega^5}} + \underline{\frac{504\bar{\alpha}_s^4}{\omega^7}} + \mathcal{O}(\alpha_s^5)$$

- **<u>All in all, very good agreement</u>**
- **Large- N_c & N_f limit next**

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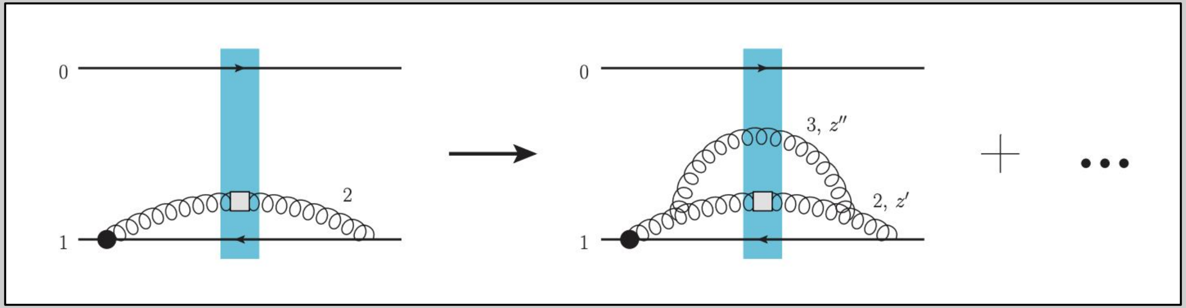
Extra Slides

Neighbor Dipole Amplitudes



One step in evolution of neighbor dipole amplitude

(1) DLA lifetime ordering $\Rightarrow$ $x_{21}^2 z' \gg x_{32}^2 z''$

(2) But also have IR cutoff for dipole 02 $\Rightarrow$ $x_{32} \ll x_{20}$

When $x_{20}^2 > x_{21}^2 \dfrac{z'}{z''}$ $\Rightarrow$ (1) is more constraining than (2)

So for everything to be ordered properly, subsequent evolution in dipole 02 (here evolving to give dipole 32) 'knows' about dipole 21

$$\frac{1}{2N_c}\left\langle\!\!\left\langle \mathrm{tr}\left[V_{\underline{0}}\,V_{\underline{1}}^{\mathrm{pol}[1]\,\dagger}\right]+\mathrm{c.c.}\right\rangle\!\!\right\rangle(zs)=\frac{1}{2N_c}\left\langle\!\!\left\langle \mathrm{tr}\left[V_{\underline{0}}\,V_{\underline{1}}^{\mathrm{pol}[1]\,\dagger}\right]+\mathrm{c.c.}\right\rangle\!\!\right\rangle_0(zs) \qquad G(x_{10}^2,zs)$$

$$+\frac{\alpha_s N_c}{2\pi^2}\int\limits_{\frac{\Lambda^2}{s}}^{z}\frac{dz'}{z'}\int d^2x_2\Bigg\{\left[\frac{1}{x_{21}^2}-\frac{\underline{x}_{21}}{x_{21}^2}\cdot\frac{\underline{x}_{20}}{x_{20}^2}\right]\frac{1}{N_c^2}\left\langle\!\!\left\langle \mathrm{tr}\left[t^bV_{\underline{0}}t^aV_{\underline{1}}^{\dagger}\right]\left(U_{\underline{2}}^{\mathrm{pol}[1]}\right)^{ba}+\mathrm{c.c.}\right\rangle\!\!\right\rangle(z's)$$

$$+\left[2\frac{\epsilon^{ij}\,x_{21}^j}{x_{21}^4}-\frac{\epsilon^{ij}\,(x_{20}^j+x_{21}^j)}{x_{20}^2\,x_{21}^2}-\frac{2\,\underline{x}_{20}\times\underline{x}_{21}}{x_{20}^2\,x_{21}^2}\left(\frac{x_{21}^i}{x_{21}^2}-\frac{x_{20}^i}{x_{20}^2}\right)\right]\frac{1}{N_c^2}\left\langle\!\!\left\langle \mathrm{tr}\left[t^bV_{\underline{0}}t^aV_{\underline{1}}^{\dagger}\right]\left(U_{\underline{2}}^{i\,\mathrm{G}[2]}\right)^{ba}+\mathrm{c.c.}\right\rangle\!\!\right\rangle(z's)\Bigg\}$$

$$+\frac{\alpha_s N_c}{4\pi^2}\int\limits_{\frac{\Lambda^2}{s}}^{z}\frac{dz'}{z'}\int\frac{d^2x_2}{x_{21}^2}\left\{\frac{1}{N_c^2}\left\langle\!\!\left\langle \mathrm{tr}\left[t^b\,V_{\underline{0}}\,t^a\,V_{\underline{2}}^{\mathrm{pol}[1]\,\dagger}\right]U_{\underline{1}}^{ba}\right\rangle\!\!\right\rangle(z's)+2\,\frac{\epsilon^{ij}\,\underline{x}_{21}^j}{x_{21}^2}\,\frac{1}{N_c^2}\left\langle\!\!\left\langle \mathrm{tr}\left[t^b\,V_{\underline{0}}\,t^a\,V_{\underline{2}}^{i\,\mathrm{G}[2]\,\dagger}\right]U_{\underline{1}}^{ba}\right\rangle\!\!\right\rangle(z's)+\mathrm{c.c.}\right\}$$

$$+\frac{\alpha_s N_c}{2\pi^2}\int\limits_{\frac{\Lambda^2}{s}}^{z}\frac{dz'}{z'}\int d^2x_2\,\frac{x_{10}^2}{x_{21}^2\,x_{20}^2}\left\{\frac{1}{N_c^2}\left\langle\!\!\left\langle \mathrm{tr}\left[t^b\,V_{\underline{0}}\,t^a\,V_{\underline{1}}^{\mathrm{pol}[1]\,\dagger}\right]U_{\underline{2}}^{ba}\right\rangle\!\!\right\rangle(z's)-\frac{C_F}{N_c^2}\left\langle\!\!\left\langle \mathrm{tr}\left[V_{\underline{0}}\,V_{\underline{1}}^{\mathrm{pol}[1]\,\dagger}\right]\right\rangle\!\!\right\rangle(z's)+\mathrm{c.c.}\right\}.$$

$$\frac{1}{2N_c}\left\langle\!\!\left\langle \mathrm{tr}\left[V_{\underline{0}}\,V_{\underline{1}}^{i\,\mathrm{G}[2]\,\dagger}\right]+\mathrm{c.c.}\right\rangle\!\!\right\rangle(zs)=\frac{1}{2N_c}\left\langle\!\!\left\langle \mathrm{tr}\left[V_{\underline{0}}\,V_{\underline{1}}^{i\,\mathrm{G}[2]\,\dagger}\right]+\mathrm{c.c.}\right\rangle\!\!\right\rangle_0(zs) \qquad G_2(x_{10}^2,zs)$$

$$+\frac{\alpha_s N_c}{4\pi^2}\int\limits_{\frac{\Lambda^2}{s}}^{z}\frac{dz'}{z'}\int d^2x_2\Bigg\{\left[\frac{\epsilon^{ij}x_{21}^j}{x_{21}^2}-\frac{\epsilon^{ij}x_{20}^j}{x_{20}^2}+2x_{21}^i\frac{\underline{x}_{21}\times\underline{x}_{20}}{x_{21}^2\,x_{20}^2}\right]\frac{1}{N_c^2}\left\langle\!\!\left\langle \mathrm{tr}\left[t^bV_{\underline{0}}t^aV_{\underline{1}}^{\dagger}\right]\left(U_{\underline{2}}^{\mathrm{pol}[1]}\right)^{ba}+\mathrm{c.c.}\right\rangle\!\!\right\rangle(z's)$$

$$+\left[\delta^{ij}\left(\frac{3}{x_{21}^2}-2\,\frac{\underline{x}_{20}\cdot\underline{x}_{21}}{x_{20}^2\,x_{21}^2}-\frac{1}{x_{20}^2}\right)-2\frac{x_{21}^i\,x_{20}^j}{x_{21}^2\,x_{20}^2}\left(2\frac{\underline{x}_{20}\cdot\underline{x}_{21}}{x_{20}^2}+1\right)+2\frac{x_{21}^i\,x_{21}^j}{x_{21}^2\,x_{20}^2}\left(2\frac{\underline{x}_{20}\cdot\underline{x}_{21}}{x_{21}^2}+1\right)+2\frac{x_{20}^i\,x_{20}^j}{x_{20}^4}-2\frac{x_{21}^i\,x_{21}^j}{x_{21}^4}\right]$$

$$\times\frac{1}{N_c^2}\left\langle\!\!\left\langle \mathrm{tr}\left[t^bV_{\underline{0}}t^aV_{\underline{1}}^{\dagger}\right]\left(U_{\underline{2}}^{j\,\mathrm{G}[2]}\right)^{ba}+\mathrm{c.c.}\right\rangle\!\!\right\rangle(z's)\Bigg\}$$

$$+\frac{\alpha_s N_c}{2\pi^2}\int\limits_{\frac{\Lambda^2}{s}}^{z}\frac{dz'}{z'}\int d^2x_2\,\frac{x_{10}^2}{x_{21}^2\,x_{20}^2}\left\{\frac{1}{N_c^2}\left\langle\!\!\left\langle \mathrm{tr}\left[t^b\,V_{\underline{0}}\,t^a\,V_{\underline{1}}^{i\,\mathrm{G}[2]\,\dagger}\right]\left(U_{\underline{2}}\right)^{ba}\right\rangle\!\!\right\rangle(z's)-\frac{C_F}{N_c^2}\left\langle\!\!\left\langle \mathrm{tr}\left[V_{\underline{0}}\,V_{\underline{1}}^{i\,\mathrm{G}[2]\,\dagger}\right]\right\rangle\!\!\right\rangle(z's)+\mathrm{c.c.}\right\}.$$

Full equations for the fundamental dipole amplitudes (don't close)

Full Solution

$$G_{2\omega\gamma} = G^{(0)}_{2\omega\gamma} + \frac{\overline{\alpha}_s}{\omega\left(\gamma-\gamma_\omega^-\right)\left(\gamma-\gamma_\omega^+\right)}\left[2\left(\gamma-\delta_\omega^+\right)\left(G^{(0)}_{\delta_\omega^+\gamma}+2G^{(0)}_{2\delta_\omega^+\gamma}\right)-2\left(\gamma_\omega^+-\delta_\omega^+\right)\left(G^{(0)}_{\delta_\omega^+\gamma_\omega^+}+2G^{(0)}_{2\delta_\omega^+\gamma_\omega^+}\right)+8\delta_\omega^-\left(G^{(0)}_{2\omega\gamma}-G^{(0)}_{2\omega\gamma_\omega^+}\right)\right]$$

$$G^{(0)}(x_{10}^2, zs) = \int \frac{\mathrm{d}\omega}{2\pi i}\int\frac{\mathrm{d}\gamma}{2\pi i} e^{\omega\ln\left(zsx_{10}^2\right)+\gamma\ln\left(\frac{1}{x_{10}^2\Lambda^2}\right)} G^{(0)}_{\omega\gamma}$$

$$G^{(0)}_2(x_{10}^2, zs) = \int \frac{\mathrm{d}\omega}{2\pi i}\int\frac{\mathrm{d}\gamma}{2\pi i} e^{\omega\ln\left(zsx_{10}^2\right)+\gamma\ln\left(\frac{1}{x_{10}^2\Lambda^2}\right)} G^{(0)}_{2\omega\gamma}$$

$$\Gamma_\omega^+(x_{10}^2) = \frac{e^{-\delta_\omega^+\ln\left(\frac{1}{x_{10}^2\Lambda^2}\right)}}{\overline{\alpha}_s\left(\delta_\omega^+-\delta_\omega^-\right)}\int\frac{\mathrm{d}\gamma}{2\pi i}e^{\gamma\ln\left(\frac{1}{x_{10}^2\Lambda^2}\right)}\frac{\omega\delta_\omega^+}{2(\gamma-\delta_\omega^+)}\left[G_{2\omega\gamma}\left(\gamma^2-\omega\gamma+4\overline{\alpha}_s-\frac{8\overline{\alpha}_s}{\omega}\delta_\omega^-\right)-G^{(0)}_{2\omega\gamma}\left(\gamma^2-\omega\gamma+4\overline{\alpha}_s\right)\right]$$

$$\Gamma_\omega^-(x_{10}^2) = \frac{e^{-\delta_\omega^-\ln\left(\frac{1}{x_{10}^2\Lambda^2}\right)}}{\overline{\alpha}_s\left(\delta_\omega^--\delta_\omega^+\right)}\int\frac{\mathrm{d}\gamma}{2\pi i}e^{\gamma\ln\left(\frac{1}{x_{10}^2\Lambda^2}\right)}\frac{\omega\delta_\omega^-}{2(\gamma-\delta_\omega^-)}\left[G_{2\omega\gamma}\left(\gamma^2-\omega\gamma+4\overline{\alpha}_s-\frac{8\overline{\alpha}_s}{\omega}\delta_\omega^+\right)-G^{(0)}_{2\omega\gamma}\left(\gamma^2-\omega\gamma+4\overline{\alpha}_s\right)\right]$$

$$\delta_\omega^\pm = \frac{\omega}{2}\left[1\pm\sqrt{1-\frac{4\overline{\alpha}_s}{\omega^2}}\right] \qquad \gamma_\omega^\pm = \frac{\omega}{2}\left[1\pm\sqrt{1-\frac{16\overline{\alpha}_s}{\omega^2}\sqrt{1-\frac{4\overline{\alpha}_s}{\omega^2}}}\right]$$

Full Solution

$$G_2(x_{10}^2, zs) = \int \frac{\mathrm{d}\omega}{2\pi i} \int \frac{\mathrm{d}\gamma}{2\pi i} e^{\omega \ln(zsx_{10}^2) + \gamma \ln\left(\frac{1}{x_{10}^2\Lambda^2}\right)} G_{2\omega\gamma}$$

$$\Gamma_2(x_{10}^2, x_{21}^2, z's) = \int \frac{\mathrm{d}\omega}{2\pi i} \int \frac{\mathrm{d}\gamma}{2\pi i} \left[e^{\omega \ln(z'sx_{21}^2) + \gamma \ln\left(\frac{1}{x_{10}^2\Lambda^2}\right)} \left(G_{2\omega\gamma} - G_{2\omega\gamma}^{(0)}\right) + e^{\omega \ln(z'sx_{10}^2) + \gamma \ln\left(\frac{1}{x_{10}^2\Lambda^2}\right)} G_{2\omega\gamma}^{(0)} \right]$$

$$G(x_{10}^2, zs) = \int \frac{\mathrm{d}\omega}{2\pi i} \int \frac{\mathrm{d}\gamma}{2\pi i} e^{\omega \ln(zsx_{10}^2) + \gamma \ln\left(\frac{1}{x_{10}^2\Lambda^2}\right)} \left[\frac{\omega\gamma}{2\overline{\alpha}_s} \left(G_{2\omega\gamma} - G_{2\omega\gamma}^{(0)}\right) - 2G_{2\omega\gamma} \right]$$

$$\begin{aligned}
\Gamma(x_{10}^2, x_{21}^2, z's) = &\int \frac{\mathrm{d}\omega}{2\pi i} e^{\omega \ln(z'sx_{21}^2)} \left[\Gamma_\omega^+(x_{10}^2) e^{\delta_\omega^+ \ln\left(\frac{1}{x_{21}^2\Lambda^2}\right)} + \Gamma_\omega^-(x_{10}^2) e^{\delta_\omega^- \ln\left(\frac{1}{x_{21}^2\Lambda^2}\right)} \right] \\
&+ \int \frac{\mathrm{d}\omega}{2\pi i} \int \frac{\mathrm{d}\gamma}{2\pi i} e^{\omega \ln(z'sx_{21}^2) + \gamma \ln\left(\frac{1}{x_{21}^2\Lambda^2}\right)} \left[\frac{\left(-\frac{3}{2}\omega\gamma + 4\overline{\alpha}_s\right) G_{2\omega\gamma} + \frac{3}{2}\omega\gamma G_{2\omega\gamma}^{(0)}}{\gamma^2 - \omega\gamma + \overline{\alpha}_s} \right] \\
&- \int \frac{\mathrm{d}\omega}{2\pi i} \int \frac{\mathrm{d}\gamma}{2\pi i} \left[2e^{\omega \ln(z'sx_{21}^2) + \gamma \ln\left(\frac{1}{x_{10}^2\Lambda^2}\right)} \left(G_{2\omega\gamma} - G_{2\omega\gamma}^{(0)}\right) + 2e^{\omega \ln(z'sx_{10}^2) + \gamma \ln\left(\frac{1}{x_{10}^2\Lambda^2}\right)} G_{2\omega\gamma}^{(0)} \right]
\end{aligned}$$

Useful properties of the large-N_c equations

Scaling between $\mathbf{G_2}$ and $\boldsymbol{\Gamma}_2$

$$\Gamma_2(s_{10}, s_{21}, \eta') - G_2^{(0)}(s_{10}, \eta') = G_2(s_{10}, \eta = \eta' + s_{10} - s_{21}) - G_2^{(0)}(s_{10}, \eta = \eta' + s_{10} - s_{21})$$

Boundary conditions for neighbors

$$\Gamma_2(s_{10}, s_{21} = s_{10}, \eta) = G_2(s_{10}, \eta)$$
$$\Gamma(s_{10}, s_{21} = s_{10}, \eta) = G(s_{10}, \eta)$$

PDE for $\boldsymbol{\Gamma}$

$$\frac{\partial^2 \Gamma(s_{10}, s_{21}, \eta')}{\partial s_{21}^2} + \frac{\partial^2 \Gamma(s_{10}, s_{21}, \eta')}{\partial s_{21} \partial \eta'} + \Gamma(s_{10}, s_{21}, \eta') = -3G(s_{21}, \eta') - 2G_2(s_{21}, \eta') - 2\Gamma_2(s_{10}, s_{21}, \eta')$$

Note the rescaled variables

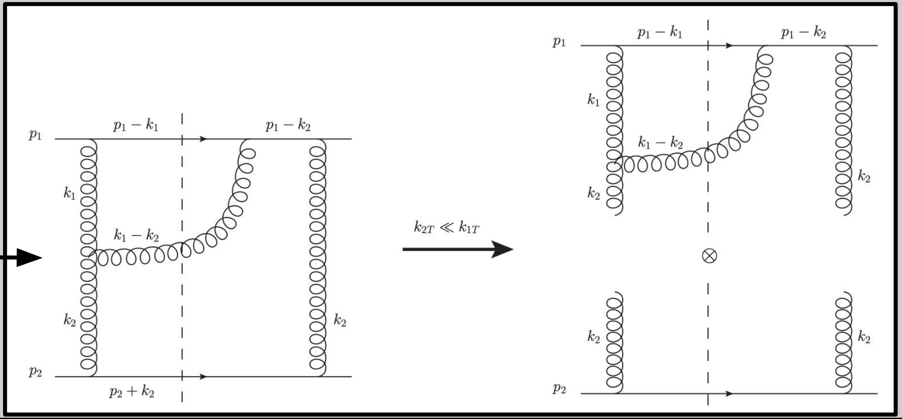
$$\eta = \sqrt{\overline{\alpha}_s} \ln \frac{zs}{\Lambda^2} \qquad \eta' = \sqrt{\overline{\alpha}_s} \ln \frac{z's}{\Lambda^2}$$
$$s_{10} = \sqrt{\overline{\alpha}_s} \ln \frac{1}{x_{10}^2 \Lambda^2} \qquad s_{21} = \sqrt{\overline{\alpha}_s} \ln \frac{1}{x_{21}^2 \Lambda^2}$$

with $\overline{\alpha}_s = \frac{\alpha_s N_c}{2\pi}$

Disagreement with BER

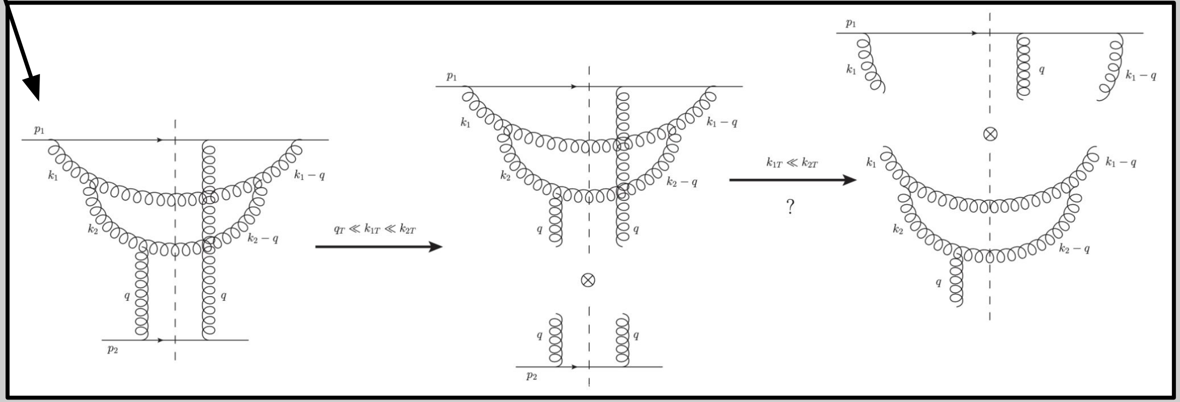
No hard non-ladder gluons in IREE

Two diagrams contained within KPS-CTT evolution



Ladder with rails k_1- k_2 & k_2 , (uncut) rung p_1- k_2 , and bremsstrahlung gluon k_1

Hard non-ladder gluon k_1 - k_2 accommodated at $\mathcal{O}(\alpha_s^3)$



3- and 5-point Green functions (BER IREE have only 4-point)

Problem at $\mathcal{O}(\alpha_s^4)$ (?)