6144 Pions from Lattice QCD

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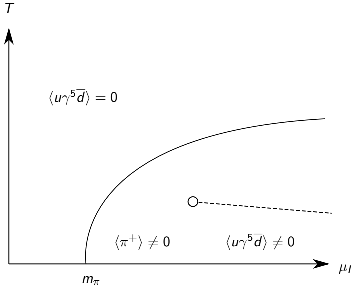
Isospin Chemical Potential

- Isospin chemical potential μ_I, similar to baryon chemical potential μ_B
  - but no sign problem
- Relevant for high isospin-systems, e.g. neutron stars
- Phase Diagram (Son and Stephanov 2000):



Multipion Correlators

- To use grand canonical ensemble, need many-pion correlators
- Goal: compute correlator C_n for $n = 1, \ldots, N$

$$C_n(t) = \sum_{\{y_i\}} \left\langle \left(\sum_x \pi^-(x,t) \right)^n \pi^+(y_1, 0) \ldots \pi^+(y_n, 0) \right\rangle$$

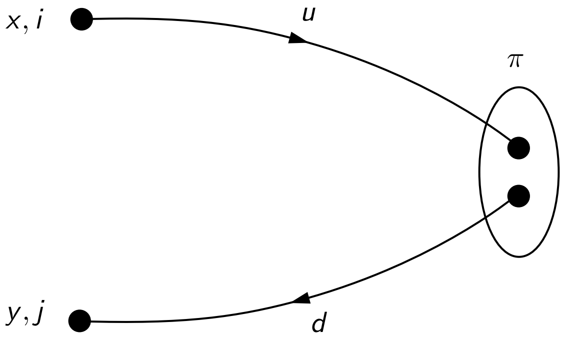
- Naive Wick contractions: $O(n!)$, works for $n \lesssim 10$
- Previous work: up to $n = 72$ (Detmold et al. 2012)

The Pion Block

Definition:

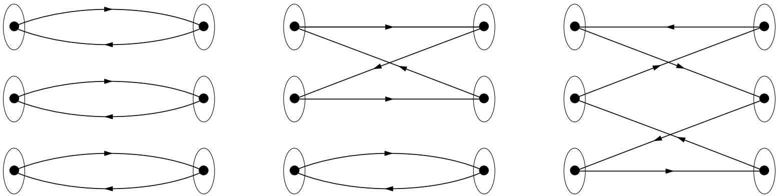
$$\Pi_{ij}(x,y) = \sum_z S_{ik}(x,0;z,t)S^{\dagger}_{kj}(z,t;y,0)$$

- Numerically: $N \times N$ matrix, $N = \max \#$ of pions



Pion Correlators from Pion blocks

- Using Π, can compute C_n for $n \leq N$:



$\mathrm{Tr}(\Pi)^3$ $\mathrm{Tr}(\Pi^2)\,\mathrm{Tr}(\Pi)$ $\mathrm{Tr}(\Pi^3)$

$$C_3 = \mathrm{Tr}(\Pi)^3 - 3\,\mathrm{Tr}(\Pi^2)\,\mathrm{Tr}(\Pi) + 2\,\mathrm{Tr}(\Pi)^3$$

Description of the Algorithm

- Key idea: change into eigenbasis of Π and use symmetry
- Eigenvalues of Π: $x_1, \ldots x_N$
- Expand via

$$\mathrm{Tr}(\Pi^k) = \sum_{i=1}^{N} x_i^k$$

Example

Example with $N = 4$:

$$\begin{aligned} C_3 &= \operatorname{Tr}(\Pi)^3 - 3\operatorname{Tr}(\Pi^2)\operatorname{Tr}(\Pi) + 2\operatorname{Tr}(\Pi^3) \\ &= (x_1 + x_2 + x_3 + x_4)^3 \\ &- 3(x_1^2 + x_2^2 + x_3^2 + x_4^2)(x_1 + x_2 + x_3 + x_4) \\ &+ 2(x_1^3 + x_2^3 + x_3^3 + x_4^3) \end{aligned}$$

Example

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$$
\begin{aligned}
C_3 &= \mathrm{Tr}(\Pi)^3 - 3\,\mathrm{Tr}(\Pi^2)\,\mathrm{Tr}(\Pi) + 2\,\mathrm{Tr}(\Pi^3) \\
&= (x_1 + x_2 + x_3 + x_4)^3 \\
&\quad - 3(x_1^2 + x_2^2 + x_3^2 + x_4^2)(x_1 + x_2 + x_3 + x_4) \\
&\quad + 2(x_1^3 + x_2^3 + x_3^3 + x_4^3) \\
&= 6(x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4)
\end{aligned}
$$

Description of the Algorithm

- General formula:

$$C_n = n!E_n(x_1, \ldots, x_N) = n! \sum_{i_1 < i_2 < \cdots < i_n} x_{i_1} \ldots x_{i_n}$$

- Problem: $\binom{N}{n}$ terms – too many
- Solution: simplify using

$$E_n(x_1, \ldots, x_k) = E_n(x_1, \ldots x_{k-1}) + x_k E_{n-1}(x_1, \ldots, x_{n-1})$$

- Need values for $1 \leq n, k \leq N \implies O(N^2)$ work for all correlators
    - Computing eigenvalues x_i is $O(N^3)$

Lattice Details

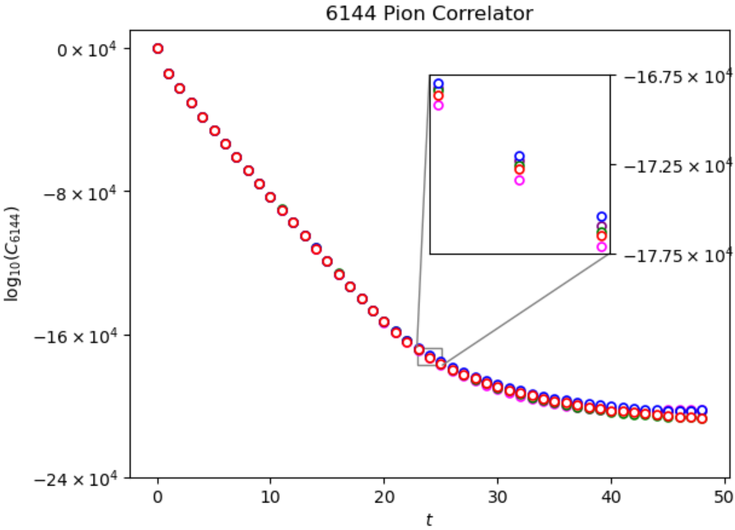
- Generated by NPLQCD collaboration
- Wilson Clover fermions
- $a = 0.091\,\mathrm{fm}$, $a^{-1} = 2.17\,\mathrm{GeV}$
- $M_\pi \sim 170\,\mathrm{MeV}$
- 200 configurations on $48^3 \times 96$ volume
- 8^3 grid of smeared + sparsened point-source propagators
  - Maximum # of pions: $N = 12 \times 8^3 = 6144$

ID	a (fm)	M_π (MeV)	β	C_{SW}	am_{ud}	am_s	$L^3 \times T$	$M_\pi L$
a091m170	0.091(1)	166(2)	6.3	1.20536588	-0.2416	-0.2050	$48^3 \times 96$	3.7
a091m170L	0.091(1)	172(6)	6.3	1.20536588	-0.2416	-0.2050	$64^3 \times 128$	5.08

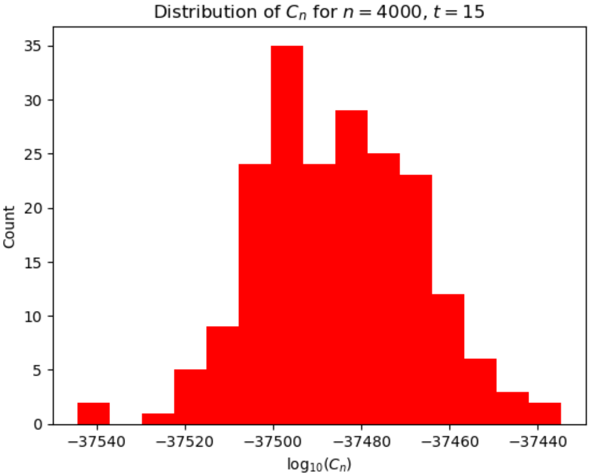
Practical Considerations

- Pion block Π ill-conditioned $\implies$ use high-precision SVD instead of eigendecomposition
  - Same result since Π positive definite
- Tested at varying precisions, chose long-double (80 bit)
  - Finite precision limits $n_\pi \leq 4655$

6144 Pion Correlator



Correlator Distribution



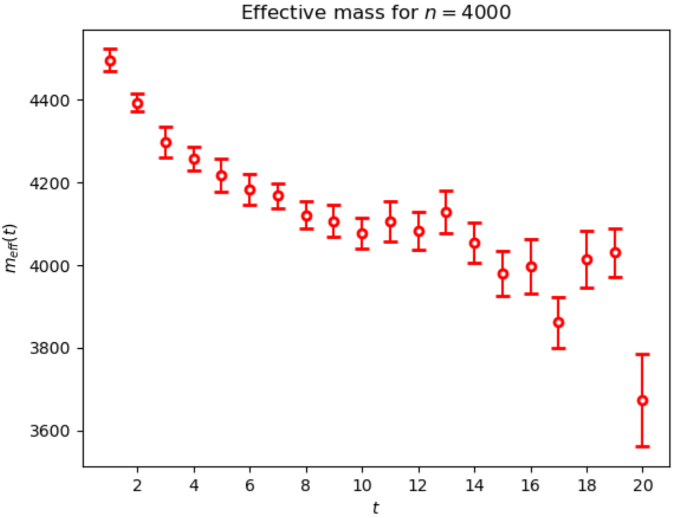
Assuming Log Normality

- If we assume $\log C_n(t)[U] \sim \mathcal{N}(\mu_n, \sigma_n^2)$ then

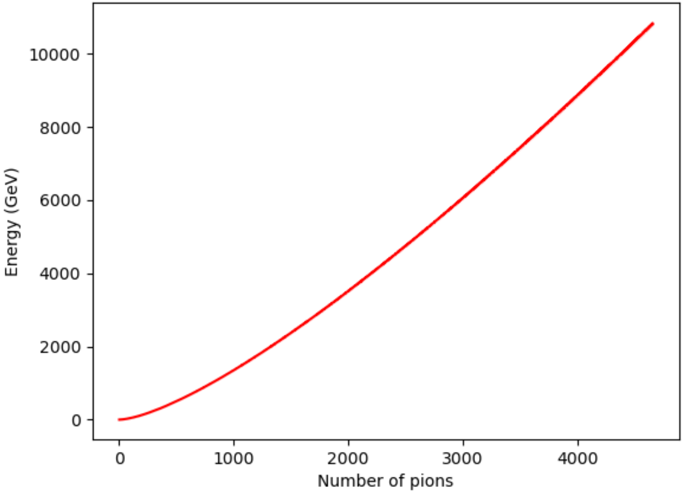
$$\log \langle C_n \rangle = \mu_n + \frac{\sigma_n^2}{2}$$

- By estimating μ_n, σ_n, can recover more information
  - Can make this systematically improvable via cumulants
- Empirically, log-normality seems good except for t in middle of lattice

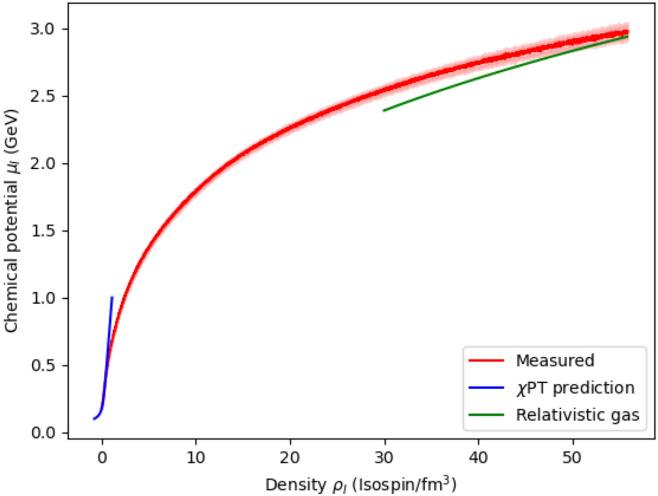
Effective Mass



Effective Mass



Chemical Potential



Conclusion/Outlook

- New method pushes the limit on multipion correlators from 72 to 6144
- Analysis of many-pion correlators difficult, but log-normality helps
- Future work:
  - Other mesons (e.g. K^+), mixed systems of mesons
  - Baryons
- Thanks to NPLQCD for the propagators

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- Thanks to NPLQCD for the propagators
- Thanks for listening! Questions?

Backup

Formulae for isospin density

- Relativistic Gas:

$$\mu_I = (6\pi^2 \rho_I)^{1/3} \tag{1}$$

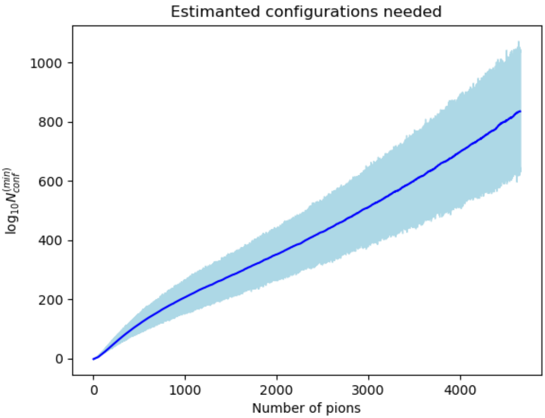
- ChiPT (Son and Stephanov, 2000)
  - (conventions from Detmold et al, 2008)

$$\rho_I = \frac{1}{2} f_\pi^2 \mu_I \left(1 - \frac{m_\pi^4}{\mu_I^4}\right) \tag{2}$$

- If $\log X \sim \mathcal{N}(\mu, \sigma^2)$, central limit theorem needs at least

$$N_{\text{conf}}^{(min)} = e^{\sigma^2} - 1$$

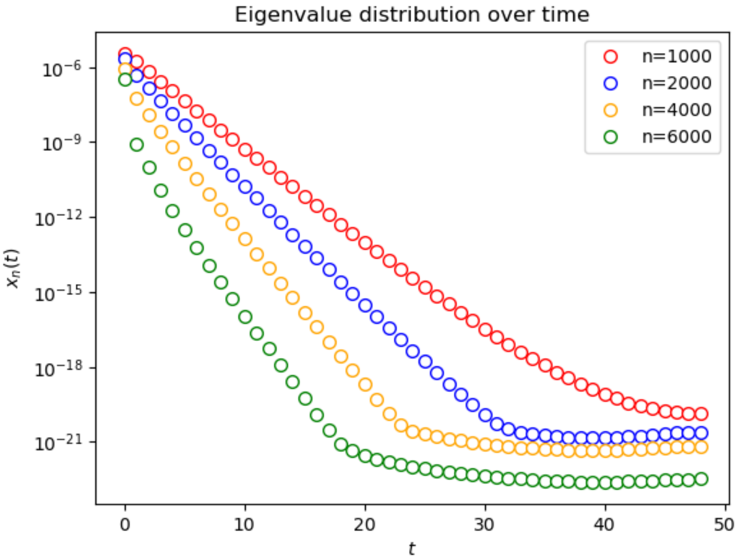
- Estimate of $N_{\text{conf}}^{(min)}$:



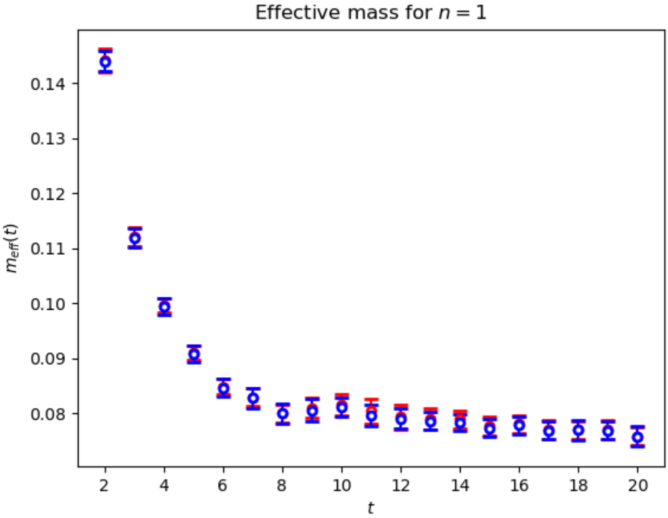
$$P_{(\lambda_1,\ldots,\lambda_n)}(x) = x_1^{\lambda_1} \ldots x_n^{\lambda_n}$$

$$\begin{aligned}
C_n &= \sum_{\sigma \in S_n} \epsilon(\sigma) P_{\lambda(\sigma)}(x) \\
&= \sum_{\sigma \in S_n} \sum_{\lambda'} \epsilon(\sigma) \chi_{\lambda'}(\sigma) S_{\lambda'}(x) \\
&= \sum_{\lambda'} S_{\lambda'}(x) \sum_{\sigma \in S_n} \epsilon(\sigma) \chi_{\lambda'}(\sigma) \\
&= n! S_{(1,\ldots,1)}(x) \\
&= n! E_n(x)
\end{aligned}$$

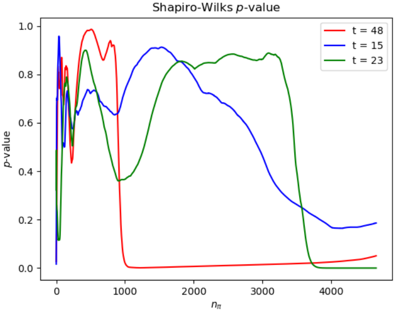
Eigenvalue Distribution

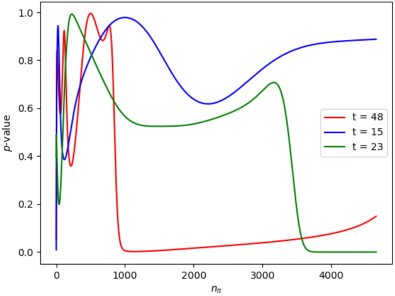


Effective Mass

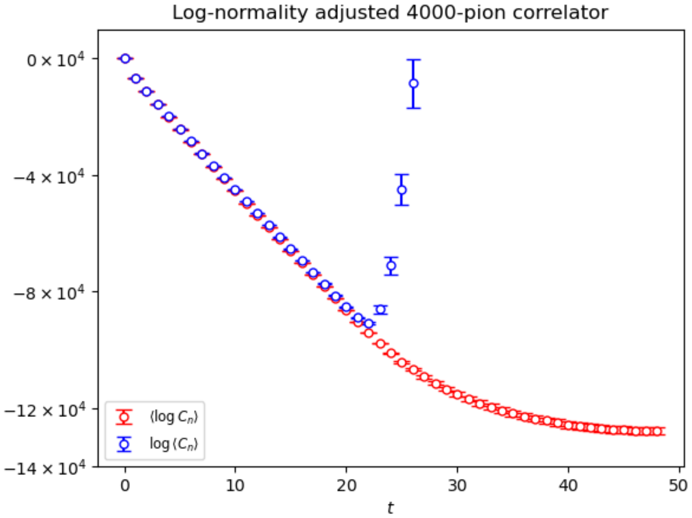


Log Normality





Assuming Log Normality



More Correlator Trace Formulae

$$
\begin{aligned}
C_4 &= \mathrm{Tr}(\Pi)^4 - 6\,\mathrm{Tr}(\Pi^2)\,\mathrm{Tr}(\Pi)^2 + 3\,\mathrm{Tr}(\Pi^2)^2 + 8\,\mathrm{Tr}(\Pi^3)\,\mathrm{Tr}(\Pi) - 6\,\mathrm{Tr}(\Pi^4) \\
C_5 &= \mathrm{Tr}(\Pi)^5 - 10\,\mathrm{Tr}(\Pi^2)\,\mathrm{Tr}(\Pi)^3 + 15\,\mathrm{Tr}(\Pi^2)^2\,\mathrm{Tr}(\Pi) + 20\,\mathrm{Tr}(\Pi^3)\,\mathrm{Tr}(\Pi)^2 \\
&\quad - 20\,\mathrm{Tr}(\Pi^3)\,\mathrm{Tr}(\Pi^2) - 30\,\mathrm{Tr}(\Pi^4)\,\mathrm{Tr}(\Pi) + 24\,\mathrm{Tr}(\Pi^5) \\
C_6 &= \mathrm{Tr}(\Pi)^6 - 15\,\mathrm{Tr}(\Pi^2)\,\mathrm{Tr}(\Pi)^4 + 45\,\mathrm{Tr}(\Pi^2)^2\,\mathrm{Tr}(\Pi)^2 - 15\,\mathrm{Tr}(\Pi^2)^3 \\
&\quad + 40\,\mathrm{Tr}(\Pi^3)\,\mathrm{Tr}(\Pi)^3 - 120\,\mathrm{Tr}(\Pi^3)\,\mathrm{Tr}(\Pi^2)\,\mathrm{Tr}(\Pi) + 40\,\mathrm{Tr}(\Pi^3)^2 \\
&\quad - 90\,\mathrm{Tr}(\Pi^4)\,\mathrm{Tr}(\Pi)^2 + 90\,\mathrm{Tr}(\Pi^4)\,\mathrm{Tr}(\Pi^2) + 144\,\mathrm{Tr}(\Pi^5)\,\mathrm{Tr}(\Pi) \\
&\quad - 120\,\mathrm{Tr}(\Pi^6)
\end{aligned}
$$

More Correlator Trace Formulae

$$\begin{aligned}C_7 = {} & \mathrm{Tr}(\Pi)^7 - 21\,\mathrm{Tr}(\Pi^2)\,\mathrm{Tr}(\Pi)^5 + 105\,\mathrm{Tr}(\Pi^2)^2\,\mathrm{Tr}(\Pi)^3 \\ & - 105\,\mathrm{Tr}(\Pi^2)^3\,\mathrm{Tr}(\Pi) + 70\,\mathrm{Tr}(\Pi^3)\,\mathrm{Tr}(\Pi)^4 - 420\,\mathrm{Tr}(\Pi^3)\,\mathrm{Tr}(\Pi^2)\,\mathrm{Tr}(\Pi)^2 \\ & + 210\,\mathrm{Tr}(\Pi^3)\,\mathrm{Tr}(\Pi^2)^2 + 280\,\mathrm{Tr}(\Pi^3)^2\,\mathrm{Tr}(\Pi) - 210\,\mathrm{Tr}(\Pi^4)\,\mathrm{Tr}(\Pi)^3 \\ & + 630\,\mathrm{Tr}(\Pi^4)\,\mathrm{Tr}(\Pi^2)\,\mathrm{Tr}(\Pi) - 420\,\mathrm{Tr}(\Pi^4)\,\mathrm{Tr}(\Pi^3) + 504\,\mathrm{Tr}(\Pi^5)\,\mathrm{Tr}(\Pi)^2 \\ & - 504\,\mathrm{Tr}(\Pi^5)\,\mathrm{Tr}(\Pi^2) - 840\,\mathrm{Tr}(\Pi^6)\,\mathrm{Tr}(\Pi) + 720\,\mathrm{Tr}(\Pi^7)\end{aligned}$$