



Exotic hadron spectroscopy with functional methods

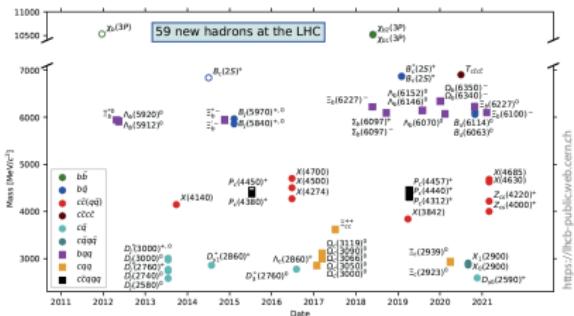
Gernot Eichmann

FDSA2022 - 4th Workshop on Future Directions in Spectroscopy Analysis, JLab
Nov 16, 2022

Many open questions

- Understanding exotic hadrons:

Hadron spectroscopy at LHC, Belle II, BES III, PANDA, JLab, ELSA, ...



- Mass generation and confinement?



- Quark-gluon structure of hadrons and nuclei:
Hadron tomography at EIC, JLab, COMPASS/AMBER, ...

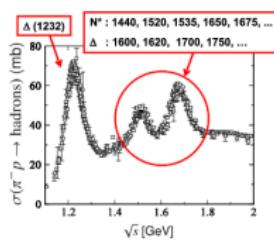


Theory tools

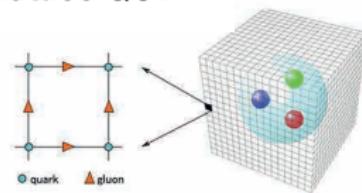
Functional methods (DSEs & BSEs, FRG, ...)



Amplitude analyses



Lattice QCD



Phenomenological models



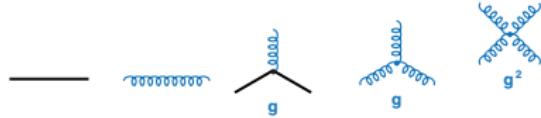
Effective theories (ChPT, ...)



Functional methods

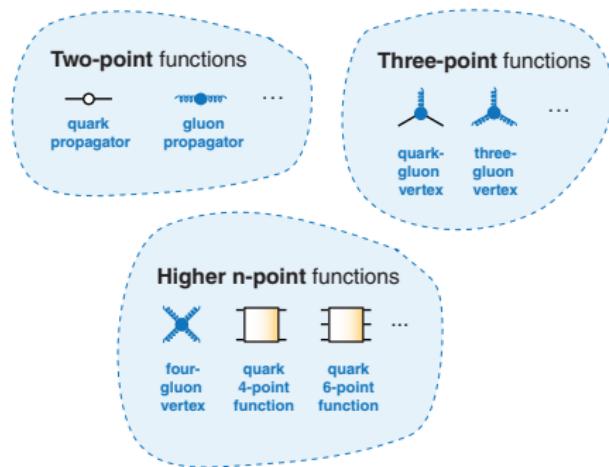
Classical Lagrangian of QCD:

$$\mathcal{L} = \bar{\psi} (i\cancel{\partial} + g\cancel{A} - \cancel{M}) \psi - \frac{1}{4} F_{\mu\nu}^a F_a^{\mu\nu}$$



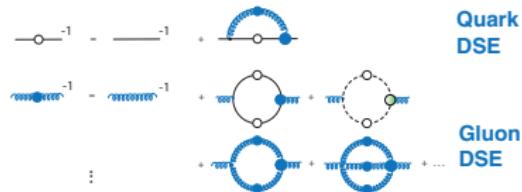
Quantum field theory

Correlation functions in QCD:



Can be calculated ...

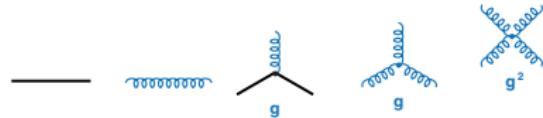
- in **perturbation theory**
- in **lattice QCD**
- with **functional methods:**
DSEs (Dyson-Schwinger equations),
FRG (functional renormalization group)



Functional methods

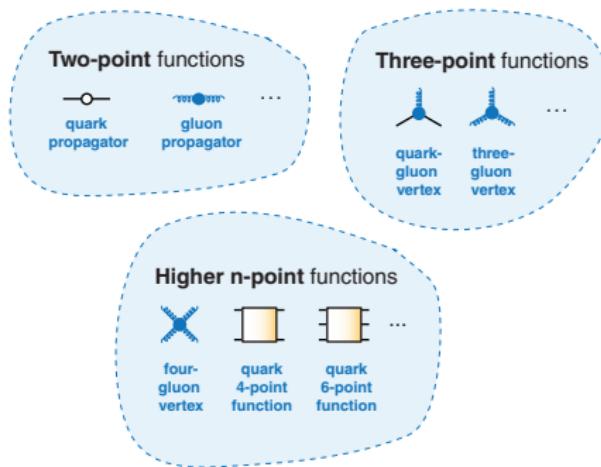
Classical Lagrangian of QCD:

$$\mathcal{L} = \bar{\psi} (i\cancel{D} + g\cancel{A} - \cancel{M}) \psi - \frac{1}{4} F_{\mu\nu}^a F_a^{\mu\nu}$$

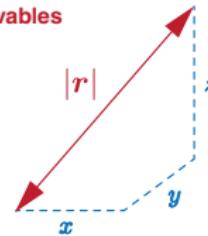


Quantum field theory

Correlation functions in QCD:



gauge-invariant observables



gauge-dependent d.o.f.:
quarks, gluons, ghosts,
diquarks, ...

Hadrons

- Hadron poles (e.g. mesons) appear in all correlation functions that can produce them:

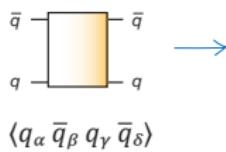


$$\langle q_\alpha \bar{q}_\beta q_\gamma \bar{q}_\delta \rangle$$

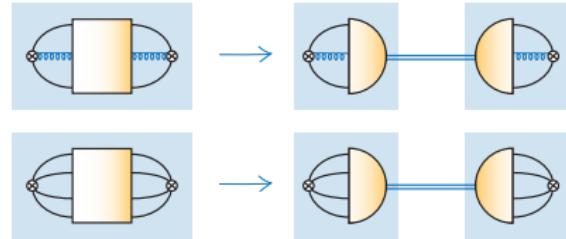
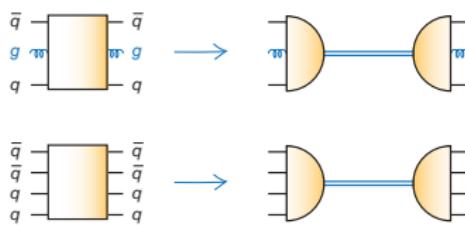
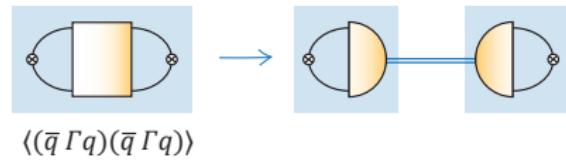


Hadrons

- Hadron poles (e.g. mesons) appear in all correlation functions that can produce them:



- **Lattice QCD:** construct gauge-invariant (e.g., 2-point) correlation functions

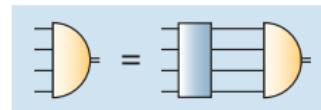
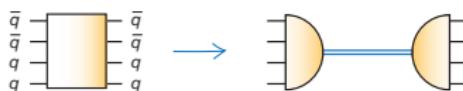
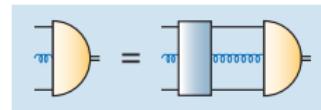
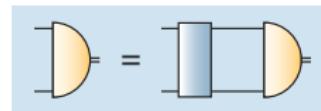


Hadrons

- Hadron poles (e.g. mesons) appear in all correlation functions that can produce them:



- **Functional methods:** solve covariant bound-state equations

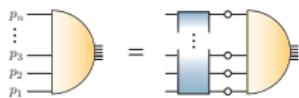


ultimately, they should all produce **same spectrum**

with simpler truncations,
 $q\bar{q}$ ($qq\bar{q}\bar{q}$, $q\bar{q}g$, ...) BSE
works better for
 $q\bar{q}$ ($qq\bar{q}\bar{q}$, $q\bar{q}g$, ...)
dominated states

Functional methods

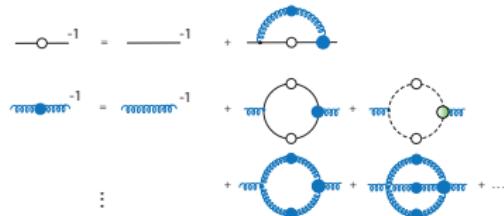
- Hadronic bound-state equations
(Bethe-Salpeter & Faddeev eqs)



"QFT analogue of Schrödinger eq."

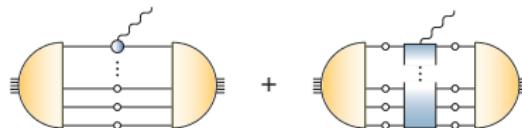
- hadron masses & "wave functions"
- **spectroscopy calculations**

- Ingredients: **QCD's n-point functions**,
Satisfy quantum eqs. of motion (DSEs)



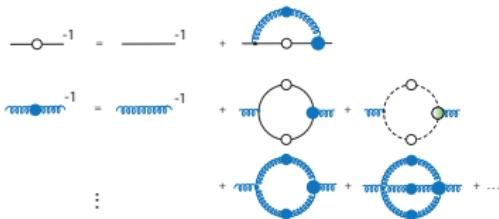
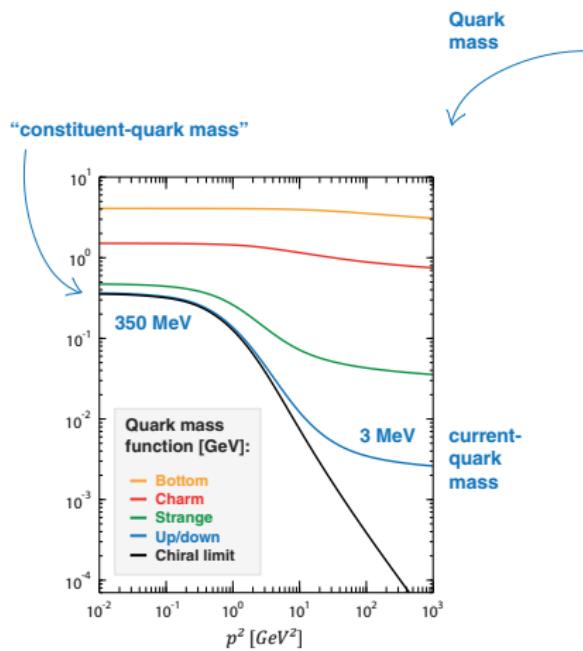
- Dynamical mass generation,
gluon mass gap, confinement, ...

- Structure calculations:** form factors, PDFs, GPDs, TMDs, two-photon processes, ...



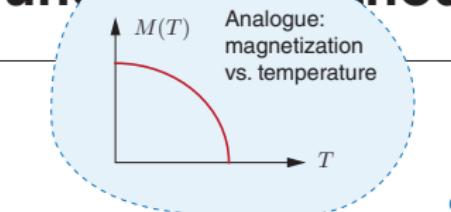
Functional methods

- Ingredients: **QCD's n-point functions**,
Satisfy quantum eqs. of motion (DSEs)

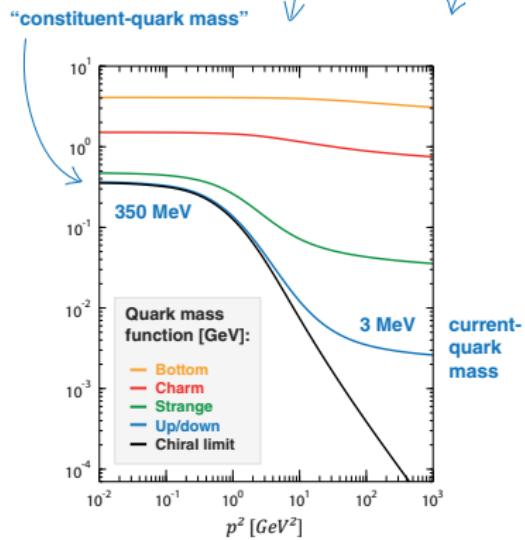


→ Dynamical mass generation,
gluon mass gap, confinement, ...

Functional methods

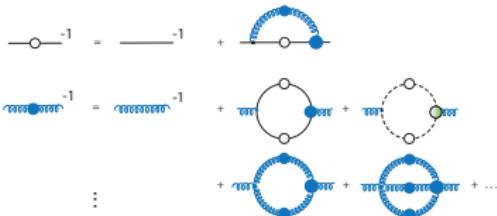


Analogue:
magnetization
vs. temperature



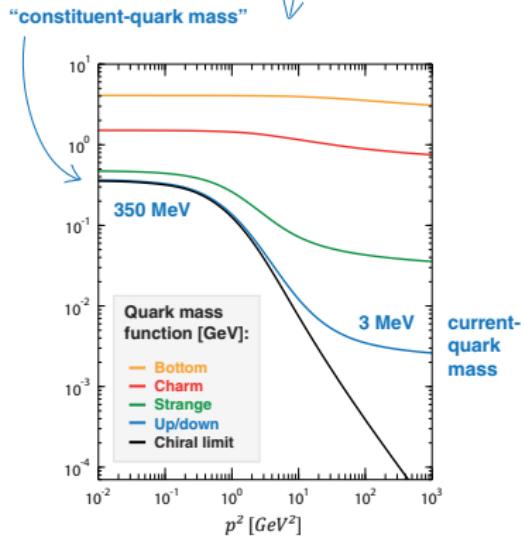
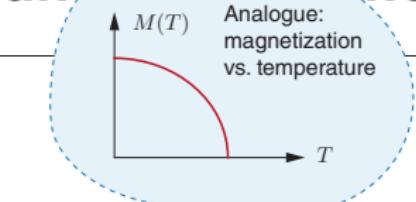
- Ingredients: **QCD's n-point functions**,
Satisfy quantum eqs. of motion (DSEs)

Quark mass



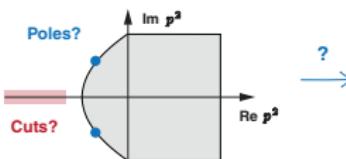
→ Dynamical mass generation,
gluon mass gap, confinement, ...

Functional methods



- Analytic structure?

Gauge-dependent correlation functions



Gauge-invariant correlation functions



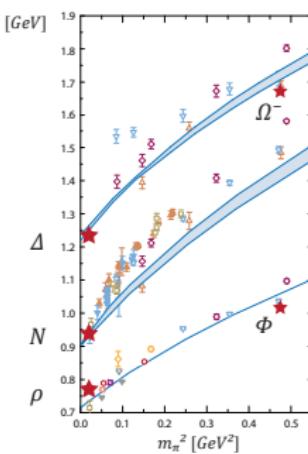
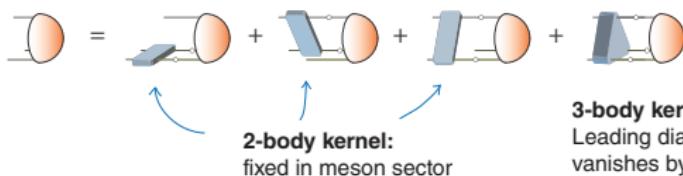
Much recent technical progress to study such questions:

- Contour deformations
- Cauchy integration
- Analytic continuations
- Spectral DSEs [Horak et al., PRD 102 \(2020\) 2006.09778](#)
- Machine Learning [Windisch et al., PRE 101 \(2020\) 1912.12322](#)
- ...

Baryons

Three-quark BSE (Faddeev equation) for baryons:

GE, Alkofer, Nicmorus, Krassnigg, PRL 104 (2010)



$$\Psi_{\alpha\beta\gamma\delta}(p, q, P) = \sum_i f_i(p^2, q^2, p \cdot q, p \cdot P, q \cdot P) \tau_i(p, q, P)_{\alpha\beta\gamma\delta} \otimes \text{Flavor} \otimes \text{Color}$$

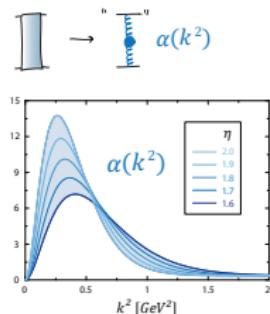
Lorentz-invariant dressing functions

Relativistically, nucleon also has p waves!



Rainbow-ladder

Maris, Tandy, PRC 60 (1999)
Qin, Chang et al., PRC 84 (2011)



Scale set by f_π ,
shape parameter → bands

Dirac-Lorentz tensors:

64 for $J = 1/2$,
128 for $J = 3/2$

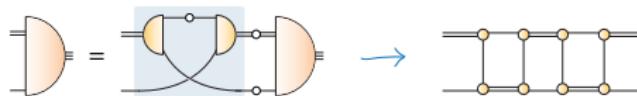
Heavy baryons:

Qin, Roberts, Schmidt, PRD 97 (2018)

Diquark correlations

- Quark-diquark (two-body) equation

Oettel et al., PRC 58 (1998), GE et al., Ann. Phys. 323 (2008), Cloet et al., FBS 46 (2009), Segovia et al., PRL 115 (2015), Chen et al., PRD 97 (2018)

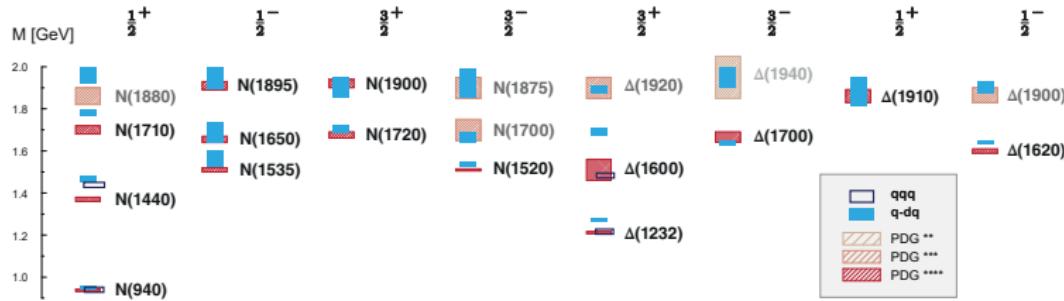


- Three-quark and quark-diquark results very similar

GE, Fischer, Sanchis-Alepuz, PRD 94 (2016)

Diquark clustering in baryons?

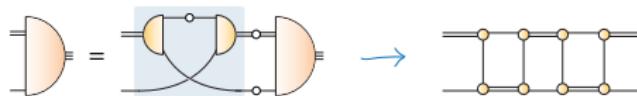
Barabanov et al., Prog. Part. Nucl. Phys. 116 (2021)



Diquark correlations

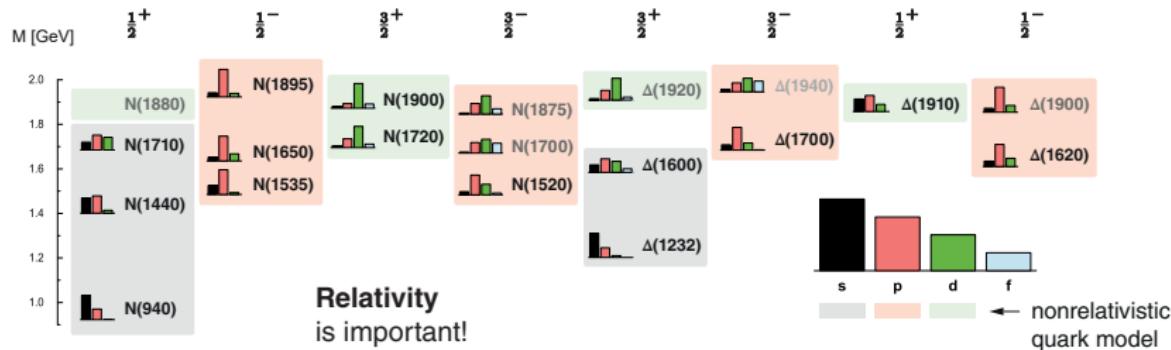
- Quark-diquark (two-body) equation

Oettel et al., PRC 58 (1998), GE et al., Ann. Phys. 323 (2008), Cloet et al., FBS 46 (2009), Segovia et al., PRL 115 (2015), Chen et al., PRD 97 (2018)



- Three-quark and quark-diquark results very similar

GE, Fischer, Sanchis-Alepuz, PRD 94 (2016), GE, FBS 58 (2017) & FBS 63 (2022)



Diquark clustering in baryons?
Barabanov et al., Prog. Part. Nucl. Phys. 116 (2021)



Diquark correlations

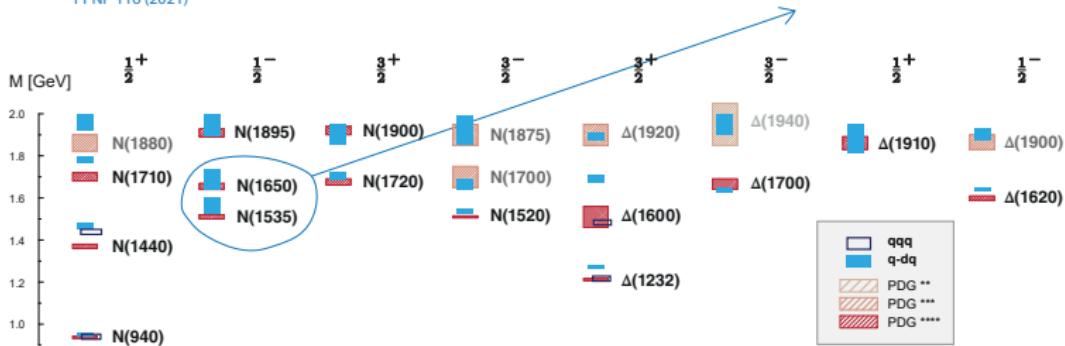
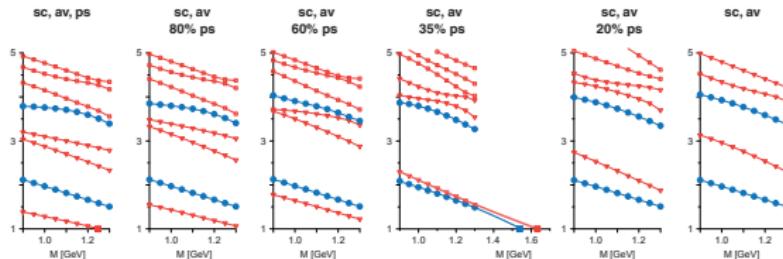
RL, sc + av + ps + v
“N(1535)” too low

“Beyond RL”:
two nearby states

RL, sc + av:
“N(1650)” too high

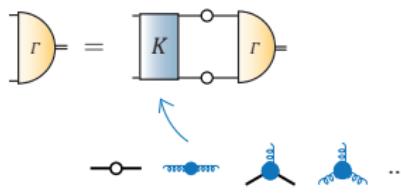
- Level ordering**
strongly affected by diquark dynamics:
sc, av: “good”, “bad”
ps, v: “ugly”
- Diquarks are not pointlike, rich spectrum!

GE, Fischer, Sanchis-Alepuz,
PRD 94 (2016), Baranov et al.,
PPNP 116 (2021)



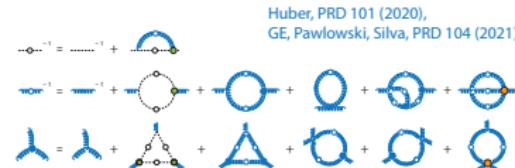
Towards ab-initio

- Going **beyond rainbow-ladder** means calculating higher n-point functions



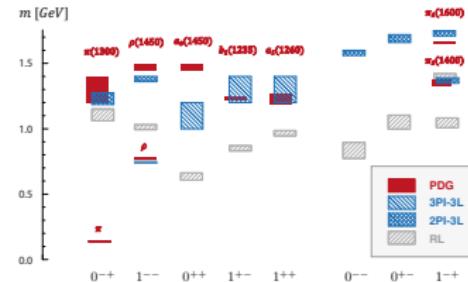
- Lots of activity with **DSEs, FRG, lattice QCD**

..., Williams, Fischer, Heupel, PRD 93 (2016), Cyrol et al., PRD 97 (2018), Oliveira, Silva, Skullerud, Sternbeck, PRD 99 (2019), Aguilar et al., EPJ C 80 (2020), Qin, Roberts, Chin. Phys. Lett. 38 (2021), ...



- Beyond rainbow-ladder calculations improve **light-meson spectrum**

Williams, Fischer, Heupel, PRD 93 (2016)



Towards ab-initio

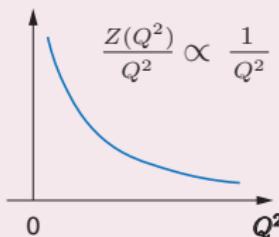
Gluon propagator:



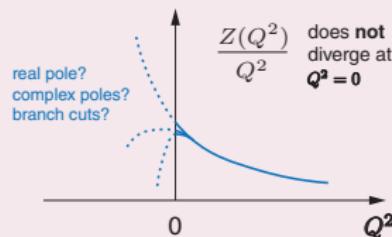
$$D^{\mu\nu}(Q) = \frac{Z(Q^2)}{Q^2} \left(\delta^{\mu\nu} - \frac{Q^\mu Q^\nu}{Q^2} \right) + \xi \frac{L(Q^2)}{Q^2} \frac{Q^\mu Q^\nu}{Q^2}$$

transverse dressing longitudinal dressing = 1

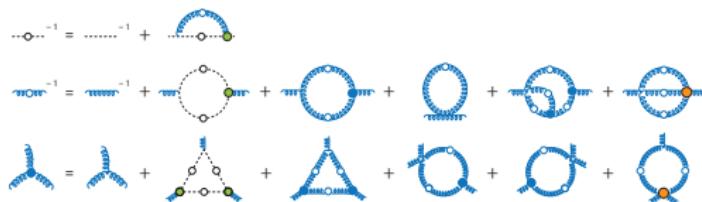
- Perturbation theory:
Massless gluon pole



- Nonperturbative calculations:
Massless pole disappears!



Coupled Yang-Mills DSEs
GE, Pawłowski, Silva, PRD 104 (2021)



Family of “decoupling” solutions,
also seen in lattice QCD

Cucchieri, Maas, Mendes, PRD 77 (2008)

Boucaud et al., JHEP 06 (2008)

Bogolubsky et al., PLB 676 (2009)

Fischer, Maas, Pawłowski, Ann. Phys. 324 (2009)

Duarte, Oliveira, Silva, PRD 94 (2016)

Aguilar et al., EPJ C 80 (2020)

Endpoint is “scaling” solution,
confinement manifest

Lerche, Smekal, PRD 65 (2002)

Fischer, Alkofer, PLB 536 (2002)

Alkofer, Fischer, Llanes-Estrada, MPLA 23 (2008)

All solutions show gluon mass gap

$$\lim_{r \rightarrow \infty} \int \frac{d^3 Q}{(2\pi)^3} \frac{Z(Q^2)}{Q^2} e^{i \mathbf{x} \cdot \mathbf{Q}} \propto e^{-m_{\text{gap}} r}$$

Towards ab-initio

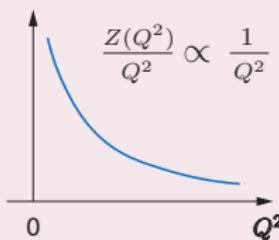
Gluon propagator:



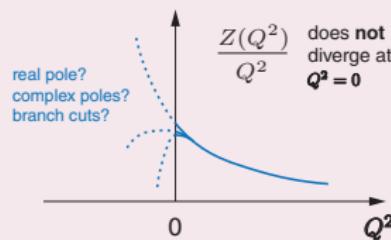
$$D^{\mu\nu}(Q) = \frac{Z(Q^2)}{Q^2} \left(\delta^{\mu\nu} - \frac{Q^\mu Q^\nu}{Q^2} \right) + \xi \frac{L(Q^2)}{Q^2} \frac{Q^\mu Q^\nu}{Q^2}$$

transverse dressing longitudinal dressing = 1

- Perturbation theory:
Massless gluon pole



- Nonperturbative calculations:
Massless pole disappears!



Family of “decoupling” solutions,
also seen in lattice QCD

Cucchieri, Maas, Mendes, PRD 77 (2008)

Boucaud et al., JHEP 06 (2008)

Bogolubsky et al., PLB 676 (2009)

Fischer, Maas, Pawłowski, Ann. Phys. 324 (2009)

Duarte, Oliveira, Silva, PRD 94 (2016)

Aguilar et al., EPJ C 80 (2020)

Endpoint is “scaling” solution,
confinement manifest

Lerche, Smekal, PRD 65 (2002)

Fischer, Alkofer, PLB 536 (2002)

Alkofer, Fischer, Llanes-Estrada, MPLA 23 (2008)

All solutions show gluon mass gap

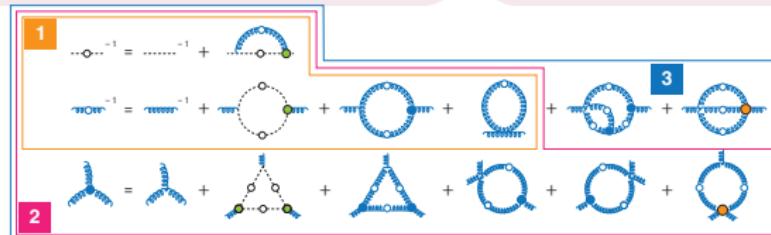
$$\lim_{r \rightarrow \infty} \int \frac{d^3 Q}{(2\pi)^3} \frac{Z(Q^2)}{Q^2} e^{i \mathbf{x} \cdot \mathbf{Q}} \propto e^{-m_{\text{gap}} r}$$

Coupled Yang-Mills DSEs

GE, Pawłowski, Silva, PRD 104 (2021)

truncation error:

1 60% **2** 10% **3** 4%

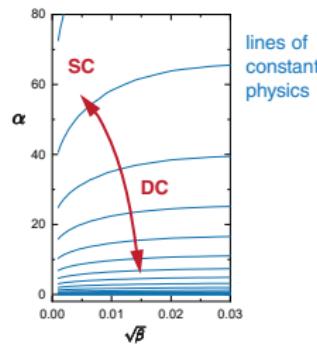


Towards ab-initio

- System depends on 2 parameters:

- α ... coupling parameter
(not physical coupling)
- β ... mass scale, arising from subtraction of quadratic divergences

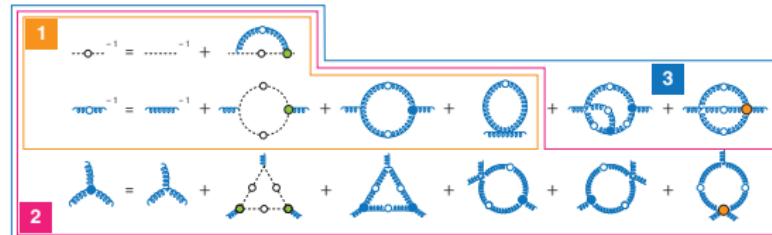
One combination changes scale,
the other distinguishes scaling &
decoupling solutions



Coupled Yang-Mills DSEs
GE, Pawłowski, Silva, PRD 104 (2021)

truncation error:

1 60% 2 10% 3 4%



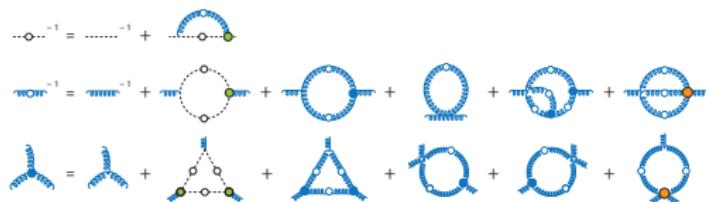
Longitudinal massless poles
(Schwinger mechanism) trigger
mass generation in transverse sector:

- SC: ghost-gluon vertex
GE, Pawłowski, Silva, PRD 104 (2021)
- DC: three-gluon vertex
Aguilar, Ferreira, Papavassiliou, PRD 105 (2022)

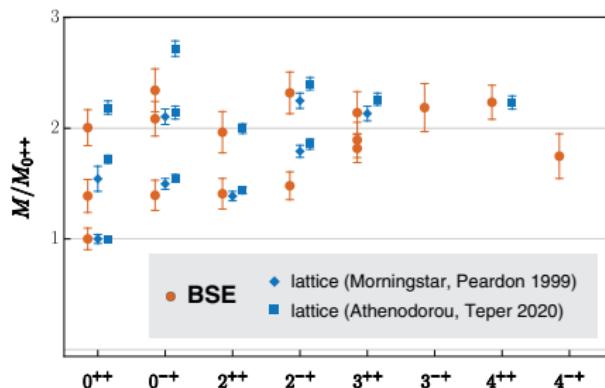
Are all massless (DC & SC)
solutions physically equivalent?

Can test confinement in
hadron observables!

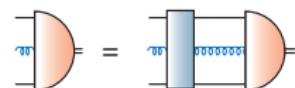
Glueballs



- Glueball spectrum agrees with lattice QCD
Huber, Fischer, Sanchis-Alepuz, EPJ C 80 (2020), EPJ C 81 (2021)

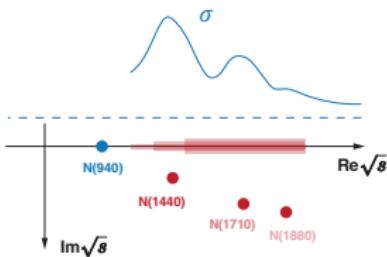


- Next up: Hybrid mesons

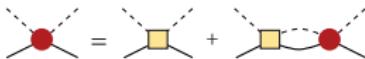


Resonances

- Most hadrons are **resonances** and decay
 \Leftrightarrow poles in complex momentum plane



- EFT:** straightforward to generate cuts & resonance poles (internal hadrons in loops)



- Lattice:** correlators calculated directly, but cuts & resonance poles only in infinite volume



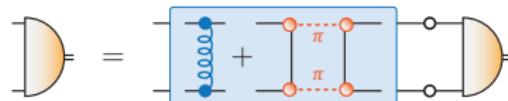
- Functional methods:** quarks & gluons should ...

- (1) dynamically generate hadrons,
- (2) which do *not* decay into quarks & gluons,
- (3) and generate hadronic thresholds

... that's a tall order!

- $q\bar{q}, qqq$: include resonance mechanism in kernels $\Rightarrow \rho$ meson becomes resonance

Williams, PLB 798 (2019), Miramontes, Sanchis-Alepuz, EPJA 55 (2019),
Santowsky, GE, Fischer, Wallbott, PRD 102 (2020),
Miramontes, Sanchis-Alepuz, Alkofer, PRD 103 (2021)

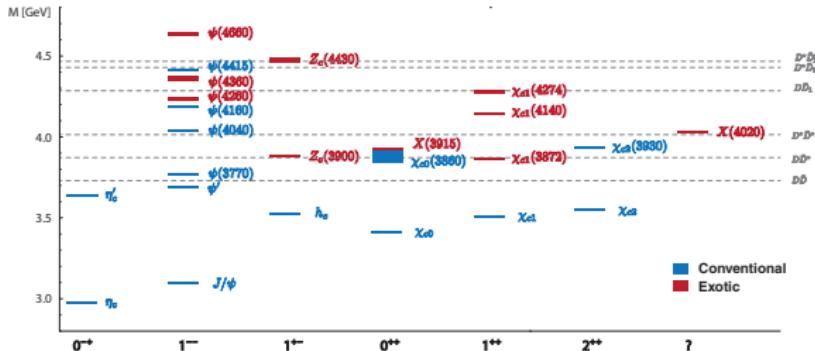


- Multiquarks:** resonance mechanism emerges dynamically (\rightarrow see later)

- Contour deformations** as tool to go beyond thresholds

GE, Duarte, Peña, Stadler, PRD 100 (2019)

Exotic mesons



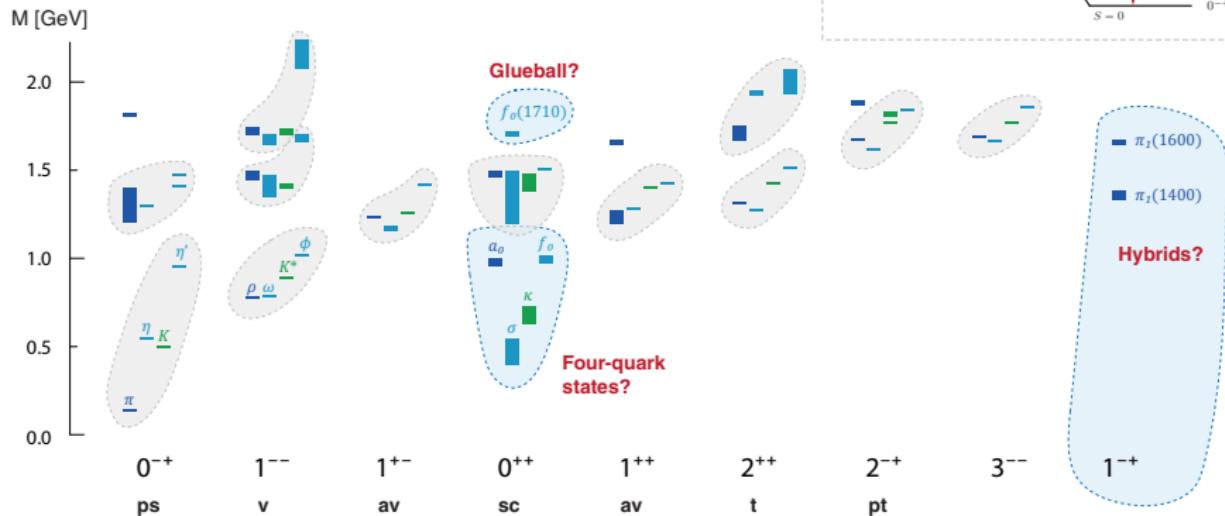
- Several tetraquark candidates in **charmonium spectrum**: $X(3872)$, $X(3915)$, $Z_c(3900)$, ...
- Z states cannot be $c\bar{c}$ since they carry charge
- Recent additions: all-charm $X(6900)$, open-charm T_{cc}^+ , ...
- Oldest tetraquark candidates: **light scalar mesons**

Reviews:

- Chen, Chen, Liu, Zhu,
Phys. Rept. 639 (2016), 1601.02092
- Lebed, Mitchell, Swanson
PPNP 93 (2017), 1610.04528
- Esposito, Pilloni, Polosa,
Phys. Rept. 668 (2017), 1611.07920
- Guo, Hanhart, Meißner et al.,
Rev. Mod. Phys. 90 (2018), 1705.00141
- Ali, Lange, Stone,
PPNP 97 (2017), 1706.00610
- Olsen, Skwarnicki, Zieminska,
Rev. Mod. Phys. 90 (2018), 1708.04012
- Liu, Chen, Chen, Liu, Zhu,
PPNP 107 (2019), 1903.11976
- Brambilla, Eidelman, Hanhart et al.,
Phys. Rept. 873 (2020)
- ...

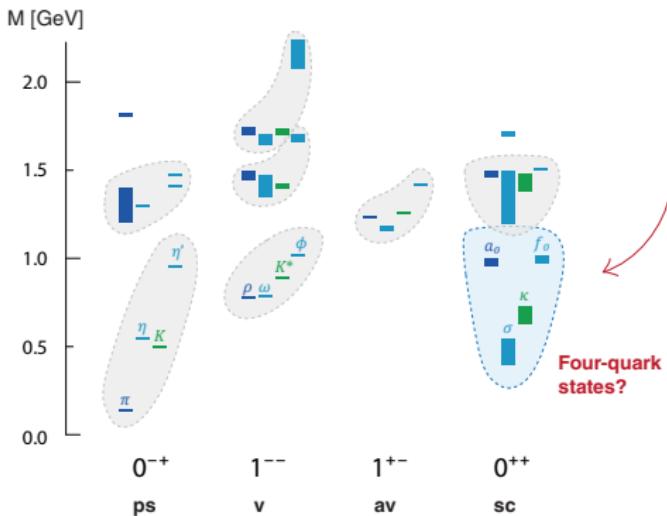
Light exotic mesons

Light meson spectrum
(PDG 2020)

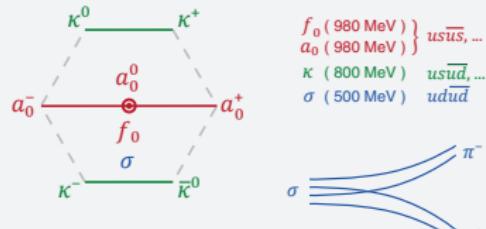


Light exotic mesons

Light meson spectrum
(PDG 2020)



- **Diquark-antidiquark?**
Explains mass ordering & decay widths
Jaffe 1977, Close, Tornqvist 2002,
Maiani, Polosa, Riquer 2004

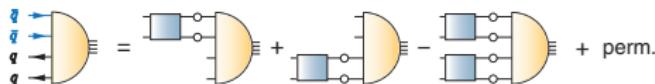


- **Meson molecules?**
Weinstein, Isgur 1982, 1990; Close, Isgur, Kumano 1993
- **Non-q \bar{q} nature supported by various approaches**
Pelaez, Phys. Rept. 658 (2016)

Four-quark states

- Light scalar mesons (σ, κ, a_0, f_0) as **four-quark states**:

GE, Fischer, Heupel, PLB 753 (2016)



$$\Gamma(p, q, k, P) = \sum_i f_i(p^2, q^2, k^2, \{\omega_j\}, \{\eta_j\}) \tau_i(p, q, k, P) \otimes \text{Color} \otimes \text{Flavor}$$

9 Lorentz invariants:

$$p^2, \quad q^2, \quad k^2, \quad P^2 = -M^2$$

$$\omega_1 = p \cdot k \quad \eta_1 = p \cdot P$$

$$\omega_2 = p \cdot k \quad \eta_2 = q \cdot P$$

$$\omega_3 = p \cdot q \quad \eta_3 = k \cdot P$$

256 Dirac-Lorentz tensors

2 Color tensors:
3 \otimes 3, 6 \otimes 6 or
1 \otimes 1, 8 \otimes 8
(Fierz-equivalent)

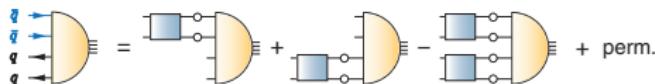
$$K \psi_i = \lambda_i \psi_i$$

	dim K	memory
Mesons	10^3	20 MB
Baryons	10^8	10^7 GB
Tetraquarks	10^{13}	10^{18} GB

Four-quark states

- Light scalar mesons (σ, κ, a_0, f_0) as **four-quark states**:

GE, Fischer, Heupel, PLB 753 (2016)



$$\Gamma(p, q, k, P) = \sum_i f_i(p^2, q^2, k^2, \{\omega_j\}, \{\eta_j\}) \tau_i(p, q, k, P) \otimes \text{Color} \otimes \text{Flavor}$$

9 Lorentz invariants:

$$p^2, \quad q^2, \quad k^2, \quad P^2 = -M^2$$

$$\omega_1 = \mathbf{k} \cdot \mathbf{p} \quad \eta_1 = \mathbf{p} \cdot \mathbf{P}$$

$$\omega_2 = \mathbf{p} \cdot \mathbf{k} \quad \eta_2 = \mathbf{q} \cdot \mathbf{P}$$

$$\omega_3 = \mathbf{p} \cdot \mathbf{q} \quad \eta_3 = \mathbf{k} \cdot \mathbf{P}$$

256 Dirac-Lorentz tensors

2 Color tensors:

$$3 \otimes \overline{3}, \quad 6 \otimes \overline{6} \quad \text{or} \\ 1 \otimes 1, \quad 8 \otimes 8 \\ (\text{Fierz-equivalent})$$

- Group momentum variables into multiplets of **permutation group S4**: can switch off groups of variables without destroying symmetries

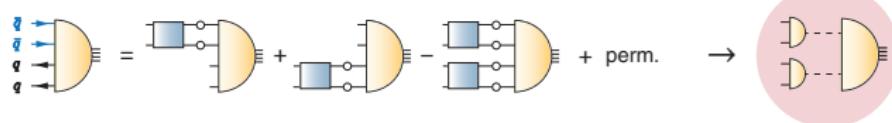
GE, Fischer, Heupel, PRD 92 (2015)

$$f_i(S_0, \nabla, \triangle, \circ)$$

Four-quark states

- Light scalar mesons (σ, κ, a_0, f_0) as **four-quark states**:

GE, Fischer, Heupel, PLB 753 (2016)



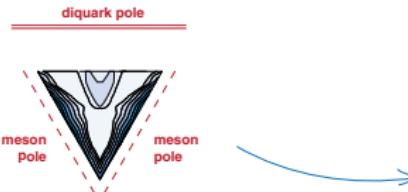
- BSE dynamically generates **meson poles** in BS amplitude:

$$f_i(S_0, \nabla, \Delta, \circ) \rightarrow 1500 \text{ MeV}$$

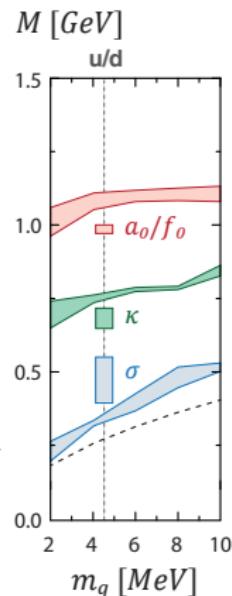
$$f_i(S_0, \nabla, \Delta, \circ) \rightarrow 1500 \text{ MeV}$$

$$f_i(S_0, \nabla, \Delta, \circ) \rightarrow 1200 \text{ MeV}$$

$$f_i(S_0, \nabla, \Delta, \circ) \rightarrow 350 \text{ MeV} !$$



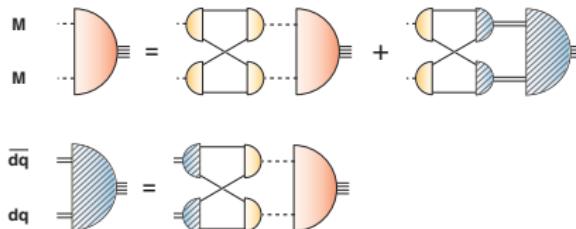
- "Light scalar mesons" look like **meson molecules**, diquark-antidiquark components almost negligible.
Lightness is inherited from pseudoscalar Goldstone bosons!



Four-quark states

Two-body formulation: **meson-meson / diquark-antidiquark**,
follows from four-quark eq. (analogue of quark-diquark for baryons)

Heupel, GE, Fischer, PLB 718 (2012)

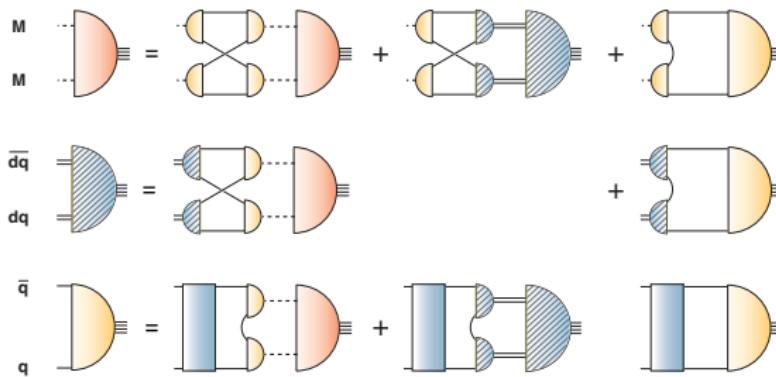


- Interaction by **quark exchange**
- System ‘wants’ to be **meson-meson-like** (no diagonal $d\bar{q}-d\bar{q}$ term)
- Similar results as in 4-quark approach:
 $m_\sigma \sim 400$ MeV, etc.

Four-quark states

Two-body formulation: **meson-meson / diquark-antidiquark**,
follows from four-quark eq. (analogue of quark-diquark for baryons)

Heupel, GE, Fischer, PLB 718 (2012)



Include mixing with $q\bar{q}$:
 $\pi\pi$ still dominant

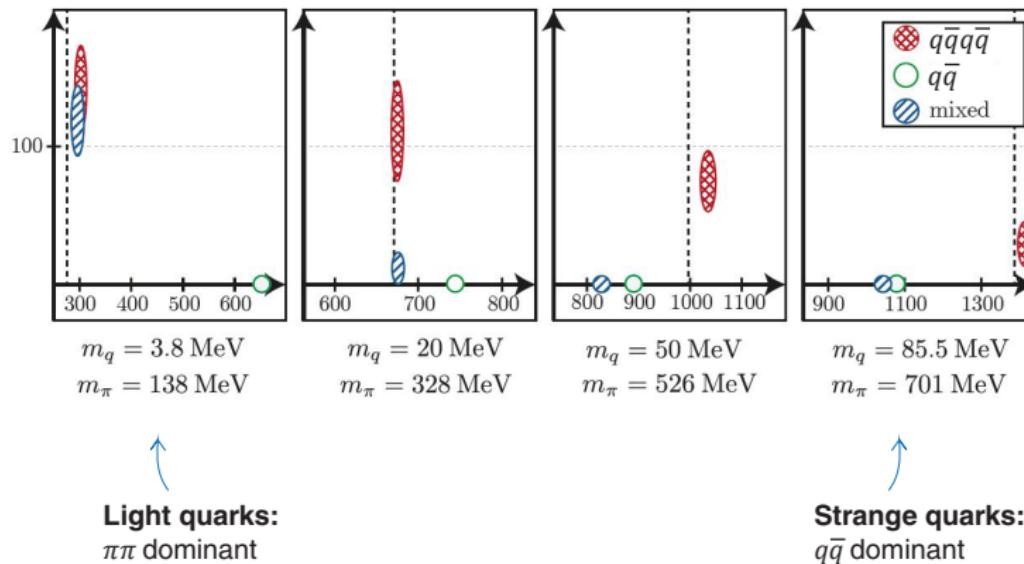
Santowsky, GE, Fischer, Wallbott,
Williams, PRD 102 (2020)

[MeV]	ground state mass	first excitation
$\pi\pi$	416 ± 26	970 ± 130
$\pi\pi + 0^+ 0^+$	416 ± 26	970 ± 130
$q\bar{q}$	667 ± 2	1036 ± 8
$\pi\pi + q\bar{q}$	472 ± 22	1080 ± 280
$\pi\pi + 0^+ 0^+ + q\bar{q}$	456 ± 24	1110 ± 110

Four-quark states

Four-quark vs. $q\bar{q}$ dominance

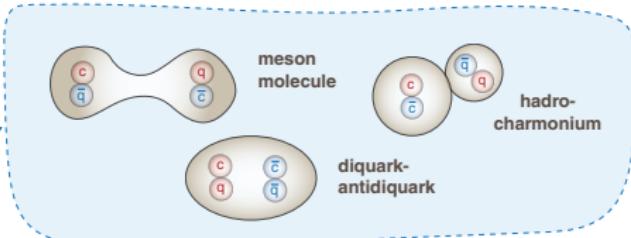
Santowsky, Fischer, PRD 105 (2022)



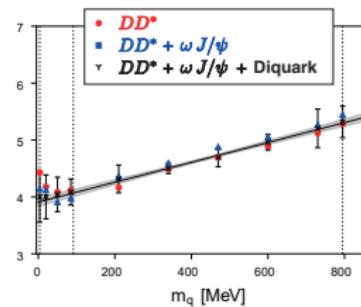
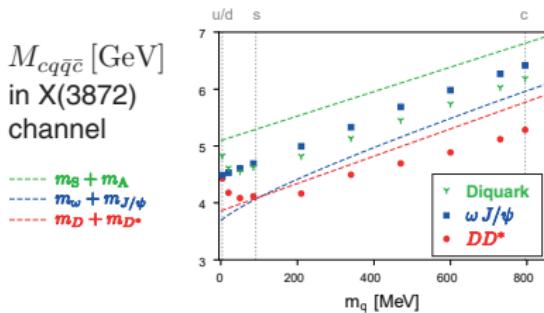
Four-quark states

- Heavy-light four-quark states:
what is their internal decomposition?

$cq\bar{q}\bar{c}$



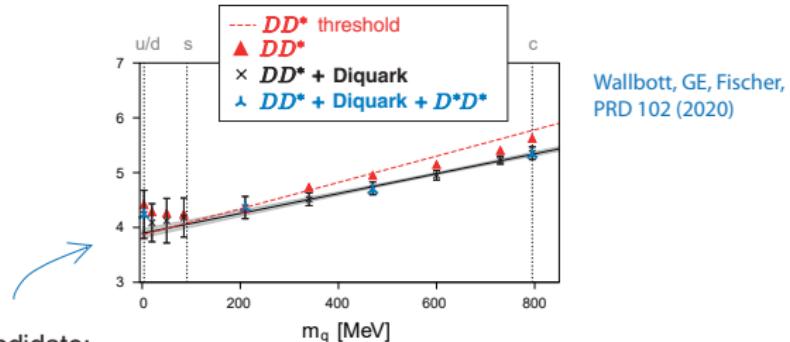
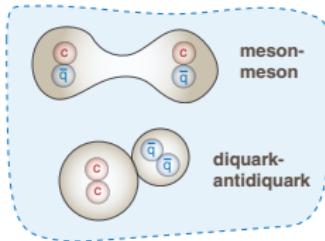
- Four-quark BSE: all mix together



$cq\bar{q}\bar{c} \rightarrow$ strong meson-meson component: DD^* for $X(3872)$, $Z_c(3900)$

Four-quark states

- Open-charm states: $cc\bar{q}\bar{q}$



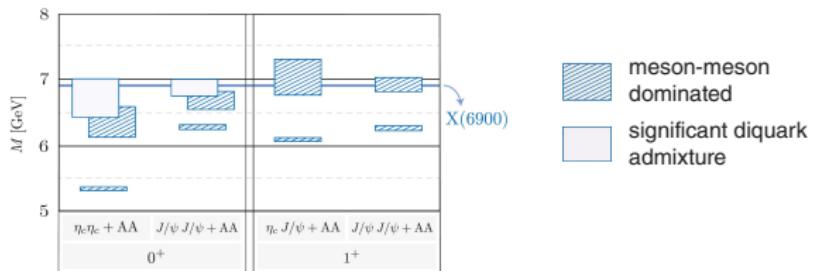
Experimental candidate:
 $T_{cc}^+, 0(1^+), 3875 \text{ MeV}$

Wallbott, GE, Fischer,
PRD 102 (2020)

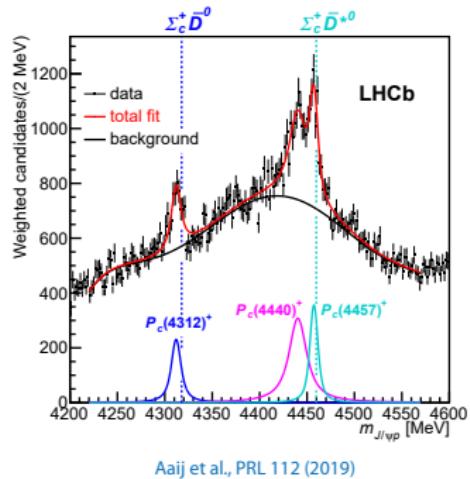
- All-charm state: $cccc$
 $X(6900)$

Results so far available
in two-body approach,
1st radial excitation?

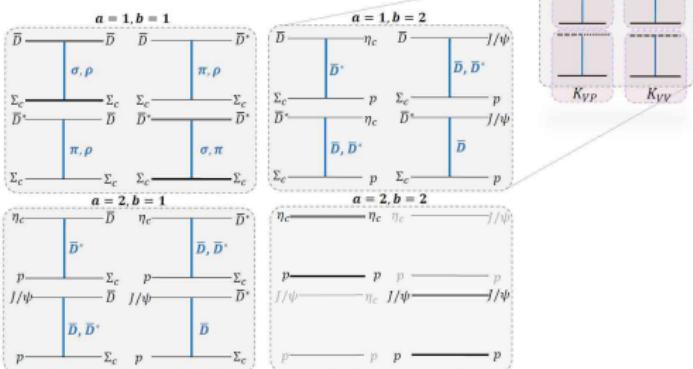
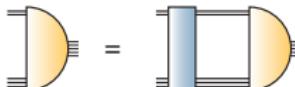
Santowsky, Fischer, EPJC 82 (2022)



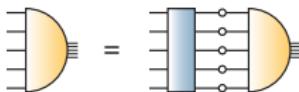
Pentaquarks?



- Meson-baryon equation with hadronic exchanges
GE, Lourenco, Peña, Stadler, Torres, in preparation

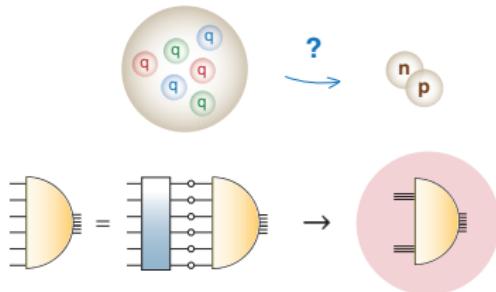


- Next up: 5-body equation



Nucleons in nuclei?

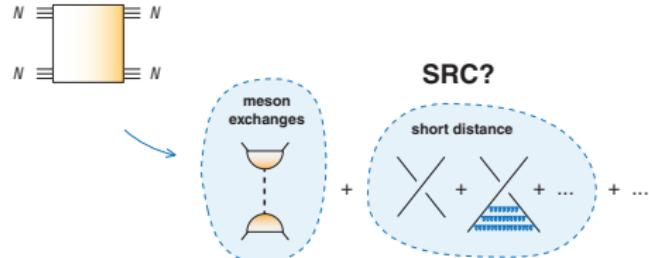
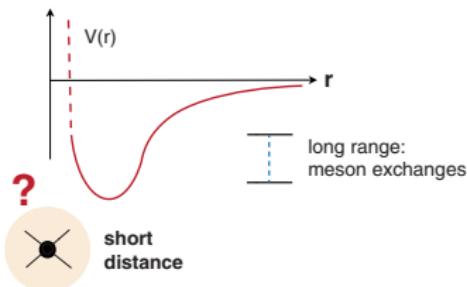
Transition from quarks & gluons to **light nuclei**:



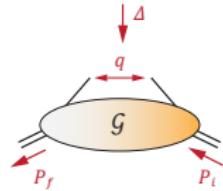
- Relativistic structure of **deuteron**
- Exotic dibaryons and hypernuclei
- **Short-range correlations**
- **EMC effect:** overlapping nucleons in nuclei?

Hen, Miller, Piasetzky, Weinstein, Rev. Mod. Phys. 89 (2017),
Cloet et al., J. Phys. G 46 (2019)

Microscopic origins of **short-range nuclear force**?

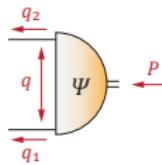


Hadron structure



Hadron-to-hadron correlator

$$\mathcal{G}(z, P, \Delta) = \langle P_f | \mathcal{T} \Phi(z) \mathcal{O} \Phi(0) | P_i \rangle$$



Bethe-Salpeter WF:
vacuum-to-hadron correlator

$$\Psi(z, P) = \langle 0 | \mathcal{T} \Phi(z) \Phi(0) | P \rangle$$

	$\mathcal{G}(q, P, \Delta = 0)$	$\mathcal{G}(q, P, \Delta)$	$\Psi(q, P)$
$\int dq^-$	TMD	GTMD	LFWF
$\int d^2\mathbf{q}_\perp \int dq^-$	PDF	GPD	PDA

Diehl, Phys. Rept. 388 (2003)

Belitsky, Radyushkin,

Phys. Rept. 418 (2005)

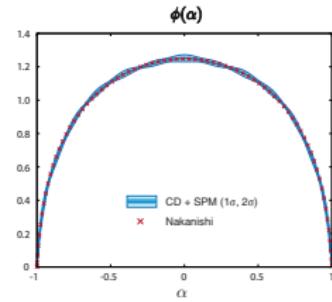
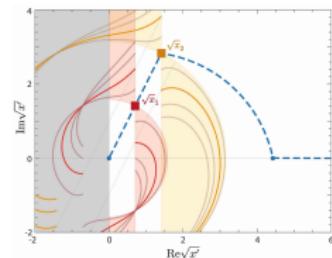
Lorcé, Pasquini, Vanderhaeghen,
JHEP 05 (2011)

...

Novel method to compute
light-front wave functions
via contour deformations

Editors' Suggestion:

GE, Ferreira, Stadler, PRD 105 (2022)



Summary & outlook

- **Baryons:**

GE, Sanchis-Alepuz, Williams, Fischer, Alkofer, PPNP 91 (2016), arXiv:1606.09602
Barabanov et al., PPNP 116 (2021), arXiv:2008.07630

- **Four-quark states:**

GE, Fischer, Heupel, Santowsky, Wallbott, FBS 61 (2020), arXiv:2008.10240

- Towards **ab-initio calculations:**

higher n-point functions, gluon mass generation, resonances

- **Exotic hadrons:** glueballs, hybrids, tetraquarks, pentaquarks

- **Hadron structure:** PDFs, GPDs, TMDs

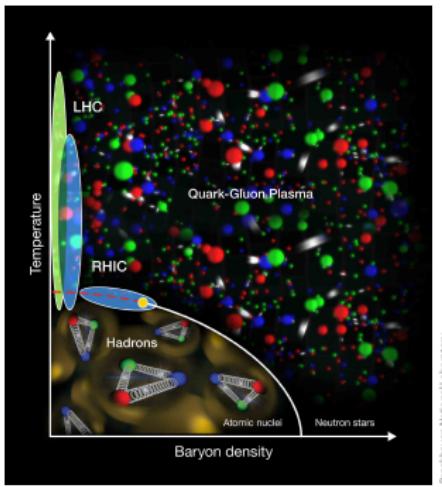
- **Flavor matrix elements:** constrain QCD contributions

- **Nuclei** from quarks and gluons

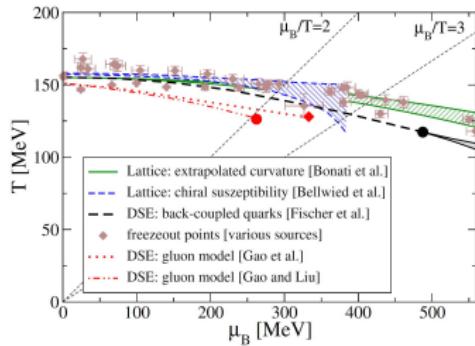
Thank you!

Backup slides

QCD phase diagram



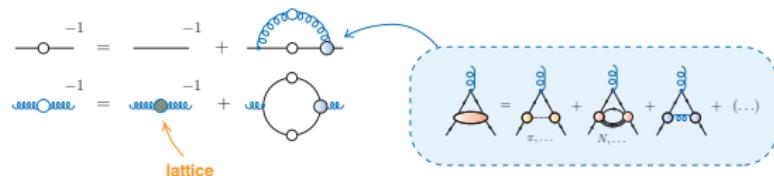
Search for **critical endpoint (CEP)** from DSEs & lattice:



Fischer, Prog. Part.
Nucl. Phys. 105 (2019)

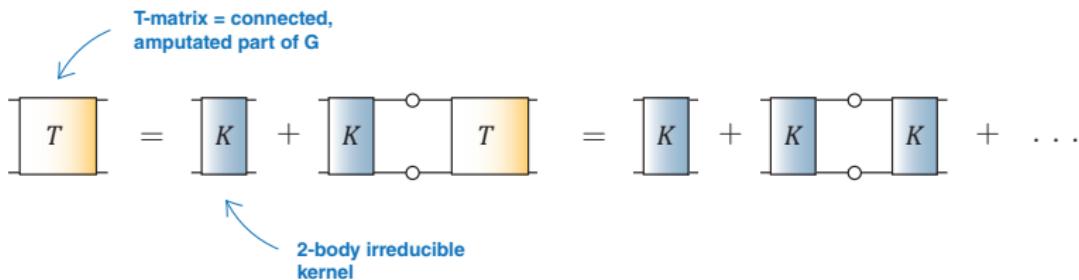
Location of CEP sensitive to baryons?

GE, Fischer, Welzbacher, PRD 93 (2016)



Bethe-Salpeter equations

Write down inhomogeneous BSE:



Analogy: **geometric series**

$$\begin{aligned} f(x) &= 1 + x f(x) \\ &= 1 + x + x^2 f(x) \quad \Rightarrow \quad f(x) = \frac{1}{1-x} \quad \left. \right\} \text{“non-perturbative”} \\ &= 1 + x + x^2 + x^3 f(x) \\ f(x) &\approx 1 + x + x^2 + x^3 + \dots \quad \text{only for } |x| < 1 \quad \left. \right\} \text{“perturbative”} \end{aligned}$$

Bethe-Salpeter equations

Write down inhomogeneous BSE:

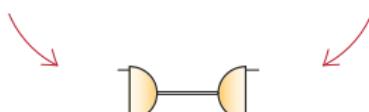
$$T = K + K \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} T$$

T-matrix = connected, amputated part of G

Homogeneous BSE at pole:

$$\Gamma = K \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \Gamma$$

compare pole residues



- $q\bar{q}$ irreducible kernel
- chiral symmetry constraints ($V + AV$ WTI)
- can be systematically derived from effective action, depends on QCD's n-point functions



- Analogue of Schrödinger equation in QFT!
- Γ = **Bethe-Salpeter amplitude**

Bethe-Salpeter equations

Write down inhomogeneous BSE:

$$T = K + K \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} T$$

T-matrix = connected, amputated part of G

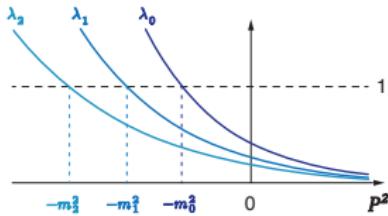
Homogeneous BSE at pole:

$$\Gamma = K \begin{array}{c} \text{---} \\ | \\ \text{---} \end{array} \Gamma$$

compare pole residues

BSE = eigenvalue equation,
pole in $T \Leftrightarrow$ eigenvalue = 1

$$KG_0\Gamma_i = \lambda_i\Gamma_i$$



Explicitly:

$$\Gamma(p, P) = \int \frac{d^4 q}{(2\pi)^4} \mathbf{K}_{\alpha\gamma,\delta\beta}(p, q, P) [S(q_+) \Gamma(q, P) S(q_-)]_{\gamma\delta}$$

Basis decomposition:

$$\Gamma(p, P) = \sum_{i=1}^n f_i(p^2, \hat{p} \cdot \hat{P}, P^2) \tau_i(p, P)$$

⇒ Coupled Lorentz-invariant equations
for the dressing functions f_i

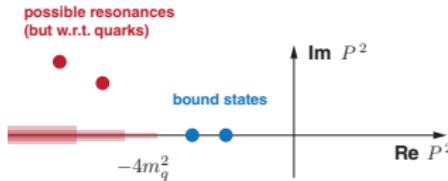
Ladder

Simplest attempt:

$$\Gamma = \text{free propagators} = \frac{-i\cancel{p} + m}{p^2 + m^2}$$
$$T = \text{free propagators} + \text{contact interaction}$$

The diagram illustrates the decomposition of a vertex function Γ into free propagators and a contact interaction term T . On the left, a yellow circle labeled Γ is equated to two parallel blue lines representing free propagators. An arrow points from the right side of this equation to the right side of the next equation. The right side shows a yellow rectangle labeled T followed by a plus sign, then a ladder diagram with three rungs, followed by another plus sign and three dots, indicating a series of higher-order terms.

Analytic structure of G , T , etc.
would look like this:



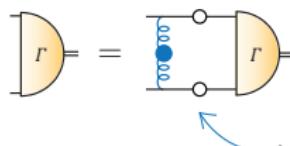
- breaks chiral symmetry
(free propagators \Leftrightarrow contact interaction)
- generates bound-state poles in G and T ,
possibly also resonances
- but also quark thresholds & cuts:
“hadrons” decay into quarks,
no confinement

would be ok if elementary d.o.f. were
not quarks but **hadrons** (\rightarrow EFTs)

Rainbow-ladder

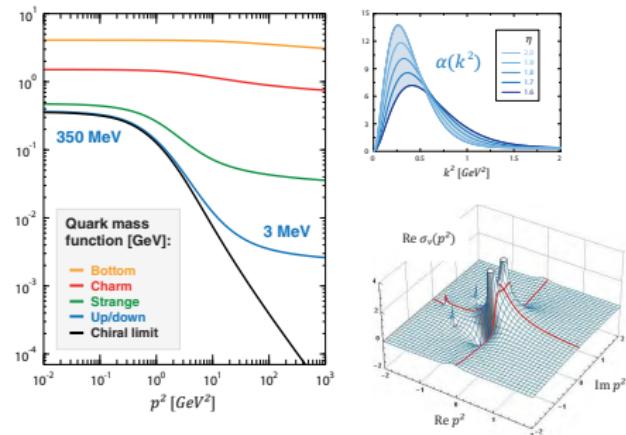
Better: rainbow-ladder truncation

Maris, Roberts, PRC 56 (1997), Maris, Tandy, PRC 60 (1999)

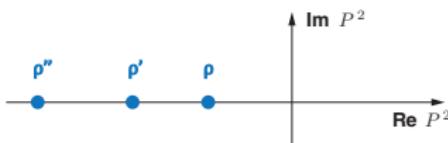


gluon = effective interaction,
dressed propagators
from quark DSE

$$\frac{1}{A(p^2)} \frac{-i\cancel{p} + \mathbf{M}(p^2)}{p^2 + \mathbf{M}(p^2)^2}$$



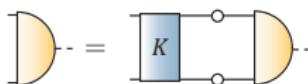
Analytic structure of G , T , etc.
would look like this:



- chiral symmetry ✓
- dynamical propagators do not have real poles \Rightarrow no quark thresholds ✓
- but no resonances: **bound states**
(need to go beyond rainbow-ladder)

Diquark correlations

Mesons and diquarks closely related through BSE
Maris, FBS 32 (2002)

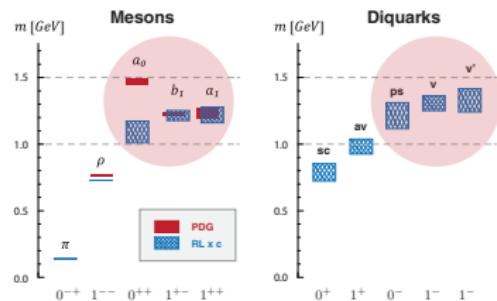


Lowest-lying diquarks are dominant for ground-state octet & decuplet baryons

$$\begin{array}{ll} \text{pseudoscalar mesons} & \Leftrightarrow \text{scalar diquarks} (\sim 0.8 \text{ GeV}) \\ \text{vector mesons} & \Leftrightarrow \text{axialvector diquarks} (\sim 1 \text{ GeV}) \end{array}$$

Higher-lying diquarks are subleading, but contribute to excited states & remaining channels

$$\begin{array}{ll} \text{scalar mesons} & \Leftrightarrow \text{pseudoscalar diquarks} (\sim 1.2 \text{ GeV}) \\ \text{axialvector mesons} & \Leftrightarrow \text{vector diquarks} (\sim 1.3 \text{ GeV}) \end{array}$$



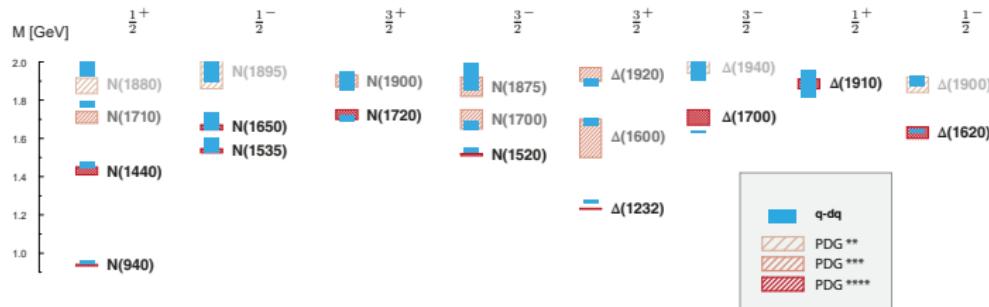
In RL, these are too strongly bound;
simulate beyond-RL effects
by (one) strength parameter c

Roberts, Chang, Cloet, Roberts, FBS 51 (2011)
GE, Fischer, Sanchis-Alepuz, PRD 94 (2016)

Diquark correlations

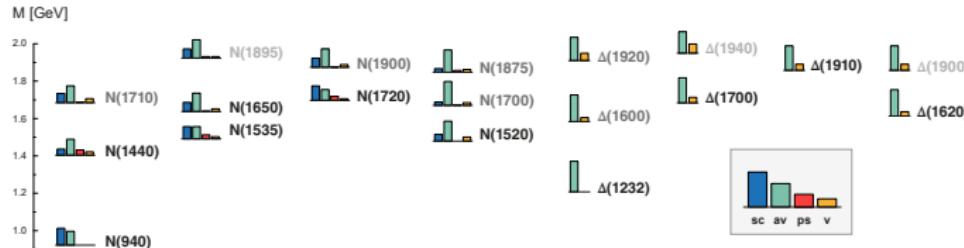
Light baryon spectrum

GE Fischer, Sanchis-Alepuz, PRD 94 (2016)



Diquark content:

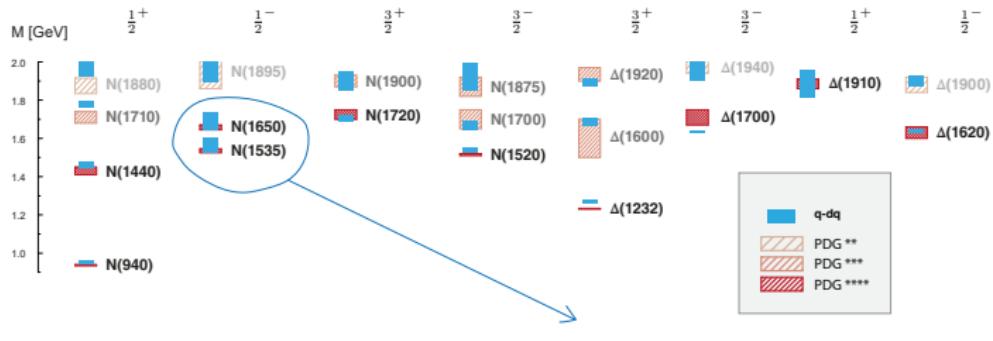
Barabanov et al., PPNP 116 (2021)



Diquark correlations

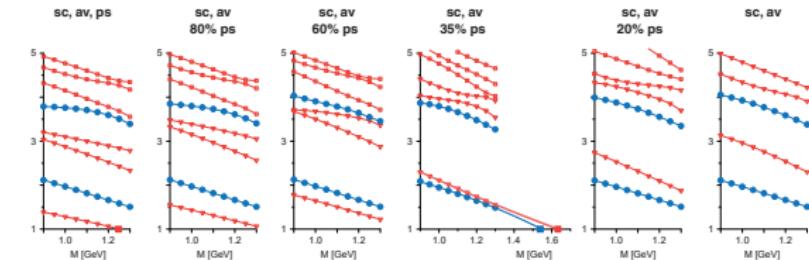
Light baryon spectrum

GE Fischer, Sanchis-Alepuz, PRD 94 (2016)



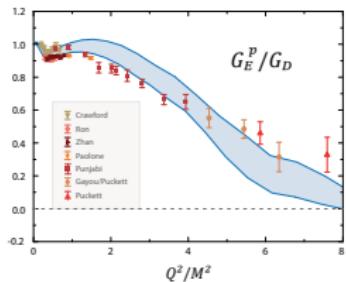
- **Level ordering** determined by diquark dynamics
- Diquarks are not pointlike, also here **rich spectrum!**

Barabanov et al., PPNP 116 (2021)

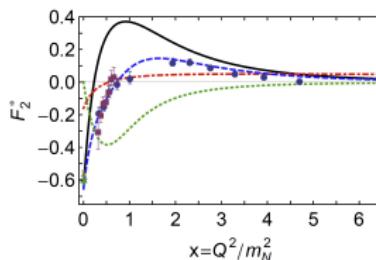


Baryon structure

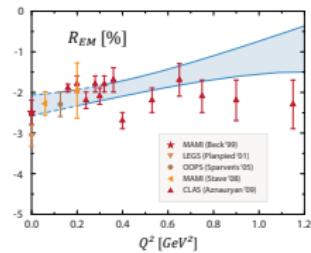
Nucleon electromagnetic FFs
GE, PRD 84 (2011)



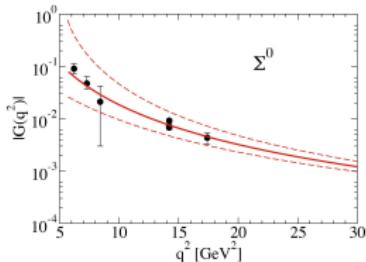
Roper em. transition FFs
Segovia et al., PRL 115 (2015)



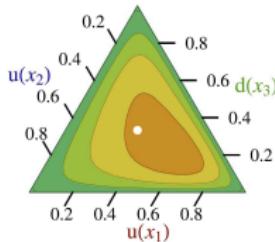
Δ em. transition FFs
GE, Nicmorus, PRD 85 (2012)



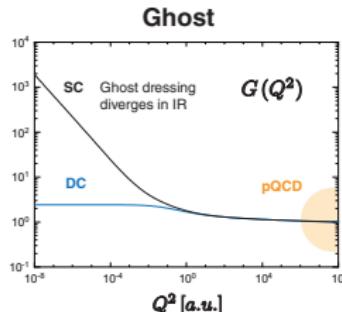
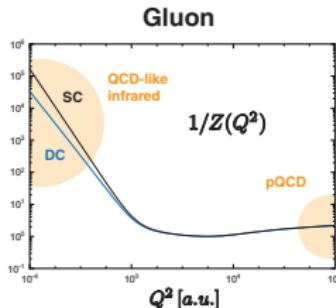
Timelike em. strangeness FFs
Ramalho, Peña, PRD 101 (2020)



Distribution amplitudes
Mezrag, Segovia, Chang, Roberts, PLB 783 (2018)



Gluon mass gap



- What distinguishes SC + DC?
- What is the “true” solution? Are all solutions physically equivalent?

Scaling (SC) solution:

- n-point functions scale with IR power laws
[Lerche, Smekal, PRD 65 \(2002\)](#), [Fischer, Alkofer, PLB 536 \(2002\)](#)
- Confinement
[Alkofer, Fischer, Llanes-Estrada, Mod. Phys. Lett. A 23 \(2008\)](#)

Decoupling (DC) solution:

- Seen in lattice QCD
[Cucchieri, Maas, Mendes, PRD 77 \(2008\)](#), [Bogolubsky et al., PLB 676 \(2009\)](#),
[Duarte, Oliveira, Silva, PRD 94 \(2016\)](#), [Aguilar et al., EPJ C 80 \(2020\)](#)
- Functional methods: family of DC solutions with SC solution as endpoint
[Boucaud et al., JHEP 06 \(2008\)](#), [Fischer, Maas, Pawłowski, Ann. Phys. 324 \(2009\)](#), [Reinosa et al., PRD 96 \(2017\)](#)

Mass generation

Gluon has T + L component, L = trivial.

$$(D^{-1})^{\mu\nu}(Q) = (D_0^{-1})^{\mu\nu}(Q) + \boxed{\Pi^{\mu\nu}(Q)} \quad \text{Expand self-energy in overcomplete basis:}$$

$$\text{---}^{-1} = \text{---}^{-1} + \dots$$

$$\Delta_T(Q^2)(Q^2\delta^{\mu\nu} - Q^\mu Q^\nu) + \Delta_0(Q^2)\delta^{\mu\nu} + \Delta_L(Q^2)Q^\mu Q^\nu$$

contains quadratic divergences, need to be subtracted
must contain longitudinal massless poles
 $\Delta_L = -\frac{\Delta_0}{Q^2}$
Mass generation must come from here!

$$Z(Q^2)^{-1} = Z_A + \Delta_T + \frac{\Delta_0}{Q^2}$$

$$L(Q^2)^{-1} = 1 + \xi \left[\Delta_L + \frac{\Delta_0}{Q^2} \right]$$

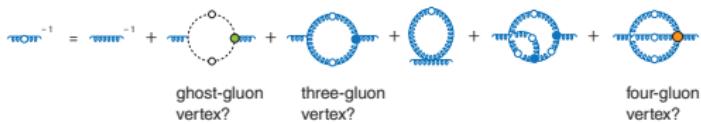
$$\stackrel{!}{=} 0$$

$$\Pi = \Delta_T - \Delta_L$$

$$\tilde{\frac{\Pi}{Q^2}} = \frac{\Delta_0}{Q^2} + \Delta_L \stackrel{!}{=} 0$$

Two possibilities:

- Scenario A: $\Delta_L = 0 \Rightarrow \Delta_0$ must be artifact (from hard cutoff and/or truncation)
- Scenario B: $\Delta_L \neq 0 \Rightarrow$ Longitudinal consistency condition, requires longitudinal massless poles in the vertices (where?), does not affect transverse equation

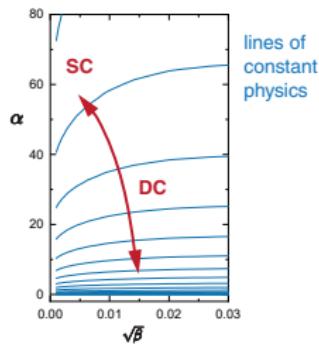


Towards ab-initio

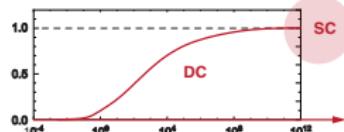
- System depends on 2 parameters:

- α ... coupling parameter
(not physical coupling)
- β ... mass scale, arising from subtraction of quadratic divergences

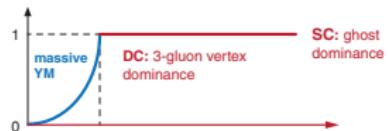
One combination changes scale,
the other distinguishes scaling &
decoupling solutions



- BSE eigenvalue: if $1 \Rightarrow L = 1$,
longitudinal massless poles
(Schwinger mechanism) in
ghost-gluon vertex



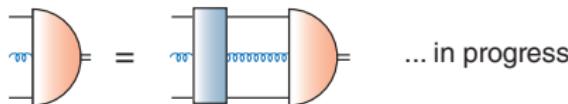
? ↗
lattice calculations are
here, also PT-BFM (Schwinger
mechanism for 3-gluon vertex)
Aguilar, Ferreira, Papavassiliou,
PRD 105 (2022)



- Are all massless (DC & SC)
solutions physically equivalent?
- Can test confinement in
hadron observables!

Hybrids

- **Three-body equation** (quark, antiquark, gluon)



- **Two-body equation** [quark–gluon]–antiquark
with model ansätze

Xu, Cui, Chang, Papavassiliou, Roberts, EPJA 55 (2019)

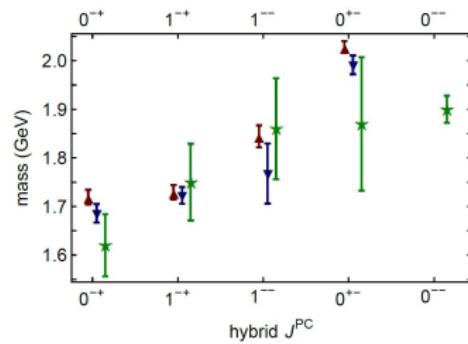
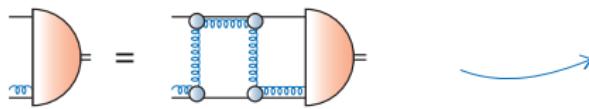
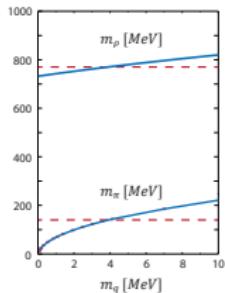


FIG. 2. Comparison between our ACM-improved spectrum (stars, green), Row 2 in Table I, and the rescaled lQCD results in Rows 3 (up-triangles, red) and 4 (down-triangles, blue).

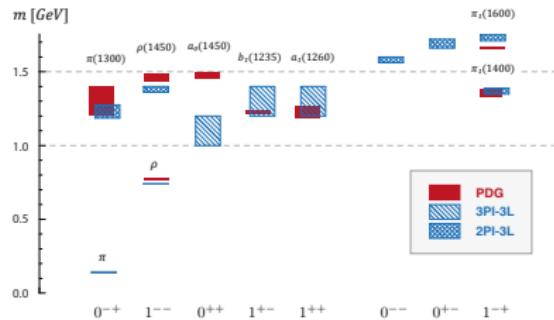
Lattice: Dudek, Edwards, Peardon, Richards, Thomas, PRD 82 (2010)

Mesons

- Pion is **Goldstone boson**: $m_\pi^2 \sim m_q$



- Light meson spectrum beyond rainbow-ladder**

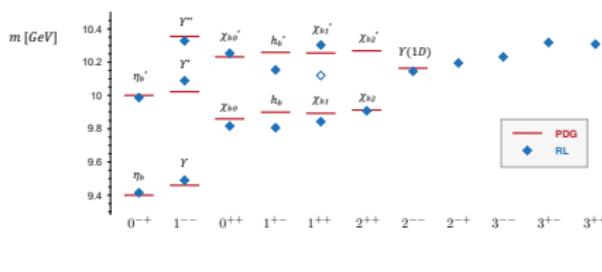


Williams, Fischer, Heupel,
PRD 93 (2016)

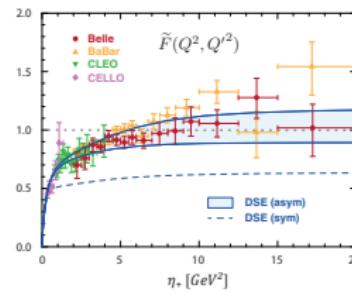
GE, Sanchis-Alepuz, Williams,
Alkofer, Fischer, PPNP 91 (2016)

- Bottomonium spectrum**

Fischer, Kubrak, Williams, EPJ A 51 (2015)

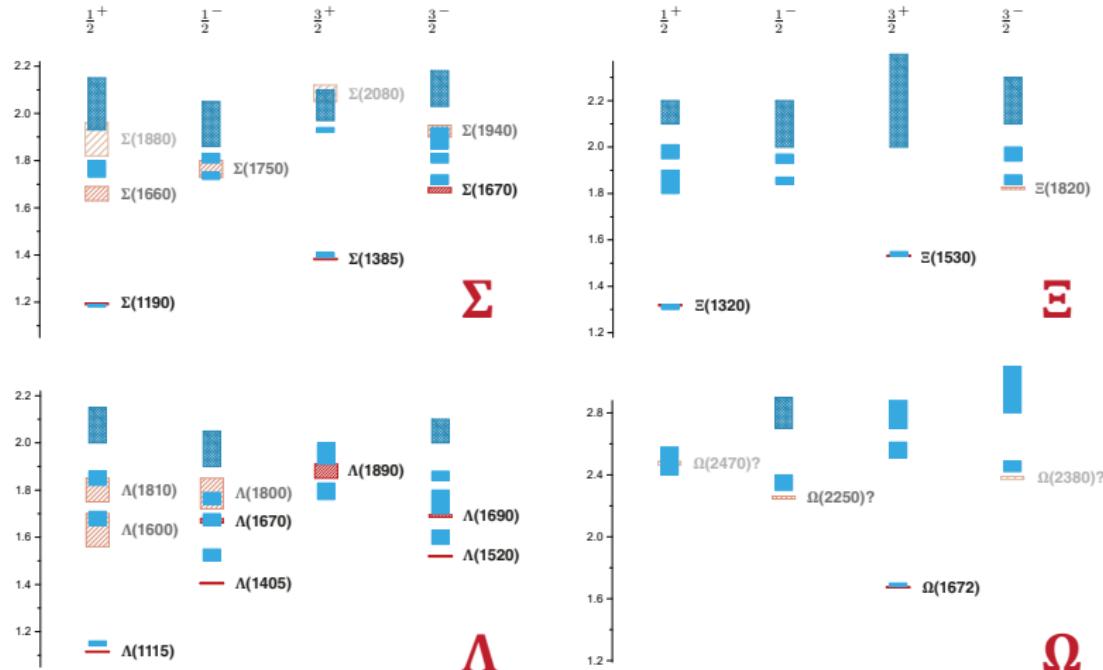


- Pion transition form factor**



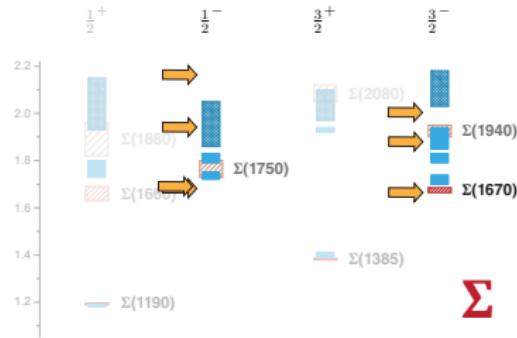
GE, Fischer, Weil, Williams,
PLB 774 (2017)

Strange baryons



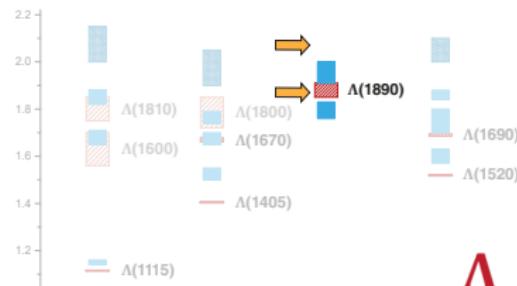
GE, Fischer, FBS 60 (2019), Fischer, GE, PoS Hadron 2017

Strange baryons



New states from Bonn-Gatchina
Sarantsev et al., 1907.13387 [nucl-ex]

Σ

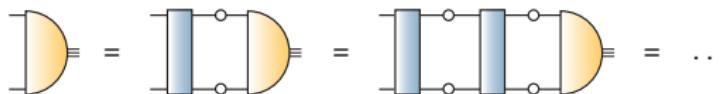


Λ

GE, Fischer, FBS 60 (2019), Fischer, GE, PoS Hadron 2017

Reply to comment

- Recent comment: BSE kernel is reducible
[Blankleider & Kvinikhidze, 2102.05818](#)
- Irrelevant for homogeneous BSE: same spectrum



- Results are identical within numerical errors
[Santowsky, GE, Fischer, Wallbott, Williams, 2103.14673](#)

