Phenomenological applications of Khuri-Treiman equations

4th Workshop on Future Directions in Spectroscopy Analysis (FDSA2022), JLab, November 14-16, 2022

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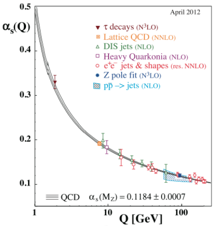
Based on:

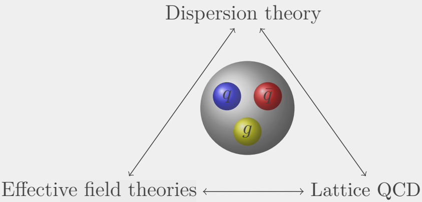
- JPAC Collaboration, Prog.Part.Nucl.Phys. 127 (2022) 103981;
- JPAC Collaboration, Eur.Phys.J.C 80 (2020) 12, 1107;
- S. Gonzàlez-Solís, P. Roig, Eur.Phys.J.C 79 (2019) 5, 436;

LA-UR-22-3197

- **Asymptotic freedom**:
  "like QED", but only at high energies
- **Confinement**:
  at low energies the gluons bind the quarks together to form the hadrons



- **Approaches** to describe the Low-energy regime of QCD:



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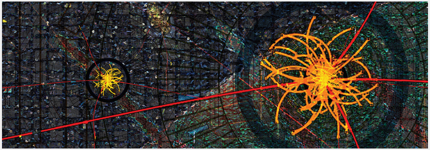
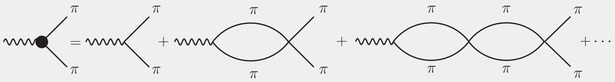
Experimental data

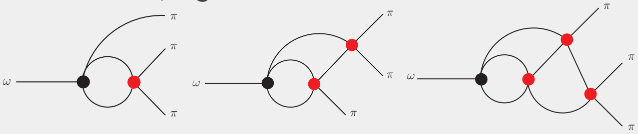


Artist: Xavier Cortada

- **One** hadron, *e.g.* $\tau \to \pi\nu_\tau$, decay constant from Lattice-QCD
- **Two** hadrons, *e.g.* $\tau \to \pi\pi\nu_\tau$

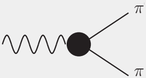


  - Good control: form factors (Omnès)
- **Three** hadrons, *e.g.* $\omega \to 3\pi$



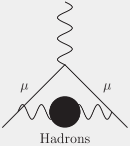
  - Reasonably $\to$ good control: Khuri-Treiman
    (see also talk by **Stamen**)
- Huge progress on few-hadron dynamics from Lattice QCD, see talks by: **Hanlon**, **Jackura**, **Doring**

- photon conversion into two pions



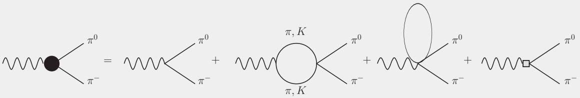
$$\longrightarrow \quad \langle \pi^+(p)\pi^-(p')|J_\mu(0)|0\rangle = i(p-p')_\mu F_\pi(s)\,,$$

- $F_\pi(s)$ contain the s dependent response of the $\pi\pi$ to $J_\mu(0)$
- Key object in many hadronic reactions, *e.g.* muon $(g-2)$



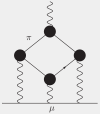
- Good pedagogic advent towards the challenges in the description of low-energy QCD

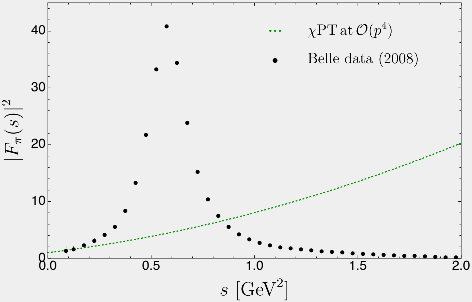
Chiral perturbation theory



$$F_\pi(s)\big|_{\chi\mathrm{PT}}^{\mathcal{O}(p^4)} \quad = \quad 1 + \frac{2L_9^r(\mu)}{F_\pi^2}s - \frac{s}{96\pi^2 F_\pi^2}\left(A_\pi(s,\mu^2) + \frac{1}{2}A_K(s,\mu^2)\right) \,,$$

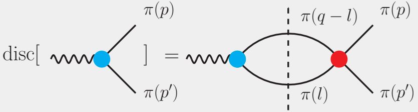
- **Drawbacks**:
  - limited energy range
  - $|F_V^\pi(s)|_{\chi\mathrm{PT}}^{\mathcal{O}(p^4)} \propto s$ vs $|F_V^\pi(s)|_{\mathrm{QCD}} \propto 1/s$
  - Non-predictive:
    divergent calculations





- **Solution**: Dispersion theory (No need of a Lagrangian)

- **Unitarity:**



$$\mathrm{disc}F_\pi(s) = 2i\mathrm{Im}F_\pi(s) = 2i\sigma_\pi(s)F_\pi(s)t_1^{1*}(s) = 2iF_\pi(s)\sin\delta_1^1(s)e^{-i\delta_1^1(s)}\,,$$

- **Watson's** theorem:

$$\mathrm{Im}F_\pi(s) = |F_\pi(s)|e^{i\delta_{F_\pi}(s)}\sin\delta_1^1(s)e^{-i\delta_1^1(s)}\theta(s-4m_\pi^2)\,,$$

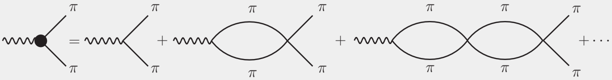
$$\Rightarrow \quad \delta_{F_\pi}(s) = \delta_1^1(s)\,,$$

- Analytic solution, **Omnès** equation:

$$F_\pi(s) = \frac{1}{2\pi i}\int_{4m_\pi^2}^\infty \frac{\mathrm{disc}F_\pi(s')}{s'-s}ds'\,, \quad \Omega(s) = \exp\left\{\frac{s}{\pi}\int_{4M_\pi^2}^\infty ds'\frac{\delta_1^1(s')}{s'(s'-s-i\varepsilon)}\right\}$$

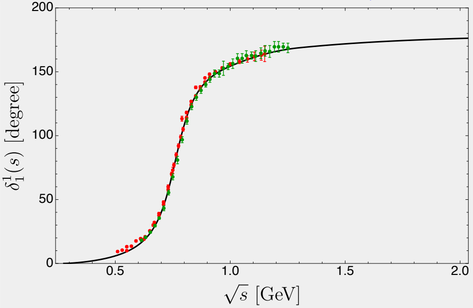
Omnès equation

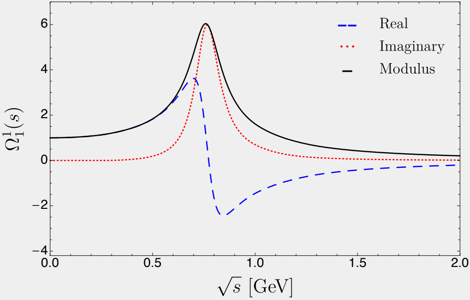
- Diagrammatic interpretation



- **Solution** depends solely on the *P*-wave phase shift of $\pi\pi$

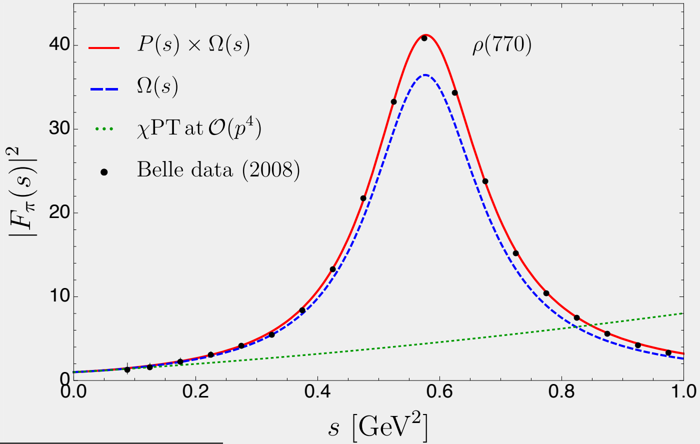
PRD 83, 074004 (2011)





Polynomial ambiguity

- Most general solution: $F_\pi(s) = P(s)\Omega_1^1(s)$
- $P(s) = 1 + \alpha s$, with $\alpha = 0.11\ \text{GeV}^{-2}$ due to inelasticities



Three-body decays: $\omega \to 3\pi$

- Decay amplitude for $\omega \to \pi^+\pi^-\pi^0$

$$\mathcal{M}(s,t,u) = i\epsilon_{\mu\nu\alpha\beta}\epsilon^{\mu}p_+^{\nu}p_-^{\alpha}p_0^{\beta}\mathcal{F}(s,t,u)\,,$$

$$|\mathcal{M}(s,t,u)|^2 = \frac{1}{4}\left(stu - m_\pi^2(m_V^2 - m_\pi^2)^2\right)|\mathcal{F}(s,t,u)|^2 = \mathcal{P}(s,t,u)|\mathcal{F}(s,t,u)|^2\,,$$

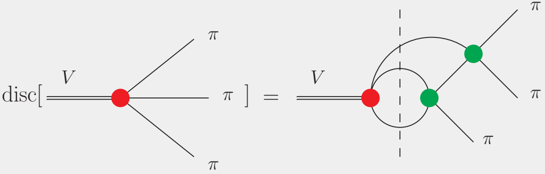
- If no dynamics: $|\mathcal{F}(s,t,u)|^2 = 1 \Rightarrow |\mathcal{M}(s,t,u)|^2$ follows *P*-wave distribution
  - In 1961, ω spin and parity from $\omega \to 3\pi$ was consistent with a *P*-wave
  - Current $\omega \to 3\pi$ precision Dalitz analyses show deviation from the *P*-wave
- Dalitz plot parameters: $X = \frac{t-u}{\sqrt{3}R_\omega} = \sqrt{Z}\cos\phi\,,\, Y = \frac{s_c - s}{R_\omega} = \sqrt{Z}\sin\phi\,,$

$$|\mathcal{F}(Z,\phi)|^2 = |N|^2\left[1 + 2\alpha Z + 2\beta Z^{3/2}\sin 3\phi + 2\gamma Z^2 + \mathcal{O}(Z^{5/2})\right]\,,$$

Reference	$\alpha \times 10^3$	$\beta \times 10^3$	$\gamma \times 10^3$
BESIII [PRD98, 112007 (2018)]	120.2(7.1)(3.8)	29.5(8.0)(5.3)	—
WASA-at-COSY [PLB770, 418 (2017)]	133(41)	37(54)	—
BESIII [PRD98, 112007 (2018)]	111(18)	25(10)	22(29)

- Decomposition of the amplitude (considering only P-waves)

$$\mathcal{F}(s,t,u) \;=\; \mathcal{F}(s)+\mathcal{F}(t)+\mathcal{F}(u)\,,$$



$$\text{disc}\mathcal{F}(s) \;=\; 2i\left(\mathcal{F}(s)+\hat{\mathcal{F}}(s)\right)\theta(s-4m_\pi^2)\sin\delta(s)e^{-i\delta(s)}\,,$$

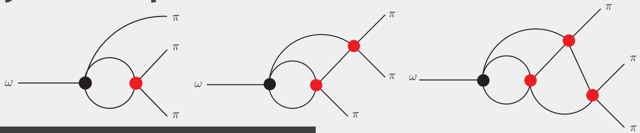
  - $\delta(s)$: P-wave $\pi\pi$ scattering phase shifts
  - $\hat{\mathcal{F}}(s)$: s-channel projections of $\mathcal{F}(t), \mathcal{F}(u)$

- **Unsubtracted** solution:

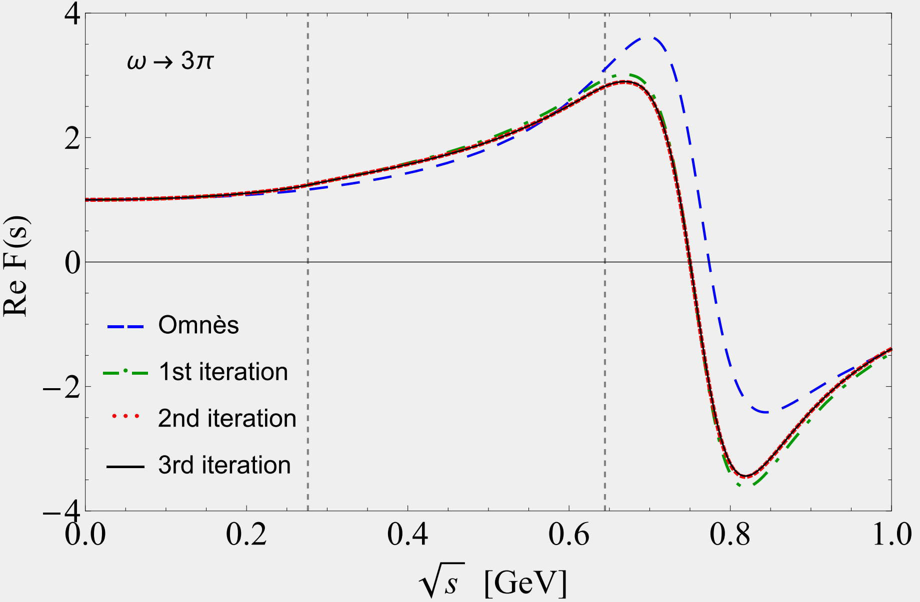
$$\mathcal{F}(s) = \Omega(s)\left(a + \frac{s}{\pi}\int_{4m_\pi^2}^{\infty}\frac{ds'}{s'}\frac{\sin\delta(s')\,\hat{\mathcal{F}}(s')}{|\Omega(s')|\,(s'-s)}\right),$$

$$\hat{\mathcal{F}}(s) = 3\int_{-1}^{1}\frac{dz_s}{2}(1-z_s^2)\mathcal{F}(t(s,z_s)),$$

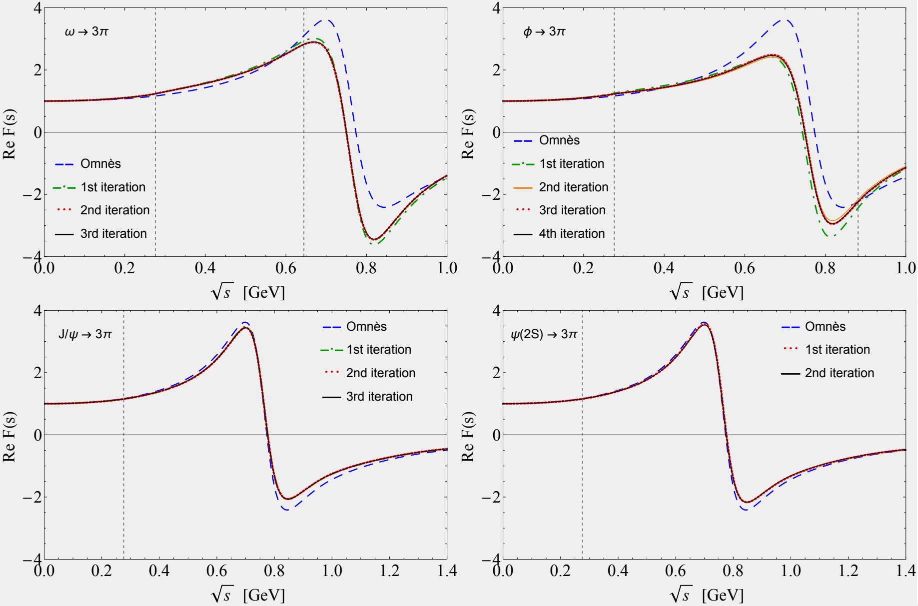
  - Subtraction constants: free parameters not fixed by unitarity
- **Complications**: Integration contour for $\hat{\mathcal{F}}(s)$
  - Generates cuts in several variables (3-particle cuts)
  - Discontinuity is complex
- **Physical interpretation**



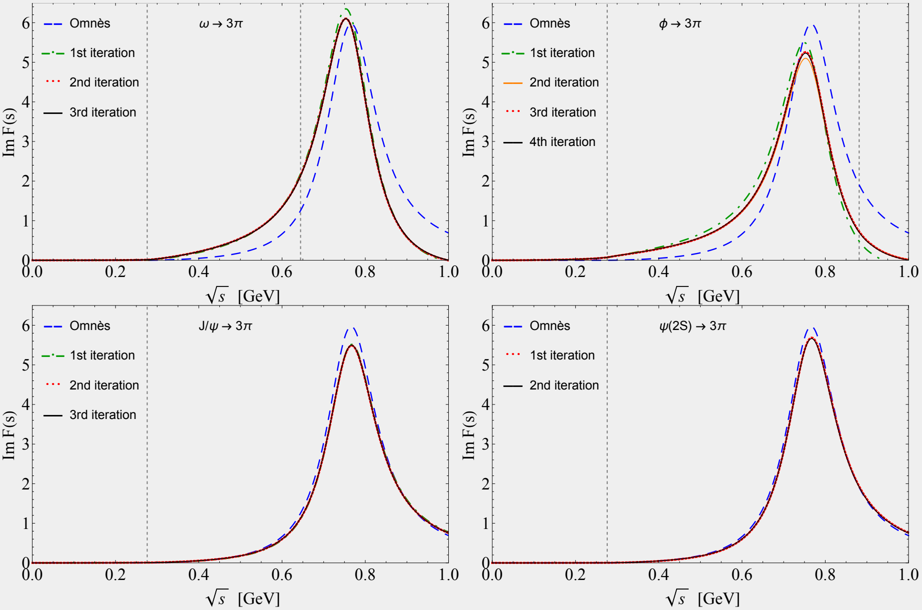
Solution by numerical iteration



Solution by numerical iteration



Solution by numerical iteration



$\omega \to 3\pi$ Dalitz plot parameters

Reference (Bonn, JPAC)	$\alpha \times 10^3$	$\beta \times 10^3$	$\gamma \times 10^3$
BESIII	111(18)	25(10)	22(29)
Omnès [EPJC C72 2014,(2012)]	116(4)	28(2)	16(2)
KT unsub [EPJC C72 2014,(2012)]	77(4)	26(2)	5(2)
Omnès [PRD91, 094029 (2015)]	113	27	24
KT unsub [PRD91, 094029 (2015)]	80	27	8

- Someone is wrong: experiment? KT. eqs? something particular?
- Solution: Once subtracted KT

$$\mathcal{F}(s) = \Omega(s)\left[a + b's + \frac{s^2}{\pi}\int_{4m_\pi^2}^{\infty}\frac{ds'}{s'^2}\frac{\sin\delta(s')\,\hat{\mathcal{F}}(s')}{|\Omega(s')|(s'-s)}\right],$$

$$b \equiv b'/a = \frac{1}{a}\frac{1}{\pi}\int_{4m_\pi^2}^{\infty}\frac{ds'}{s'^2}\frac{\sin\delta(s')\,\hat{\mathcal{F}}(s')}{|\Omega(s')|} = 0.55e^{0.15i}\,\mathrm{GeV}^{-2}$$

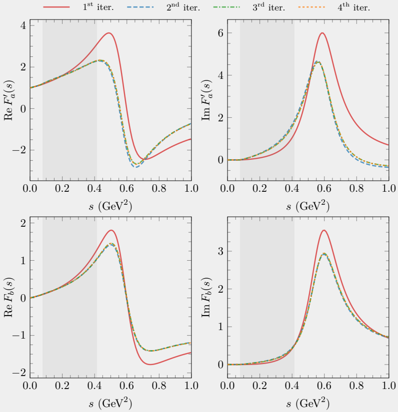
Khuri-Treiman representation

JPAC Coll, EPJ C80 (2020) no.12, 1107

- $\mathcal{F}(s) = a\,[F_a(s) + b\,F_b(s)]$

$$F_a(s) = \Omega(s)\left[1 + \frac{s^2}{\pi}\int_{4m_\pi^2}^{\infty}\frac{ds'}{s'^2}\frac{\sin\delta(s')\,\hat{F}_a(s')}{|\Omega(s')|(s'-s)}\right],$$

$$F_b(s) = \Omega(s)\left[s + \frac{s^2}{\pi}\int_{4m_\pi^2}^{\infty}\frac{ds'}{s'^2}\frac{\sin\delta(s')\,\hat{F}_b(s')}{|\Omega(s')|(s'-s)}\right].$$



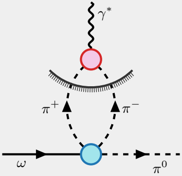
- Subtraction constant from Dalitz parameters

$$b_{\rm Fit} \simeq 2.9e^{1.90(1)i}\,{\rm GeV}^{-2} \quad {\rm vs} \quad b_{\rm sr} = 0.55e^{0.15i}\,{\rm GeV}^{-2}$$

$\omega \to \pi^0\gamma^*$ FORM FACTOR

- Dispersive representation

JPAC Coll, EPJ C80 (2020) no.12, 1107

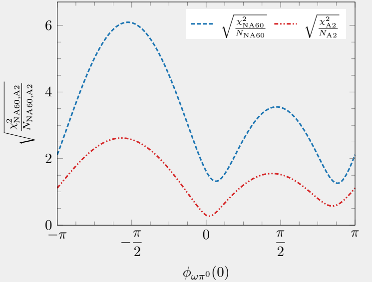


$$f_{\omega\pi^0}(s) = |f_{\omega\pi^0}(0)|\, e^{i\phi_{\omega\pi^0}(0)} + \frac{s}{12\pi^2}\int_{4m_\pi^2}^{\infty} \frac{ds'}{(s')^{3/2}} \frac{p^3(s')\, F_\pi^{V*}(s')\, f_1^{\omega\to 3\pi}(s')}{(s'-s)},$$

- $|f_{\omega\pi^0}(0)|$ from data:

$$\Gamma(\omega \to \pi^0\gamma) = \frac{e^2(m_\omega^2 - m_{\pi^0}^2)^3}{96\pi m_\omega^3}\, |f_{\omega\pi^0}(0)|^2,$$

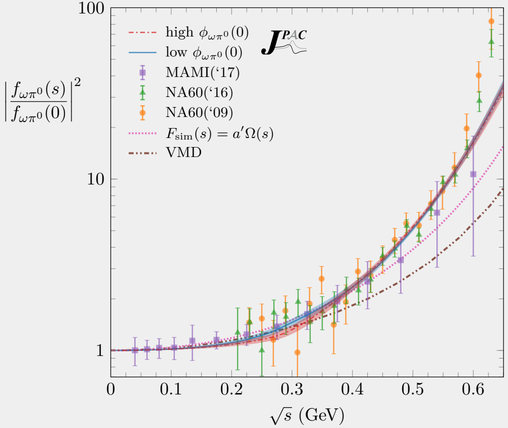
- Data: $F_{\omega\pi}(s) = \frac{f_{\omega\pi}(s)}{f_{\omega\pi}(0)}$ [NA60('09,'16), A2('17)]
- $\phi_{\omega\pi^0}(0)$ only free parameter
- Two minima are found (low and high)



COMBINED $\omega \to 3\pi$ AND $\omega \to \pi^0\gamma^*$ ANALYSIS

Reference	$\alpha \times 10^3$	$\beta \times 10^3$	$\gamma \times 10^3$
BESIII	111(18)	25(10)	22(29)
low	112(15)	23(6)	29(6)
high	109(14)	26(6)	19(5)

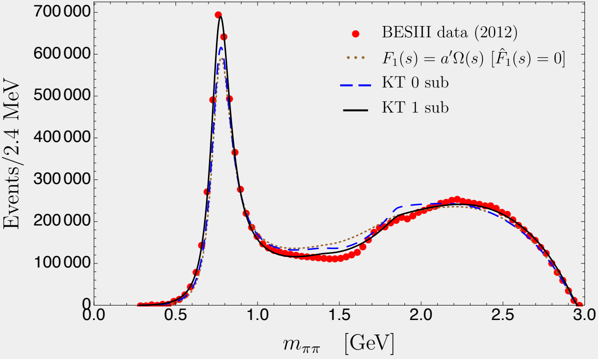
JPAC Coll, EPJ C80 (2020) no.12, 1107



- Only 1 sub KT describes **both** Dalitz-param and form factor

$J/\psi \to 3\pi$ DECAYS

- Completely analogous formalism
- Larger phase space, but the decay is still dominated by the ρ
- BESIII data [PLB 710 (2012)]
- JPAC (preliminary) analysis:
  - Two-body
  - KT unsub basic features
  - KT 1-sub improves the description



$$b_{\text{Fit}} = 0.20(1)e^{2.68(1)i}\,\text{GeV}^{-2}\,,$$

$$b_{\text{sr}} = 0.16e^{2.43i}\,\text{GeV}^{-2}$$

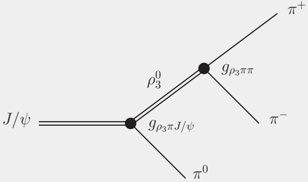
Inclusion of F-wave

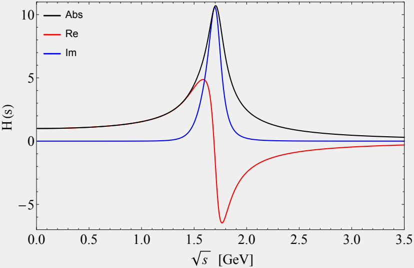
- Isobar decomposition of the amplitude

$$\begin{aligned}\mathcal{F}(s,t,u) &= \mathcal{F}(s)+\mathcal{F}(t)+\mathcal{F}(u)\\ &+ \kappa^2(s)P_3'(z_s)\mathcal{H}(s)+\kappa^2(t)P_3'(z_t)\mathcal{H}(t)+\kappa^2(u)P_3'(z_u)\mathcal{H}(u)\,,\end{aligned}$$

- Exchange of a $\rho_3(1690)$ in the s-channel:

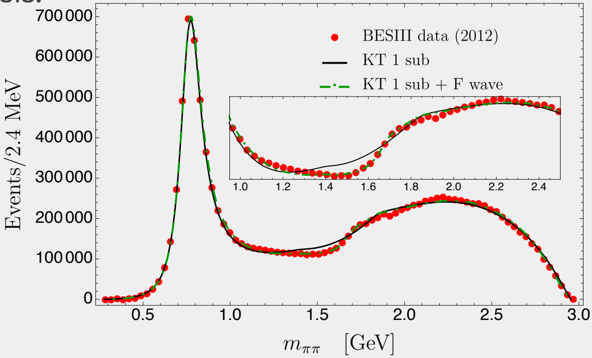
$$\mathcal{H}(s) = P(s)\frac{m_{\rho_3}^2}{m_{\rho_3}^2 - s - i m_{\rho_3}\Gamma_{\rho_3}(s)}\,,$$





$J/\psi \to 3\pi$ DECAYS

- Larger phase space, but the decay is still dominated by the ρ
- BESIII data [PLB 710 (2012)]
- JPAC (preliminary) analysis:
  - Two-body
  - KT unsub basic features
  - KT 1-sub improves the description
  - KT 1-sub+F-wave describe better $m_{\pi\pi} \sim 1.5$ GeV.



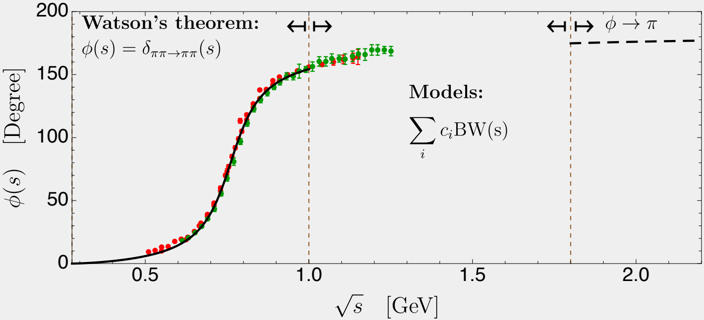
- **Khuri-Treiman equations**:
  - Dispersive representation for **3-particle Final-State Interactions**
  - Based on fundamental principles: **analyticity, unitarity and crossing symmetry**
  - Input: $\pi\pi$ scattering **phase shifts**
  - Resonance shape affected by **left-hand cuts / 3-body effects**
  - **Predictive power** (subtraction constants)
  - Experimental **data** well described

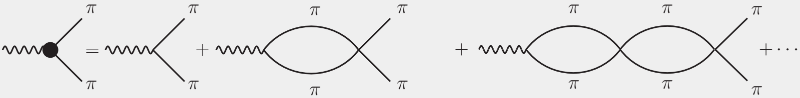
Dispersive representation

- Dispersion relation with subtractions:

$$F_V^\pi(s) = \exp\left[\alpha_1 s + \frac{\alpha_2}{2}s^2 + \frac{s^3}{\pi}\int_{4m_\pi^2}^{s_{\rm cut}} ds' \frac{\phi(s')}{(s')^3(s'-s-i\varepsilon)}\right],$$

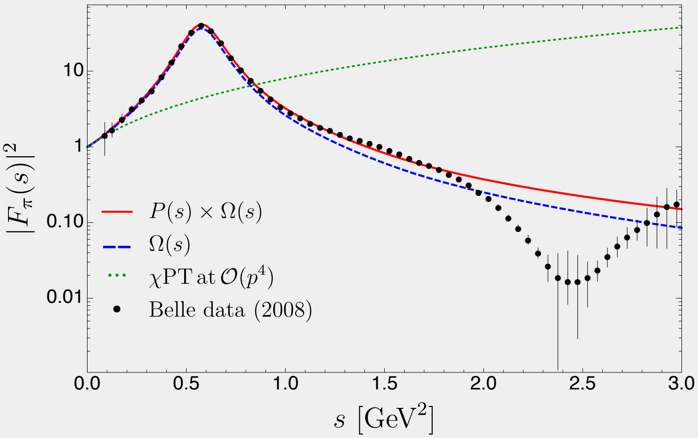
- Form Factor phase $\phi(s)$:





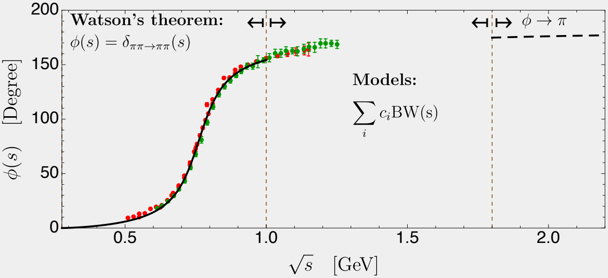
Beyond the elastic region

- $P(s) = 1 + \alpha s$, with $\alpha = 0.11\ \text{GeV}^{-2}$ due to inelasticities

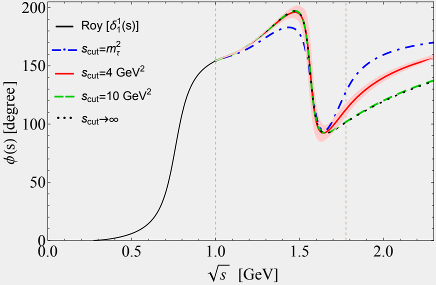


Beyond the elastic approximation

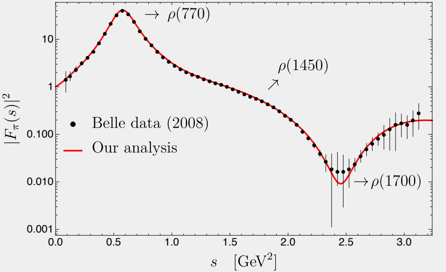
- Built an effective phase:



- Fit to data:



Gonzàlez-Solís and Roig, Eur.Phys.J C79 (2019) no.5, 436



POLYNOMIAL AMBIGUITY

- Most general solution: $F_\pi(s) = P(s)\Omega_1^1(s)$
- To find the constraint on $P(s)$, we need $\lim_{s\to\infty}\Omega_1^1(s)$
- Assume $\delta_1^1(s > \Lambda^2) = n\pi$

$$\Omega_1^1(s) = \exp\left\{\frac{s}{\pi}\int_{4m_\pi^2}^{\Lambda^2} ds' \frac{\delta_1^1(s')}{s'(s'-s)} + \frac{s}{\pi}\int_{\Lambda^2}^{\infty} ds' \frac{\delta_1^1(s')}{s'(s'-s)}\right\}, \quad (1)$$

$$= \exp\left\{-\frac{1}{\pi}\int_{4m_\pi^2}^{\Lambda^2} ds' \frac{\delta_1^1(s')}{s'} + n\int_{\Lambda^2}^{\infty} ds' \left(\frac{1}{s'-s} - \frac{1}{s'}\right)\right\}, \quad (2)$$

$$= \exp\left\{\text{constant} - n\log\left(\frac{\Lambda^2 - s}{\Lambda^2}\right)\right\}, \quad (3)$$

$$\lim_{s\to\infty}\Omega_1^1(s) \propto s^{-n} \quad (4)$$

- Assume $\lim_{s\to\infty} F_\pi(s) \propto s^m$,

$$F_\pi(s) = P^{m+n}(s)\Omega_1^1(s), \quad (5)$$

Low-energy observables

- $\langle r^2 \rangle_V^\pi = 6\alpha_1$

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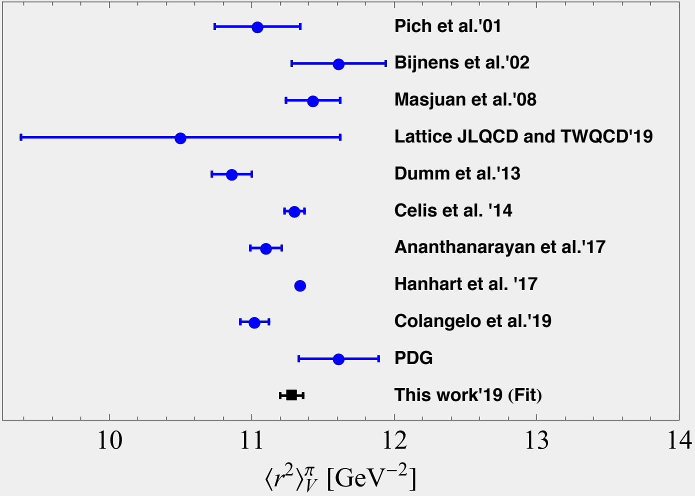

Khuri-Treiman representation

- Once subtracted solution:

$$\mathcal{F}(s) \;=\; \Omega(s)\left[a + b's + \frac{s^2}{\pi}\int_{4m_\pi^2}^{\infty}\frac{ds'}{s'^2}\frac{\sin\delta(s')\,\hat{\mathcal{F}}(s')}{|\Omega(s')|(s'-s)}\right], \tag{6}$$

$$b \;\equiv\; b'/a = \frac{1}{a}\frac{1}{\pi}\int_{4m_\pi^2}^{\infty}\frac{ds'}{s'^2}\frac{\sin\delta(s')\,\hat{\mathcal{F}}(s')}{|\Omega(s')|} = 0.55e^{0.15i}\,\mathrm{GeV}^{-2}$$

$$\mathcal{F}(s) \;=\; a\left[F_a(s) + b\,F_b(s)\right], \tag{7a}$$

$$F_a(s) \;=\; \Omega(s)\left[1 + \frac{s^2}{\pi}\int_{4m_\pi^2}^{\infty}\frac{ds'}{s'^2}\frac{\sin\delta(s')\,\hat{F}_a(s')}{|\Omega(s')|(s'-s)}\right], \tag{7b}$$

$$F_b(s) \;=\; \Omega(s)\left[s + \frac{s^2}{\pi}\int_{4m_\pi^2}^{\infty}\frac{ds'}{s'^2}\frac{\sin\delta(s')\,\hat{F}_b(s')}{|\Omega(s')|(s'-s)}\right]. \tag{7c}$$

$\omega \to 3\pi$ DECAYS AND $\omega \to \pi^0\gamma^*$ FORM FACTOR

- $\omega \to \pi^0\gamma^*$ form factor

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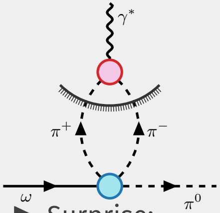
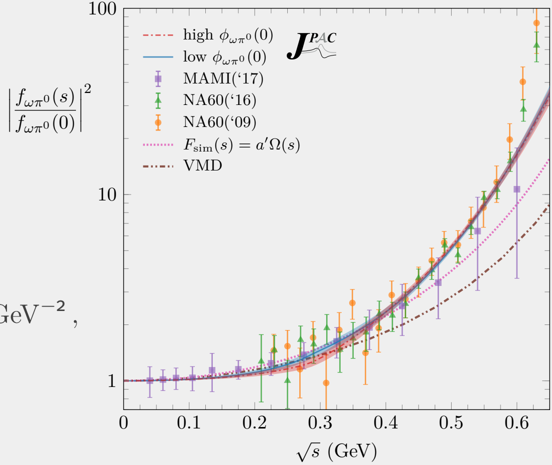


- Surprise:

$$b_{\rm Fit} = 3.15(22)e^{2.03(14)i}\,{\rm GeV}^{-2}\,,$$

$$b_{\rm Sum-rule} = 0.55e^{0.15i}\,{\rm GeV}^{-2}$$



- Only 1 sub KT describes both Dalitz-parameters and form factor

$\omega \to 3\pi$

- Dalitz plot parameters:

$$X = \frac{t-u}{\sqrt{3}R_\omega} = \sqrt{Z}\cos\phi\,, \quad Y = \frac{s_c - s}{R_\omega} = \sqrt{Z}\sin\phi\,,$$

$$|\mathcal{F}(Z,\phi)|^2 = |N|^2\left[1 + 2\alpha Z + 2\beta Z^{3/2}\sin 3\phi + 2\gamma Z^2 + \mathcal{O}(Z^{5/2})\right]\,,$$

Reference	$\alpha \times 10^3$	$\beta \times 10^3$	$\gamma \times 10^3$
BESIII	120.2(8.1)	29.5(9.6)	—
Omnès	130(5)	31(2)	—
KT unsub	79(5)	26(2)	—
KT 1 sub (this work)	120.1(7.7)(0.7)	30.2(4.3)(2.5)	—
BESIII	111(18)	25(10)	22(29)
Omnès	116(4)	28(2)	16(2)
KT unsub	77(4)	26(2)	5(2)
KT 1 sub (this work)	109(14)(2)	26(6)(2)	19(5)(4)

- γ is non zero at a $3 \sim \sigma$ level

KT 1 sub vs Omnès

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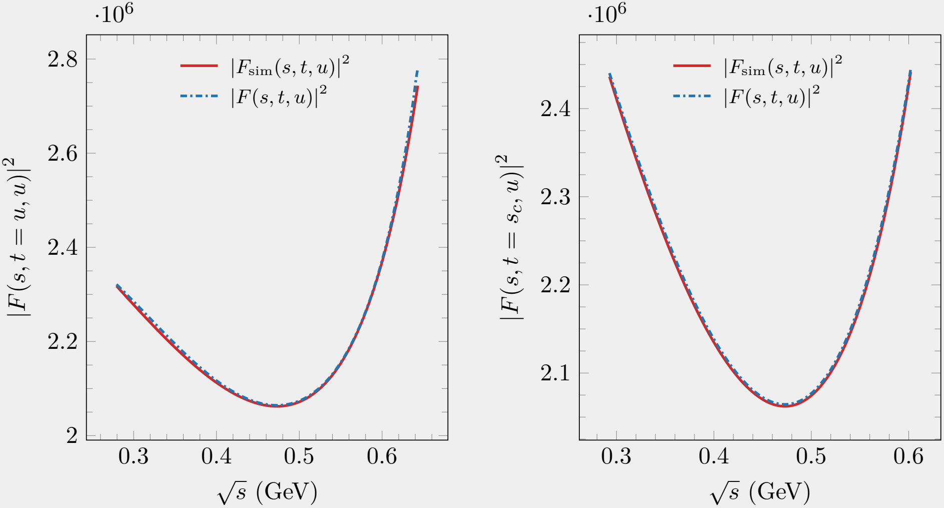
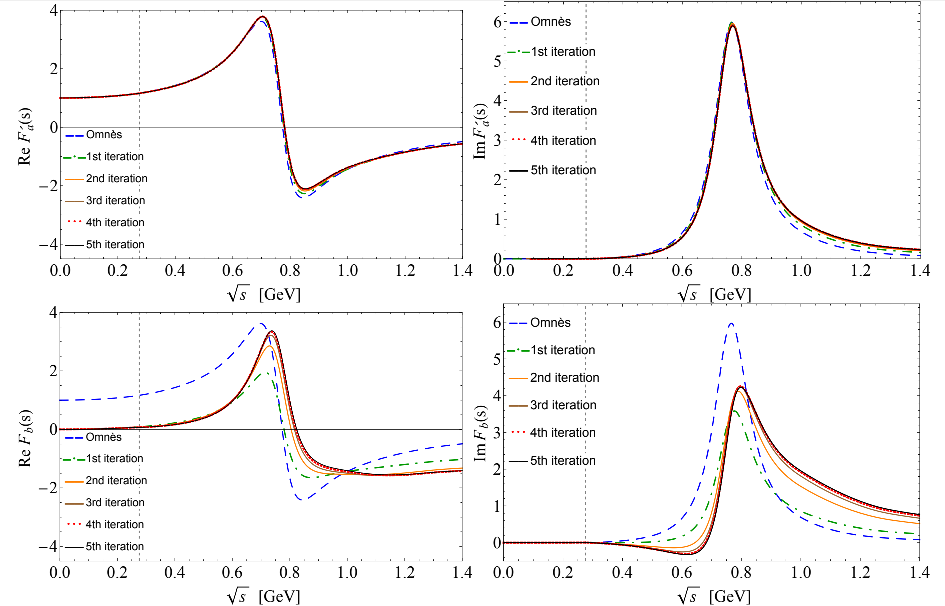
- Accidental cancellation of $|F(s,t,u)|^2$

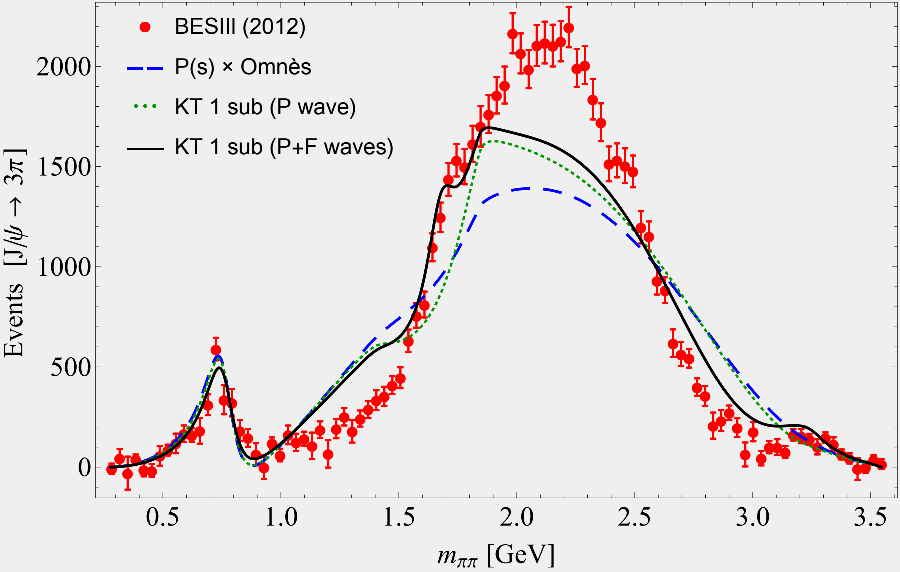


ONCE SUB. ISOBAR AMPLITUDES $F(s)$: $\psi(2S) \to 3\pi$

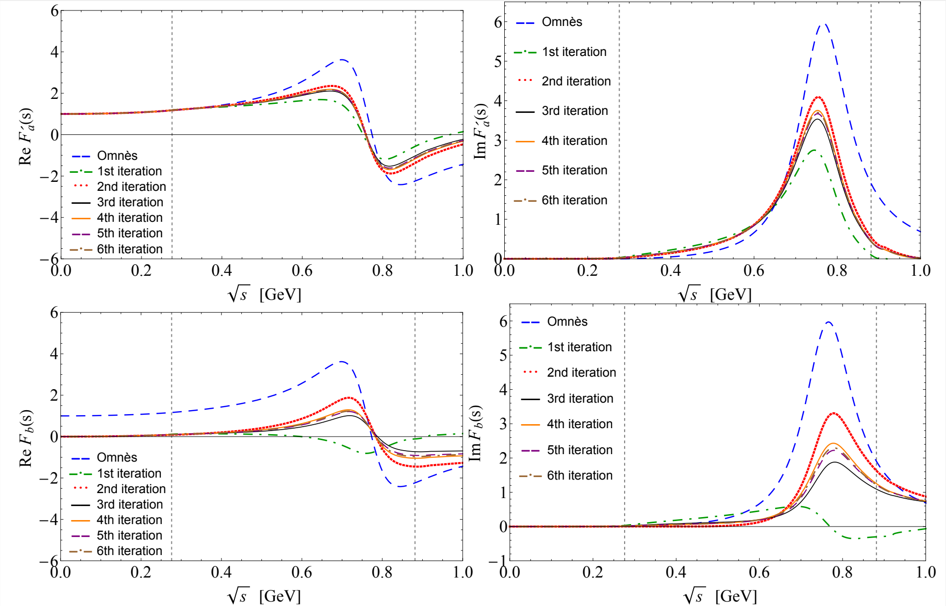


Fits to the $\psi(2S) \to 3\pi$ di-pion distribution

JPAC Collaboration, in progress



Once subtracted isobar amplitudes $F(s)$: $\phi \to 3\pi$

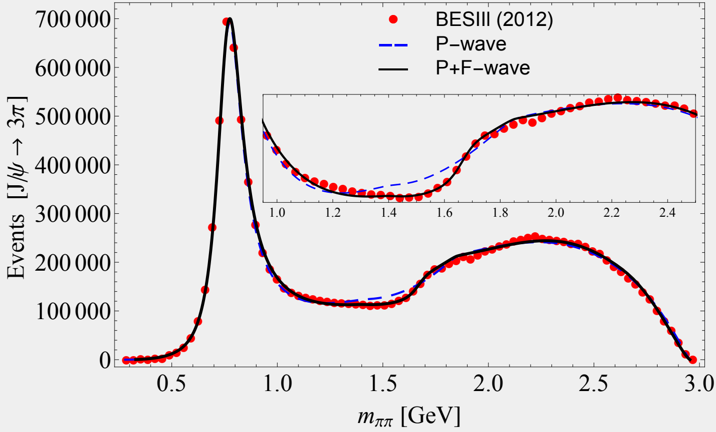


Fit results

- P+F waves

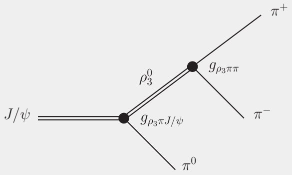
Data set	b	ϕ_b	$c \times 10^3$	ϕ_c	$d \times 10^3$	ϕ_d	m_{ρ_3} [MeV]	Γ_{ρ_3} [MeV]
BESIII 2012	0.20(1)	2.88(1)	3.87(2)	3.72(1)	1.42(1)	0.59(1)	$= 1689$	$= 161$

JPAC Collaboration, to appear soon



CONTRIBUTION OF THE F-WAVE: $\rho_3(1690)$ EXCHANGE

- Exchange of a $\rho_3(1690)$ in the s-channel:



$$\mathcal{H}(s) = P(s)\frac{m_{\rho_3}^2}{m_{\rho_3}^2 - s - i m_{\rho_3}\Gamma_{\rho_3}(s)},$$

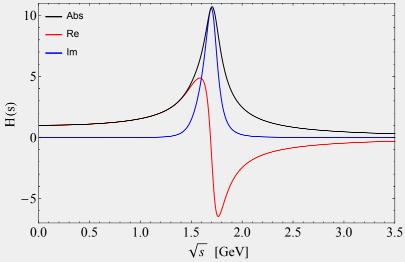
  - Energy-dependent width

$$\Gamma_{\rho_3}(s) = \frac{\Gamma_{\rho_3} m_{\rho_3}}{\sqrt{s}}\left(\frac{p_\pi(s)}{p_\pi(m_{\rho_3}^2)}\right)^{2\ell+1}\left(F_R^\ell(s)\right)^2,$$

$$p_\pi(s) = \frac{\sqrt{s}}{2}\sigma_\pi(s),$$

$$F_R^{\ell=3}(s) = \sqrt{\frac{z_0(z_0-15)^2 + 9(2z_0-5)^2}{z(z-15)^2 + 9(2z-5)^2}},$$

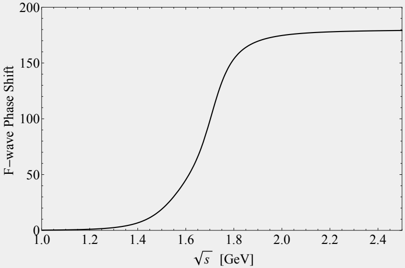
$$z = r_R^2 p_\pi^2(s), z_0 = r_R^2 p_\pi^2(m_{\rho_3}^2),$$



OMNÈS-LIKE CONTRIBUTION OF THE F-WAVE

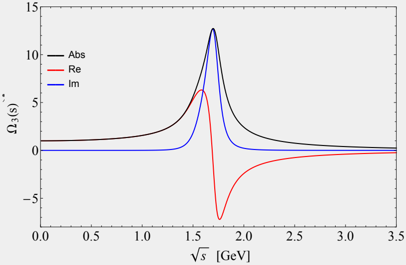
- Extraction of the F-wave phase:

$$\tan \delta_3(s) = \frac{\mathrm{Im}\mathcal{H}(s)|_{\mathrm{model}}}{\mathrm{Re}\mathcal{H}(s)|_{\mathrm{model}}},$$



- Omnès:

$$\mathcal{H}(s) = \Omega_3(s) \equiv \exp\left[\frac{s}{\pi}\int_{4m_\pi^2}^{\infty}\frac{ds'}{s'}\frac{\delta_3(s')}{s'-s}\right].$$



1-SUB SOLUTION FOR $J/\psi \to 3\pi$