Application of p.w. dispersive techniques in the analysis of experimental and lattice data

Igor Danilkin

4th Workshop on Future Directions in Spectroscopy Analysis (FDSA2022)

November 16, 2022

Deutsche Forschungsgemeinschaft
FOR5327

JOHANNES GUTENBERG UNIVERSITÄT MAINZ

Introduction and Motivation

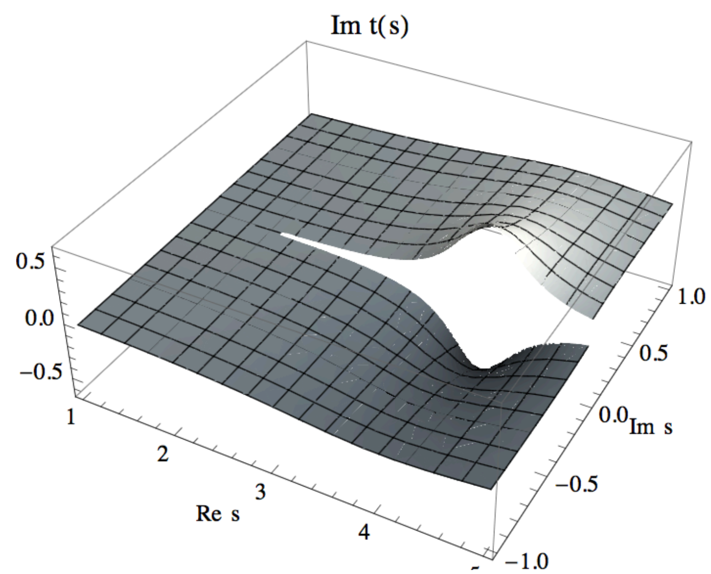
- New era of hadron spectroscopy, motivated by recent discoveries of unexpected exotic hadron resonances: LHCb, BESIII, COMPASS, Belle, …
- Significant progress in lattice QCD studies of poorly known hadronic states

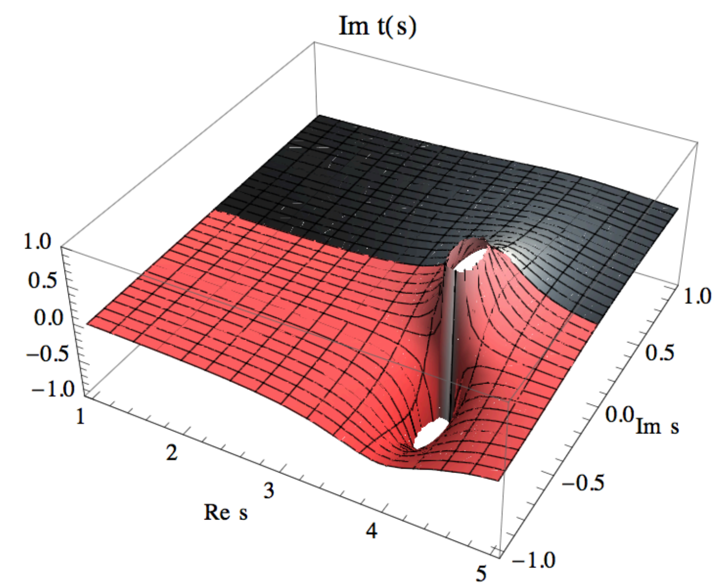
To correctly identify resonance parameters one has to search for poles of the S-matrix in the complex plane

It is particularly important when

➢ there is an interplay between several inelastic channels

➢ the pole is lying very deep in the complex plane





S-matrix constraints

- Call for a framework which complies with the main principles of the S-matrix theory:
  ➢ Unitarity
  ➢ Analyticity (causality)
  ➢ Crossing symmetry

S-matrix constraints

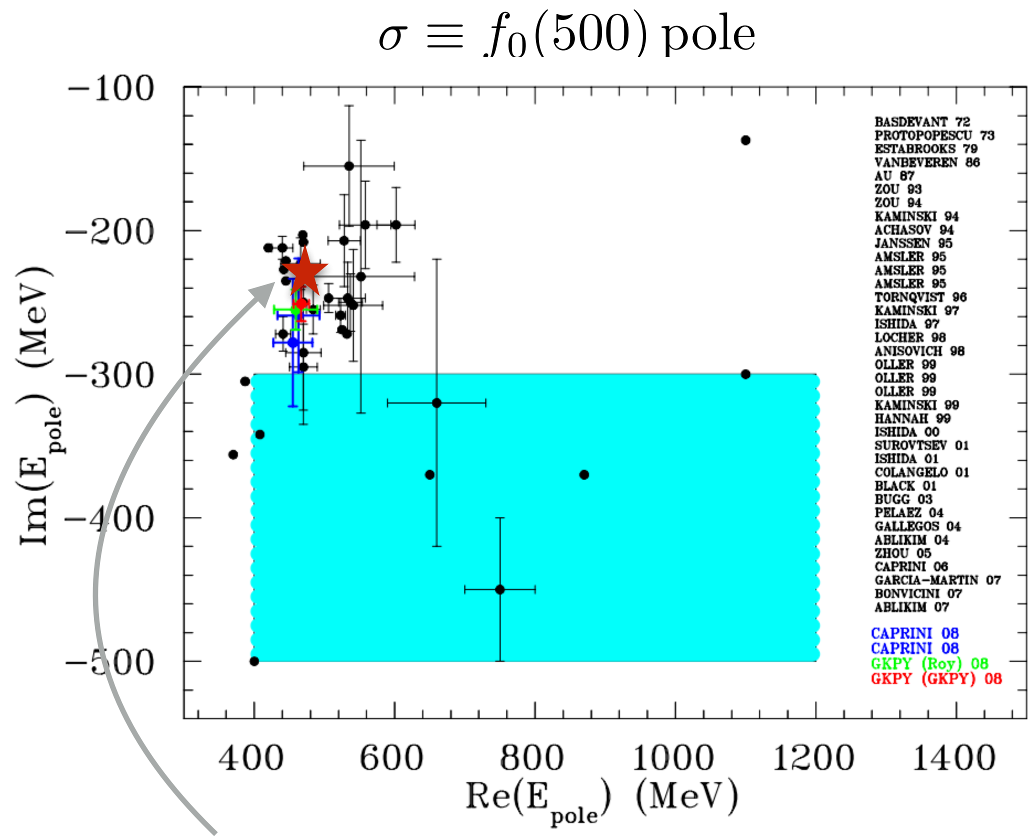
- Call for a framework which complies with the main principles of the S-matrix theory:
  - ➢ Unitarity
  - ➢ Analyticity (causality)
  - ➢ Crossing symmetry

Roy (Roy-Steiner) equations

$$t_J^I(s) = k_J^I(s) + \sum_{I'=0}^{2}\sum_{J'=0}^{\infty}\int_{4m_\pi^2}^{\infty} ds'\, K_{JJ'}^{II'}(s,s')\, \mathrm{Im}\, t_{J'}^{I'}(s')$$

subtraction polynomial

kernel functions known analytically



$$\sqrt{s_\sigma} = \left(449^{+22}_{-16}\right) \pm i\left(275 \pm 15\right)\ \mathrm{MeV}$$

[Roy (1971)]
[Colangelo et al. (2001)]
[Caprini et al. (2006)]
[Garcia-Martin et al. (2011)]
[Pelaez (2016)]

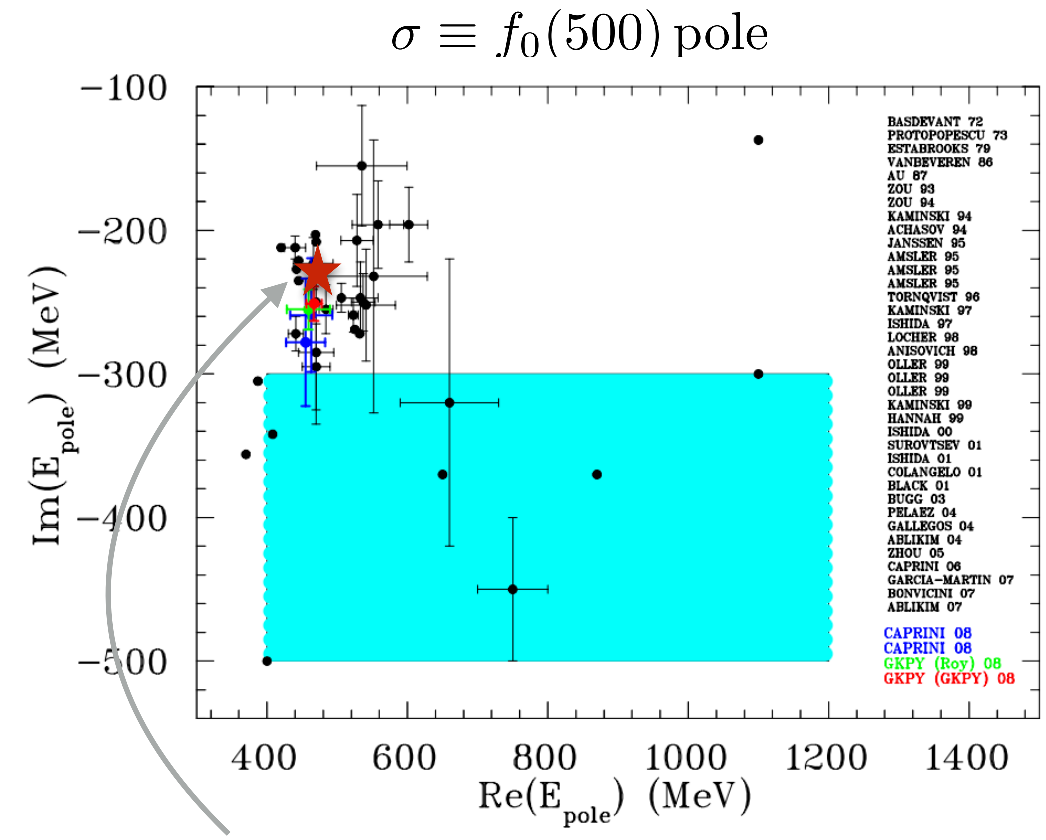
- Call for a framework which complies with the main principles of the S-matrix theory:
  ➢ Unitarity
  ➢ Analyticity (causality)
  ➢ Crossing symmetry

Roy (Roy-Steiner) equations

$$t^I_J(s) = k^I_J(s) + \sum_{I'=0}^{2} \sum_{J'=0}^{\infty} \int_{4m_\pi^2}^{\infty} ds' \, K^{II'}_{JJ'}(s,s') \, \mathrm{Im}\, t^{I'}_{J'}(s')$$

subtraction polynomial

kernel functions known analytically

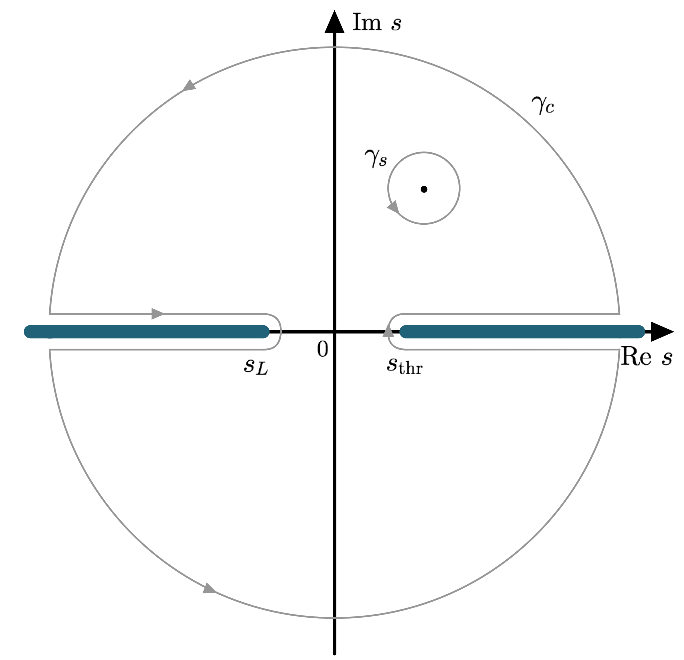


$$\sqrt{s_\sigma} = \left(449^{+22}_{-16}\right) \pm i \left(275 \pm 15\right) \text{ MeV}$$

- Limitations of **Roy-like equations**
  ➢ requires experimental knowledge of many partial waves in direct and crossed channels
  ➢ finite truncation limits results to a given kinematical region
  ➢ coupled-channel treatment is very complicated

[Roy (1971)]
[Colangelo et al. (2001)]
[Caprini et al. (2006)]
[Garcia-Martin et al. (2011)]
[Pelaez (2016)]

partial wave dispersion relation (S wave)



- Full **p.w. dispersion relation** (unitarity, analyticity, crossing)

$$t_{ab}(s) = \int_{-\infty}^{s_L} \frac{ds'}{\pi} \frac{\text{Im}\, t_{ab}(s')}{s'-s} + \int_{s_{th}}^{\infty} \frac{ds'}{\pi} \frac{\text{Im}\, t_{ab}(s')}{s'-s}$$

- **Unitarity relation** (above threshold)

$$\text{Im}\, t_{ab}(s) = \sum_c t_{ac}(s)\,\rho_c(s)\, t^*_{cb}(s), \quad \rho_c(s) = \frac{2\,p_c(s)}{\sqrt{s}}$$

$$-\frac{1}{2\rho_1} \leq \text{Re}\, t_{11}(s) \leq \frac{1}{2\rho_1}, \quad 0 < \text{Im}\, t_{11}(s) \leq \frac{1}{\rho_1}, \quad \ldots$$

- Assuming $t(\infty) \to const$ we subtract the dispersion relation once

$$t_{ab}(s) = t_{ab}(s_M) + \frac{s-s_M}{\pi} \int_{-\infty}^{s_L} \frac{ds'}{s'-s_M} \frac{\text{Im}\, t_{ab}(s')}{s'-s} + \frac{s-s_M}{\pi} \sum_c \int_{s_{th}}^{\infty} \frac{ds'}{s'-s_M} \frac{t_{ac}(s')\,\rho_c(s')\, t^*_{cb}(s')}{s'-s}$$

partial wave dispersion relation (S wave)

- Once-subtracted p.w. dispersion relation

$$t_{ab}(s) = \underbrace{t_{ab}(s_M) + \frac{s-s_M}{\pi}\int_{-\infty}^{s_L} \frac{ds'}{s'-s_M}\frac{\mathrm{Im}\, t_{ab}(s')}{s'-s}}_{U_{ab}(s)} + \frac{s-s_M}{\pi}\sum_c \int_{s_{th}}^{\infty} \frac{ds'}{s'-s_M}\frac{t_{ac}(s')\,\rho_c(s')\,t^*_{cb}(s')}{s'-s}$$

partial wave dispersion relation (S wave)

- Once-subtracted p.w. dispersion relation

$$t_{ab}(s) = \underbrace{t_{ab}(s_M) + \frac{s-s_M}{\pi}\int_{-\infty}^{s_L} \frac{ds'}{s'-s_M}\frac{\text{Im}\, t_{ab}(s')}{s'-s}}_{U_{ab}(s)} + \frac{s-s_M}{\pi}\sum_c \int_{s_{th}}^{\infty} \frac{ds'}{s'-s_M}\frac{t_{ac}(s')\,\rho_c(s')\,t^*_{cb}(s')}{s'-s}$$

can be solved using N/D method with input from $U_{ab}(s)$ **above threshold**

[Chew, Mandelstam (1960)]
[Luming (1964)]
[Johnson, Warnock (1981)]

$$t_{ab}(s) = \sum_c D^{-1}_{ac}\, N_{cb}(s)$$

$$N_{ab}(s) = U_{ab}(s) + \frac{s-s_M}{\pi}\sum_c \int_{s_{th}}^{\infty} \frac{ds'}{s'-s_M}\frac{N_{ac}(s')\,\rho_c(s')\,(U_{cb}(s') - U_{cb}(s))}{s'-s}$$

$$D_{ab}(s) = \delta_{ab} - \frac{s-s_M}{\pi}\int_{s_{th}}^{\infty} \frac{ds'}{s'-s_M}\frac{N_{ab}(s')\,\rho_b(s')}{s'-s}$$

the obtained N/D solution can be checked that it **fulfils** the p.w. dispersion relation

partial wave dispersion relation (S wave)

- Once-subtracted p.w. dispersion relation

$$t_{ab}(s) = \underbrace{t_{ab}(s_M) + \frac{s-s_M}{\pi}\int_{-\infty}^{s_L} \frac{ds'}{s'-s_M}\frac{\mathrm{Im}\, t_{ab}(s')}{s'-s}}_{U_{ab}(s)} + \frac{s-s_M}{\pi}\sum_c \int_{s_{th}}^{\infty} \frac{ds'}{s'-s_M}\frac{t_{ac}(s')\,\rho_c(s')\,t^*_{cb}(s')}{s'-s}$$

can be solved using N/D method with input from $U_{ab}(s)$ **above threshold**

[Chew, Mandelstam (1960)]
[Luming (1964)]
[Johnson, Warnock (1981)]

$$t_{ab}(s) = \sum_c D^{-1}_{ac}\, N_{cb}(s)$$

$$N_{ab}(s) = U_{ab}(s) + \frac{s-s_M}{\pi}\sum_c \int_{s_{th}}^{\infty} \frac{ds'}{s'-s_M}\frac{N_{ac}(s')\,\rho_c(s')\,(U_{cb}(s')-U_{cb}(s))}{s'-s}$$

$$D_{ab}(s) = \delta_{ab} - \frac{s-s_M}{\pi}\int_{s_{th}}^{\infty} \frac{ds'}{s'-s_M}\frac{N_{ab}(s')\,\rho_b(s')}{s'-s}$$

- Bound state case

the obtained N/D solution can be checked that it **fulfils** the p.w. dispersion relation

$$\det(D_{ab}(s_B)) = 0$$

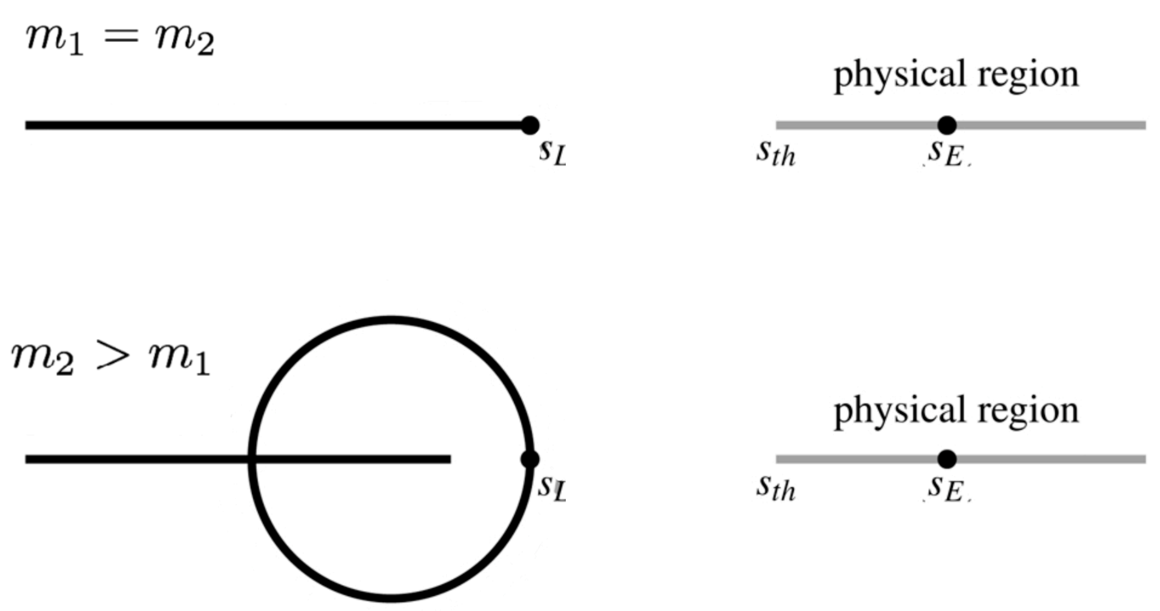
$$t_{ab}(s) = U_{ab}(s) + \frac{s-s_M}{s_B-s_M}\frac{g_a\, g_b}{s_B-s} + \frac{s-s_M}{\pi}\sum_c \int_{s_{th}}^{\infty} \frac{ds'}{s'-s_M}\frac{t_{ac}(s')\,\rho_c(s')\,t^*_{cb}(s')}{s'-s}$$

Left-hand cuts

- In general scattering problem, little is known about left-hand cuts, except their analytical structure in the complex plane.

(asymptotically bounded unknown function)

$$U_{ab}(s) = t_{ab}(s_M) + \frac{s - s_M}{\pi} \int_{-\infty}^{s_L} \frac{ds'}{s' - s_M} \frac{\mathrm{Im}\, t_{ab}(s')}{s' - s}$$



Left-hand cuts

- In general scattering problem, little is known about left-hand cuts, except their analytical structure in the complex plane. We approximate $U_{ab}(s)$ as an expansion in a **conformal mapping variable** $\omega(s)$

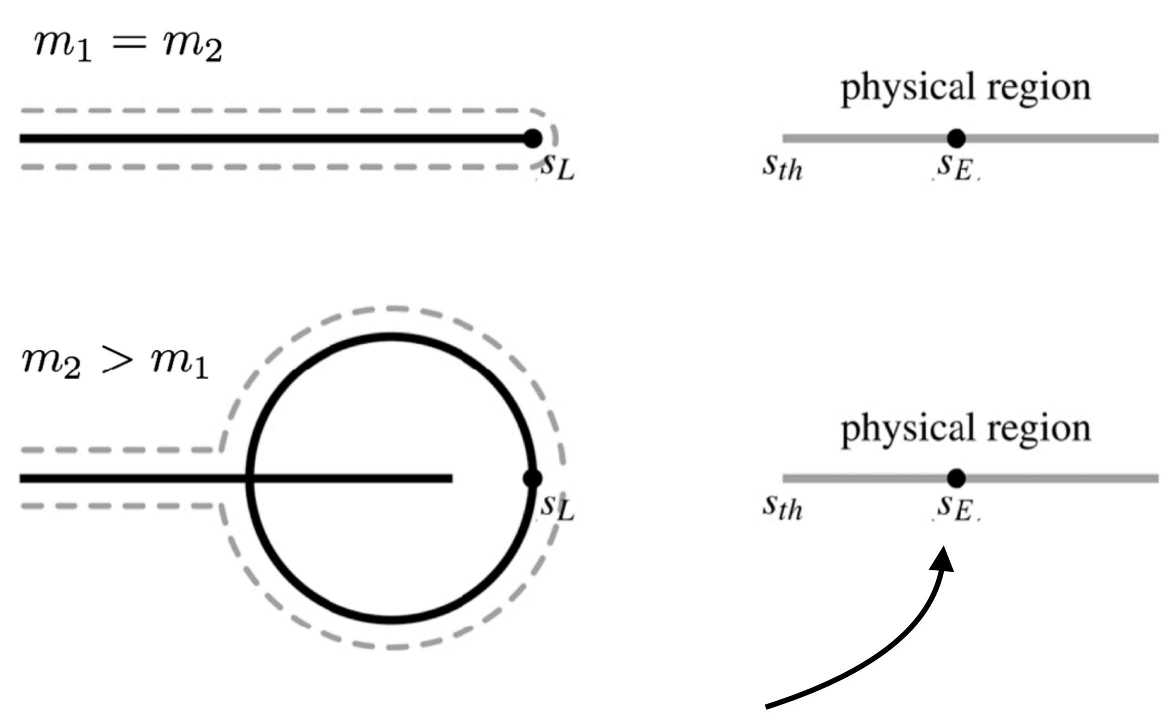
[Gasparyan, Lutz (2010)]

(asymptotically bounded unknown function)

$$U_{ab}(s) = t_{ab}(s_M) + \frac{s - s_M}{\pi} \int_{-\infty}^{s_L} \frac{ds'}{s' - s_M} \frac{\mathrm{Im}\, t_{ab}(s')}{s' - s}$$

$$\simeq \sum_{n=0}^{\infty} C_{ab,n} \, (\omega_{ab}(s))^n$$

unknown coefficients fitted to data or/and EFT



$\omega(s_E) = 0$

$\omega(s_L) = -1$

$$\sqrt{s_E} = \frac{1}{2}\left(\sqrt{s_{th}} + \sqrt{s_{max}}\right)$$

source of the systematic uncertainties

Recent research activities

- **Lattice**: Analytical dispersive parameterisation for elastic scattering of spin-less particles

  I.D, V. Biloshytskyi, X.-L. Ren, M. Vanderhaeghen 2206.15223 (2022)
  I.D, O. Deineka, M. Vanderhaeghen, Phys. Rev. D 103 11, 114023, (2021)

- **(g-2)$_\mu$**:
  A dispersive estimate of scalar contributions to hadronic light-by-light scattering

  I.D, M. Hoferichter, P. Stoffer, Phys. Lett. B 820, 136502, (2021)

  Dispersive analysis of the $\gamma^*\gamma^* \to \pi\pi\,(K\bar{K})$ process

  I.D, O. Deineka, M. Vanderhaeghen, Phys. Rev.D 101 5, 054008, (2020)

- **XYZ**:
  Dispersive analysis of the $\gamma\gamma \to D\bar{D}$ and $e^+e^- \to J/\psi D\bar{D}$ data

  O. Deineka, I.D, M. Vanderhaeghen, Phys. Lett. B 827, 136982, (2022)

  Simultaneous description of the $e^+e^- \to J/\psi\,\pi\pi\,(K\bar{K})$ processes

  I.D, D. Molnar, M. Vanderhaeghen, Phys. Rev. D 102 1, 016019, (2020)

$f_0(500)$ pole extraction from lattice data

- In [Briceno et al. 2016] the first lattice QCD determination of the energy dependence of the isoscalar $\pi\pi$ elastic scattering phase shift was presented. Lattice spectra was described using

$$\boxed{[t(s)]^{-1} = K^{-1} + I(s), \quad \mathrm{Im}\, I(s) = -\rho(s)}$$

1) Chew-Mandelstam phase space

   a) $K(s) = \dfrac{g^2}{m^2 - s} + c$

   b) $K(s) = \dfrac{g^2}{m^2 - s}$

   c) $K(s) = \dfrac{g^2}{m^2 - s} + a\, s$

2) $K(s) = \dfrac{g^2}{m^2 - s} + c, \quad I(s) = -i\rho(s)$

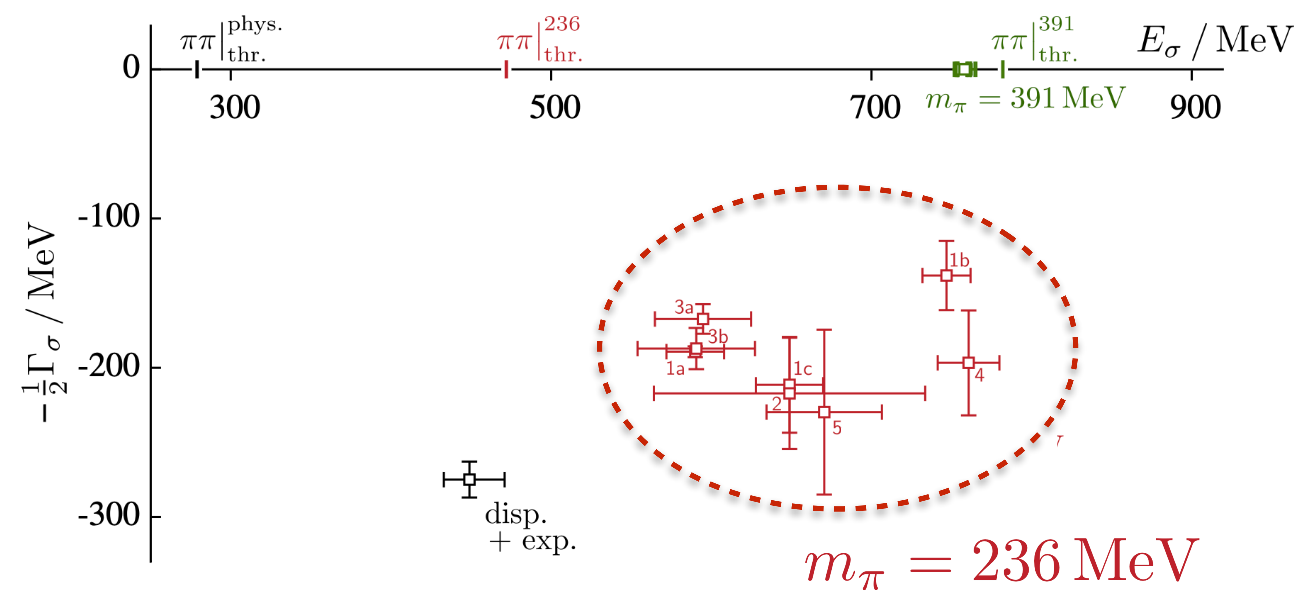
3) The K-matrix with Adler zero

   a) $K(s) = (s - s_A)\dfrac{g^2}{m^2 - s}$ with Chew Mandelstam

   b) $K(s) = (s - s_A)\dfrac{g^2}{m^2 - s}$ with $I(s) = -i\rho(s)$

4) A relativistic Breit-Wigner formula

5) An effective range expansion



$f_0(500)$ pole extraction from lattice data

- In [Briceno et al. 2016] the first lattice QCD determination of the energy dependence of the isoscalar $\pi\pi$ elastic scattering phase shift was presented. Lattice spectra was described using

$$\boxed{[t(s)]^{-1} = K^{-1} + I(s), \quad \text{Im}\, I(s) = -\rho(s)}$$

1) Chew-Mandelstam phase space

   a) $K(s) = \dfrac{g^2}{m^2 - s} + c$

   b) $K(s) = \dfrac{g^2}{m^2 - s}$

   c) $K(s) = \dfrac{g^2}{m^2 - s} + a\, s$

2) $K(s) = \dfrac{g^2}{m^2 - s} + c, \quad I(s) = -i\rho(s)$

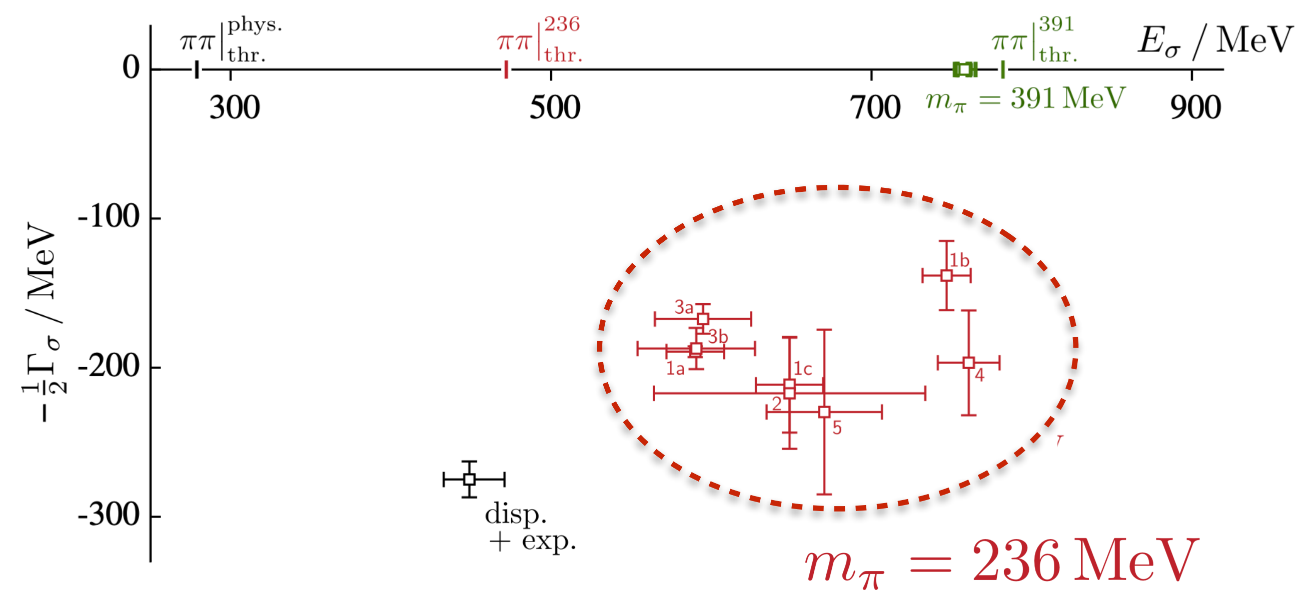
3) The K-matrix with Adler zero

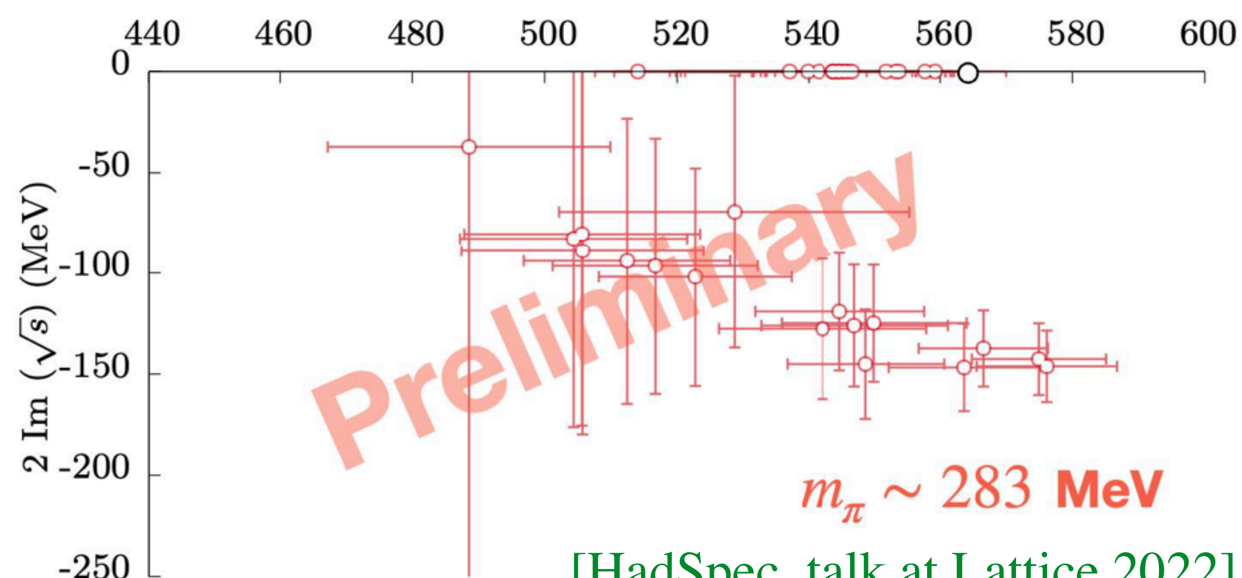
   a) $K(s) = (s - s_A)\dfrac{g^2}{m^2 - s}$ with Chew Mandelstan

   b) $K(s) = (s - s_A)\dfrac{g^2}{m^2 - s}$ with $I(s) = -i\rho(s)$

4) A relativistic Breit-Wigner formula

5) An effective range expansion





[HadSpec, talk at Lattice 2022]

+ [CLS ensembles on πK for $m_\pi = 200, 280$ MeV]

Can other S-matrix constrains help? - YES

partial wave dispersion relation for $t(s)$ (S wave)

- Once-subtracted p.w. dispersion relation (single channel approximation)

$$t(s) = \underbrace{t(s_M) + \frac{s - s_M}{\pi} \int_{-\infty}^{s_L} \frac{ds'}{s' - s_M} \frac{\mathrm{Im}\, t(s')}{s' - s}}_{U(s)} + \frac{s - s_M}{\pi} \int_{s_{th}}^{\infty} \frac{ds'}{s' - s_M} \frac{\rho(s') \, |t(s')|^2}{s' - s}$$

can be solved using the N/D method

$$t^{N/D}(s) = \frac{N(s)}{D(s)}$$

- **Adler zero** constraint $t(s_A) = 0$

partial wave dispersion relation for $t(s)$ (S wave)

- Once-subtracted p.w. dispersion relation (single channel approximation)

$$t(s) = \underbrace{t(s_M) + \frac{s-s_M}{\pi}\int_{-\infty}^{s_L}\frac{ds'}{s'-s_M}\frac{\text{Im}\, t(s')}{s'-s}}_{U(s)} + \frac{s-s_M}{\pi}\int_{s_{th}}^{\infty}\frac{ds'}{s'-s_M}\frac{\rho(s')\,|\,t(s')\,|^2}{s'-s}$$

can be solved using the N/D method

$$t^{N/D}(s) = \frac{N(s)}{D(s)}$$

- **Adler zero** constraint $t(s_A) = 0$ can be incorporated as a **zero of** $N(s)$ (e.g. $s_M = s_A$)

[I.D, Deineka, Vanderhaeghen (2021)]

$$D(s) = 1 - \frac{s-s_A}{\pi}\int_{s_{th}}^{\infty}\frac{ds'}{s'-s_A}\frac{N(s')\,\rho(s')}{s'-s}$$

$$N(s) = U(s) + \frac{s-s_A}{\pi}\int_{s_{th}}^{\infty}\frac{ds'}{s'-s_A}\frac{N(s')\,\rho(s')\,(U(s')-U(s))}{s'-s}$$

$$U(s_A) = 0,$$

partial wave dispersion relation for $t(s)$ (S wave)

- Once-subtracted p.w. dispersion relation (single channel approximation)

$$t(s) = \underbrace{t(s_M) + \frac{s - s_M}{\pi} \int_{-\infty}^{s_L} \frac{ds'}{s' - s_M} \frac{\mathrm{Im}\, t(s')}{s' - s}}_{U(s)} + \frac{s - s_M}{\pi} \int_{s_{th}}^{\infty} \frac{ds'}{s' - s_M} \frac{\rho(s')\,|\,t(s')\,|^2}{s' - s}$$

can be solved using the N/D method

$$t^{N/D}(s) = \frac{N(s)}{D(s)}$$

- **Adler zero** constraint $t(s_A) = 0$ can be incorporated as a **zero of** $N(s)$ (e.g. $s_M = s_A$)

[I.D, Deineka, Vanderhaeghen (2021)]

$$D(s) = 1 - \frac{s - s_A}{\pi} \int_{s_{th}}^{\infty} \frac{ds'}{s' - s_A} \frac{N(s')\,\rho(s')}{s' - s}$$

$$N(s) = U(s) + \frac{s - s_A}{\pi} \int_{s_{th}}^{\infty} \frac{ds'}{s' - s_A} \frac{N(s')\,\rho(s')\,(U(s') - U(s))}{s' - s}$$

$$U(s_A) = 0, \quad U(s) \approx \sum_{n=1}^{\infty} C_n\,(\omega^n(s) - \omega^n(s_A))$$

one needs to solve the integral equation numerically

partial wave dispersion relation for $t(s)$ (S wave)

- Once-subtracted p.w. dispersion relation (single channel approximation)

$$t(s) = \underbrace{t(s_M) + \frac{s - s_M}{\pi} \int_{-\infty}^{s_L} \frac{ds'}{s' - s_M} \frac{\mathrm{Im}\, t(s')}{s' - s}}_{U(s)} + \frac{s - s_M}{\pi} \int_{s_{th}}^{\infty} \frac{ds'}{s' - s_M} \frac{\rho(s')\,|\,t(s')\,|^2}{s' - s}$$

can be solved using the N/D method

$$t^{N/D}(s) = \frac{N(s)}{D(s)}$$

- **Adler zero** constraint $t(s_A) = 0$

partial wave dispersion relation for $t(s)$ (S wave)

- Once-subtracted p.w. dispersion relation (single channel approximation)

$$t(s) = \underbrace{t(s_M) + \frac{s - s_M}{\pi} \int_{-\infty}^{s_L} \frac{ds'}{s' - s_M} \frac{\mathrm{Im}\, t(s')}{s' - s}}_{U(s)} + \frac{s - s_M}{\pi} \int_{s_{th}}^{\infty} \frac{ds'}{s' - s_M} \frac{\rho(s')\,|\,t(s')\,|^2}{s' - s}$$

can be solved using the N/D method

$$t^{N/D}(s) = \frac{N(s)}{D(s)}$$

- **Adler zero** constraint $t(s_A) = 0$ can be incorporated as a **pole of** $D(s)$ (any $s_M \neq s_A$)

$$D(s) = 1 - \frac{s - s_M}{\pi} \int_{s_{th}}^{\infty} \frac{ds'}{s' - s_M} \frac{N(s')\,\rho(s')}{s' - s} + (s - s_M)\frac{g}{s - s_A}$$

$$N(s) = U(s) + \frac{s - s_M}{\pi} \int_{s_{th}}^{\infty} \frac{ds'}{s' - s_M} \frac{N(s')\,\rho(s')\,(U(s') - U(s))}{s' - s} + (U(s) - U(s_A))\frac{g}{s - s_A}$$

partial wave dispersion relation for $t(s)$ (S wave)

- Once-subtracted p.w. dispersion relation (single channel approximation)

$$t(s) = t(s_M) + \underbrace{\frac{s-s_M}{\pi}\int_{-\infty}^{s_L}\frac{ds'}{s'-s_M}\frac{\mathrm{Im}\, t(s')}{s'-s}}_{U(s)} + \frac{s-s_M}{\pi}\int_{s_{th}}^{\infty}\frac{ds'}{s'-s_M}\frac{\rho(s')\,|t(s')|^2}{s'-s}$$

can be solved using the N/D method

$$t^{N/D}(s) = \frac{N(s)}{D(s)}$$

- **Adler zero** constraint $t(s_A)=0$ can be incorporated as a **pole of** $D(s)$ (any $s_M \neq s_A$)

$$D(s) = 1 - \frac{s-s_M}{\pi}\int_{s_{th}}^{\infty}\frac{ds'}{s'-s_M}\frac{N(s')\,\rho(s')}{s'-s} + (s-s_M)\frac{g}{s-s_A}$$

$$N(s) = U(s) + \frac{s-s_M}{\pi}\int_{s_{th}}^{\infty}\frac{ds'}{s'-s_M}\frac{N(s')\,\rho(s')\,(U(s')-U(s))}{s'-s} + (U(s)-U(s_A))\frac{g}{s-s_A}$$

$$U(s) \approx \sum_{n=0}^{\infty} C_n\,\omega^n(s)$$

for $U(s)=C_0 \Rightarrow N(s)=C_0$ we get analytical formula 🙂

partial wave dispersion relation for $t(s)$ (S wave)

- Once-subtracted p.w. dispersion relation (single channel approximation)

$$\underbrace{t(s) = t(s_M) + \frac{s-s_M}{\pi}\int_{-\infty}^{s_L} \frac{ds'}{s'-s_M}\frac{\mathrm{Im}\, t(s')}{s'-s}}_{U(s)} + \frac{s-s_M}{\pi}\int_{s_{th}}^{\infty} \frac{ds'}{s'-s_M}\frac{\rho(s')\,|t(s')|^2}{s'-s}$$

can be solved using the N/D method

$$t^{N/D}(s) = \frac{N(s)}{D(s)}$$

- **Adler zero** constraint $t(s_A) = 0$ can be incorporated as a **pole of** $D(s)$ (any $s_M \neq s_A$)

$$\boxed{[t^{N/D}(s)]^{-1} = \frac{D(s)}{N(s)} \approx \frac{1}{C_0} + \frac{s-s_M}{\pi}\int_{s_{th}}^{\infty}\frac{ds'}{s'-s_M}\frac{-\rho(s')}{s'-s} + (s-s_M)\frac{g/C_0}{s-s_A}}$$

Can we obtain this formula differently? - YES

partial wave dispersion relation for $[t(s)]^{-1}$ (S wave)

- Once-subtracted p.w. dispersion relation for the inverse amplitude $[t(s)]^{-1}$

$$[t(s)]^{-1} = [t(s_M)]^{-1} + \frac{s-s_M}{\pi}\int_{-\infty}^{s_L}\frac{ds'}{s'-s_M}\frac{\mathrm{Im}\,[t(s')]^{-1}}{s'-s} + \frac{s-s_M}{\pi}\int_{s_{th}}^{\infty}\frac{ds'}{s'-s_M}\frac{-\rho(s')}{s'-s} + \frac{s-s_M}{s_A-s_M}\frac{g_A}{s-s_A}$$

- **Adler zero** constraint $t(s_A)=0$ is incorporated as a **pole of** $[t(s)]^{-1}$

partial wave dispersion relation for $[t(s)]^{-1}$ (S wave)

- Once-subtracted p.w. dispersion relation for the inverse amplitude $[t(s)]^{-1}$

$$[t(s)]^{-1} = \underbrace{[t(s_M)]^{-1} + \frac{s-s_M}{\pi}\int_{-\infty}^{s_L}\frac{ds'}{s'-s_M}\frac{\mathrm{Im}\,[t(s')]^{-1}}{s'-s}}_{u(s)\approx\sum_{n=0}^{\infty}c_n\,\omega^n(s)} + \frac{s-s_M}{\pi}\int_{s_{th}}^{\infty}\frac{ds'}{s'-s_M}\frac{-\rho(s')}{s'-s} + \frac{s-s_M}{s_A-s_M}\frac{g_A}{s-s_A}$$

- **Adler zero** constraint $t(s_A)=0$ is incorporated as a **pole of** $[t(s)]^{-1}$

partial wave dispersion relation for $[t(s)]^{-1}$ (S wave)

- Once-subtracted p.w. dispersion relation for the inverse amplitude $[t(s)]^{-1}$

$$[t(s)]^{-1} = \underbrace{[t(s_M)]^{-1} + \frac{s-s_M}{\pi}\int_{-\infty}^{s_L} \frac{ds'}{s'-s_M}\frac{\mathrm{Im}\,[t(s')]^{-1}}{s'-s}}_{u(s) \approx \sum_{n=0}^{\infty} c_n\,\omega^n(s)} + \frac{s-s_M}{\pi}\int_{s_{th}}^{\infty} \frac{ds'}{s'-s_M}\frac{-\rho(s')}{s'-s} + \frac{s-s_M}{s_A-s_M}\frac{g_A}{s-s_A}$$

- **Adler zero** constraint $t(s_A) = 0$ is incorporated as a **pole of** $[t(s)]^{-1}$

- **simple formula** - no need to solve integral equation for more complicated left-hand cuts
- **easy extension to** $J \neq 0$ (write $J+1$ subtracted DR for $p(s)^{2J}/t_J(s)$)

partial wave dispersion relation for $[t(s)]^{-1}$ (S wave)

- Once-subtracted p.w. dispersion relation for the inverse amplitude $[t(s)]^{-1}$

$$[t(s)]^{-1} = \underbrace{[t(s_M)]^{-1} + \frac{s-s_M}{\pi}\int_{-\infty}^{s_L} \frac{ds'}{s'-s_M}\frac{\mathrm{Im}\,[t(s')]^{-1}}{s'-s}}_{u(s)\approx\sum_{n=0}^{\infty} c_n\,\omega^n(s)} + \frac{s-s_M}{\pi}\int_{s_{th}}^{\infty} \frac{ds'}{s'-s_M}\frac{-\rho(s')}{s'-s} + \frac{s-s_M}{s_A-s_M}\frac{g_A}{s-s_A}$$

- **Adler zero** constraint $t(s_A) = 0$ is incorporated as a **pole of** $[t(s)]^{-1}$

- **simple formula** - no need to solve integral equation for more complicated left-hand cuts
- **easy extension to** $J \neq 0$ (write $J+1$ subtracted DR for $p(s)^{2J}/t_J(s)$)

- **cannot be extended** to the **coupled-channel case** (due to mixing of the left and right hand cuts) $\rightarrow$ need to come back to N/D

Comparison to common K-matrix parameterisations

- **D**ispersive **I**nverse **A**mplitude (**DIA**) method for S-wave

$$\boxed{[t^{DIA}(s)]^{-1} \approx \sum_{n=0}^{\infty} c_n\, \omega^n(s) + \frac{s-s_M}{\pi}\int_{s_{th}}^{\infty} \frac{ds'}{s'-s_M}\frac{-\rho(s')}{s'-s} + \frac{s-s_M}{s_A-s_M}\frac{g_A}{s-s_A}}$$

- K-matrix parameterisations

$$[t(s)]^{-1} = K^{-1} + I(s), \quad I(s) = \begin{cases} -i\,\rho(s) \\ I(s_{th}) + \frac{s-s_{th}}{\pi}\int_{s_{th}}^{\infty} \frac{ds'}{s'-s_{th}}\frac{-\rho(s')}{s'-s} \end{cases}$$

1) Standard implementation

$$K(s) = \frac{g^2}{m^2-s} + \sum_n \gamma_n s^n$$

[PDG (2022)]

2) Standard implementation
+ Adler zero

$$K(s) = (s-s_A)\left(\frac{g^2}{m^2-s} + \sum_n \gamma_n s^n\right)$$

[Briceno et al. (2016)]

3) Implementations with
+ Adler zero + left-hand cut

$$K^{-1}(s) = \frac{m_\pi^2}{s-s_A}\left(\frac{2\,s_A}{m_\pi\sqrt{s}} + \sum_{n=0}^{\infty} C_n\omega^n(s)\right)$$

[Yndurain et al. (2007)]

$$K^{-1}(s) = \frac{m_\pi^2}{s-s_A}\left(\sum_{n=0}^{\infty} C_n\omega^n(s)\right)$$

[Caprini et al. (2008)]
[Pelaez et al. (2016)]

Comparison to common K-matrix parameterisations

- **D**ispersive **I**nverse **A**mplitude (**DIA**) method for S-wave

$$\boxed{[t^{DIA}(s)]^{-1} \approx \sum_{n=0}^{\infty} c_n\, \omega^n(s) + \frac{s-s_M}{\pi}\int_{s_{th}}^{\infty} \frac{ds'}{s'-s_M}\frac{-\rho(s')}{s'-s} + \frac{s-s_M}{s_A-s_M}\frac{g_A}{s-s_A}}$$

- K-matrix parameterisations

$$[t(s)]^{-1} = K^{-1} + I(s), \quad I(s) = \begin{cases} -i\,\rho(s) \\ \boxed{I(s_{th}) + \frac{s-s_{th}}{\pi}\int_{s_{th}}^{\infty}\frac{ds'}{s'-s_{th}}\frac{-\rho(s')}{s'-s}} \end{cases}$$

1) Standard implementation

$$K(s) = \frac{g^2}{m^2-s} + \sum_n \gamma_n s^n$$

[PDG (2022)]

2) Standard implementation
+ Adler zero

$$\boxed{K(s) = (s-s_A)\left(\frac{g^2}{m^2-s}\right.} \left. + \sum_n \gamma_n s^n\right)$$

[Briceno et al. (2016)]

3) Implementations with
+ Adler zero + left-hand cut

$$K^{-1}(s) = \frac{m_\pi^2}{s-s_A}\left(\frac{2\,s_A}{m_\pi\sqrt{s}} + \sum_{n=0}^{\infty} C_n\omega^n(s)\right)$$

[Yndurain et al. (2007)]

$$K^{-1}(s) = \frac{m_\pi^2}{s-s_A}\left(\sum_{n=0}^{\infty} C_n\omega^n(s)\right)$$

[Caprini et al. (2008)]
[Pelaez et al. (2016)]

only (2) with $\gamma_n = 0$ can be matched to DIA under following assumptions:
1) constant left-hand cut: c_0
2) g^2 and m^2 do NOT need to be positive

Comparison to common K-matrix parameterisations

- **D**ispersive **I**nverse **A**mplitude (**DIA**) method for S-wave

$$\boxed{[t^{DIA}(s)]^{-1} \approx \sum_{n=0}^{\infty} c_n\, \omega^n(s) + \frac{s-s_M}{\pi}\int_{s_{th}}^{\infty} \frac{ds'}{s'-s_M}\frac{-\rho(s')}{s'-s} + \frac{s-s_M}{s_A-s_M}\frac{g_A}{s-s_A}}$$

- K-matrix parameterisations

$$[t(s)]^{-1} = K^{-1} + I(s), \quad I(s) = \begin{cases} -i\,\rho(s) \\ \boxed{I(s_{th}) + \frac{s-s_{th}}{\pi}\int_{s_{th}}^{\infty}\frac{ds'}{s'-s_{th}}\frac{-\rho(s')}{s'-s}} \end{cases}$$

1) Standard implementation

$$K(s) = \frac{g^2}{m^2-s} + \sum_n \gamma_n s^n$$

[PDG (2022)]

2) Standard implementation
+ Adler zero

$$K(s) = (s-s_A)\left(\frac{g^2}{m^2-s} + \sum_n \gamma_n s^n\right)$$

[Briceno et al. (2016)]

3) Implementations with
+ Adler zero + left-hand cut

$$K^{-1}(s) = \frac{m_\pi^2}{s-s_A}\left(\frac{2\,s_A}{m_\pi\sqrt{s}} + \sum_{n=0}^{\infty} C_n\omega^n(s)\right)$$

[Yndurain et al. (2007)]

$$K^{-1}(s) = \frac{m_\pi^2}{s-s_A}\left(\sum_{n=0}^{\infty} C_n\omega^n(s)\right)$$

[Caprini et al. (2008)]
[Pelaez et al. (2016)]

only (2) with $\gamma_n = 0$ can be matched to DIA under following assumptions:
1) constant left-hand cut: c_0
2) g^2 and m^2 do NOT need to be positive

Example: [Colangelo et al. (2001)]

$a_{I=0} = 0.220, b_{I=0} = 0.276$

$g^2 = -3.78, m^2 = -1.47$

Comparison to mIAM

- **D**ispersive **I**nverse **A**mplitude (**DIA**) method for S-wave

$$\boxed{[t^{DIA}(s)]^{-1} \approx \sum_{n=0}^{\infty} c_n\, \omega^n(s) + \frac{s-s_M}{\pi}\int_{s_{th}}^{\infty} \frac{ds'}{s'-s_M}\frac{-\rho(s')}{s'-s} + \frac{s-s_M}{s_A-s_M}\frac{g_A}{s-s_A}}$$

- **m**odified **I**nverse **A**mplitude **M**ethod for the S-wave: [Truong et al. (1988-1997)], [Gomez Nicola et al. (2008)]

$$t^{\mathrm{mIAM}}(s) = \frac{(t^{\mathrm{LO}}(s))^2}{t^{\mathrm{LO}}(s) - (t^{\mathrm{NLO}}(s) - t^{\mathrm{LO}}(s)) + A^{\mathrm{mIAM}}(s)}$$

Comparison to mIAM

- **D**ispersive **I**nverse **A**mplitude (**DIA**) method for S-wave

$$\boxed{[t^{DIA}(s)]^{-1} \approx \sum_{n=0}^{\infty} c_n\,\omega^n(s) + \frac{s-s_M}{\pi}\int_{s_{th}}^{\infty}\frac{ds'}{s'-s_M}\frac{-\rho(s')}{s'-s} + \frac{s-s_M}{s_A-s_M}\frac{g_A}{s-s_A}}$$

- **m**odified **I**nverse **A**mplitude **M**ethod for the S-wave: [Truong et al. (1988-1997)], [Gomez Nicola et al. (2008)]

$$t^{\mathrm{mIAM}}(s) = \frac{(t^{\mathrm{LO}}(s))^2}{t^{\mathrm{LO}}(s) - (t^{\mathrm{NLO}}(s) - t^{\mathrm{LO}}(s)) + A^{\mathrm{mIAM}}(s)}$$

$$g_A = \left(\left.\frac{\mathrm{d}t_0^{\mathrm{mIAM}}(s)}{\mathrm{d}s}\right|_{s=s_A}\right)^{-1}$$

$$[t^{\mathrm{mIAM}}(s)]^{-1} = [t^{\mathrm{mIAM}}(s_M)]^{-1} + \frac{s-s_M}{\pi}\int_{-\infty}^{s_L}\frac{ds'}{s'-s_M}\frac{\mathrm{Im}\,[t^{\mathrm{mIAM}}(s')]^{-1}}{s'-s} + \frac{s-s_M}{\pi}\int_{s_{th}}^{\infty}\frac{ds'}{s'-s_M}\frac{-\rho(s')}{s'-s} + \frac{s-s_M}{s_A-s_M}\frac{g_A}{s-s_A}$$

- mIAM is a special case of DIA
- DIA is not limited to the Lagrangian based resummation scheme
- If needed, DIA can be matched to EFT where it works the best

Comparison to mIAM

- **D**ispersive **I**nverse **A**mplitude (**DIA**) method for S-wave

$$\boxed{[t^{DIA}(s)]^{-1} \approx \sum_{n=0}^{\infty} c_n\,\omega^n(s) + \frac{s-s_M}{\pi}\int_{s_{th}}^{\infty}\frac{ds'}{s'-s_M}\frac{-\rho(s')}{s'-s} + \frac{s-s_M}{s_A-s_M}\frac{g_A}{s-s_A}}$$

- **m**odified **I**nverse **A**mplitude **M**ethod for the S-wave: [Truong et al. (1988-1997)], [Gomez Nicola et al. (2008)]

$$t^{\text{mIAM}}(s) = \frac{(t^{\text{LO}}(s))^2}{t^{\text{LO}}(s) - (t^{\text{NLO}}(s) - t^{\text{LO}}(s)) + A^{\text{mIAM}}(s)}$$

$$g_A = \left(\left.\frac{dt_0^{\text{mIAM}}(s)}{ds}\right|_{s=s_A}\right)^{-1}$$

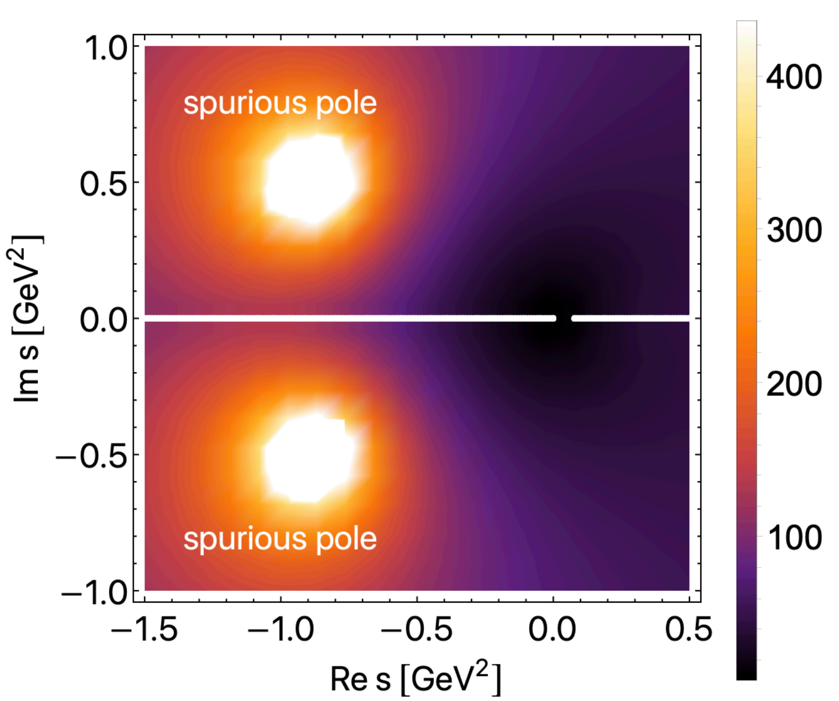
$$[t^{\text{mIAM}}(s)]^{-1} = [t^{\text{mIAM}}(s_M)]^{-1} + \frac{s-s_M}{\pi}\int_{-\infty}^{s_L}\frac{ds'}{s'-s_M}\frac{\text{Im}\,[t^{\text{mIAM}}(s')]^{-1}}{s'-s} + \frac{s-s_M}{\pi}\int_{s_{th}}^{\infty}\frac{ds'}{s'-s_M}\frac{-\rho(s')}{s'-s} + \frac{s-s_M}{s_A-s_M}\frac{g_A}{s-s_A}$$

$$t^{\text{mIAM}}(s) \neq t^{\text{mIAM}}(s_M) + \frac{s-s_M}{\pi}\int_{L,R}\frac{ds'}{s'-s_M}\frac{\text{Im}\,t^{\text{mIAM}}(s')}{s'-s}$$

in principle any DIA is NOT immune to that

- mIAM is a special case of DIA
- DIA is not limited to the Lagrangian based resummation scheme
- If needed, DIA can be matched to EFT where it works the best

Riemann sheet I



DIA and chiral extrapolation

- **D**ispersive **I**nverse **A**mplitude (**DIA**) method for S-wave

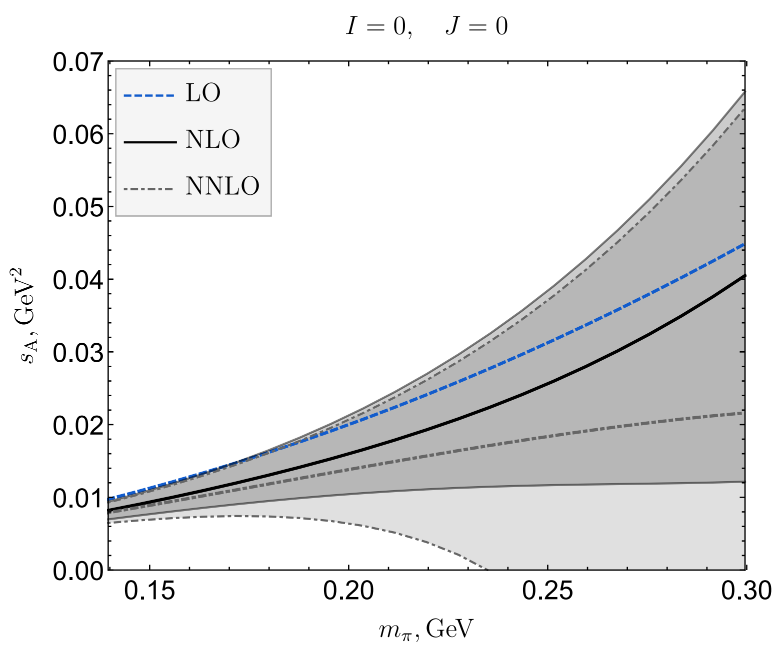
$$\boxed{[t^{DIA}(s)]^{-1} \approx \sum_{n=0}^{\infty} c_n \, \omega^n(s) + \frac{s - s_M}{\pi} \int_{s_{th}}^{\infty} \frac{ds'}{s' - s_M} \frac{-\rho(s')}{s' - s} + \frac{s - s_M}{s_A - s_M} \frac{g_A}{s - s_A}}$$

DIA and chiral extrapolation

- **D**ispersive **I**nverse **A**mplitude (**DIA**) method for S-wave

$$\boxed{[t^{DIA}(s)]^{-1} \approx \sum_{n=0}^{\infty} c_n\,\omega^n(s) + \frac{s-s_M}{\pi}\int_{s_{th}}^{\infty}\frac{ds'}{s'-s_M}\frac{-\rho(s')}{s'-s} + \frac{s-s_M}{s_A-s_M}\frac{g_A}{s-s_A}}$$

- Input from ChPT
  $s_{A,\mathrm{NLO}}(m_\pi = 236\,\mathrm{MeV}) = 0.023(12)\,\mathrm{GeV}^2$



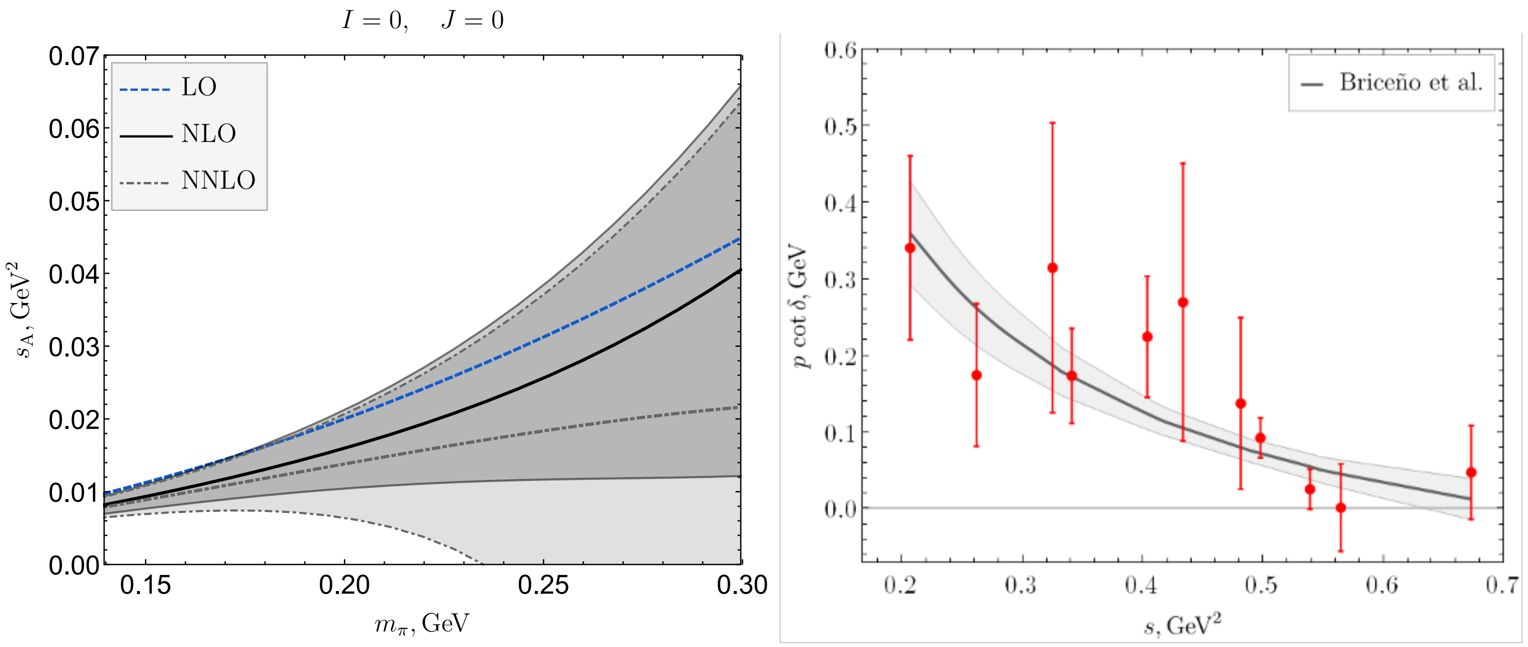
DIA and chiral extrapolation

- **D**ispersive **I**nverse **A**mplitude (**DIA**) method for S-wave

$$\boxed{[t^{DIA}(s)]^{-1} \approx \sum_{n=0}^{\infty} c_n\,\omega^n(s) + \frac{s-s_M}{\pi}\int_{s_{th}}^{\infty} \frac{ds'}{s'-s_M}\frac{-\rho(s')}{s'-s} + \frac{s-s_M}{s_A-s_M}\frac{g_A}{s-s_A}}$$

- Input from ChPT
  $s_{A,\mathrm{NLO}}(m_\pi = 236\,\mathrm{MeV}) = 0.023(12)\,\mathrm{GeV}^2$

- Fitted parameters (choose $s_M = s_{th}$)
  $g_A = 0.56(9)$
  $c_0 = 1.38(26)$



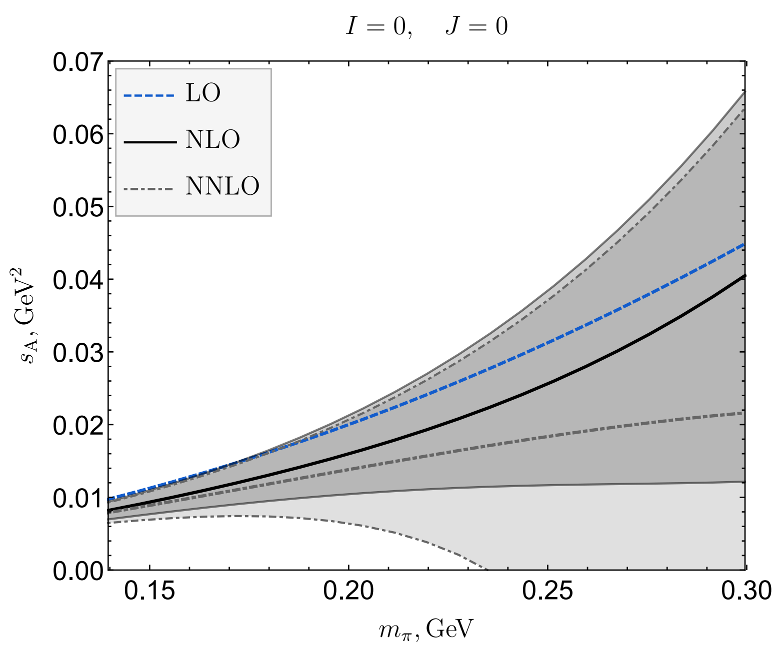
DIA and chiral extrapolation

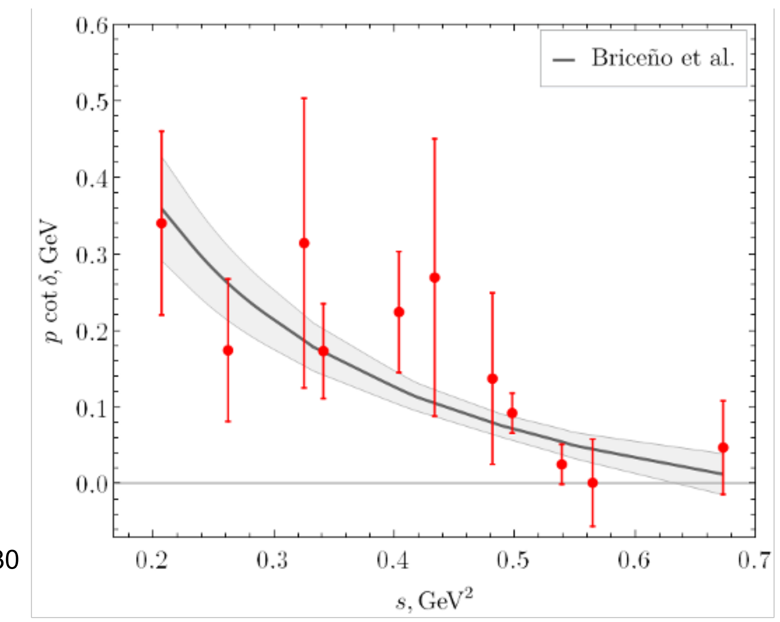
- **D**ispersive **I**nverse **A**mplitude (**DIA**) method for S-wave

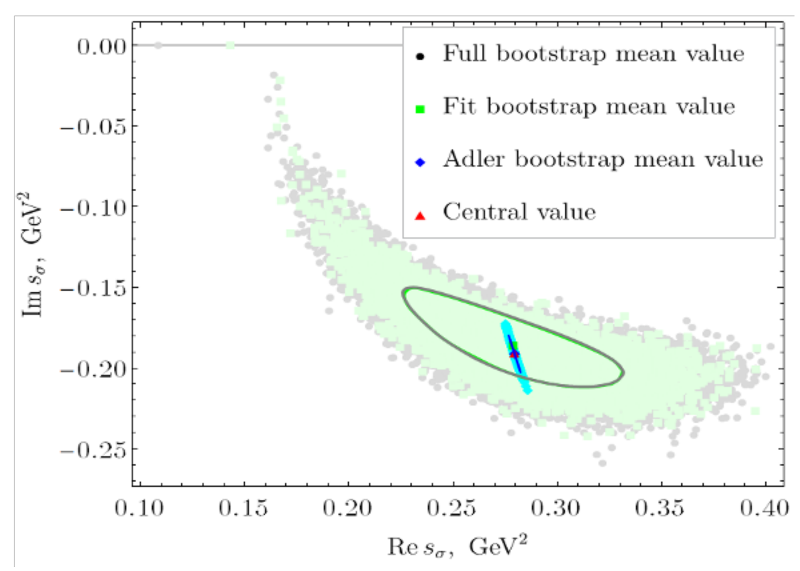
$$\boxed{[t^{DIA}(s)]^{-1} \approx \sum_{n=0}^{\infty} c_n \,\omega^n(s) + \frac{s-s_M}{\pi}\int_{s_{th}}^{\infty}\frac{ds'}{s'-s_M}\frac{-\rho(s')}{s'-s} + \frac{s-s_M}{s_A-s_M}\frac{g_A}{s-s_A}}$$

- Input from ChPT
  $s_{A,\mathrm{NLO}}(m_\pi = 236\,\mathrm{MeV}) = 0.023(12)\,\mathrm{GeV}^2$
- Fitted parameters (choose $s_M = s_{th}$)
  $g_A = 0.56(9)$
  $c_0 = 1.38(26)$

$$\sqrt{s_\sigma} = 553^{+47}_{-54} \pm i\,168^{+19}_{-15}\,\mathrm{MeV}$$







DIA and chiral extrapolation

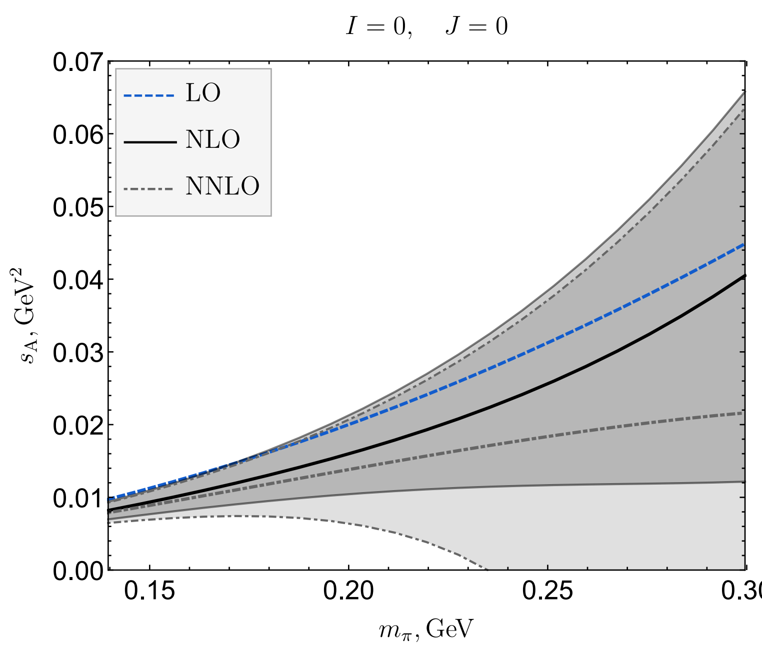
- **D**ispersive **I**nverse **A**mplitude (**DIA**) method for S-wave

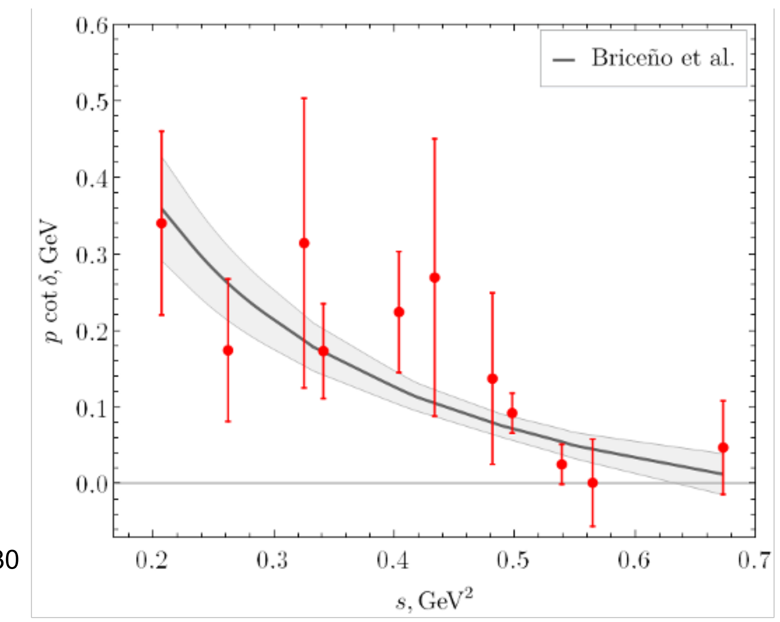
$$\boxed{[t^{DIA}(s)]^{-1} \approx \sum_{n=0}^{\infty} c_n\, \omega^n(s) + \frac{s-s_M}{\pi}\int_{s_{th}}^{\infty} \frac{ds'}{s'-s_M}\,\frac{-\rho(s')}{s'-s} + \frac{s-s_M}{s_A-s_M}\,\frac{g_A}{s-s_A}}$$

- Input from ChPT
  $s_{A,\mathrm{NLO}}(m_\pi = 236\,\mathrm{MeV}) = 0.023(12)\,\mathrm{GeV}^2$

- Fitted parameters (choose $s_M = s_{th}$)
  $g_A = 0.56(9)$
  $c_0 = 1.38(26)$

$$\sqrt{s_\sigma} = 553^{+47}_{-54} \pm i\,168^{+19}_{-15}\,\mathrm{MeV}$$



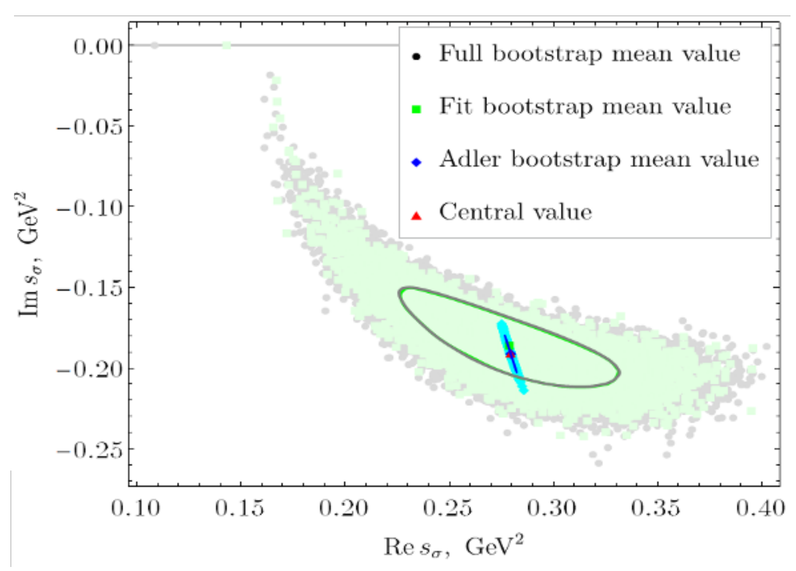


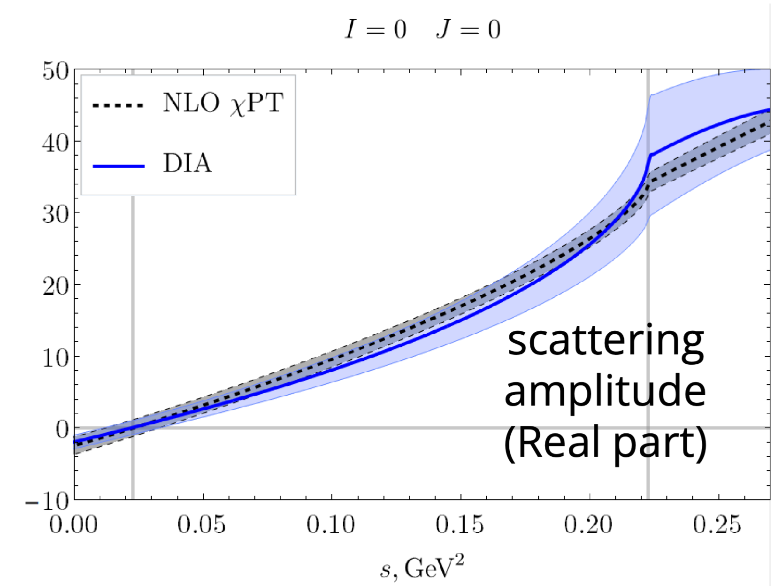
- Fitted parameters consistent with NLO

$$g_{A,\mathrm{NLO}} = \left(\left.\frac{\mathrm{d}t^{\mathrm{NLO}}(s)}{\mathrm{d}s}\right|_{s=s_A} \right)^{-1} = 0.45(3)$$

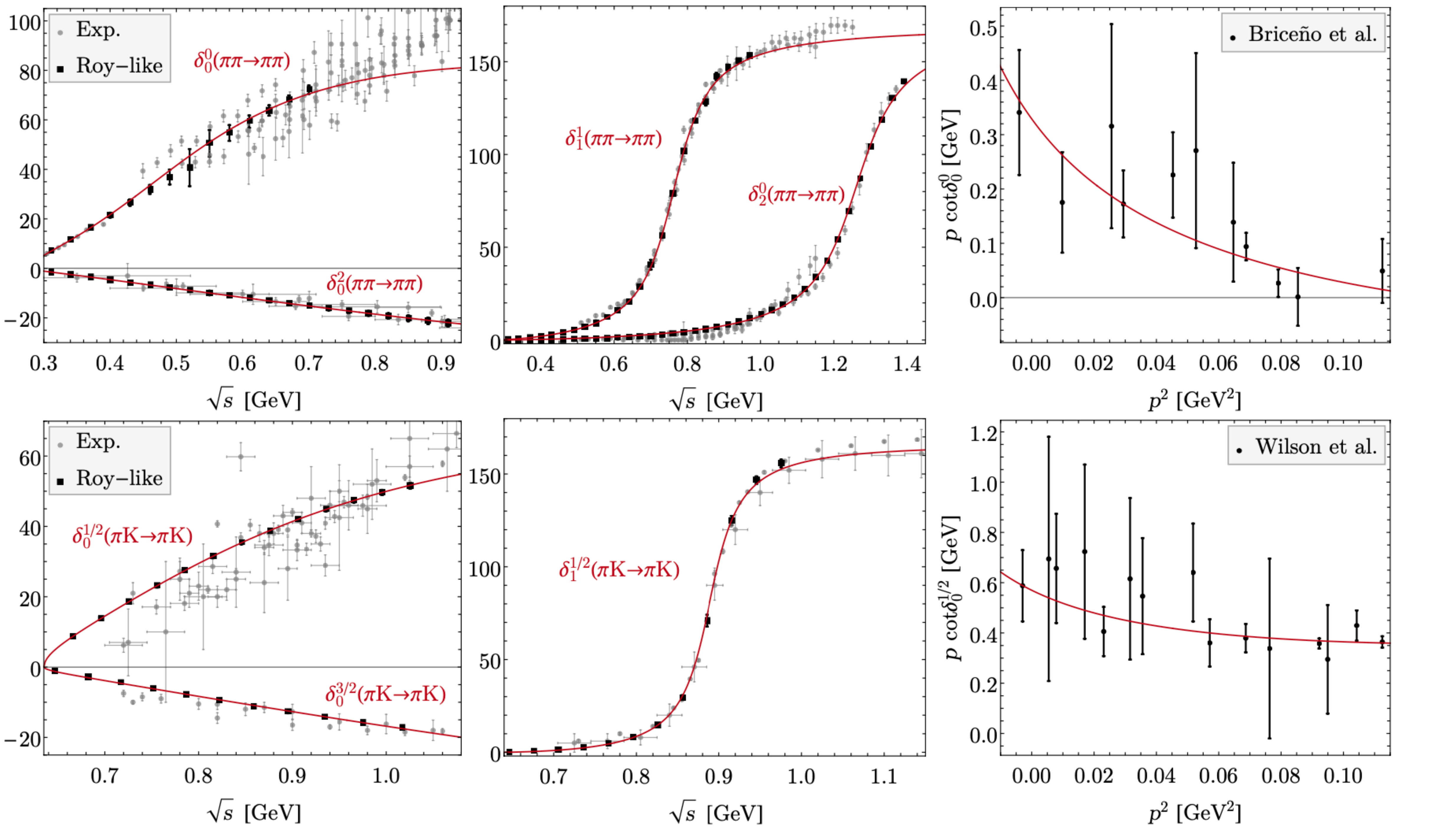
$$c_{0,\mathrm{NLO}} = \left[t^{\mathrm{NLO}}(s_{th})\right]^{-1} = 1.46(6)$$

perturbative ChPT LECs from
[Bijnens, Ecker 2014]





Numerical applications



- We have also derived parameterisations for $J \neq 0$ and $m_1 \neq m_2$ and tested them on $\pi\pi$ and πK scattering
- Potential application to $\pi\pi, \pi K, \pi D, KD, \ldots$ with $m_\pi \neq m_\pi^{\text{phys}}$

[I.D, V. Biloshytskyi, X.-L. Ren, M. Vanderhaeghen 2206.15223 (2022)]

Summary

- New parametrisations for the scattering amplitudes for spinless particles were derived
- Derivation from the general principles – unitarity, analiticity, crossing
- The test on the well-studied cases for $\pi\pi \to \pi\pi$ and $\pi K \to \pi K$ scattering were performed

- Ongoing collaboration with Lattice group (Mohler et al.)
- Determination of perturbative ChPT LEC (up to NNLO) using S and P wave lattice data