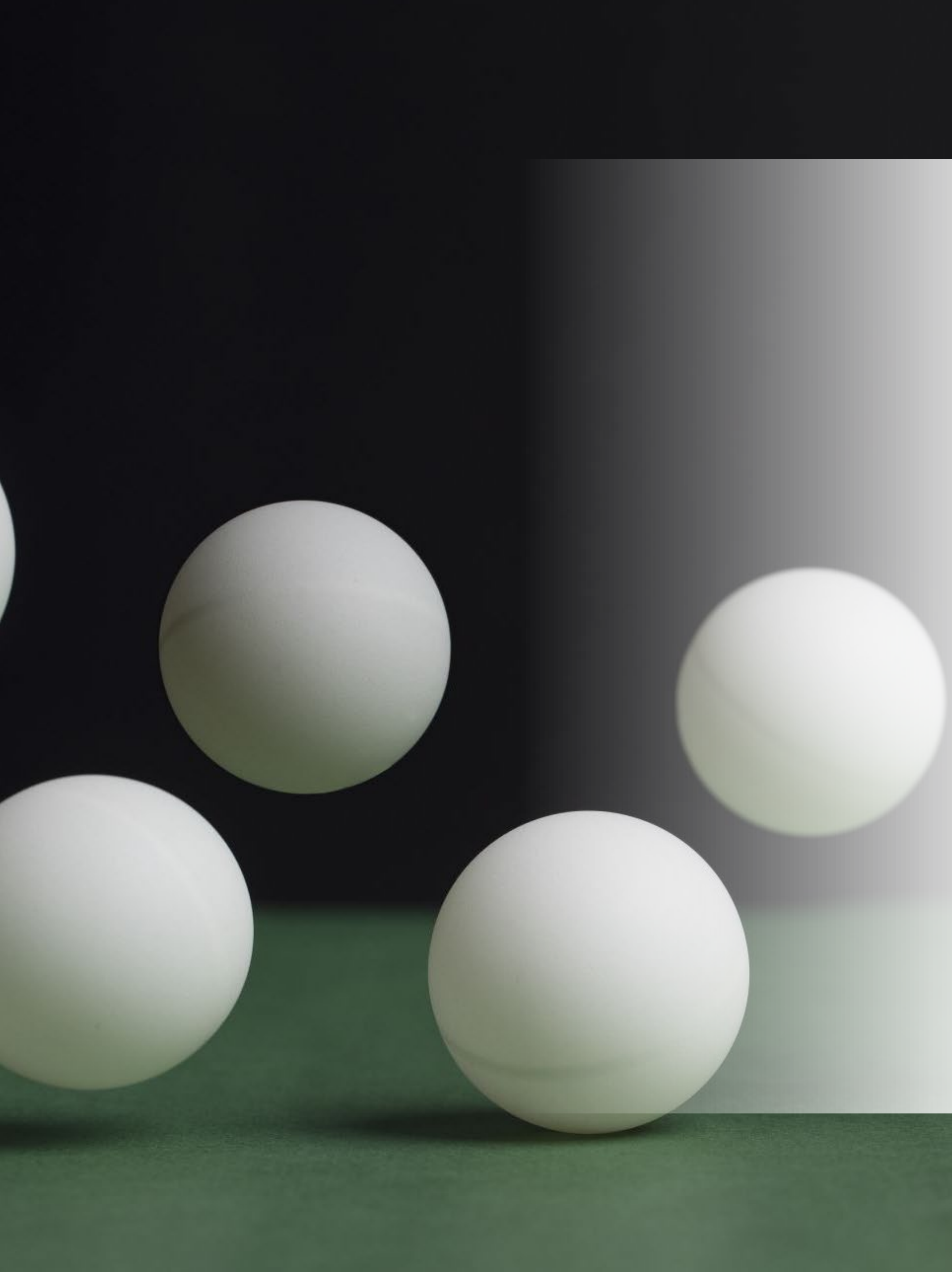

Three-body dynamics of the $a_1(1260)$

from lattice QCD

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4th Workshop on Future Directions in Spectroscopy Analysis (FDSA2022)

Review 2B-lattice: [Briceno]
Reviews 3B-lattice: [Hansen] [Mai]
Review hadron resonances: [Mai]

Key publications Finite-Volume Unitary (FVU) approach:
- Three-body unitarity [Mai]
- Three-body unitarity finite volume [Mai]
- a_1 in finite volume & results from lQCD [Mai]

Talk outline:
- 3-body unitarity
- a_1 in infinite volume
- a_1 in finite volume

Work supported by: National Science Foundation
Grant No. PHY 2012289

Progress in last three years alone (narrowly defined for 3B)

- **Whitepapers**: Snowmass whitepaper amplitude analysis: [1], Snowmass whitepaper lattice: [2]
- **FVU papers**: a_1 pole phenomenological: [3], $a_1 \to \pi\sigma$ inf. volume: [4], a_1 lQCD/PRL: [5], Review 3B lattice: [6], 3B force: [7], $3K^+$: [8], a_1 Dalitz: [9], $3\pi^+$ GWQCD data: [10] $3\pi^+$ interpretation Hanlon Data: [11], cross channel $\pi\pi$: [12], Resonance review (preprint): [13], (ρ with ETMC [14], φ^4 equivalence FVU/RFT [15])
- **RFT papers**: $3\pi^+$ HadSpec "Dalitz"/inf. vol. amplitude: [16], Decay amplitude to 3 hadrons: [17], 3 pions all isospins: [18], Review 3B fin vol Hansen: [19], QC $\pi^+\pi^+K^+$: [20], Higher-spin isobars: [21], Non-degenerate scalars 3B: [22] Alternative derivation 3B QC [23], ETMC/Bonn $3\pi^+$: [24]. $3\pi^+$ PRL analysis [25] of Hanlon/Hoerz data: [26]
- **(N)REFT**: Resonance form factor from corr functions [27], Spurious poles [28], EFT Book [29], Rel.-inv. formulation [30], ϕ^4 test scattering [31], Lüscher-Lellouch analog 3-body [32], Analytic energy shift 3B ground state [33], N-particle energy shift [34], Rusetsky Mini-review 3-body [35] Latest (schematic) effort for Roper fin vol [36].
- **Peng/Pang/Koenig, others:** Fin-vol extrapolation eigenvector continuation [37]. 3B resonances pionless EFT [38], Few-body bound states Fin Vol [39], Few-body resonances fin-vol [40], DDK system finite volume [41], Finite volume magnetic field [42, 43], Different fin vol geometries [44], Few-body resonances finite volume [45], Visualization three-body resonances (analytic cont. of L-dependence) [46], Multi-π^+ and analysis of lattice data [47], Threshold expansion N-particle Fin Vol [48], Propagation particle torus [49]
- **inf. vol./Equivalence 3B formalisms:** Equivalence different 3B QC [50], Jackura 3B unitarity PW [51], JPAC hadron physics review [52], 3B unitarity in RFT: [53].

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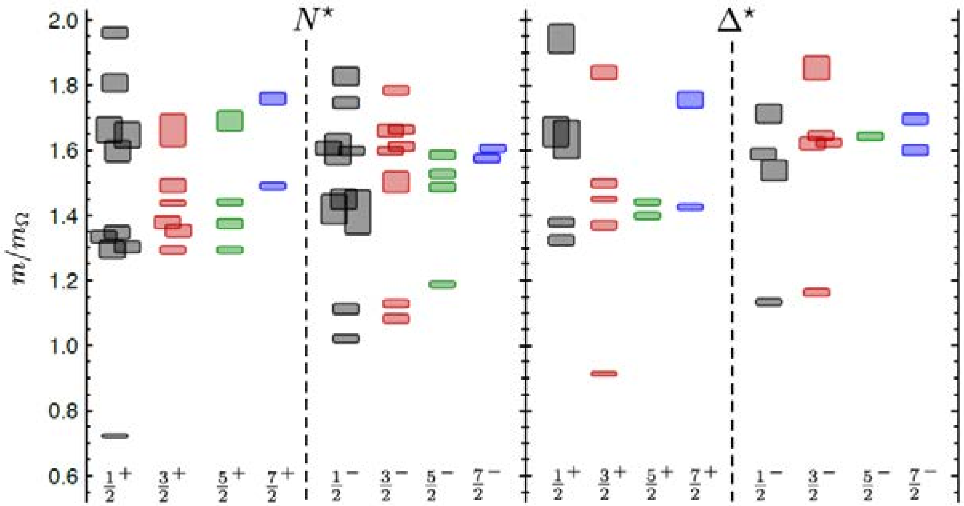
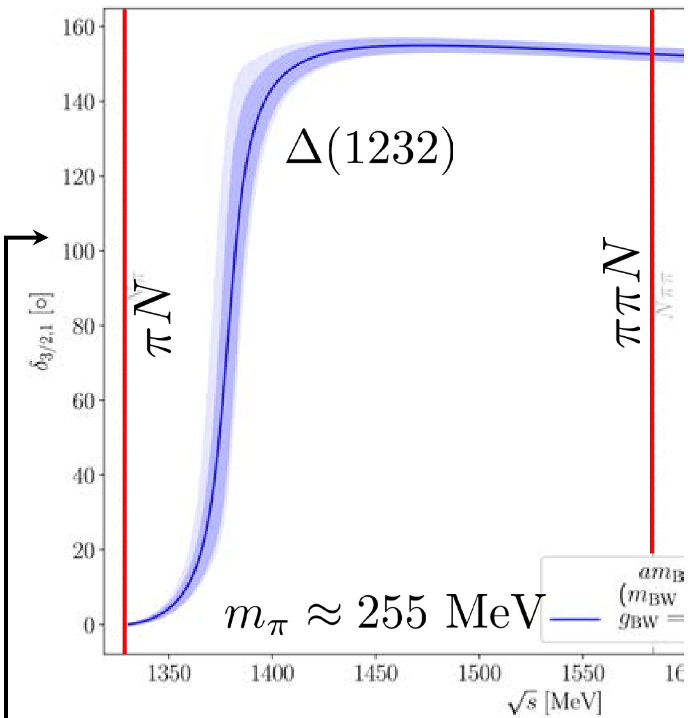
Lattice QCD for excited baryons



$m_\pi = 396$ MeV [Edwards *et al.*, Phys.Rev. D84 (2011)]

- Pioneering spectroscopic calculations
- Information on existence, width & properties of resonances requires
  - Meson-baryon interpolating operators
  - Detailed finite-volume analysis



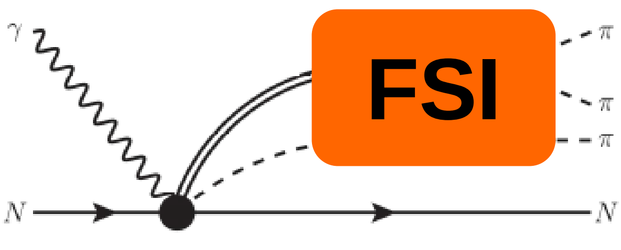
[G. Silvi et. al., arXiv: 2101.00689]

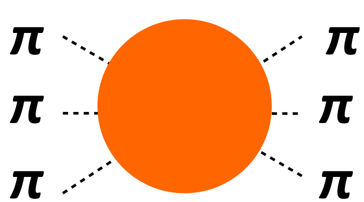
See also: Bulava et al., [2208.03867]

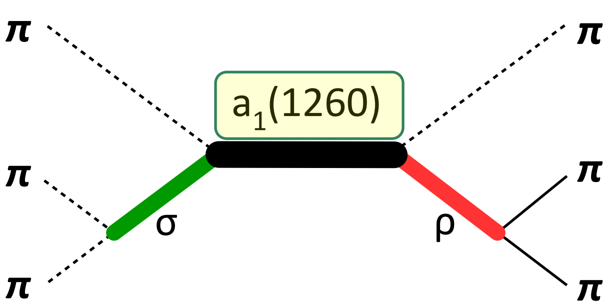
How about $\pi\pi N$? Roper resonance?

Three-body aspects: $\pi\pi N$ vs. $\pi\pi\pi$

Light mesons



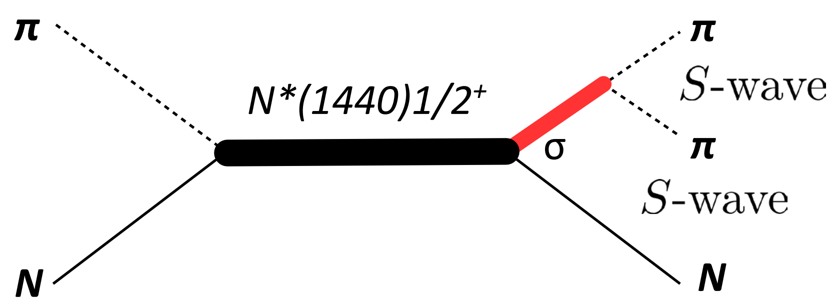




- COMPASS @ CERN: $\pi_1(1600)$ discovery
- GlueX @ Jlab in search of hybrids and exotics,
- Finite volume spectrum from lattice QCD:

  Lang (2014), Woss [HadronSpectrum] (2018)
  Hörz (2019), Culver (2020), Fischer (2020),
  Hansen (2020),...

Light baryons



- Roper resonance is debated for ~50 years in experiment. Can only be seen in PWA.
- 1[st] calculation w. meson-baryon operators on the lattice: **Lang et al. (2017)**

Three-body unitarity with isobars * [Mai 2017]

$$\langle q_1,q_2,q_3|(\hat{T}-\hat{T}^\dagger)|p_1,p_2,p_3\rangle = i\int_P \langle q_1,q_2,q_3|\hat{T}^\dagger|k_1,k_2,k_3\rangle\langle k_1,k_2,k_3|\hat{T}|p_1,p_2,p_3\rangle$$

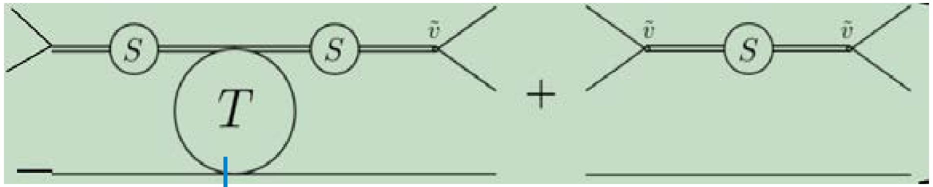
$$\times\prod_{\ell=1}^{3}\left[\frac{\mathrm{d}^4k_\ell}{(2\pi)^4}(2\pi)\boxed{\delta^+(k_\ell^2-m^2)}\right](2\pi)^4\delta^4\left(P-\sum_{\ell=1}^{3}k_\ell\right)$$

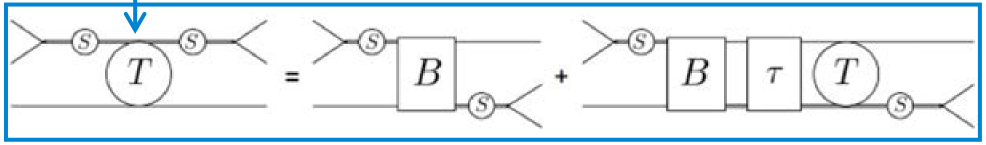
delta function sets all intermediate particles on-shell; no anti-particles!

* "Isobar" stands for two-body sub-amplitude which can be resonant or not; can be matched to CHPT expansion to one loop if desired. Isobars are re-parametrization of full 2-body amplitude [Bedaque] [Hammer]

Three-body unitarity

$$\langle q_1, q_2, q_3|(\hat{T}-\hat{T}^\dagger)|p_1, p_2, p_3\rangle \quad = \quad i\int_P \langle q_1, q_2, q_3|\hat{T}^\dagger|k_1, k_2, k_3\rangle\langle k_1, k_2, k_3|\hat{T}|p_1, p_2, p_3\rangle$$

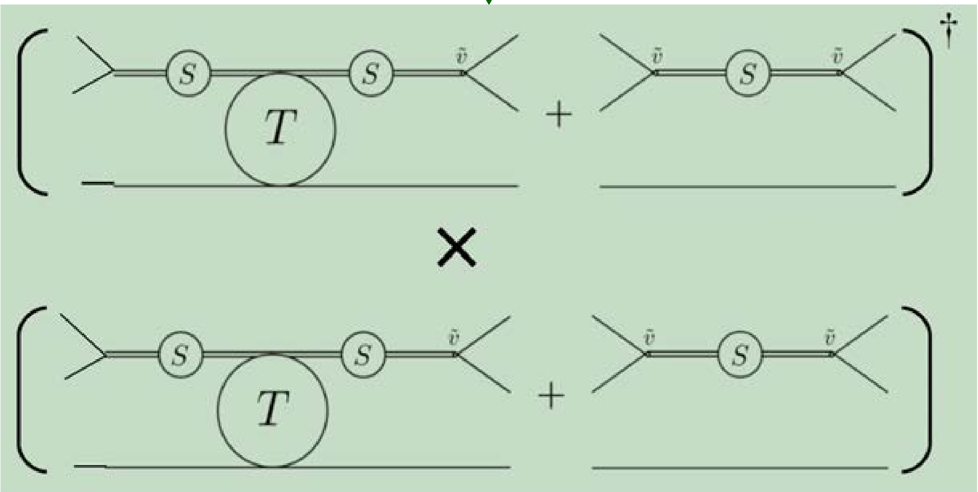




General Ansatz for the isobar-spectator interaction
→ **B & τ** are **new** unknown functions

Three-body unitarity

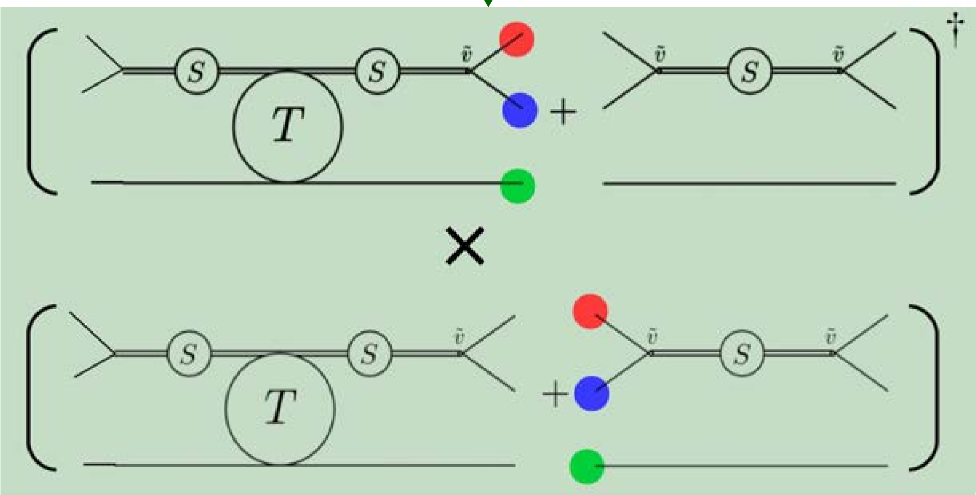
$$\langle q_1,q_2,q_3|(\hat{T}-\hat{T}^\dagger)|p_1,p_2,p_3\rangle = i\int_P \langle q_1,q_2,q_3|\hat{T}^\dagger|k_1,k_2,k_3\rangle\langle k_1,k_2,k_3|\hat{T}|p_1,p_2,p_3\rangle$$

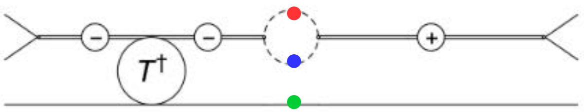


General connected-disconnected structure

Three-body unitarity

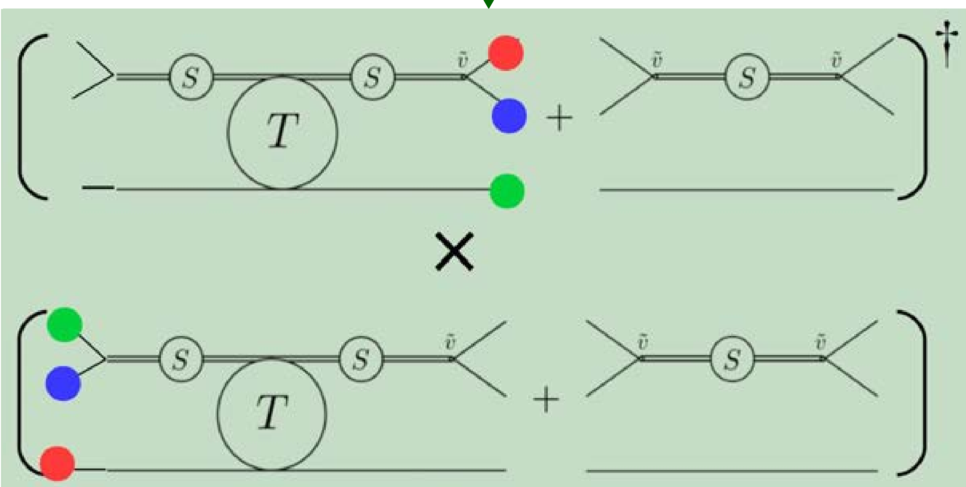
$$\langle q_1, q_2, q_3|(\hat{T}-\hat{T}^\dagger)|p_1, p_2, p_3\rangle = i\int_P \langle q_1, q_2, q_3|\hat{T}^\dagger|k_1, k_2, k_3\rangle\langle k_1, k_2, k_3|\hat{T}|p_1, p_2, p_3\rangle$$

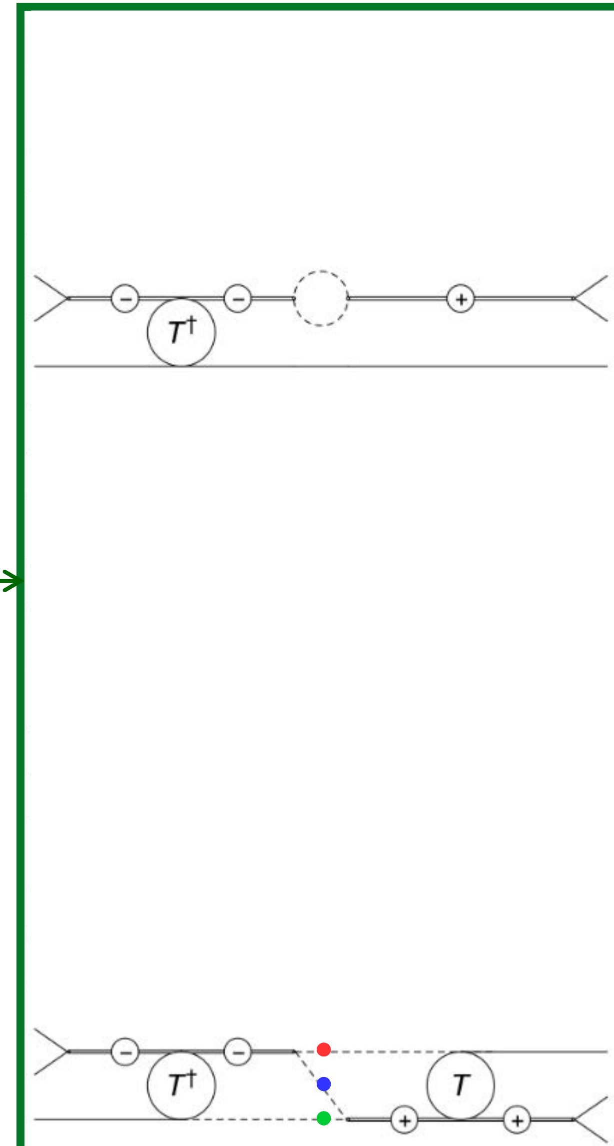




Three-body unitarity

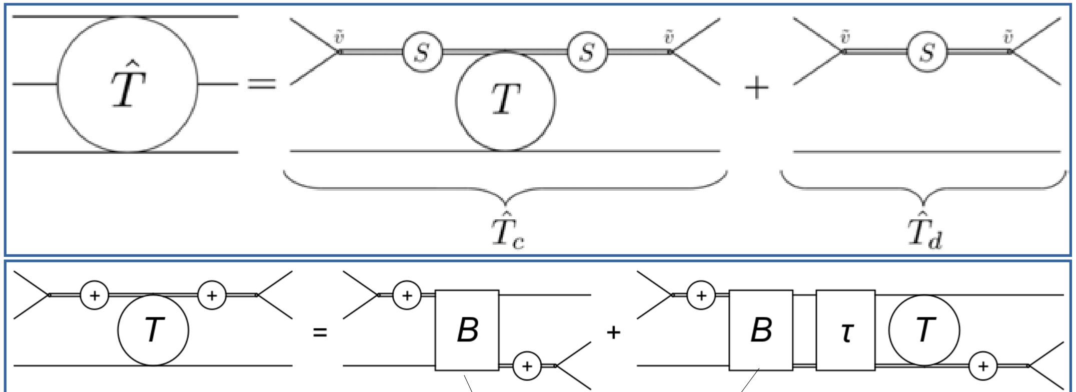
$$\langle q_1, q_2, q_3|(\hat{T}-\hat{T}^\dagger)|p_1, p_2, p_3\rangle \quad = \quad i\int_P \langle q_1, q_2, q_3|\hat{T}^\dagger|k_1, k_2, k_3\rangle\langle k_1, k_2, k_3|\hat{T}|p_1, p_2, p_3\rangle$$





Scattering amplitude

$3 \to 3$ scattering amplitude is a 3-dimensional integral equation



- Imaginary parts of ***B, S*** are fixed by **unitarity/matching**
- B, S are determined **consistently** through 8 different relations

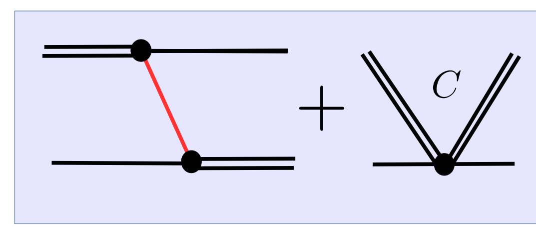
Matching → $$\mathrm{Disc}\, B(u) = 2\pi i \lambda^2 \frac{\delta\left(E_Q - \sqrt{m^2 + \mathbf{Q}^2}\right)}{2\sqrt{m^2 + \mathbf{Q}^2}}$$

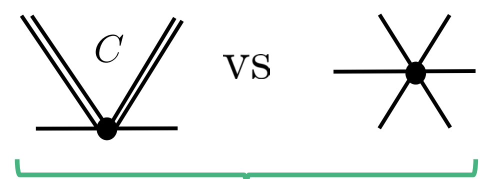
- un-subtracted dispersion relation

$$\langle q|B(s)|p\rangle = -\frac{\lambda^2}{2\sqrt{m^2+\mathbf{Q}^2}\left(E_Q - \sqrt{m^2+\mathbf{Q}^2} + i\epsilon\right)} + C$$

- one-**π** exchange in TOPT → *RESULT, NOT INPUT !*

- One <u>can</u> map to field theory but does not have to. Result is a-priori dispersive.



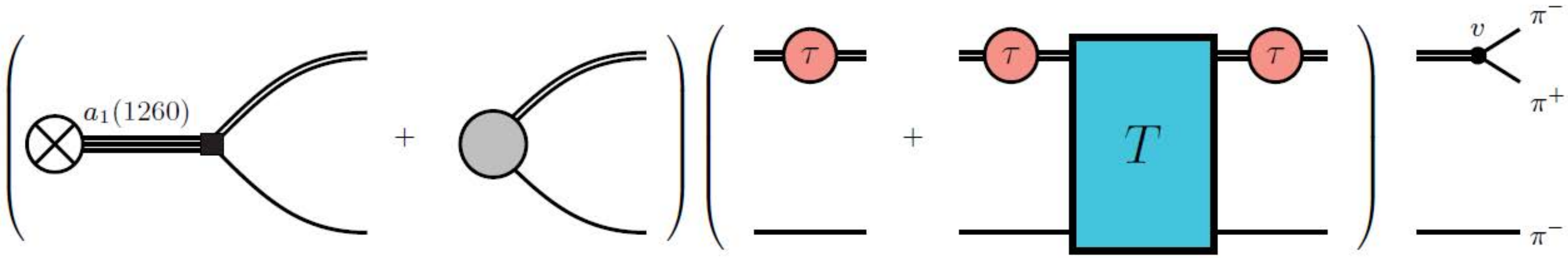


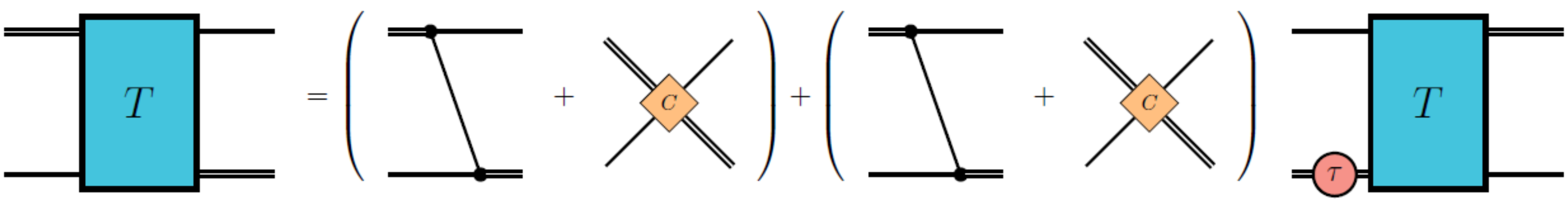
Add. Steps to map to theory might be needed [Brett (2021)]

The a_1(1260) and its Dalitz plots

[Sadasivan 2020]

- Disconnected and connected decays for three-body untarity





- New complication: the rho has spin:

$$T_{\lambda'\lambda}(p, q_1) = (B_{\lambda'\lambda}(p, q_1) + C) + \sum_{\lambda''} \int \frac{d^3 l}{(2\pi)^3 2E_l} (B_{\lambda'\lambda''}(p, l) + C)\, \tau(\sigma(l)) T_{\lambda''\lambda}(l, q_1)$$

Partial-wave decomposition

- Plane-wave basis

$$T_{\lambda'\lambda}(\boldsymbol{p},\boldsymbol{q}_1) = (B_{\lambda'\lambda}(\boldsymbol{p},\boldsymbol{q}_1) + C) + \sum_{\lambda''}\int \frac{d^3\boldsymbol{l}}{(2\pi)^3 2E_l}(B_{\lambda'\lambda''}(\boldsymbol{p},\boldsymbol{l}) + C)\,\tau(\sigma(l))T_{\lambda''\lambda}(\boldsymbol{l},\mathbf{q}_1)$$

$$B^J_{\lambda\lambda'}(q_1,p) = 2\pi\int_{-1}^{+1}\mathrm{d}x\, d^J_{\lambda\lambda'}(x)B_{\lambda\lambda'}(\boldsymbol{q}_1,\boldsymbol{p}) \qquad B^J_{LL'}(q_1,p) = U_{L\lambda}B^J_{\lambda\lambda'}(q_1,p)U_{\lambda' L'}$$

- JLS basis:

$$T^J_{LL'}(q_1,p) = \left(B^J_{LL'}(q_1,p) + C_{LL'}(q_1,p)\right) + \int_0^\Lambda \frac{\mathrm{d}l\, l^2}{(2\pi)^3 2E_l}\left(B^J_{LL''}(q_1,l) + C_{LL''}(q_1,l)\right)\tau(\sigma(l))T^J_{L''L'}(l,p)$$

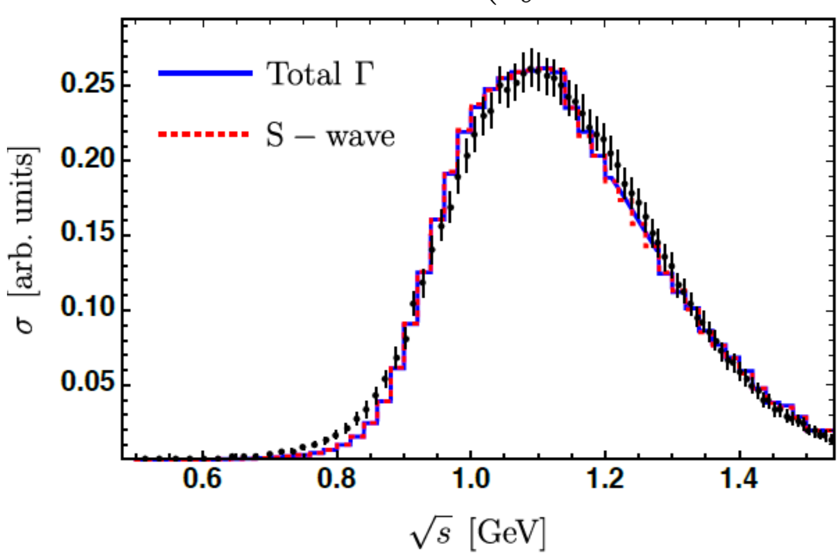
Fitting the lineshape & predicting Dalitz plots

[Sadasivan 2020]

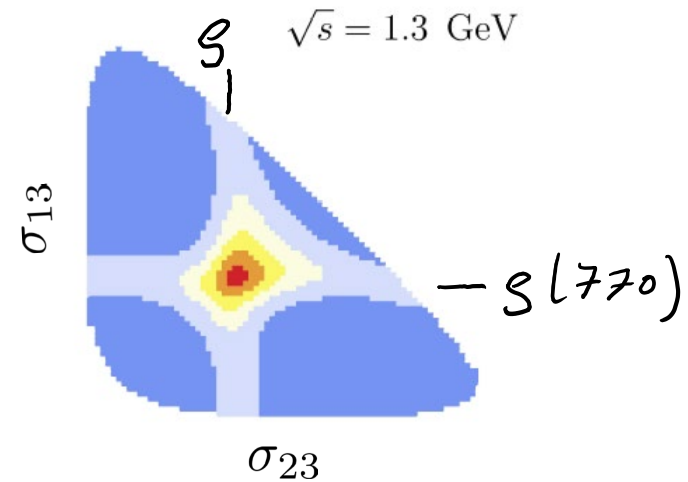
- One can have $\pi\rho$ in S- and D-wave coupled channels
- "Line shape": integrate all three final-state momenta,

$$\mathcal{L}(\sqrt{s}) = \frac{1}{\sqrt{s}} \int \frac{d^3\boldsymbol{q}_1}{(2\pi)^3} \frac{d^3\boldsymbol{q}_2}{(2\pi)^3} \frac{d^3\boldsymbol{q}_3}{(2\pi)^3} \frac{1}{2E_{q_1} 2E_{q_2} 2E_{q_3}} \times (2\pi)^4 \delta^4(P_3 - q_1 - q_2 - q_3) \overline{|\Gamma(\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3)|}^2 . \quad (18)$$

$a_1 \to \pi^-\pi^-\pi^+$ (symmetrize π^-'s!)



predict →

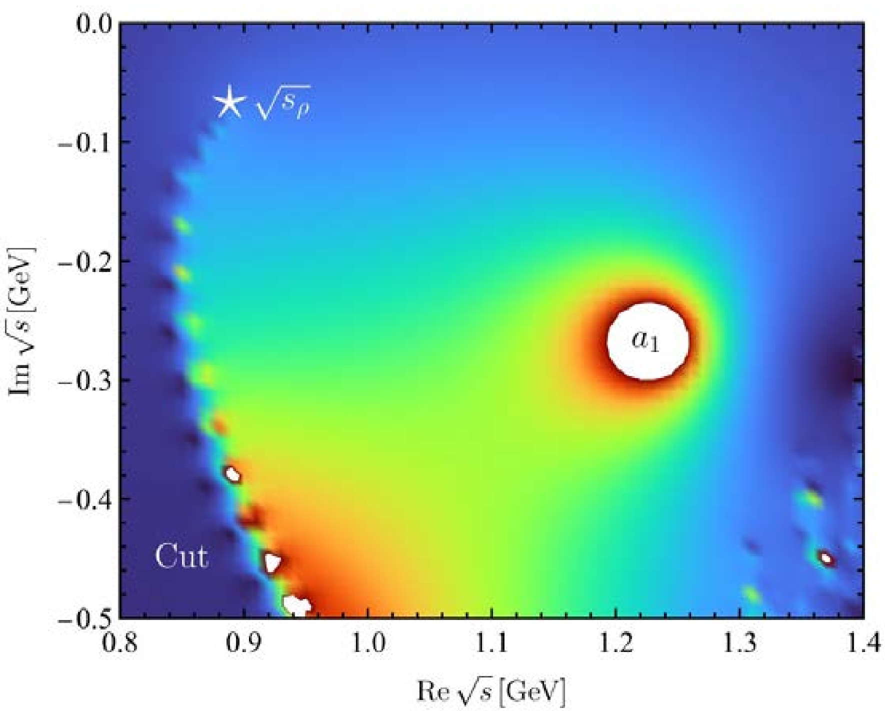


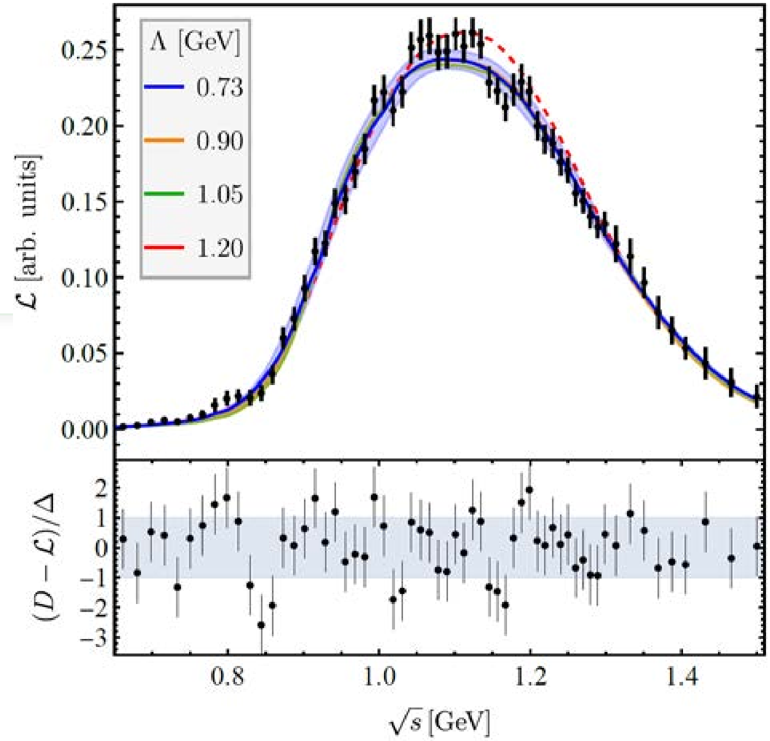
Where is the resonance pole in s?

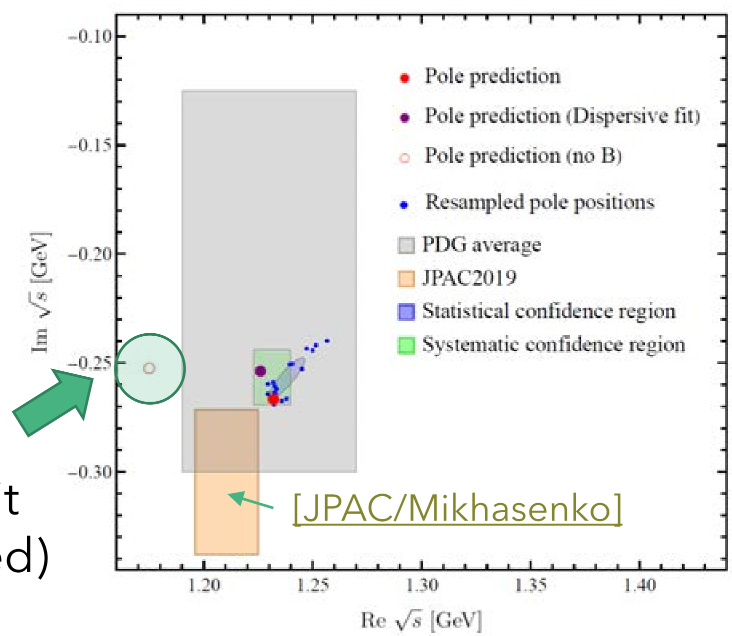
Result: Pole position

[Sadasivan 2022]

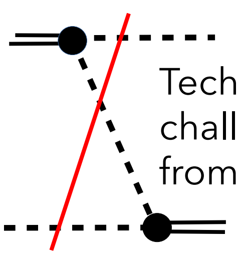
$|T_{00}|$







[JPAC/Mikhasenko]



Technical challenges from 3B cuts

If the B-term is neglected + refit (unitarity violated)

Finite volume quantization condition

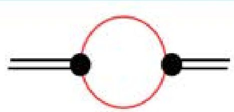
- General procedure:
  - Formulate an amplitude in infinite volume
  - Discretize all momenta
  - Partial-wave projection → project to irreps according to the symmetries of the problem:
    - Three-body system at rest
    - Three-body system with finite momenta P=(1,0,0),…
    - Cubic box; elongated boxes

Three-body quantization condition

Two-body unitarity

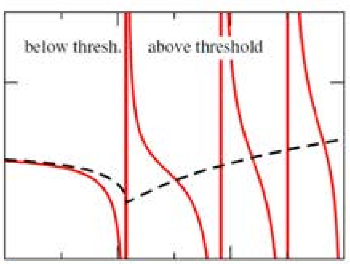
↓

On-shell condition



↓

Imaginary parts



Infinite → Fin. Vol

↓

Power-law fin-vol. effects

↓

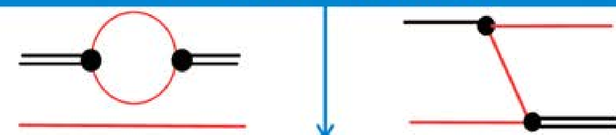
Lüscher

$$p\cot\delta(p) = -8\pi\sqrt{s}\left(\tilde{G}(E) - \mathrm{Re}\,G(E)\right)$$

Three-body unitarity

↓

On-shell condition



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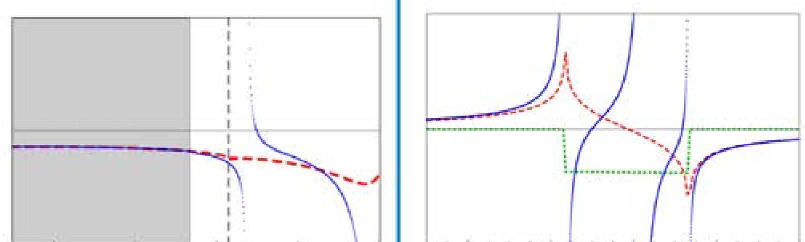
Imaginary parts



↓

Power-law fin-vol. effects

↓

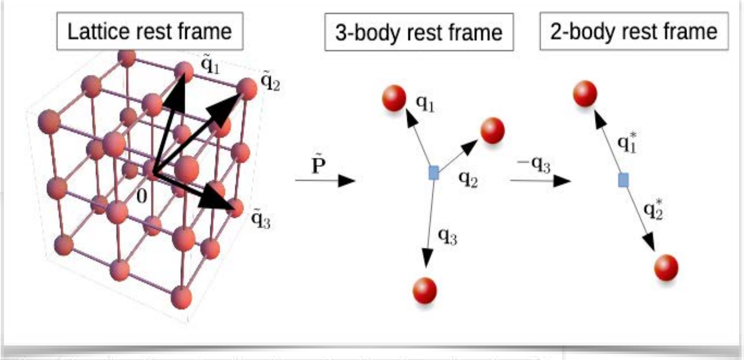
Quantization Condition

$$\mathrm{Det}\left(\mathbf{B}^{\Gamma\mathbf{ss'}}_{\mathbf{uu'}}(\mathbf{W^2}) + \frac{\mathbf{2E_s\,L^3}}{\vartheta(\mathbf{s})}\tau_{\mathbf{s}}(\mathbf{W^2})^{-1}\delta_{\mathbf{ss'}}\delta_{\mathbf{uu'}}\right) = \mathbf{0}$$

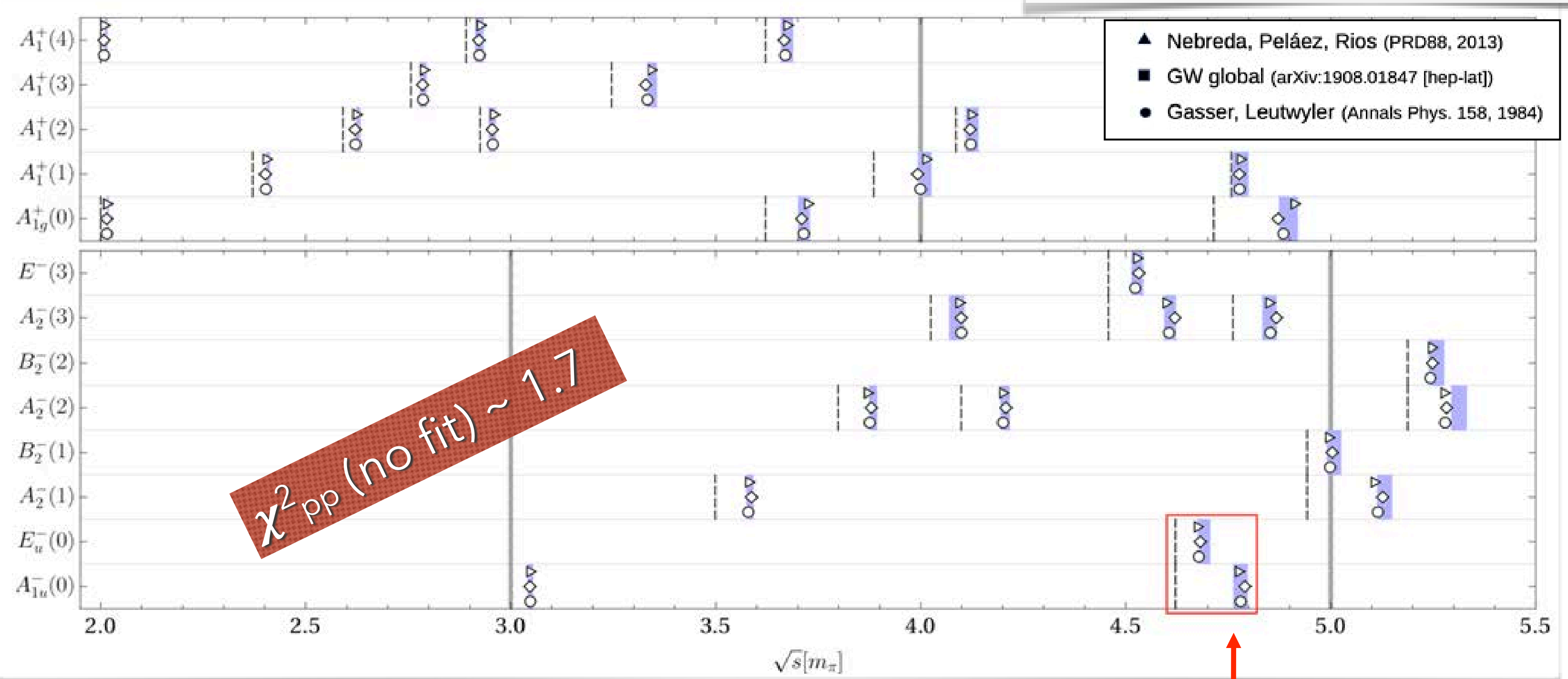
[Review Mai et al.]

The 3π⁺ System

[Hoerz 2019][GWUQCD/Culver 2019]

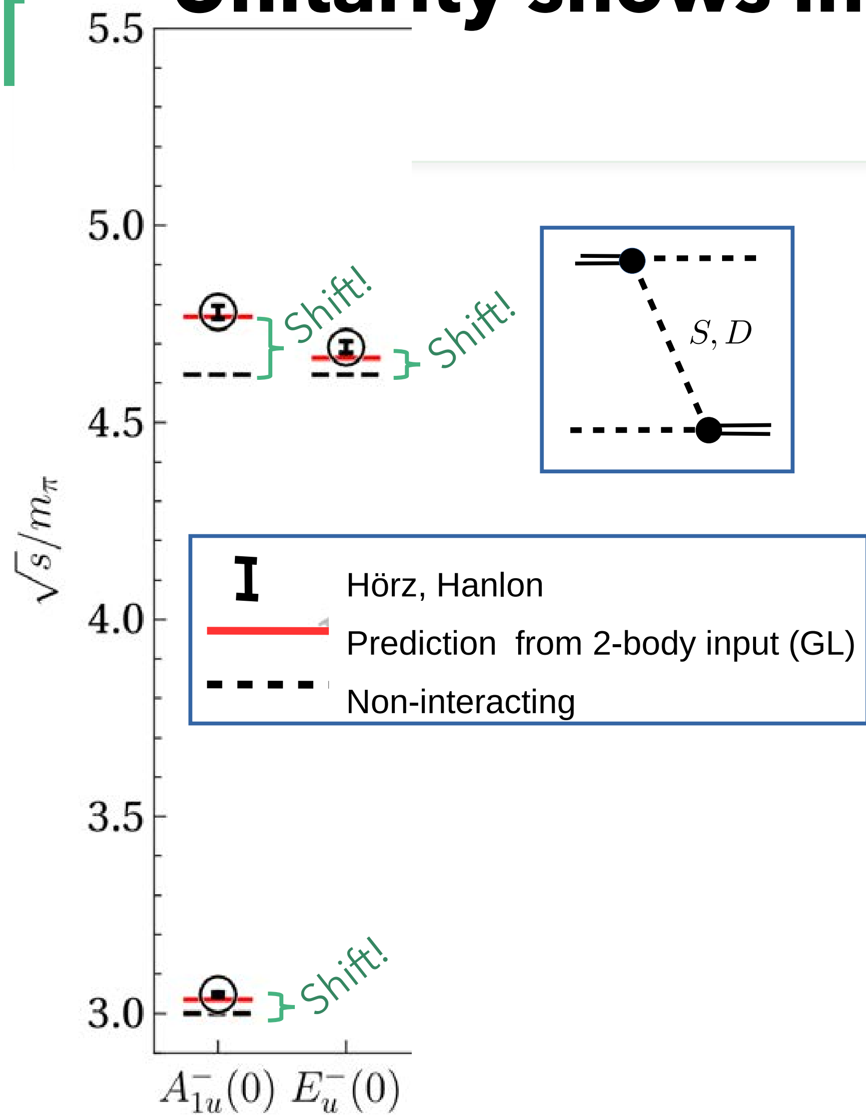


- Energy eigenvalues



- *Uncertainties dominated by the 2-body input. C~0*
- *Rel. strength between S/D waves fixed dominantly by 3b unitarity*

Unitarity shows in FV spectrum

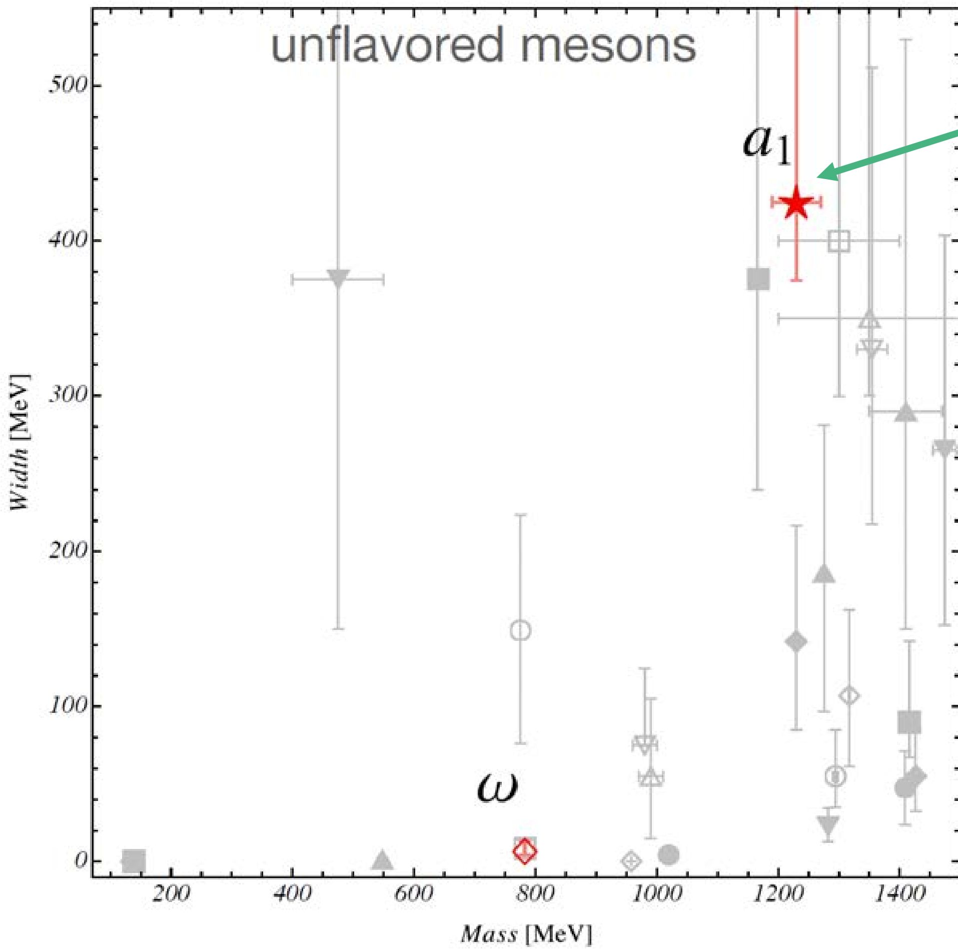


S **D** (lowest participating wave)

- S-wave shift good at threshold (like for NPLQCD data)
- S-wave prediction good at high energies
  → Energy dependence matched
- 3-body "force" set to zero
- D-wave prediction qualitatively good
  → Relative/absolute strength between S- and D-wave matched
  → Consequence that 3-body interaction dominated by exchange
  → Consequence of 3-body Unitarity

- Three-body unitarity directly visible in the eigenvalue spectrum of lattice QCD

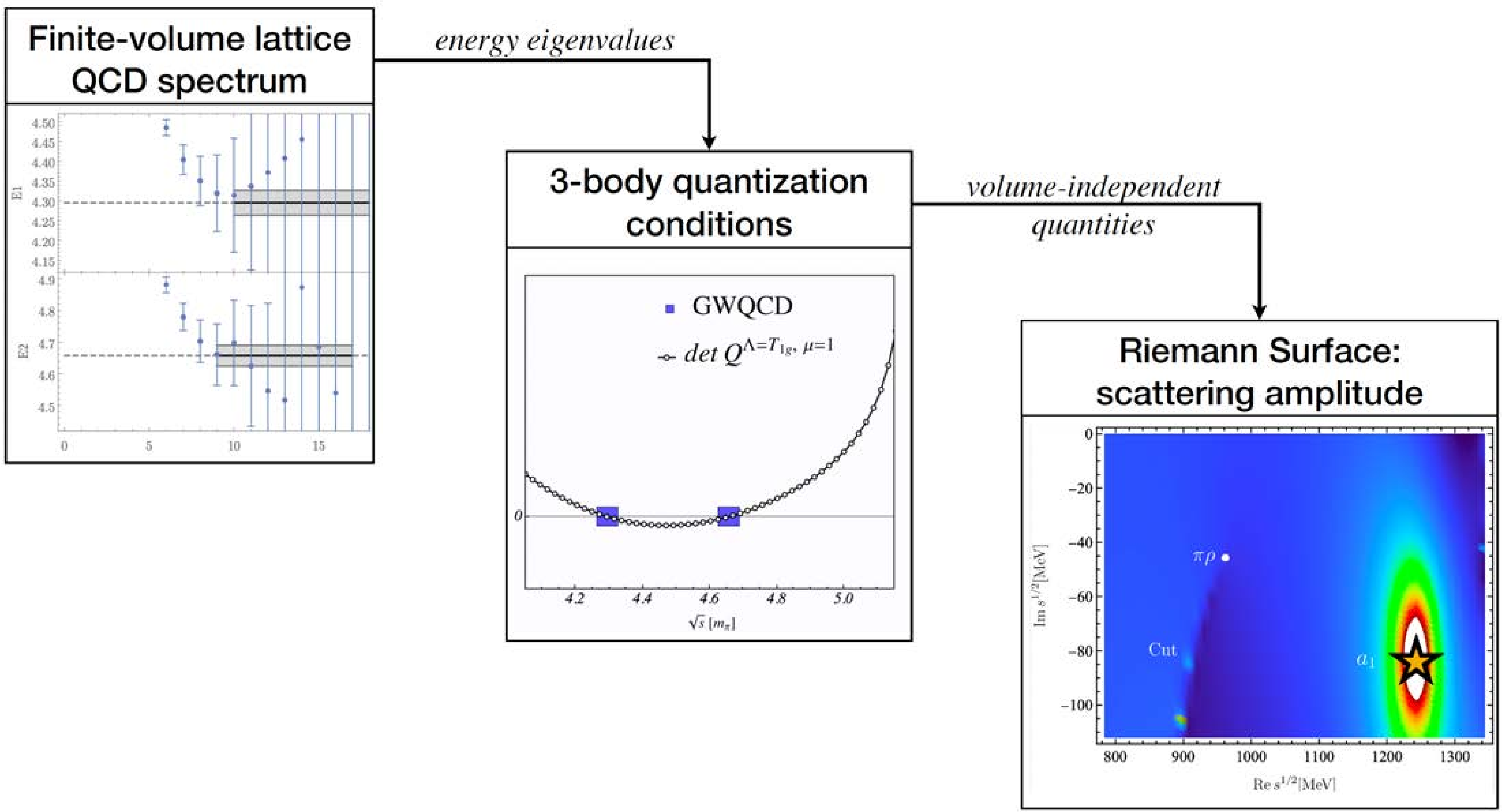
Light unflavored mesons: the a_1



We concentrate on this resonance!
(because 3-body)

Extraction of $a_1(1260)$ from lQCD [Mai/GWQCD]

- First-ever three-body resonance from 1[st] principles (with explicit three-body dynamics).



Three-particle propagation with helicities

- 2-body unitarity fixes only part of the interaction;

$$\tau^{-1}_{\lambda'\lambda}(\sigma_p) = \delta_{\lambda'\lambda}\tilde{K}^{-1}_n(s,\boldsymbol{p}) - \Sigma_{n,\lambda'\lambda}(s,\boldsymbol{p})\,,$$

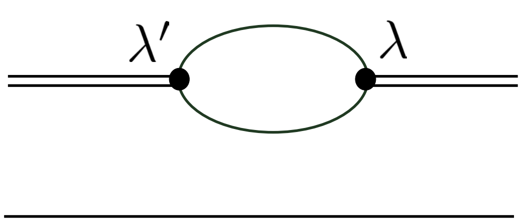
$$\tilde{K}^{-1}_n(s,\boldsymbol{p}) = \sum_{i=0}^{n-1} a_i\sigma_p^i \quad \text{and} \quad \Sigma_{n,\lambda'\lambda}(s,\boldsymbol{p}) =$$

$$\int \frac{d^3k}{(2\pi)^3}\frac{\sigma_p^n}{(4E_k^2)^n}\frac{\hat{v}^*_{\lambda'}(P-p-k,k)\hat{v}_\lambda(P-p-k,k)}{2E_k(\sigma_p-4E_k^2+i\epsilon)}$$

s : 3B energy
$\mathbf{p}$: spect. mom.

n=2-times subtracted for convergence

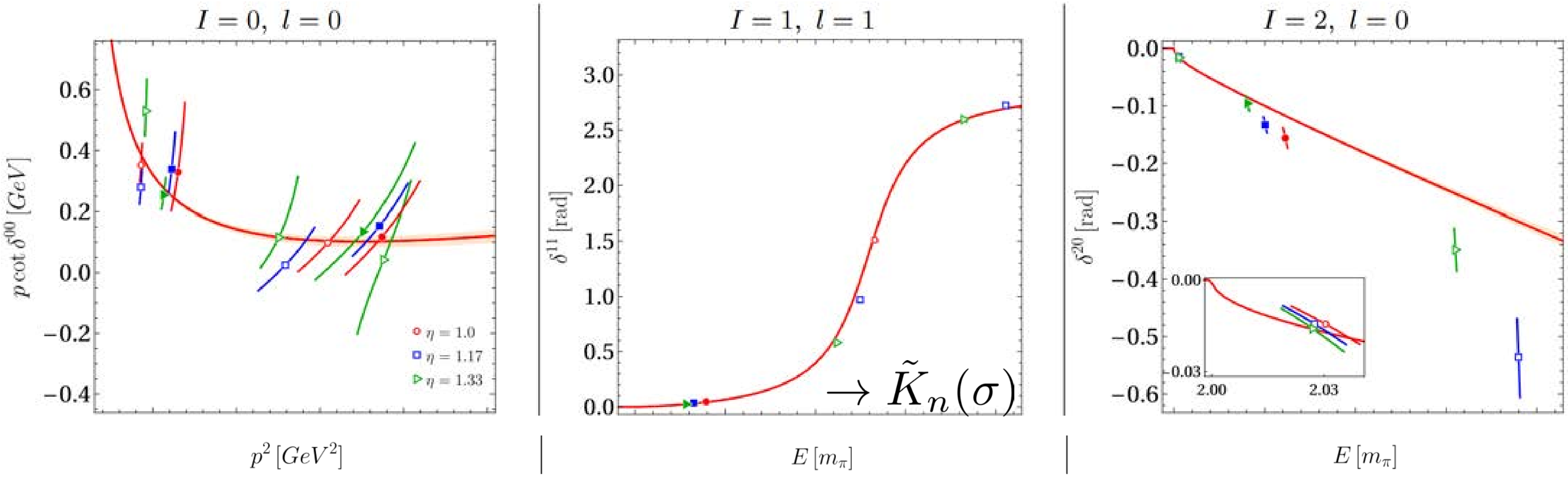
- The helicity of the *ρ* in flight can change!



$\Sigma_{\lambda'\lambda} \neq 0$ if $\lambda \neq \lambda'$

2-body input & lattice parameters

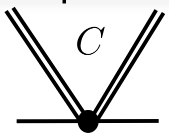
- Global fit of 2-body sector across different isospins, including correlations across isospin (IAM to NLO) [Mai 2019]



- Two mass-degenerate light quarks (u,d); valence strange quark
- nHYP-smeared clover action
- quark propagation is treated using the LapH method with optimized inverters
- Lattice spacing determined from Wilson flow parameter w_0

4 different fits to 2 energy eigenvalues

- Fitted isobar-spectator interaction (case 1, 2) for $|\boldsymbol{p}| \leq 2\pi/L|(1,1,0)| \approx 2.69\ m_\pi$



$$C_{\ell'\ell}(s, \boldsymbol{p}', \boldsymbol{p}) = \sum_{i=-1}^{\infty} c_{\ell'\ell}^{(i)}(\boldsymbol{p}', \boldsymbol{p})(s - m_{a_1}^2)^i$$

- a_1 can be generated as pole even though no built-in singularity

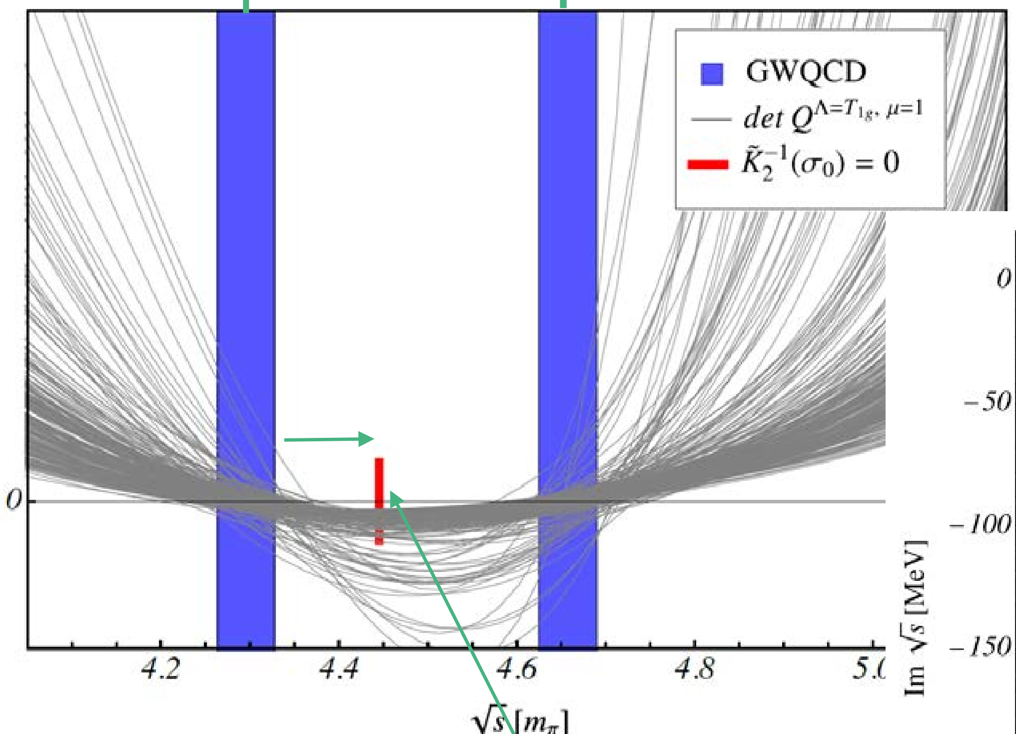
Non-zero coefficients	No of fit parameters	χ^2
c_{00}^{0} (no built-in pole)	1	9
c_{00}^{0}, c_{00}^{1} (no built-in pole)	2	0.15
g_0, g_2, m_{a1}, c	4	10^{-7}

$$C_{\ell'\ell}(s, \boldsymbol{p}', \boldsymbol{p}) = g_{\ell'} \left(\frac{|\boldsymbol{p}'|}{m_\pi}\right)^{\ell'} \frac{m_\pi^2}{s - m_{a_1}^2} g_\ell \left(\frac{|\boldsymbol{p}|}{m_\pi}\right)^{\ell} + c\,\delta_{\ell' 0}\delta_{\ell 0}$$

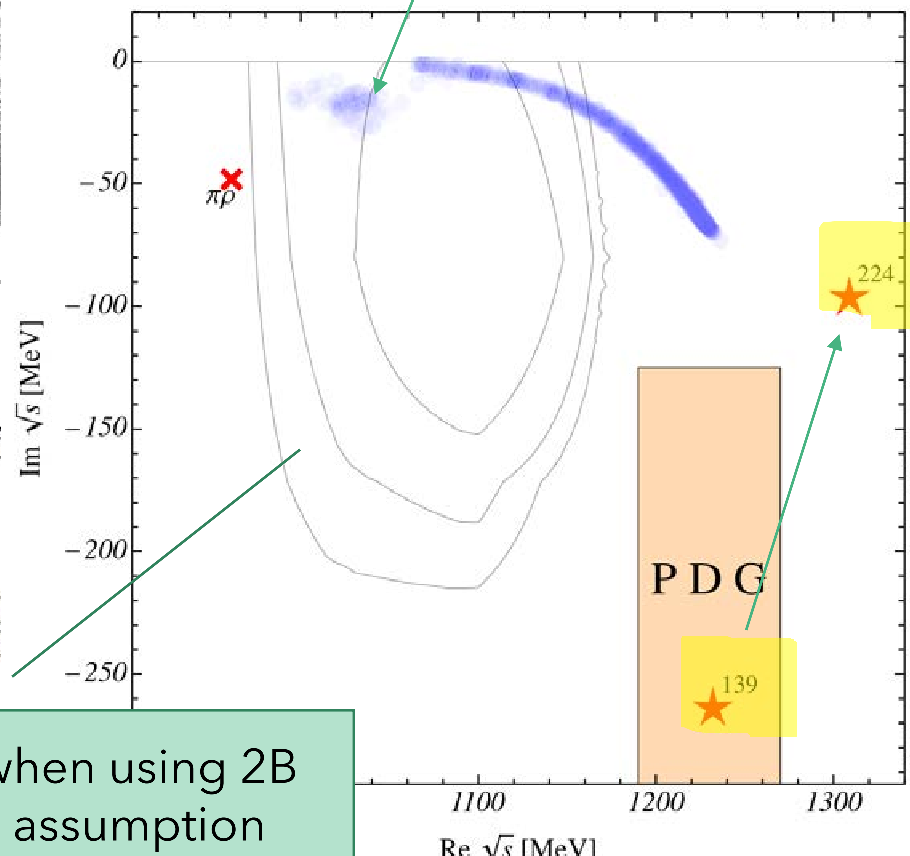
- In these cases, there is a built-in singularity, leading to resonance poles

Results – overview (4 parms)

Cluster: Large c and g_s compensate; disappears with c=0



$$0 = \det\left[B(s) + C(s) - E_L\left(\tilde{K}_2^{-1}(s) - \Sigma_2^L(s)\right)\right]_{(\lambda'\lambda)(p'p)}$$



PDF for pole when using 2B Breit-Wigner assumption

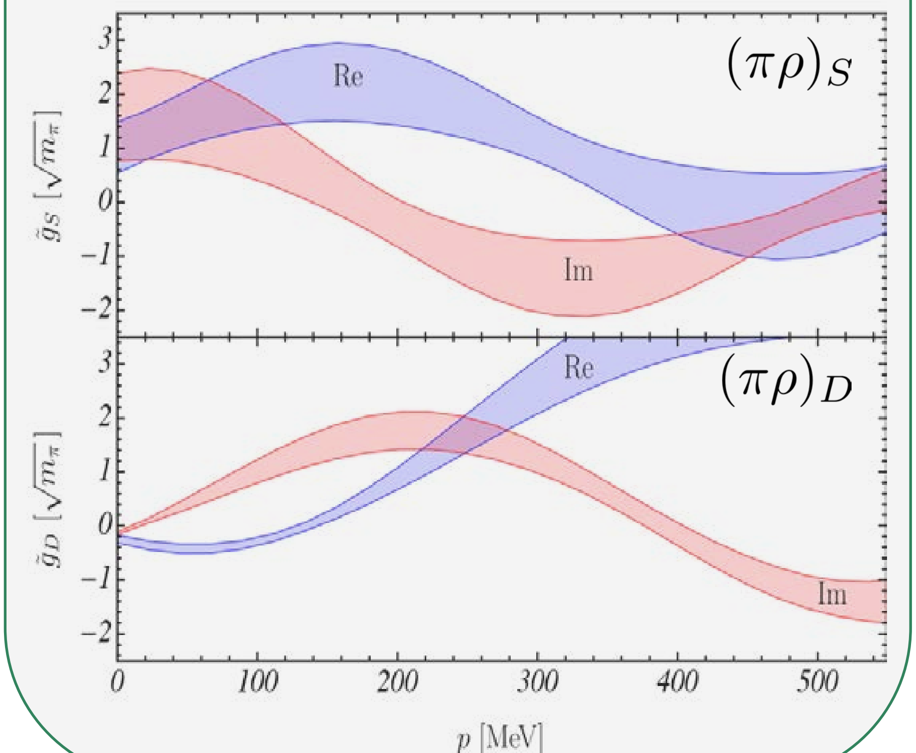
Branching ratios

- Calculate the residue at the pole:

$$\mathrm{Res}(T^{c}_{\ell'\ell}(\sqrt{s})) = \tilde{g}_{\ell'}\tilde{g}_{\ell}$$

- This result is not as reliable as pole position/existence of a_1
- More energy eigenvalues needed to better pin down the decay channels
- Other isospins needed, e.g., $(\pi\sigma)_P$ [Molina 2021]

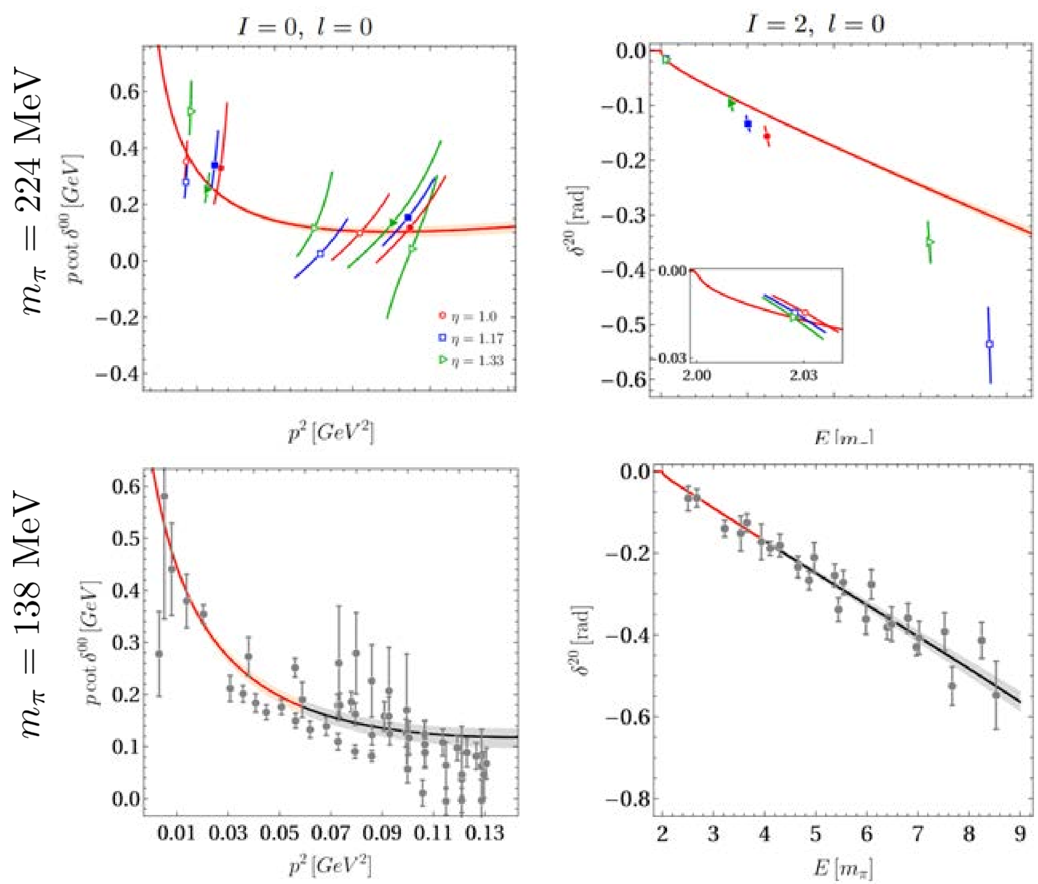
"**Branching ratios**" in 3B decays are momentum -dependent, complex pole residues

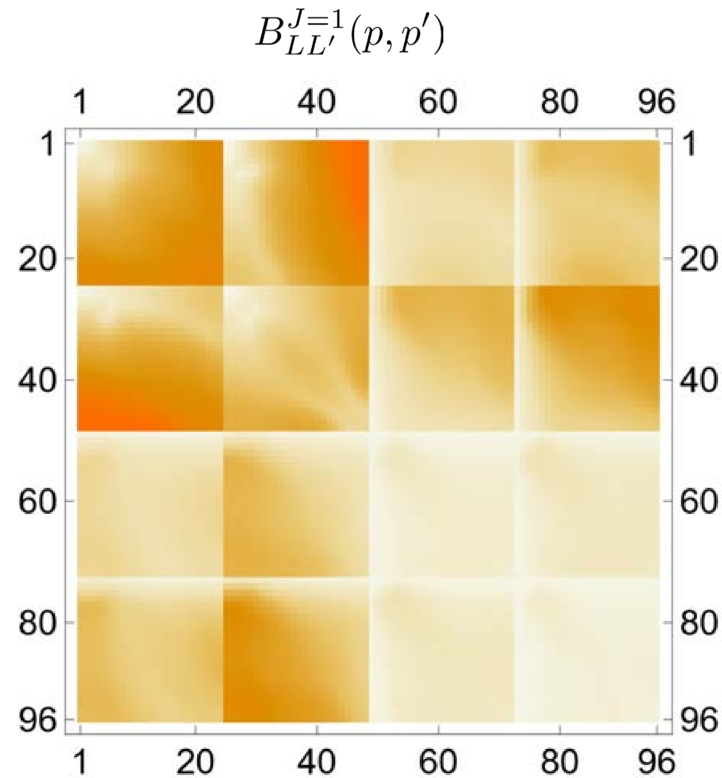


Outlook: four coupled channels

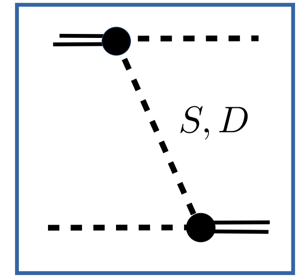
$a_1 \leftrightarrow (\pi\rho)_S \leftrightarrow (\pi\rho)_D \leftrightarrow (\pi\sigma)_P \leftrightarrow (\pi(\pi\pi)_{S,I=2})$

- Inclusion of all S- and P-wave isobars (from 2B lQCD input)
- Current status: physical point/inf. volume from experiment

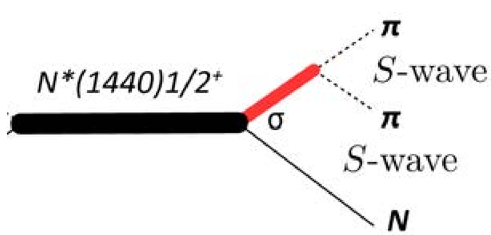




Summary



- Lattice QCD progress in determining the explicit dynamics of three-body systems:
  - Three pions at maximal isospin well understood (FVU, RFT, Peng,…)
  - First determination of existence and properties of a three-body resonance – the $a_1(1260)$ - in coupled channels, isobars with spin, and using three-body unitarity
- **Outlook:** More (isospin) channels and more energy eigenvalues to assess uncertainties and put limits on decay properties.
- **Outlook:** Roper resonance as an interesting case of a baryon which has strong 3-body dynamics due to absence of centrifugal barriers

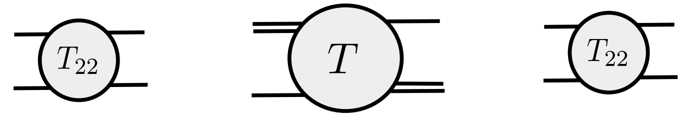


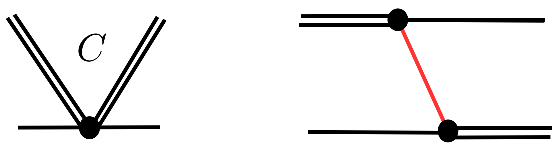
Spare slides

Scattering amplitude (Details)

Here: Version in which isobar rewritten in on-shell $2 \to 2$ scattering amplitude T_{22}

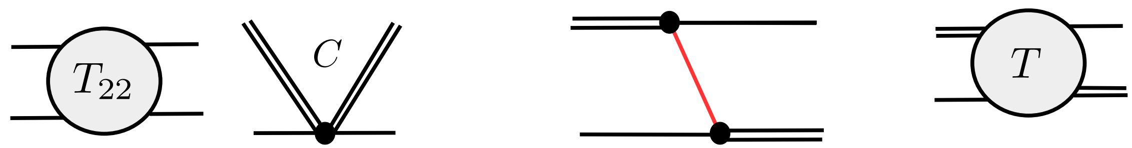
$$\langle q_1, q_2, q_3 | \hat{T}_c(s) | p_1, p_2, p_3 \rangle = \frac{1}{3!} \sum_{n=1}^{3} \sum_{m=1}^{3} T_{22}(\sigma(q_n)) \langle q_n | T(s) | p_m \rangle T_{22}(\sigma(p_m))$$

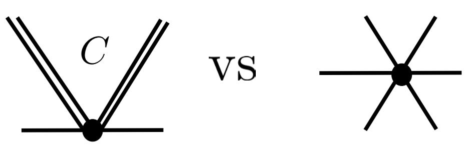




$$\langle q | T(s) | p \rangle = \langle q | C(s) | p \rangle + \frac{1}{m^2 - (P - p - q)^2 - i\epsilon}$$

$$- \int \frac{\mathrm{d}^3 \boldsymbol{\ell}}{(2\pi)^3} \frac{1}{2E_\ell} T_{22}(\sigma(\ell)) \left(\langle p | C(s) | \ell \rangle + \frac{1}{m^2 - (P - p - \ell)^2 - i\epsilon} \right) \langle \ell | T(s) | p \rangle$$



Technical Detail: 

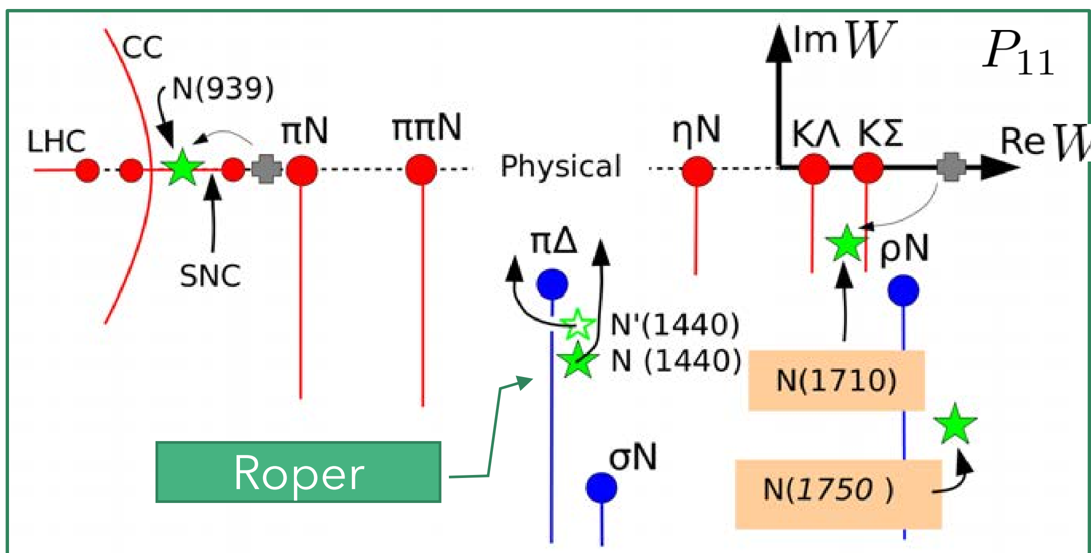
(S-wave)

Scheme-dependent 3-body force requires a mapping [Brett (2021)]

Hadronic resonances as poles

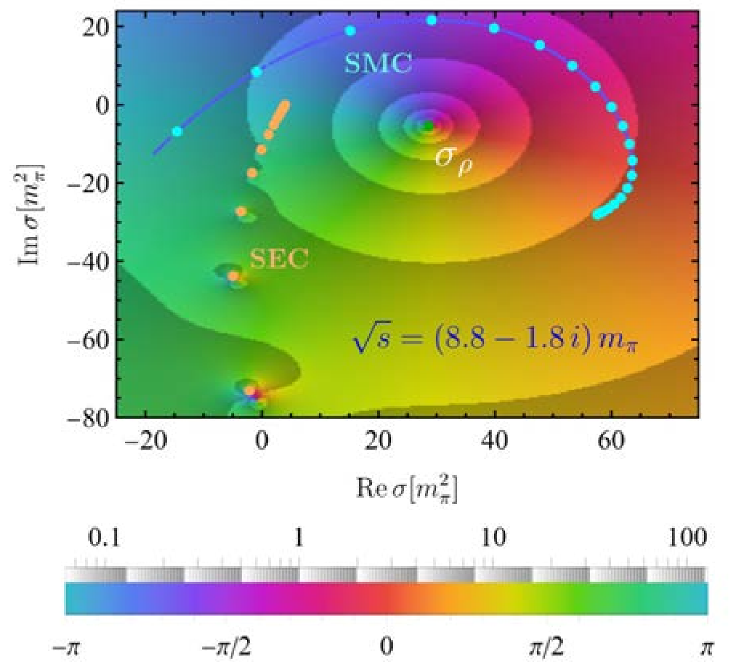
- Defining resonances as poles in amplitudes at complex energies resolves all mentioned problems
  - Real part of pole position ⟺ Mass
  - 2x Imaginary part of pole position ⟺ Width
  - Pole residue ⟺ Branching ratio into different channels because amplitudes factorize at poles [Doring 2009]

- Next goal: What is this?
  - Red: Real thresholds
  - Blue: resonanat sub-channel thres.
  - Double pole Roper
  - Note partial-wave cuts (CC, SNC) that disappear in plane-wave amplitude



Analytic cont. 3-body

[Sadasivan (2021)]
[Doering (2009)]



SMC

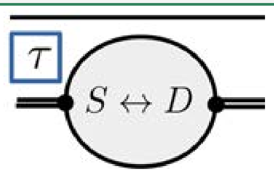
$$T^J_{LL'}(q_1,p) = \left(B^J_{LL'}(q_1,p) + C_{LL'}(q_1,p)\right) + \int_0^\Lambda \frac{\boxed{dl}\, l^2}{(2\pi)^3 2E_l}\left(B^J_{LL''}(q_1,l) + C_{LL''}(q_1,l)\right)\tau(\sigma(l))T^J_{L''L'}(l,p)$$

$$\tau^{-1}(\sigma) = K^{-1} - \Sigma\,,$$

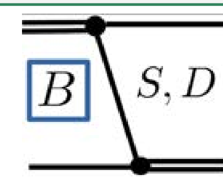
$$\Sigma = \int_0^\infty \frac{\boxed{dk}\, k^2}{(2\pi)^3}\frac{1}{2E_k}\frac{\sigma^2}{\sigma'^2}\frac{\tilde{v}(k)^*\tilde{v}(k)}{\boxed{\sigma - 4E_k^2 + i\epsilon}}$$

$$B_{\lambda\lambda'}(\boldsymbol{p},\boldsymbol{p}') = \frac{v^*_\lambda(P-p-p',p)v_{\lambda'}(P-p-p',p')}{2E_{p'+p}\boxed{\left(\sqrt{s} - E_p - E_{p'} - E_{p'+p} + i\epsilon\right)}}$$

SEC



Singularities

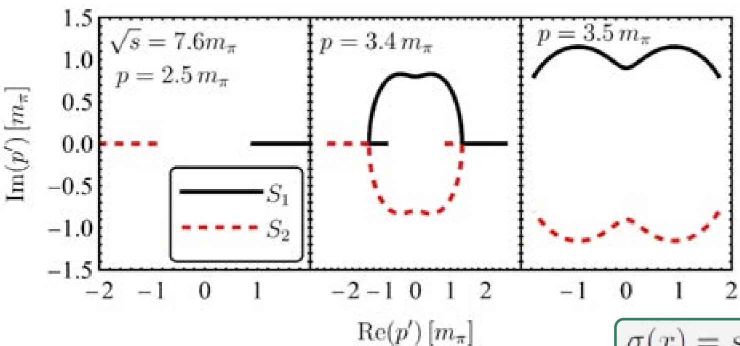


- Two contours (SMC and SEC)
- Deform both “adiabatically” to go to complex s
- Set of rules:
  - Contours cannot intersect with each others
  - Contours cannot intersect with (3-body) cuts
- Passing singularities left or right determines sheet

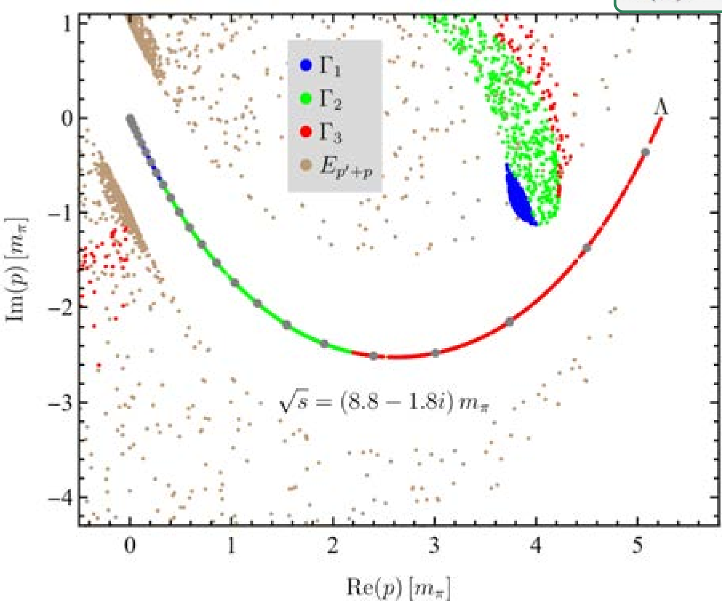
Analytic continuation 3-body (contd.)

- Three-body cuts

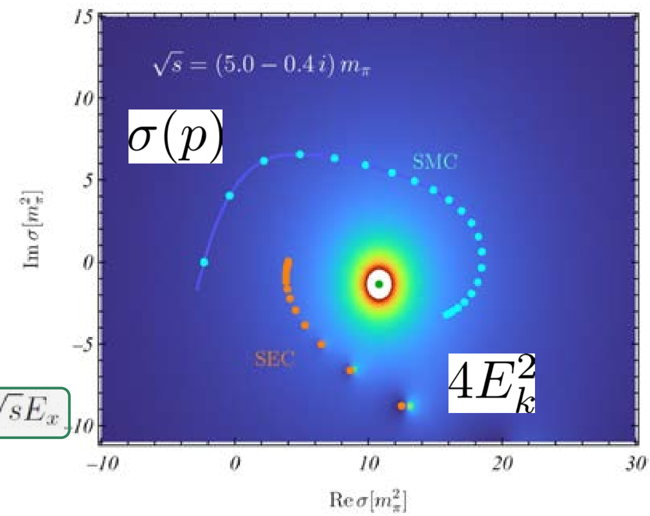
$$\sqrt{s} - E_p - E_{p'} - E_{p+p'} + i\epsilon = 0$$



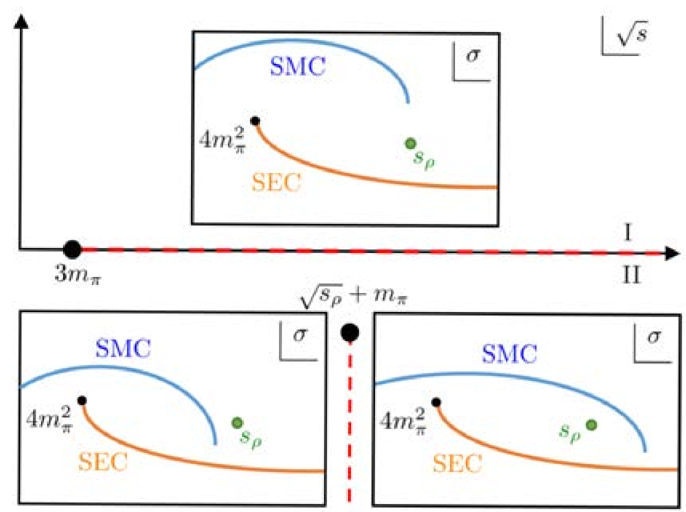
$$\sigma(x) = s + m_\pi^2 - 2\sqrt{s}E_x$$



- Complex branch points



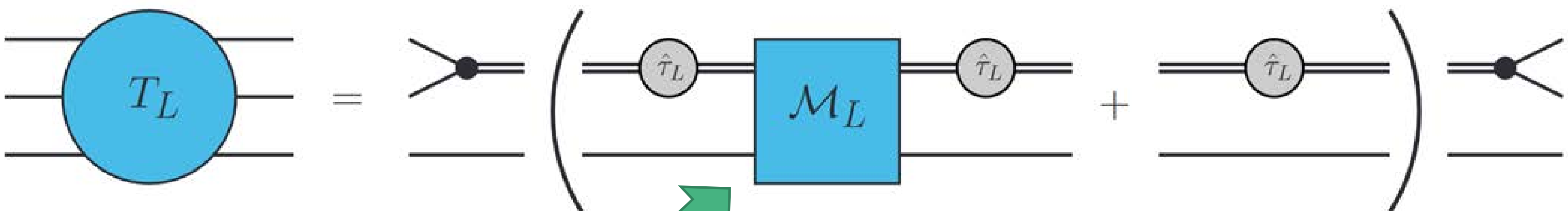
Integration limits at poles induce branch points



Finite-volume quantization

[Doering 2018]

- Connectedness structure



$$f(\mathbf{p}) = \sqrt{4\pi} \sum_{\ell m} Y_{\ell m}(\hat{\mathbf{p}}) f_{\ell m}(p)\,,$$

$$f_{\ell m}(p) = \frac{1}{\sqrt{4\pi}} \int d\Omega Y^*_{\ell m}(\hat{\mathbf{p}}) f(\mathbf{p})$$

(spinless isobar)

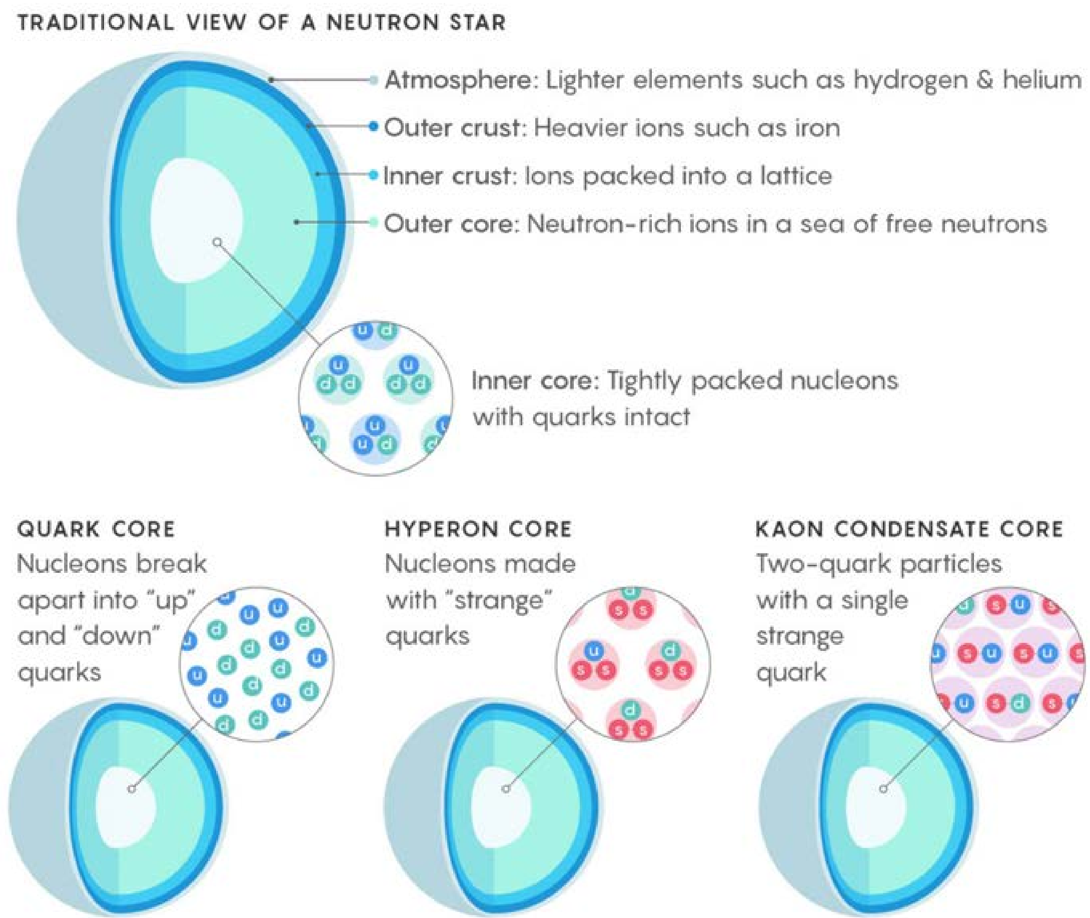
$\chi_u^{\Gamma\alpha s}$ are orthonormal
basis functions for irrep Γ
Shell index s, row α

$$f^s(\hat{\mathbf{p}}_j) = \sqrt{4\pi} \sum_{\Gamma\alpha} \sum_u f_u^{\Gamma\alpha s} \chi_u^{\Gamma\alpha s}(\hat{\mathbf{p}}_j)\,,$$

$$f_u^{\Gamma\alpha s} = \frac{\sqrt{4\pi}}{\vartheta(s)} \sum_{j=1}^{\vartheta(s)} f^s(\hat{\mathbf{p}}_j) \chi_u^{\Gamma\alpha s}(\hat{\mathbf{p}}_j) \quad \text{for} \quad \chi_u^{\Gamma\alpha s}(\hat{\mathbf{p}}) \in U_s\,,$$

Multi-kaon systems

- Relevant for heavy-ion collisions (ALICE)
- Matter with strangeness in neutron stars and equation of state:
  - Kaon condensate can soften the equation of state



[quantamagazine.org]

Three kaons at maximal isospin

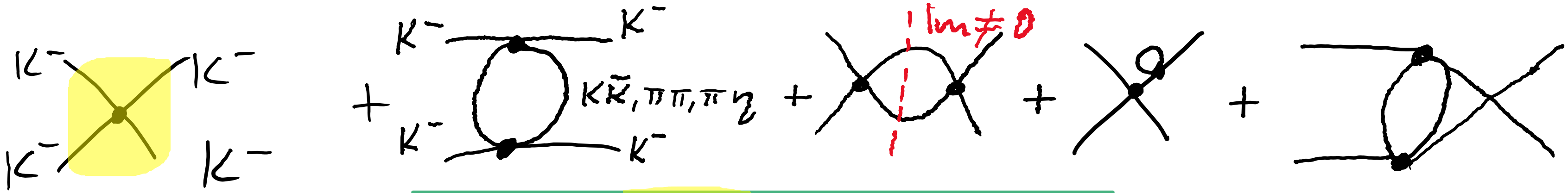
[Alexandru 2020]

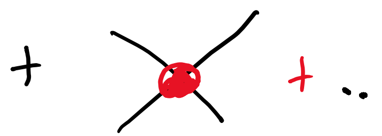
- First study of three kaons from lattice QCD with chiral amplitudes
- Other groups have improved on this in the meantime:
  - Max. isospin, non-identical masses ($\pi^+\pi^+K^+$, $\pi^+K^+K^+$)
    [Blanton 2021]
  - Pions and kaons at maximal isospin with unprecedented accuracy and no. of levels ($\pi^+\pi^+\pi^+$, $K^+K^+K^+$) [Blanton 2021]

- Two mass-degenerate light quarks (u,d); valence strange quark
- nHYP-smeared clover action
- quark propagation is treated using the LapH method with optimized inverters
- Lattice spacing determined from Wilson flow parameter w_0

Two kaons

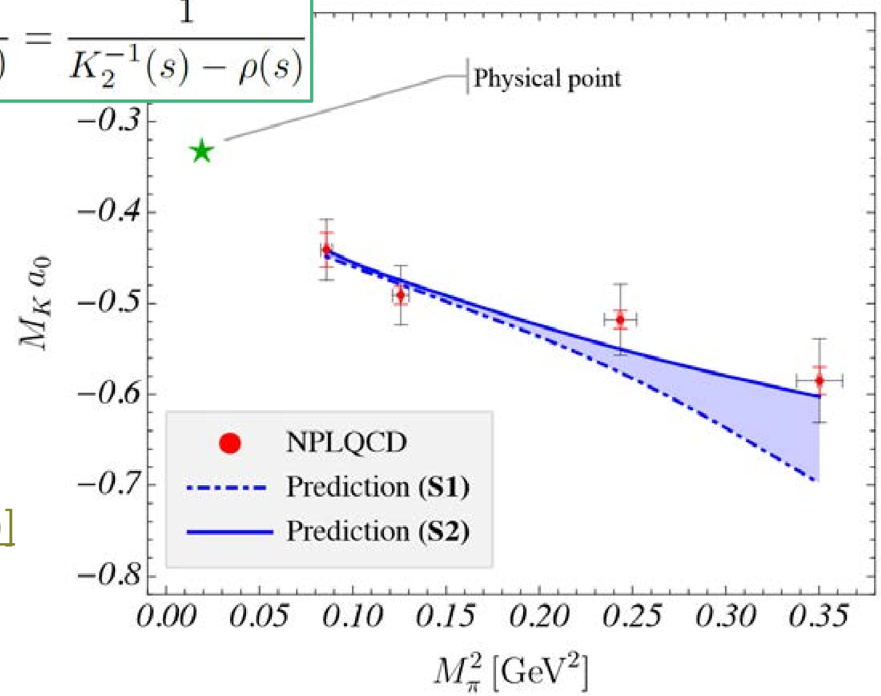
- Crossing symmetry allows to get the amplitude $K^- K^- \to K^- K^-$ from $K^+ K^- \to K^+ K^-$
- SU(3) CHPT unitarized with inverse amplitude method





$$T_2(s) = \frac{(T_{\mathrm{LO}}(s))^2}{T_{\mathrm{LO}}(s) - T_{\mathrm{NLO}}(s)} = \frac{1}{K_2^{-1}(s) - \rho(s)}$$

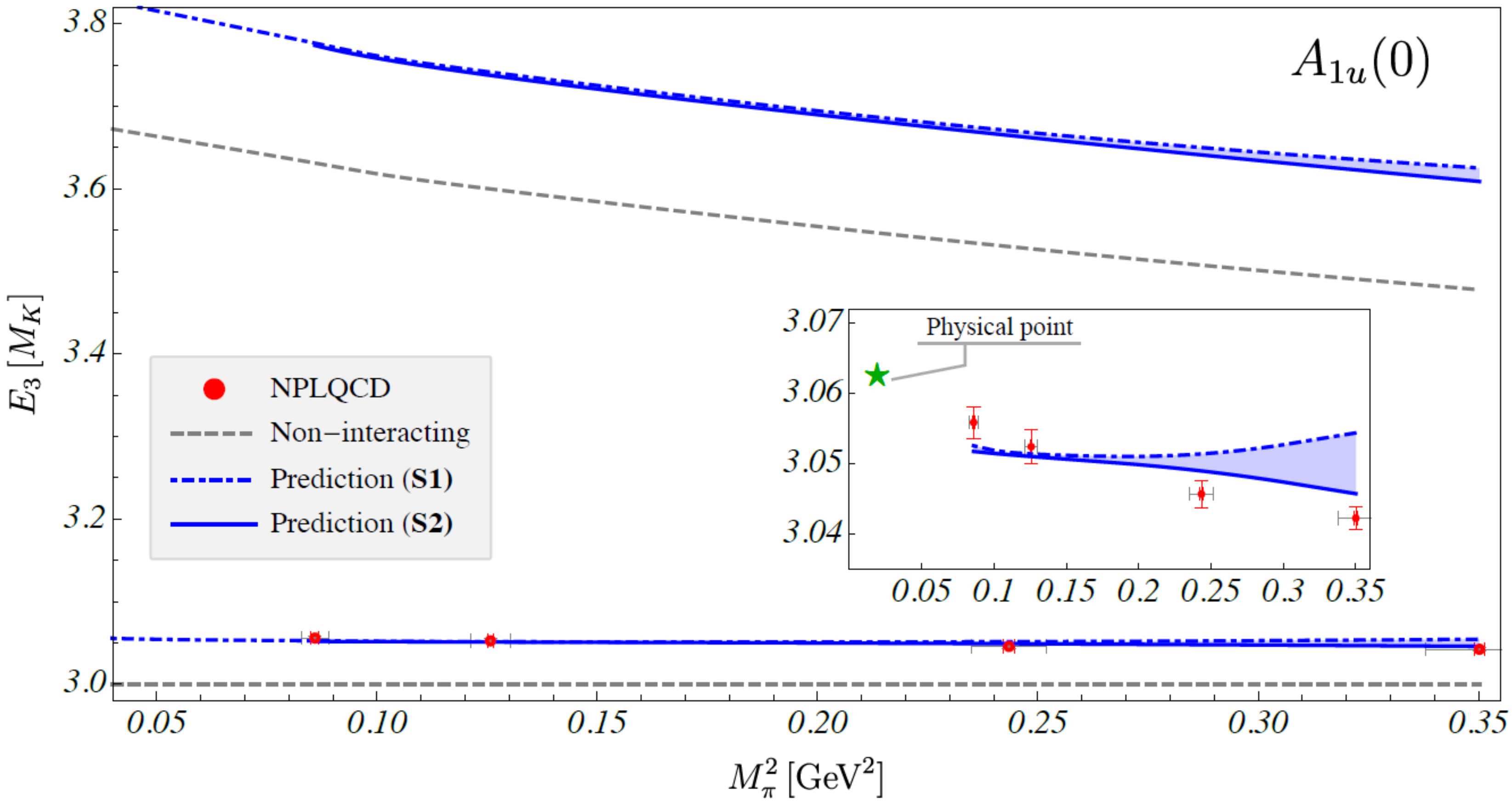
- Prediction of the I=1 KK scattering length: Data: [NPLQCD (2007)]



- LECs taken from most recent Global fit to lattice QCD [Molina (2020)]

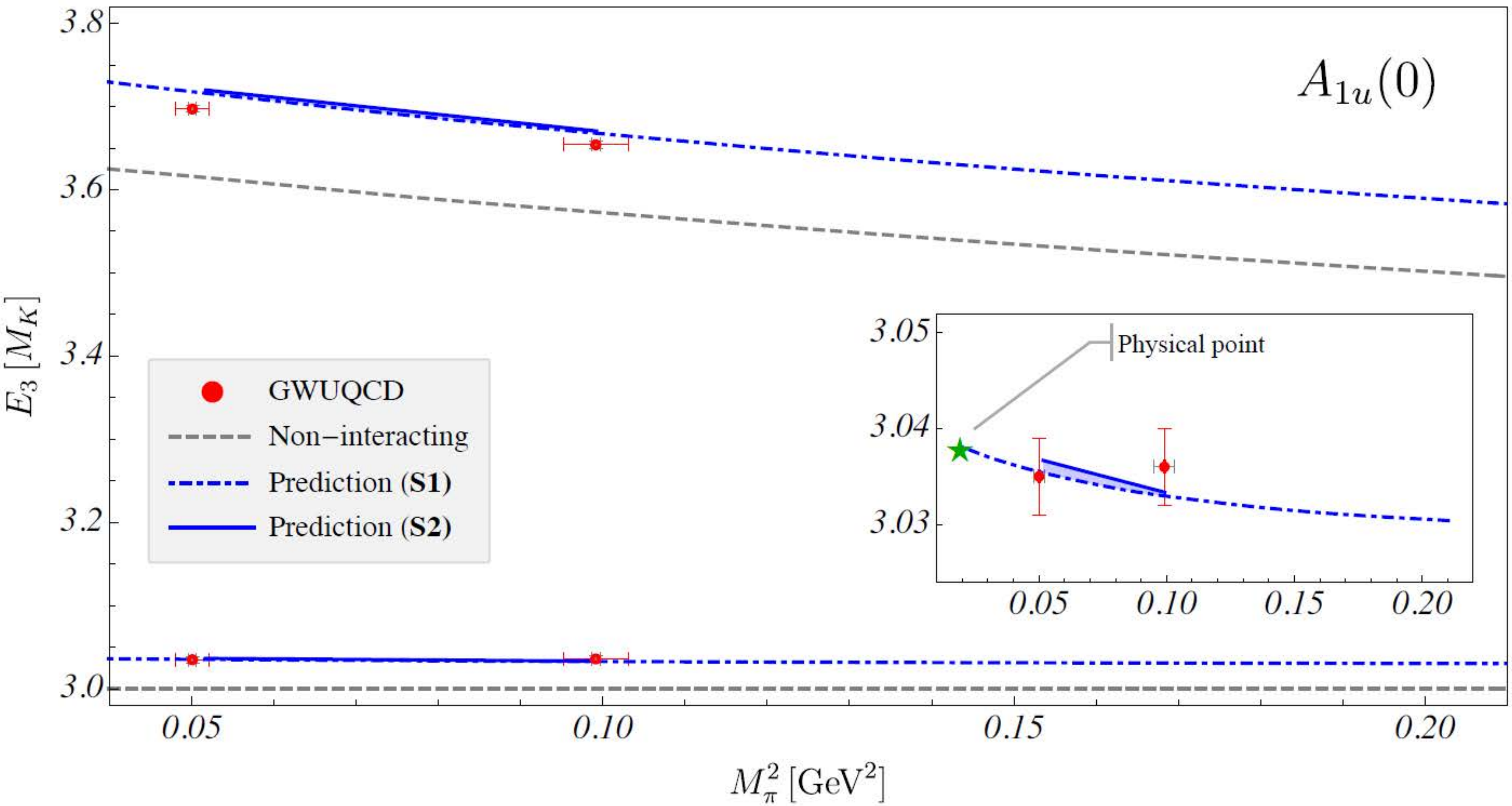
Three kaons – predictions vs lattice

- NPLQCD data – three-body term set to zero



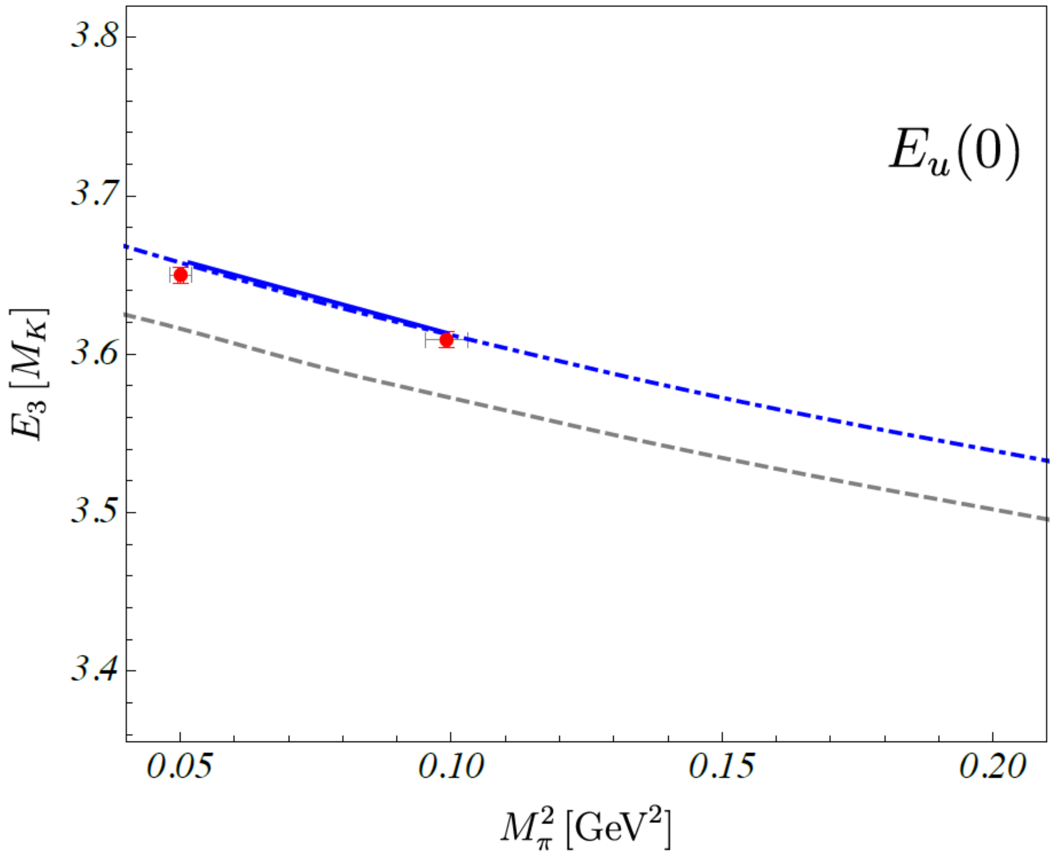
Three-kaons – predictions vs. lattice

- GWUQCD data – three-body term set to zero



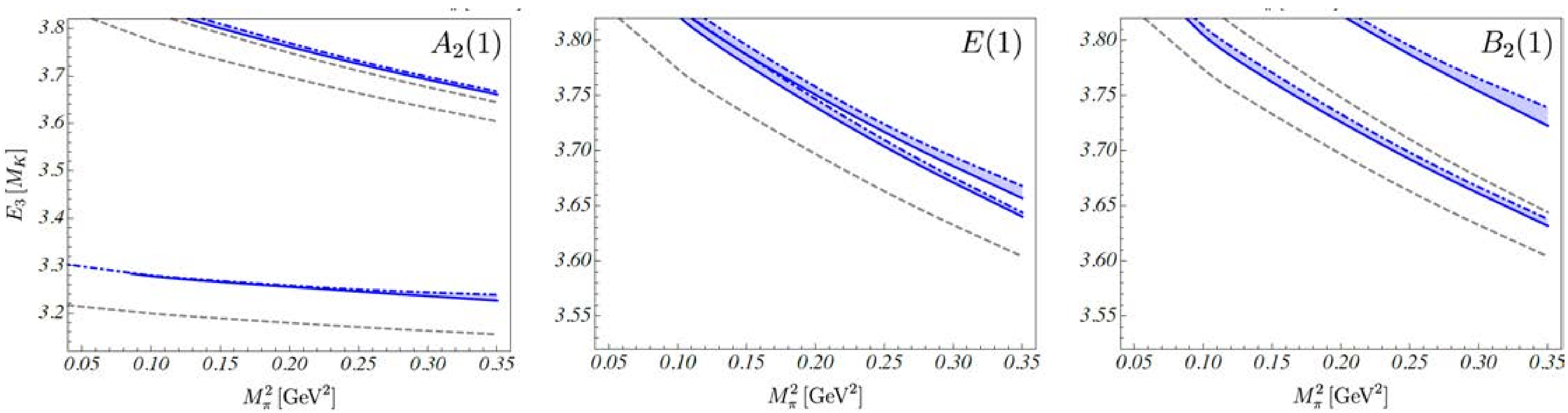
Predictions vs. lattice contd. (P=0)

- GWUQCD data – three-body term set to zero



Predictions for 3-body system with momentum P=(1,0,0)

- NPLQCD lattice setup – three-body term set to zero



- Summary:
  - First lattice QCD & chiral calculation of three kaons at maximal isospin
  - At different masses, different irreps & different total momenta
  - Relativistic 3B quantization condition for strangeness sector,
  - IAM to NLO along different trajectories predicts data from two different lattice QCD calculations well (NPLQCD and GWUQCD)

Plane-wave implementation of the C-term

- **Step 1**: JM-basis → Helicity basis
- **Step 2**: partial-wave basis → Plane-wave basis
- **Step 3**: C (and B, and 3B propagator) from plane-wave basis to irreps by suitable rotations

$$\mathcal{A}_{\lambda'\lambda}(s,\boldsymbol{p}',\boldsymbol{p}) = \sum_{M=-J}^{J} \frac{2J+1}{4\pi}\,\mathfrak{D}^{J*}_{M\lambda'}(\phi_{\boldsymbol{p}'},\theta_{\boldsymbol{p}'},0)\,\mathcal{A}^{J}_{\lambda'\lambda}(s,p',p)\,\mathfrak{D}^{J}_{M\lambda}(\phi_{\boldsymbol{p}},\theta_{\boldsymbol{p}},0)\,,$$

Step 2

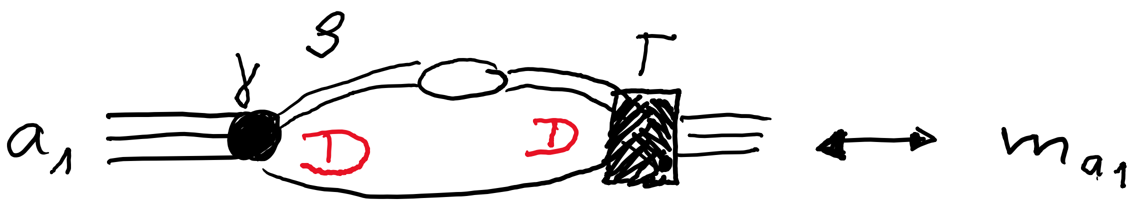
$$\mathcal{A}^{J}_{\lambda'\lambda}(s,p',p) = U_{\lambda'\ell'}\mathcal{A}_{\ell'\ell}(s,p',p)U_{\ell\lambda}\,,$$

$$U_{\ell\lambda} := \sqrt{\frac{2\ell+1}{2J+1}}(\ell 0 1\lambda|J\lambda)(1\lambda 00|1\lambda)) = \begin{pmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}, \\ \frac{1}{\sqrt{6}} & -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{6}}, \end{pmatrix},$$

Step 1

Properties of 4-parameter fit

- 4 parameters for 2 energy eigenvalues produce remarkably stable results with resonance poles in a well-defined region and not all over the place.
- The D-wave coupling and a_1-mass are strongly correlated
  - Makes sense because D-wave a_1 self energy is mostly real



$$T^{Pole} = \frac{\Gamma\,\Gamma^{\dagger}}{s - m_{a_1}^2 - \Sigma_S - \Sigma_D}$$

"Two-potential formalism"

$\Gamma =$