Introduction to Small-x Physics

(Part 3)

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HUGS 2022 *6/3/2022*

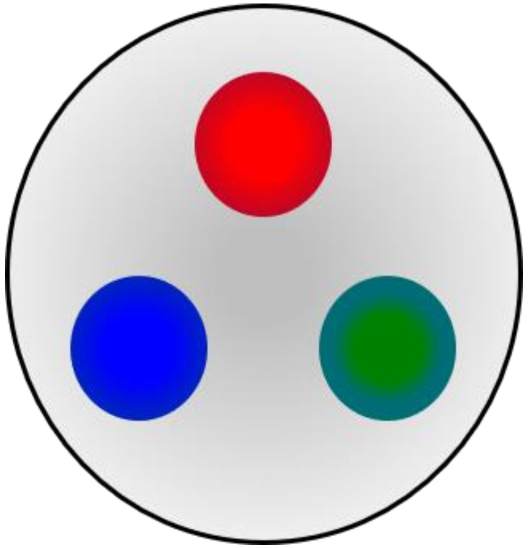
Previously...

- Light-front coordinates
- $2 \to 3$ radiative process $q + q \to q + q + g$
- Initial-state radiation vs. final-state radiation
  - ➢ Identical kinematic dependence
  - ➢ Amplitudes differ only by a sign and by the order of color matrices
  - ➢ Sum gives 0 for an Abelian theory, commutator for non-Abelian
- Non-Abelian theories like QCD produce a spectrum of soft radiation which is uniform in rapidity in the mid-rapidity region
  - ➢ Competition of $\alpha_s \ll 1$ and $\ln \frac{s}{m^2} \gg 1$ requires leading-log resummation

Deep Inelastic Scattering:

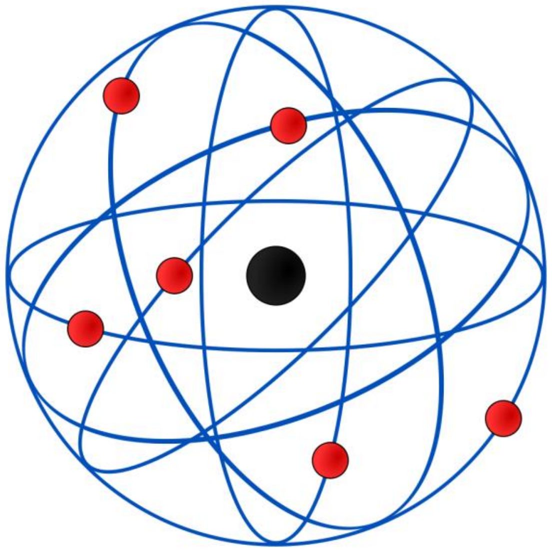
Accessing Proton Structure

The Proton



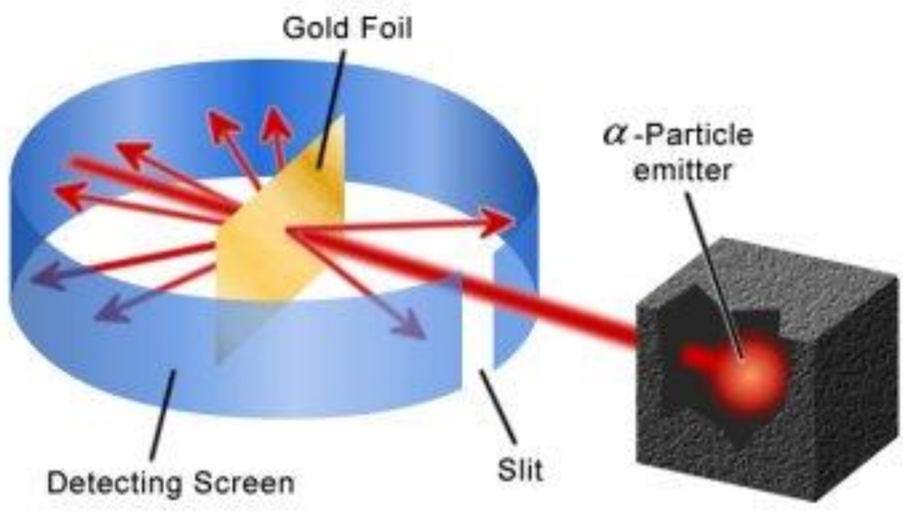
Bound together by
the **strong nuclear force**

The Atom



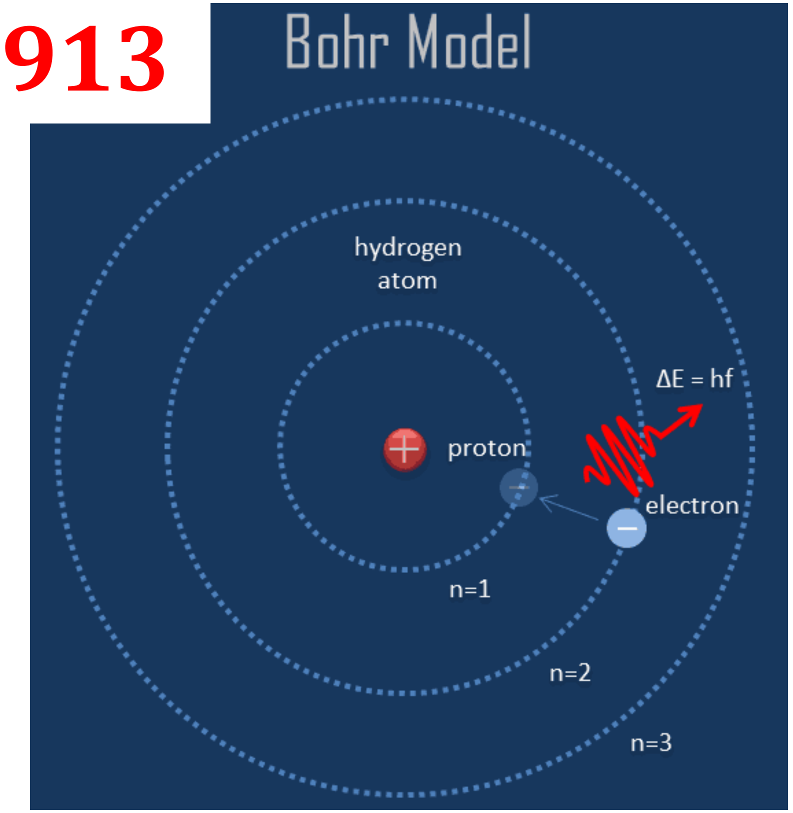
Bound together by
the **electromagnetic force**

1908 - 1917



Rutherford Gold Foil Experiment

1913



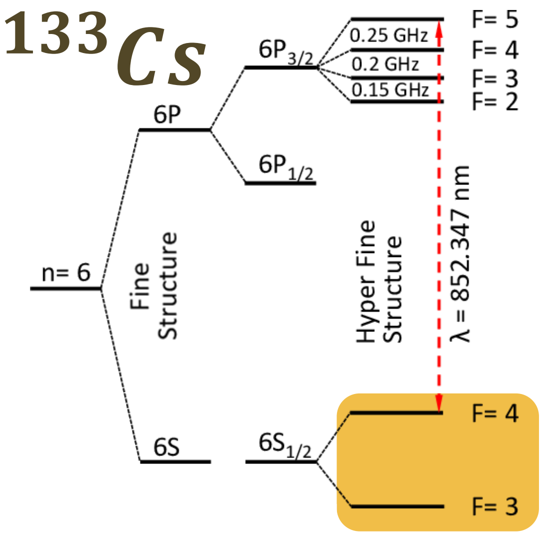
Bohr Model of the Atom

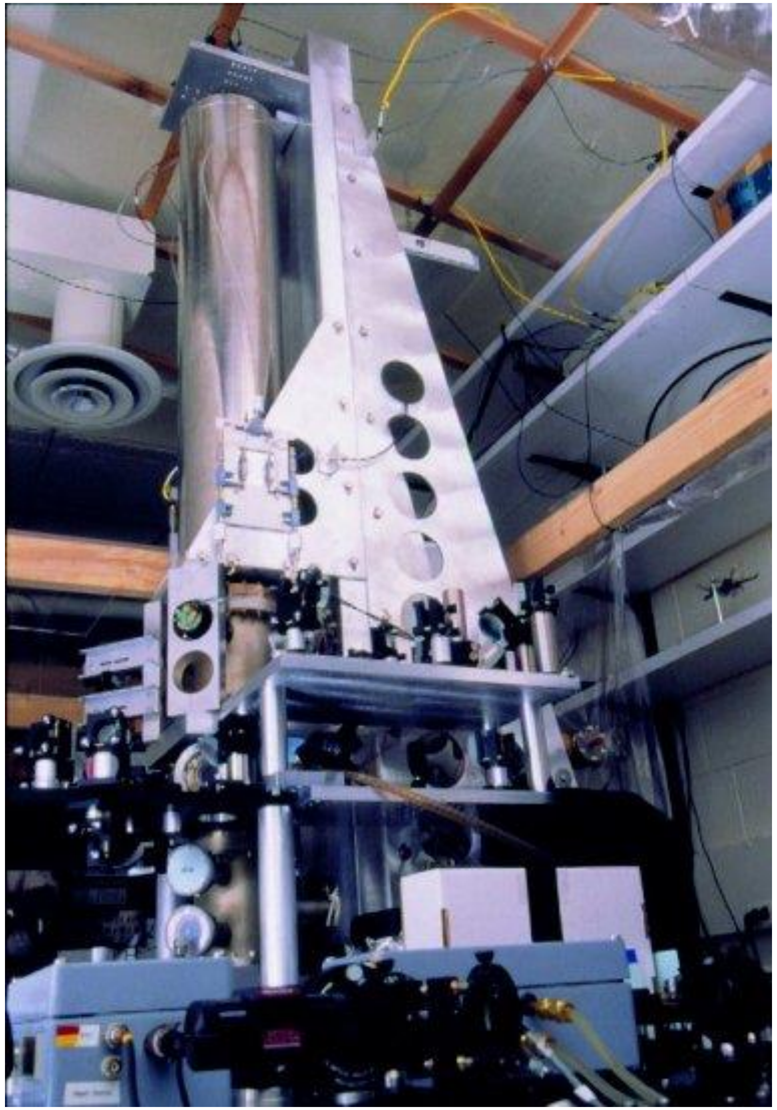
The Atom: All Grown Up

- The atom is used as the **definition of the second**

Combined Fractional Uncertainty: 0.52×10^{-15} **!!!**

Mar. 2016, NIST-F1 BIPM Report





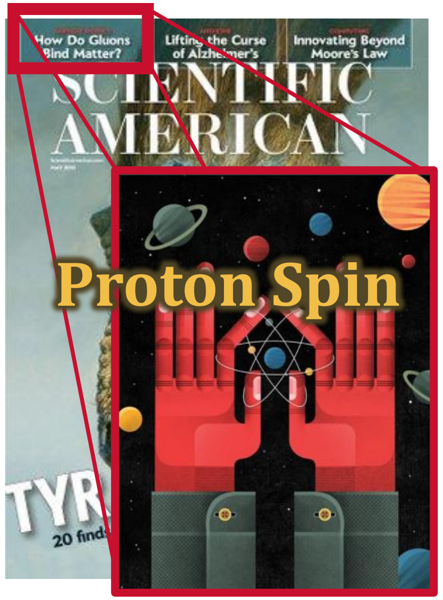
- Calculations to **extreme precision**

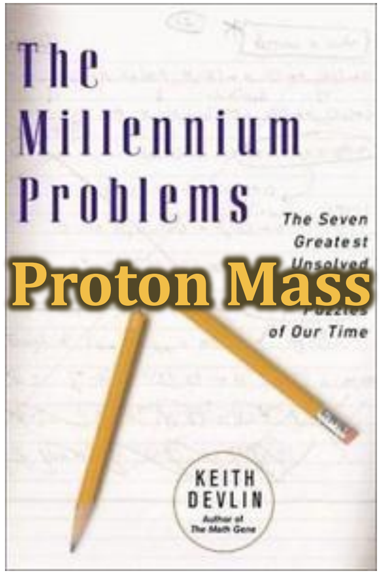
I have evaluated up to 1100 digits of precision the contribution of the 891 4-loop Feynman diagrams contributing to the electron g-2 in QED. The total mass-independent 4-loop contribution is

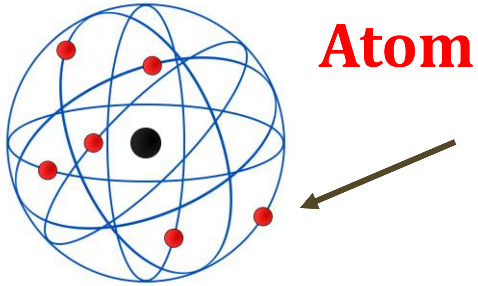
$$a_e = -1.912245764926445574152647167439830054060873390658725345\ldots\left(\frac{\alpha}{\pi}\right)^4 .$$

S. Laporta, Phys. Lett. ***B772*** *(2017) 232*

- Even **fundamental questions** about the **structure of the proton** remain **unanswered**



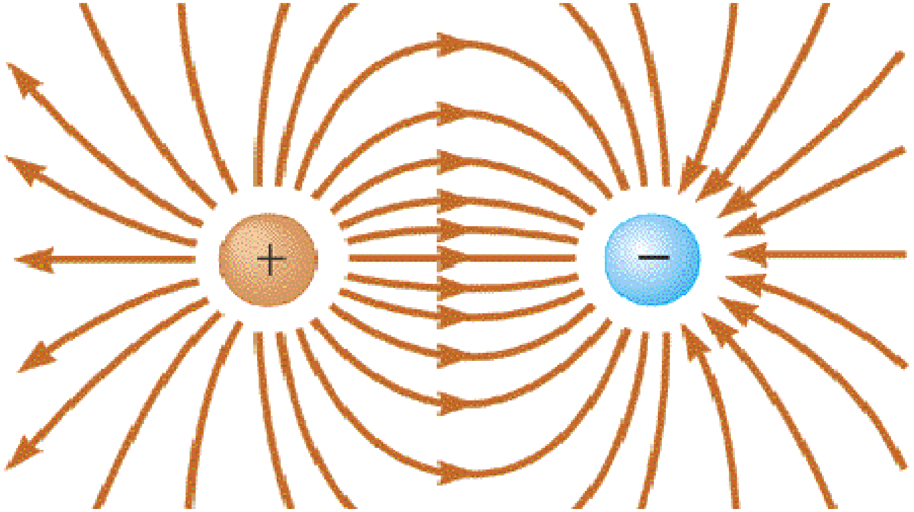


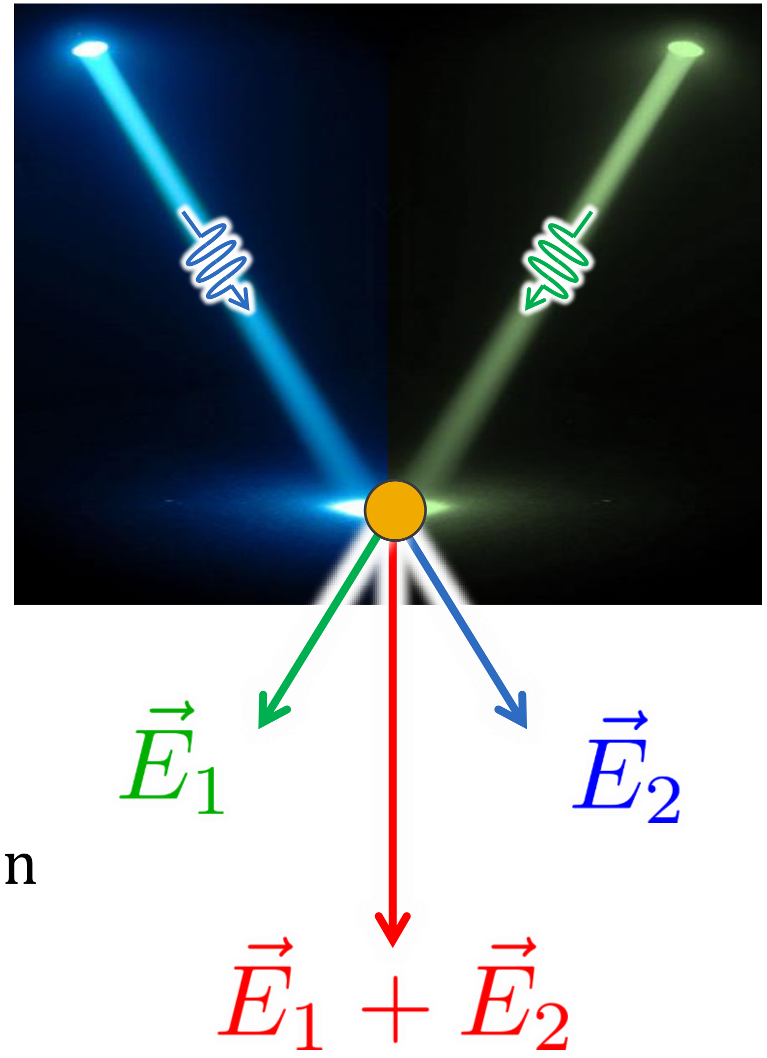


Atom: Electrodynamics

electrons

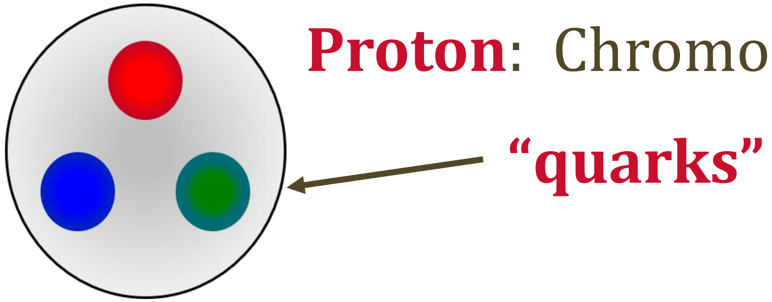
- **Charges** (electrons) radiate **fields** (photons)
- Electric charge is a **scalar** (+/−)



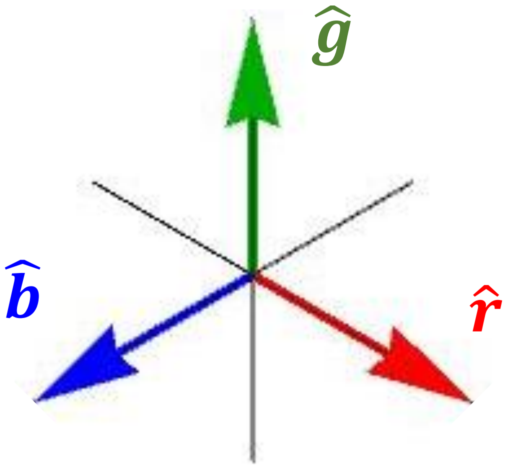


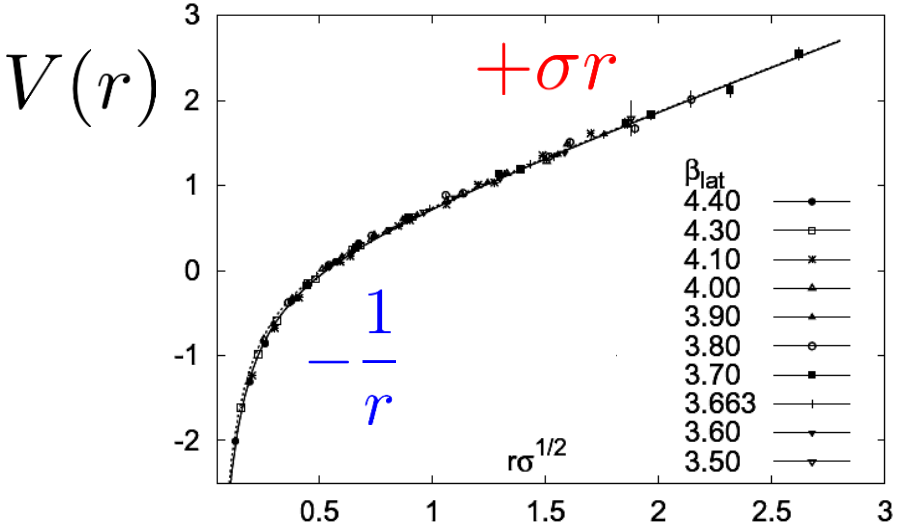
Linear:
Superposition Principle

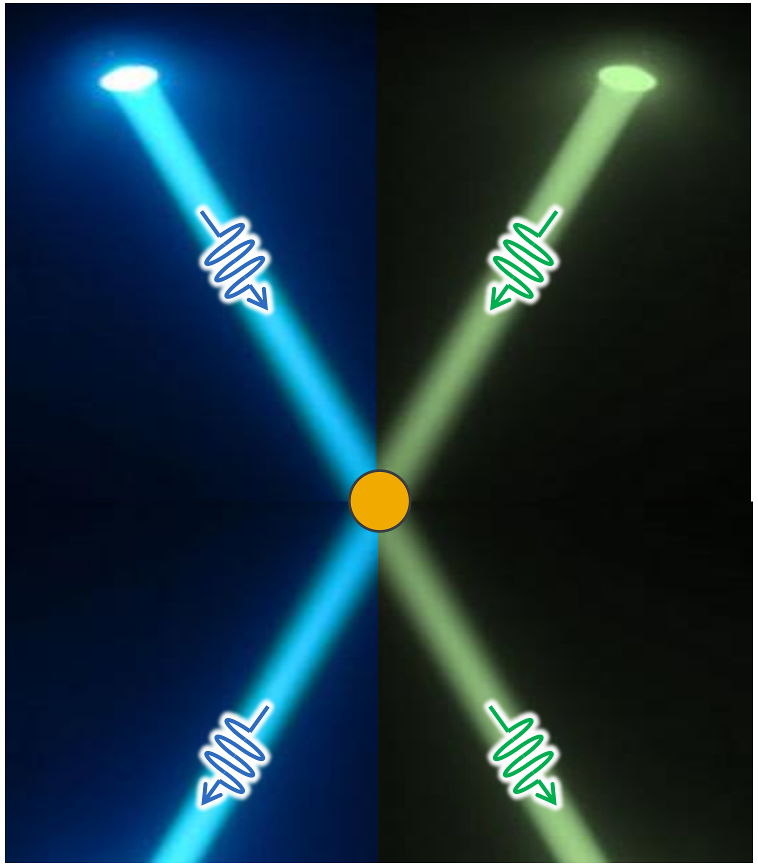
Chromo-Dynamics: One Crucial Difference



- **Charges** (quarks) radiate **fields** (gluons)
- Color charge is a **vector**



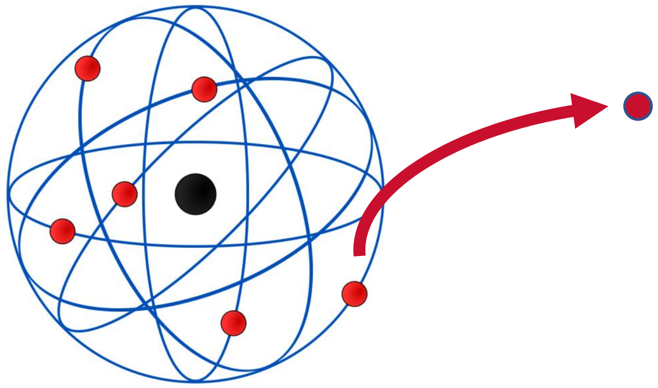


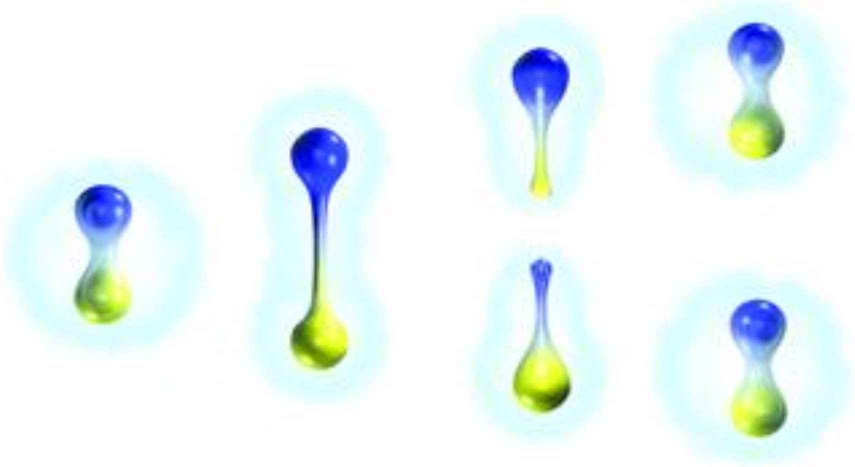


Non-Linear:
Self-interactions of fields

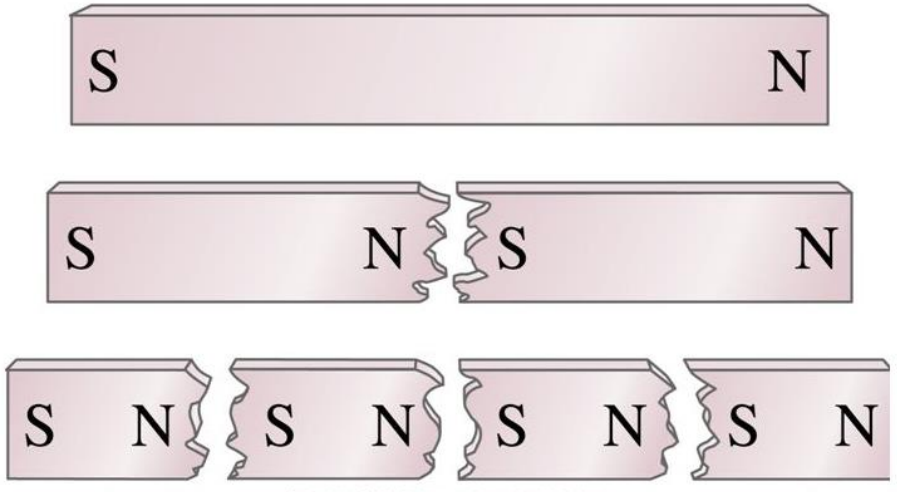
Dissecting the Proton is Hard…

- You **can't** just **"ionize" a quark** out of the proton
  - ➢ If you **hit it hard enough** to knock out a quark…
  - ➢ You create a **shower of new particles** instead!



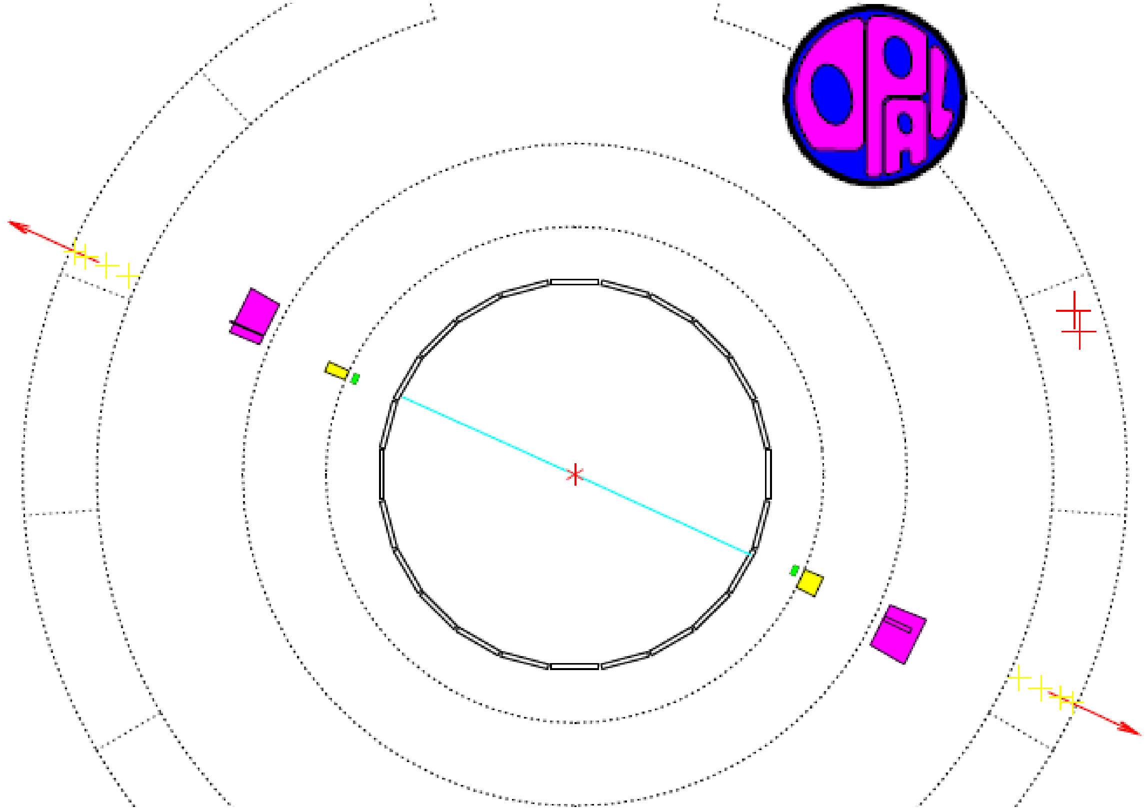


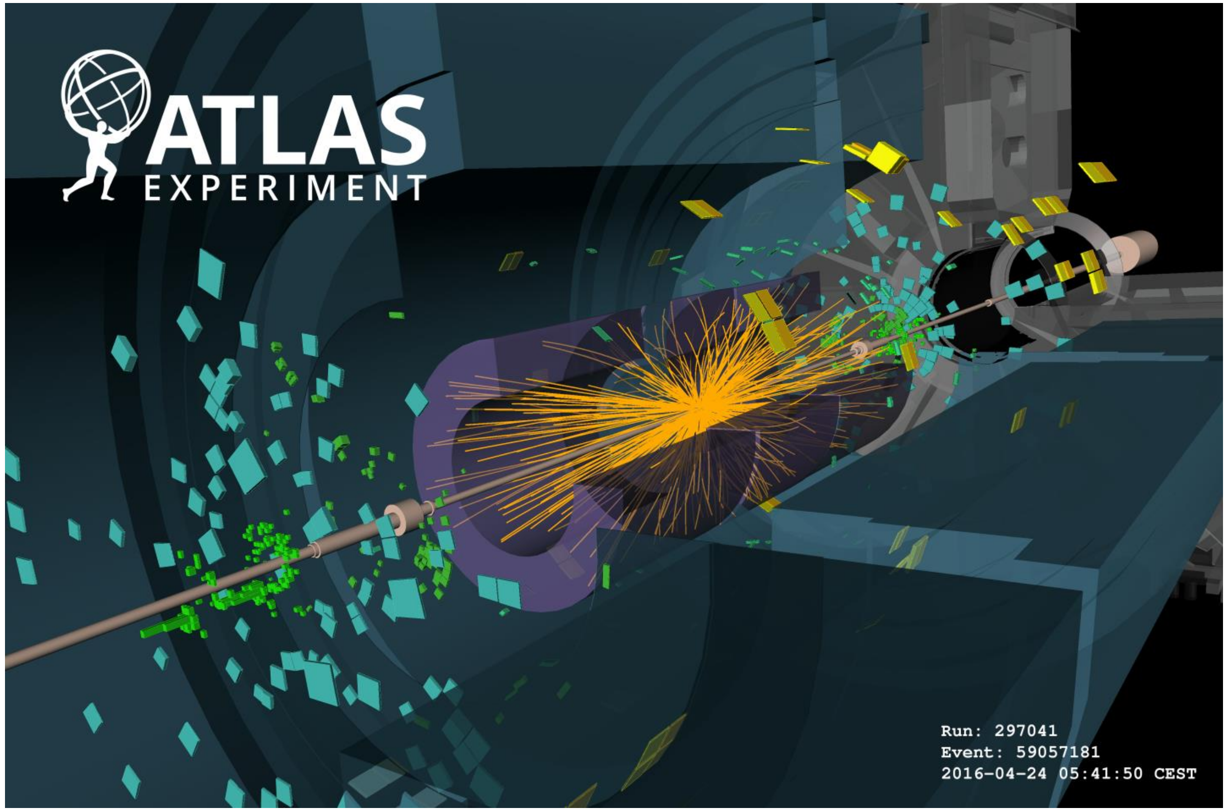
No Isolated Quarks



No Magnetic Monopoles

…and Messy!





E&M: Very **clean**

QCD: **TONS** of radiation

- An **optical microscope** is limited by the **wavelength of visible light**:

$$E \sim 2\ eV \qquad \Delta x \geq 100\ nm$$

- A **scanning electron microscope** uses a thermally produced **beam of electrons**:

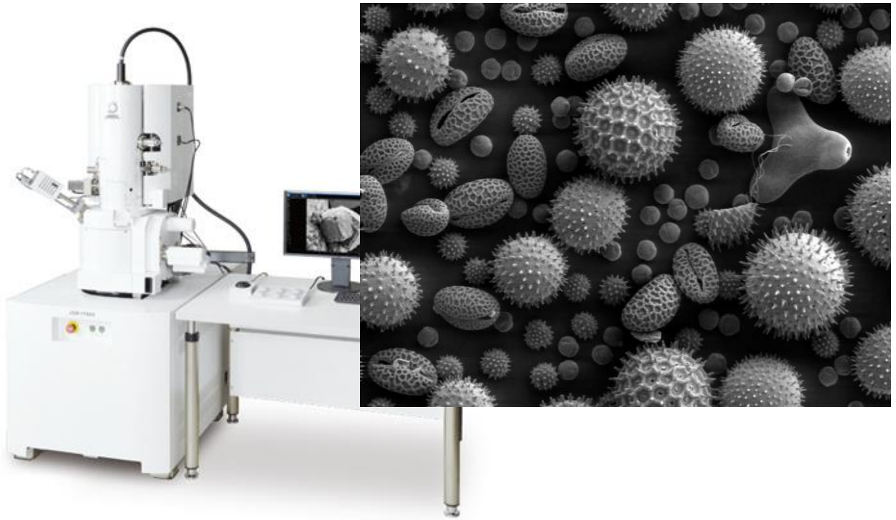
$$E \sim 10\ keV \qquad \Delta x \geq 1\ nm$$

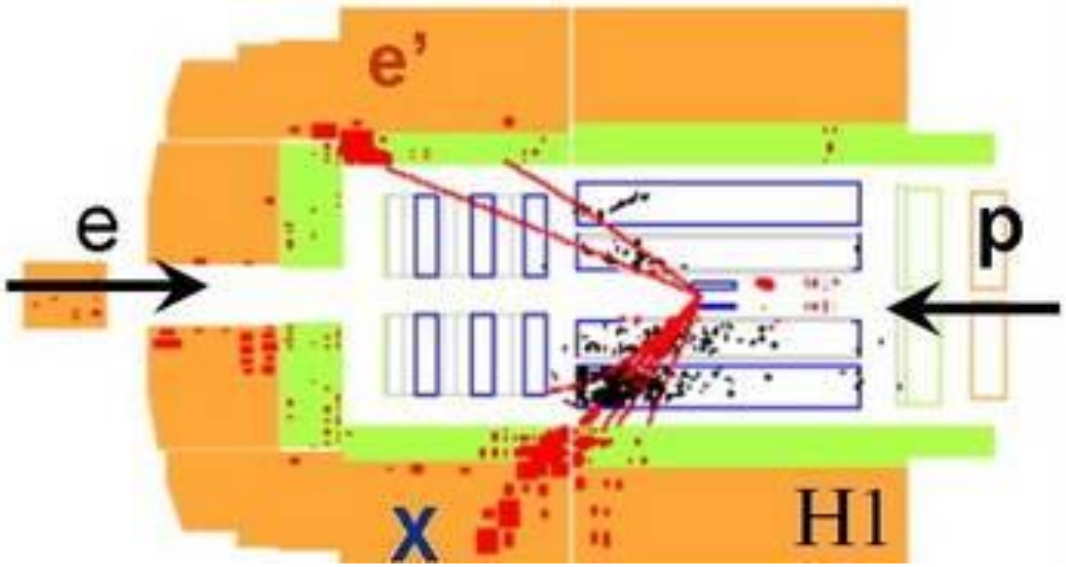
- An **electron-proton collider** can use the same principles at **top collider energy**.

$$e + p \rightarrow e' + \boldsymbol{X}$$

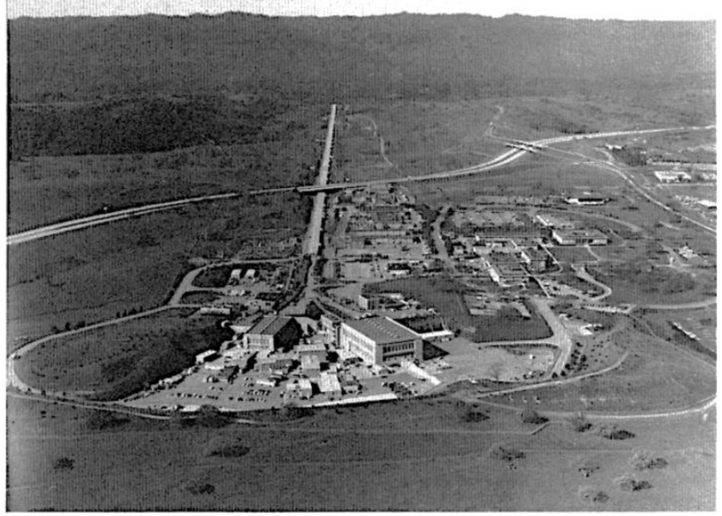
$$E \gg 1\ GeV \qquad \Delta x \ll 1\ fm$$

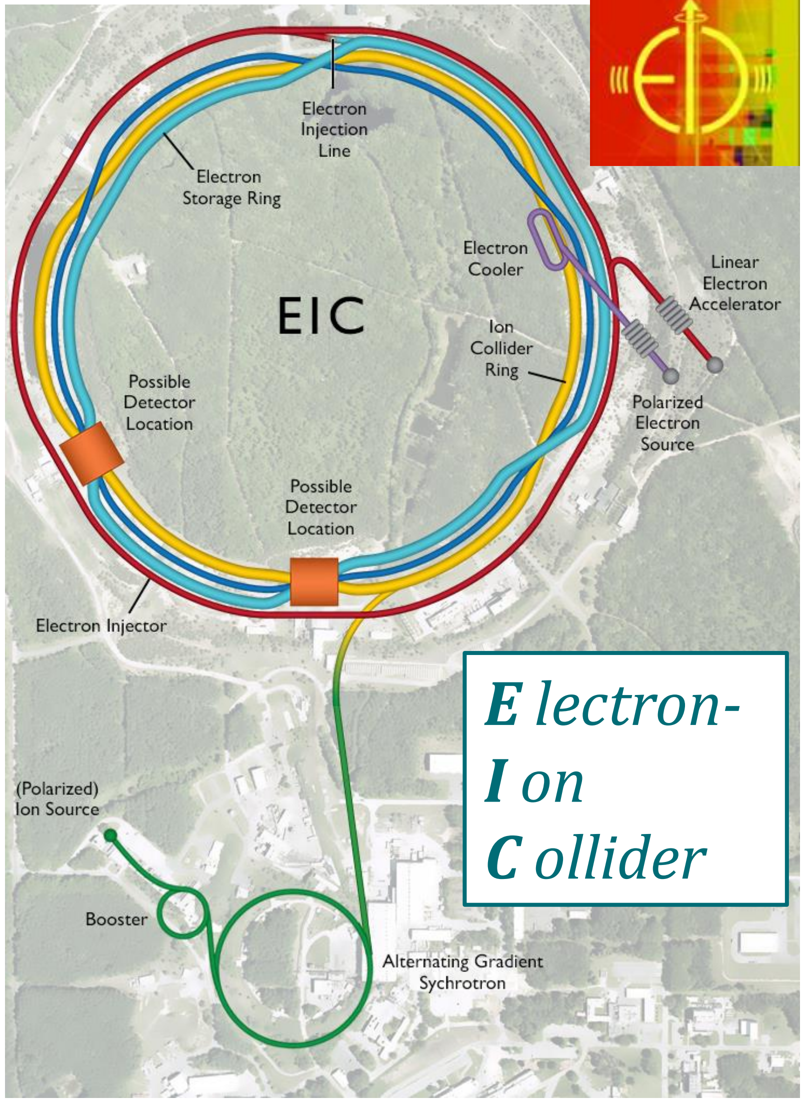
D *eep*
I *nelastic*
S *cattering*





Evolution of the DIS Femtoscope





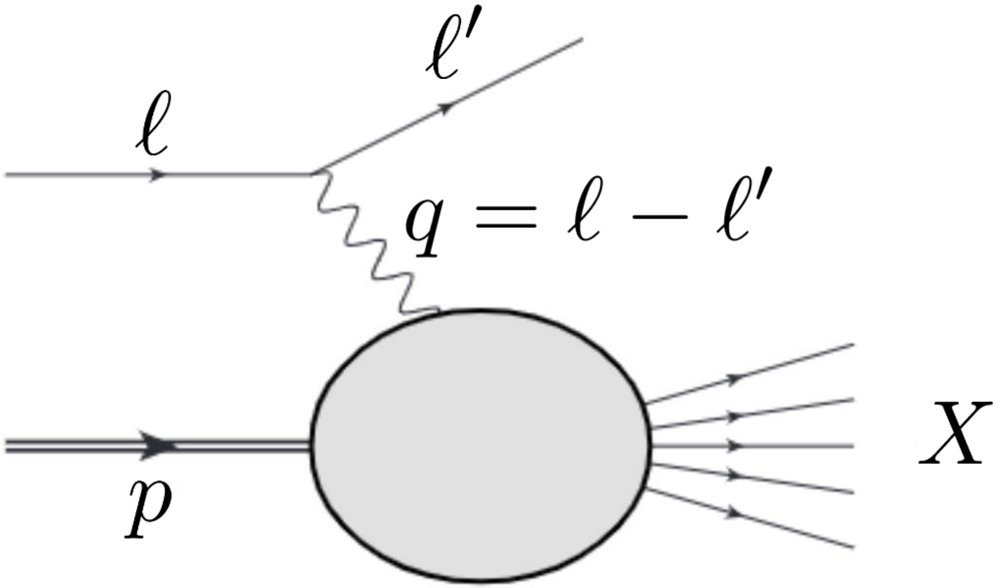
***E** lectron-*
***I** on*
***C** ollider*

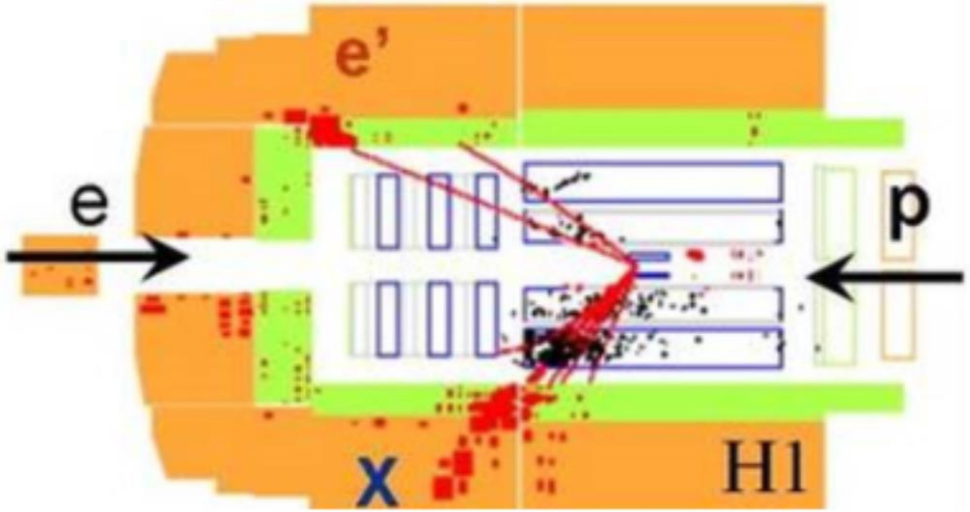
• 1968:	**SLAC-MIT Experiment**	Fixed target $E_{beam} \sim 20\ GeV$
• 1992 – 2007:	**HERA**	$E_{CM} = 318\ GeV$
• 2030+	**Electron-Ion Collider**	$E_{CM} = 140\ GeV$ + polarization + ions

Theory of DIS

(What does “x” mean, anyway?)

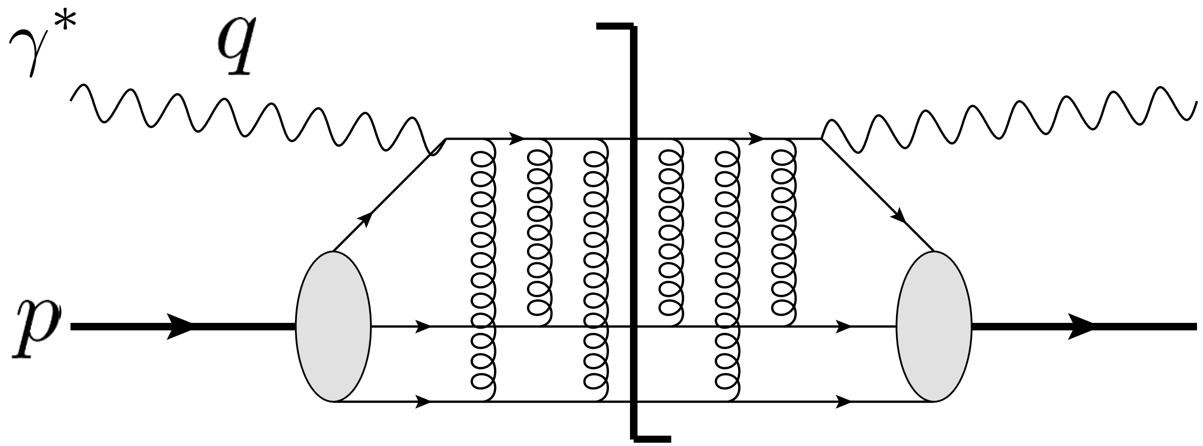
Using QED to Measure QCD





- **Inclusive** electron/proton scattering: $e + p \to e' + \boldsymbol{X}$
  - Electron acts as a **source of virtual photons** via well-controlled **QED vertex**
  - Couples to **electromagnetic currents** $\sim A_\mu \langle X | j^\mu | p \rangle$ **inside the proton**

Field-Theoretic Description of DIS

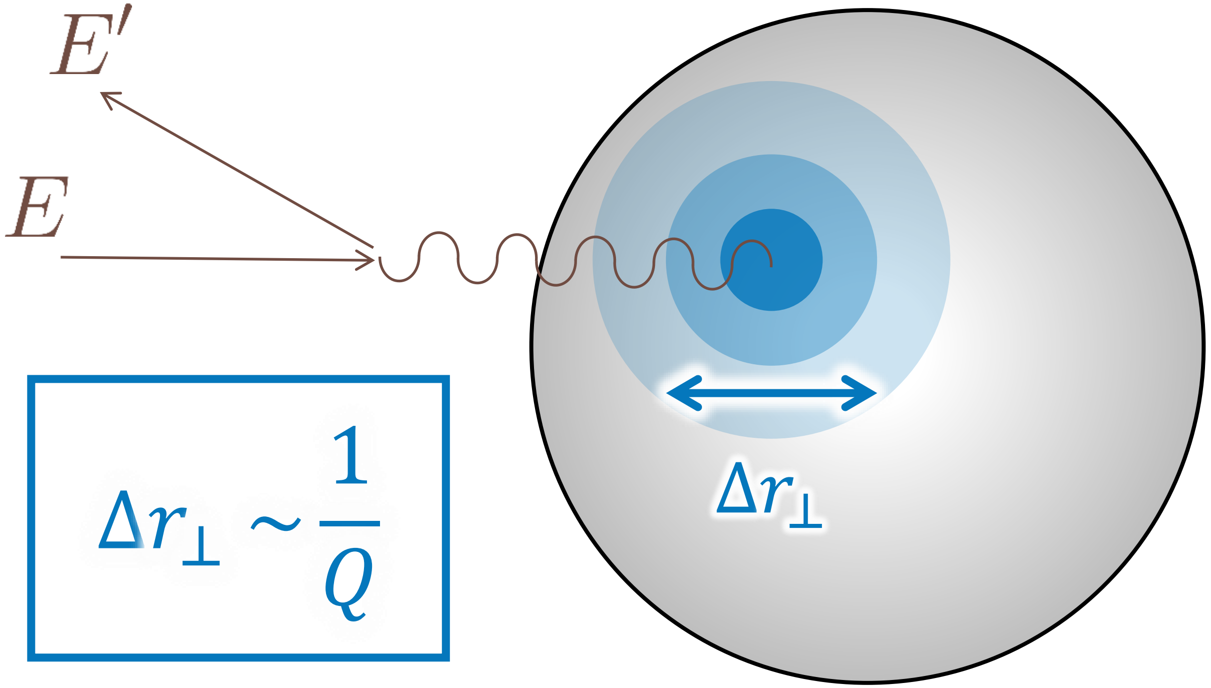


$$E'_\ell \frac{d\sigma}{d^3\ell'} = \frac{\alpha_{EM}^2}{E_\ell Q^4} L_{\mu\nu} W^{\mu\nu}$$

- Trivial **QED part** can be calculated and **removed**: $L_{\mu\nu} = \frac{1}{2} \sum_{ss'} [\bar{u}_{p's'} \gamma_\mu u_{ps}][\bar{u}_{ps} \gamma_\nu u_{p's'}]$
- Nontrivial QCD info from $\boldsymbol{\gamma^* p \to X}$ **subprocess** : $W^{\mu\nu} = \frac{1}{4\pi m} \int d^4x \, e^{iq\cdot x} \langle p | \, j^\mu(x) j^\nu(0) \, | p \rangle$
  - **Tensor-valued** function of **two momenta**: p^μ, q^μ
  - **Decompose** $W^{\mu\nu}$ into invariant **structure functions** depending on **invariants**

Image Resolution:

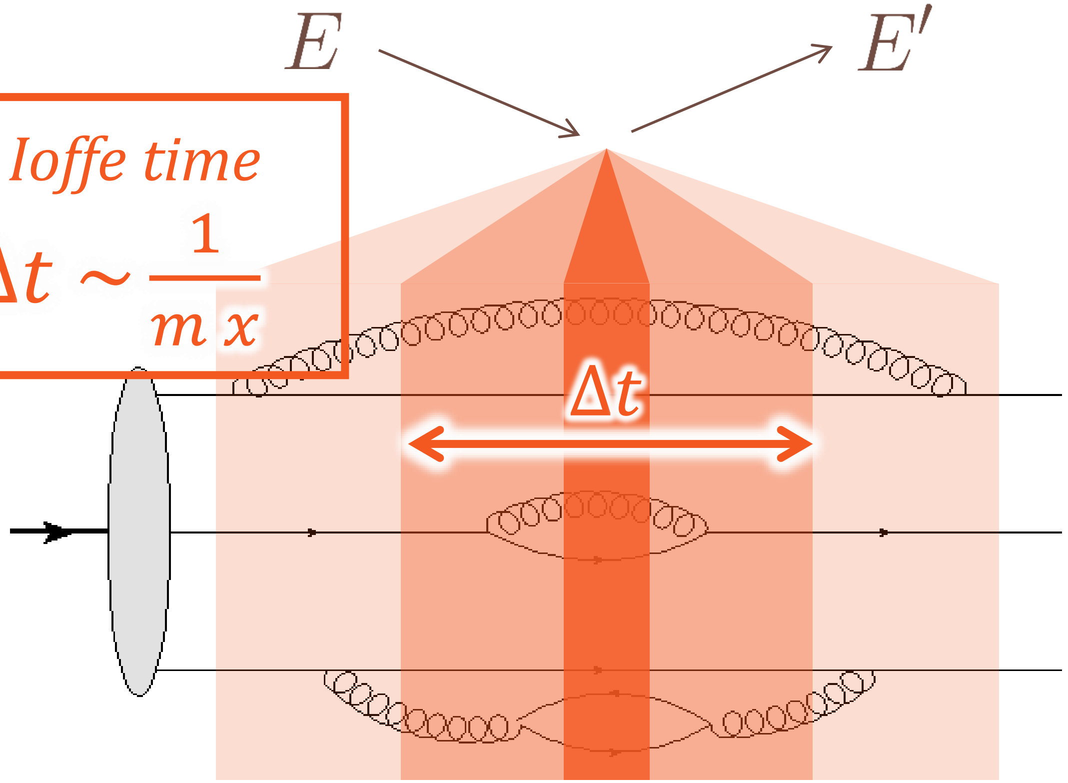
$$Q^2 \equiv -q_\mu q^\mu = 4EE' \sin^2\frac{\theta}{2}$$



$$\Delta r_\perp \sim \frac{1}{Q}$$

Exposure Time:

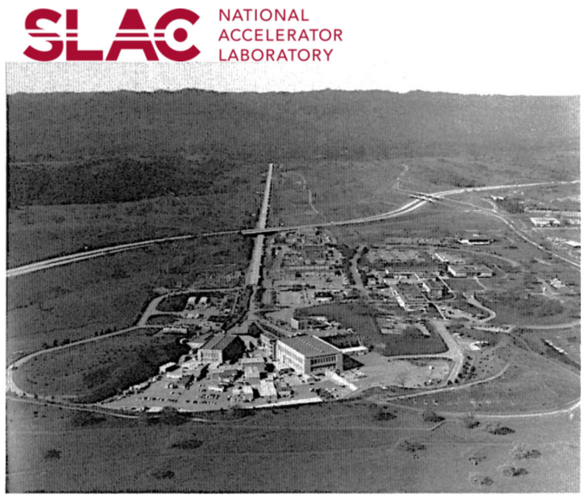
$$x_B \equiv \frac{Q^2}{2p\cdot q} \overset{R.F}{=} \frac{Q^2}{2m(E-E')}$$



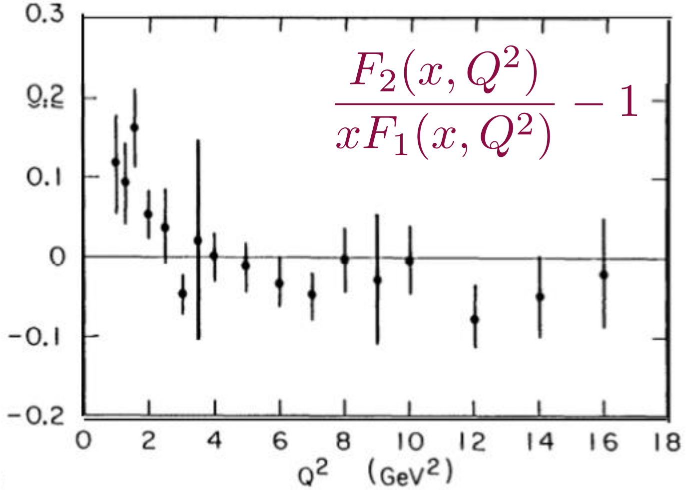
Ioffe time

$$\Delta t \sim \frac{1}{m\,x}$$

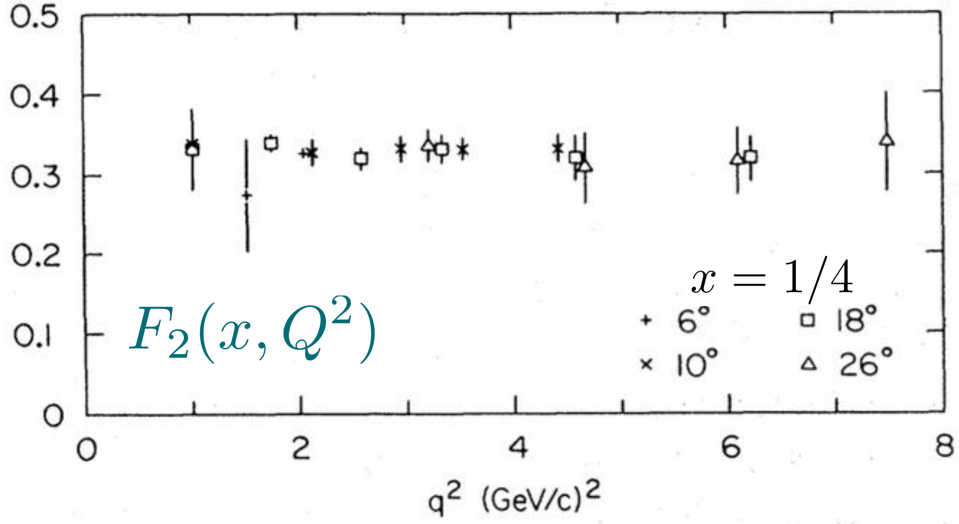
Quarks are For Real



Callan-Gross Relation



Bjorken Scaling

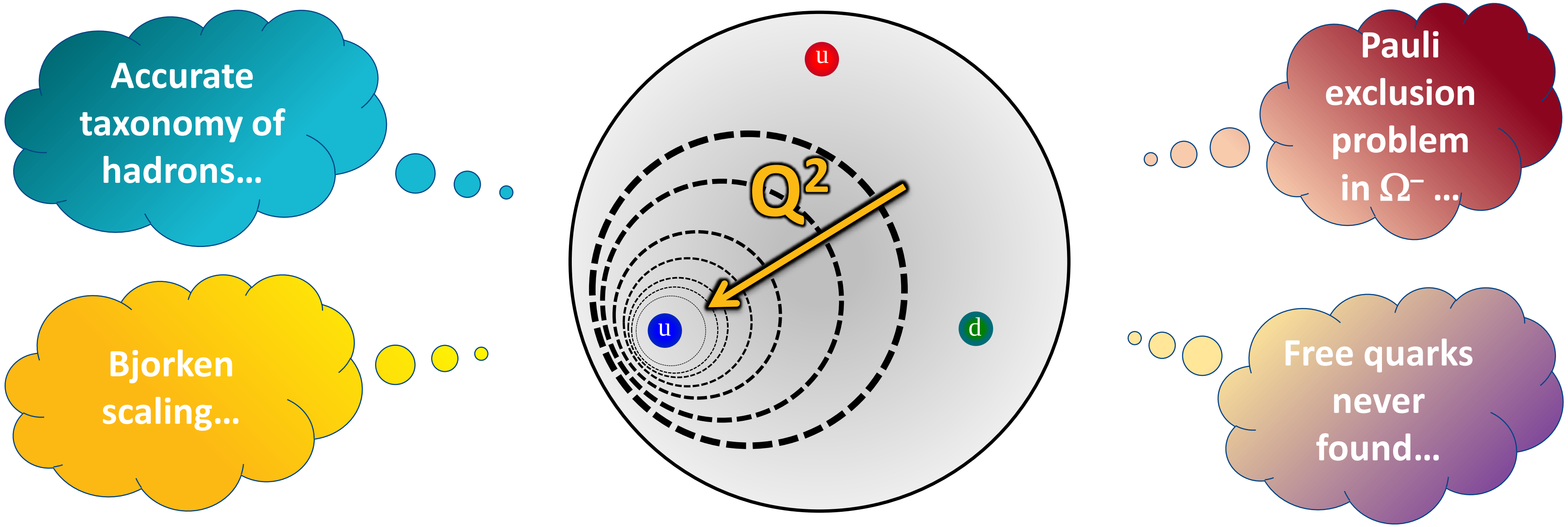


- Proton described by two **structure functions:**

$$\frac{d\sigma}{d\Omega\, dE'} = \sigma_{point} \left[\frac{2mx}{Q^2} F_2(x,Q^2) + \frac{1}{m} F_1(x,Q^2) \tan^2 \frac{\theta}{2} \right]$$

  - Proportional to scattering on **point-like particles**
  - F_1 and F_2 are not independent: **Spin ½ fermions (Quarks!!)**
  - F_1 and F_2 are independent of Q^2: **Point-like at any resolution (???)**

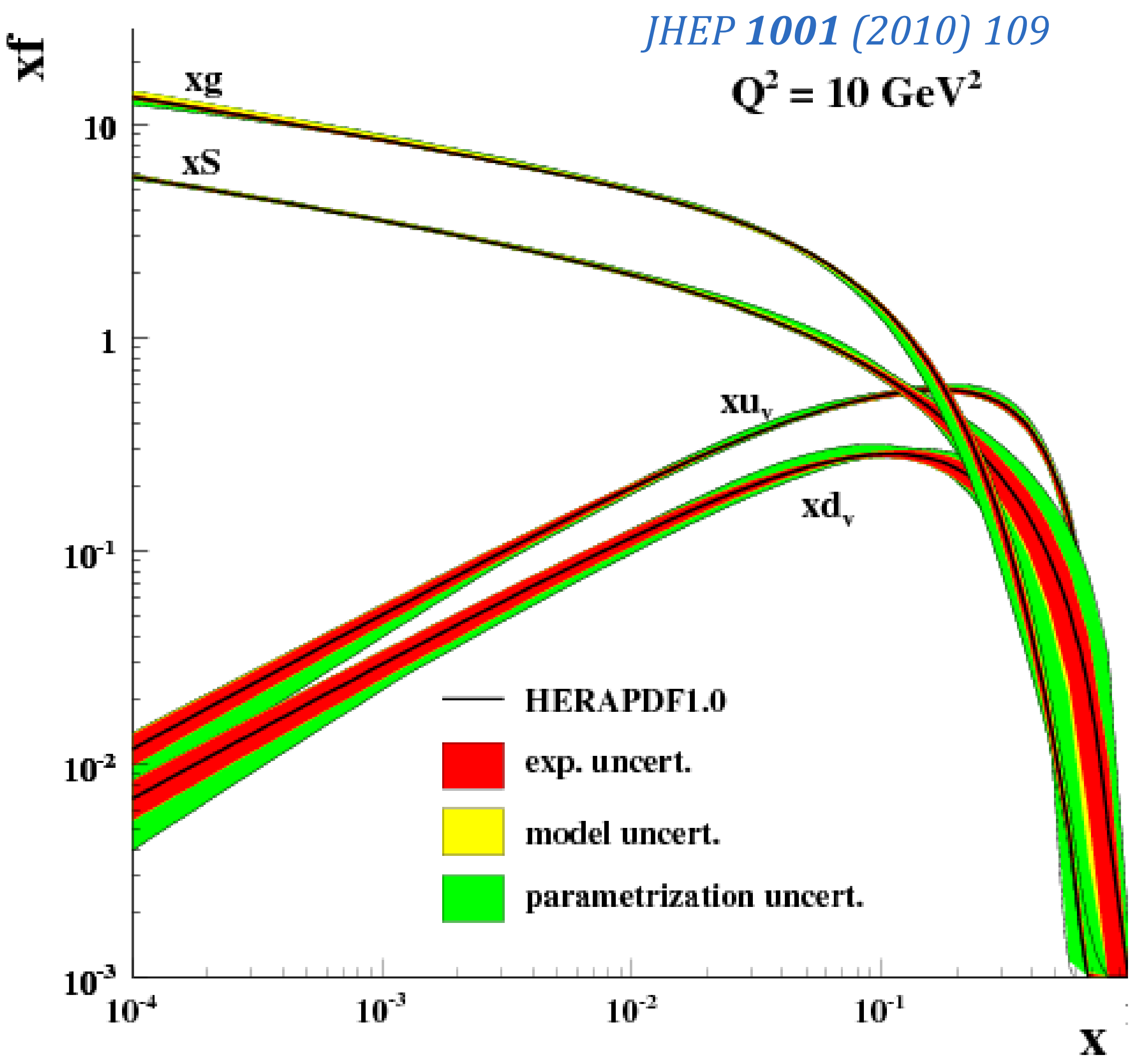
The Missing Link: Asymptotic Freedom



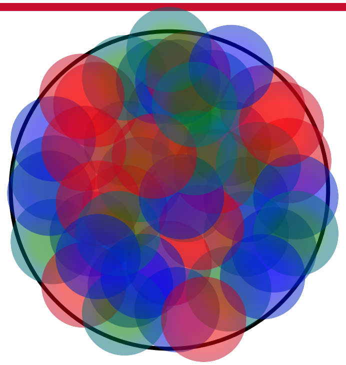
- **Bjorken scaling: No change** in proton structure with **increased resolution**

- **Uncertainty principle:** the **largest** quantum fluctuations happen over **short times and distances**…

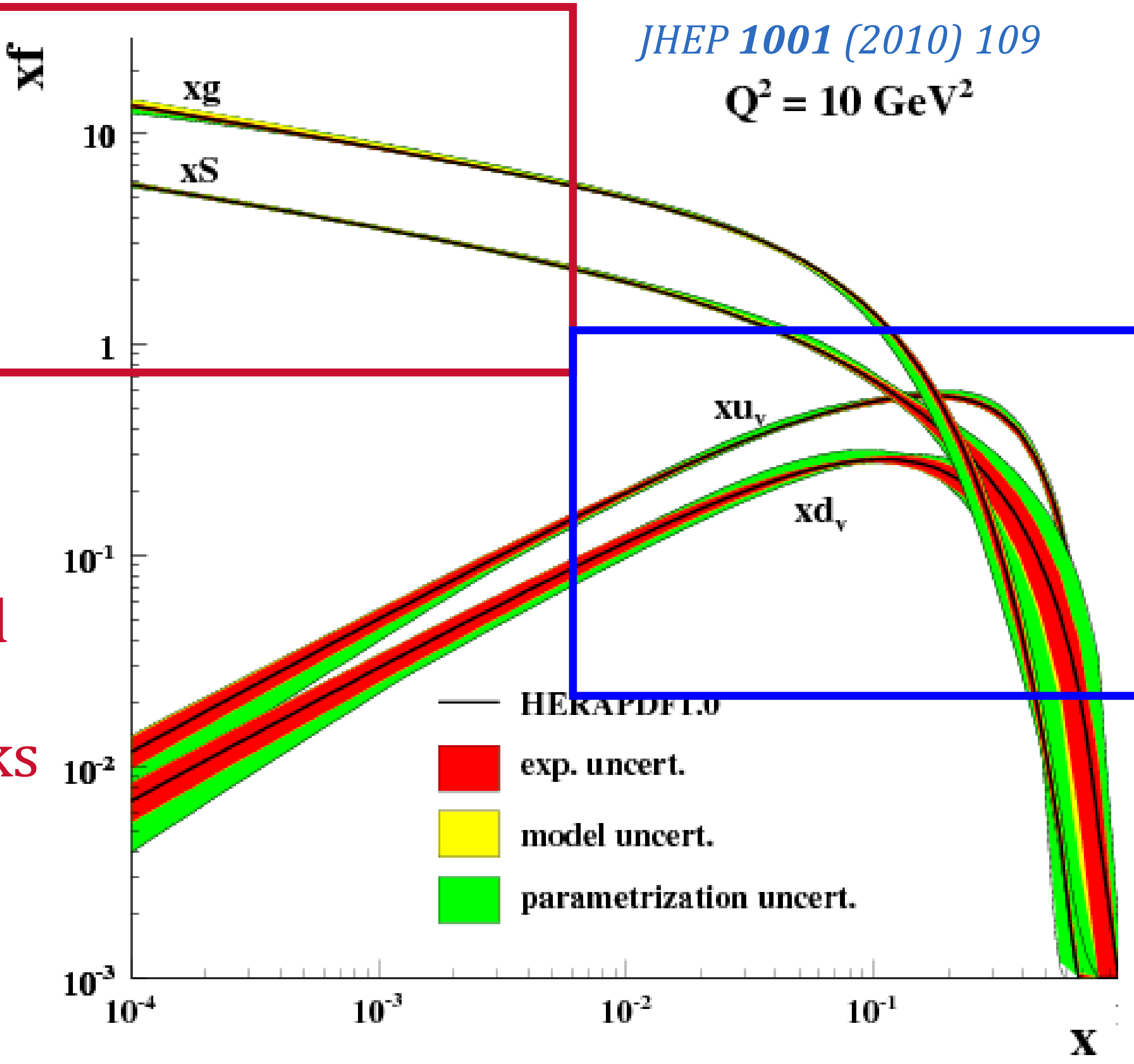
  - ➢ … Unless the **interactions** of the nuclear force **go to zero** at short distances!
  - ❖ Bjorken scaling demands **asymptotic freedom** – and QCD

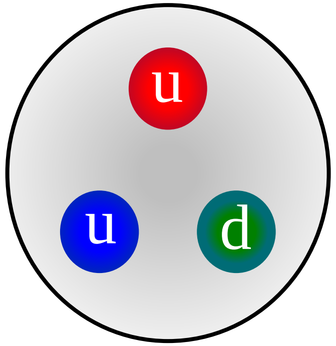

JHEP 1001 (2010) 109
Q^2 = 10 GeV^2
xf
xg
xS
xu_v
xd_v
10
1
10^{-1}
10^{-2}
10^{-3}
10^{-4}
10^{-3}
10^{-2}
10^{-1}
x
HERAPDF1.0
exp. uncert.
model uncert.
parametrization uncert.

Two Very Different Regimes of Proton Structure

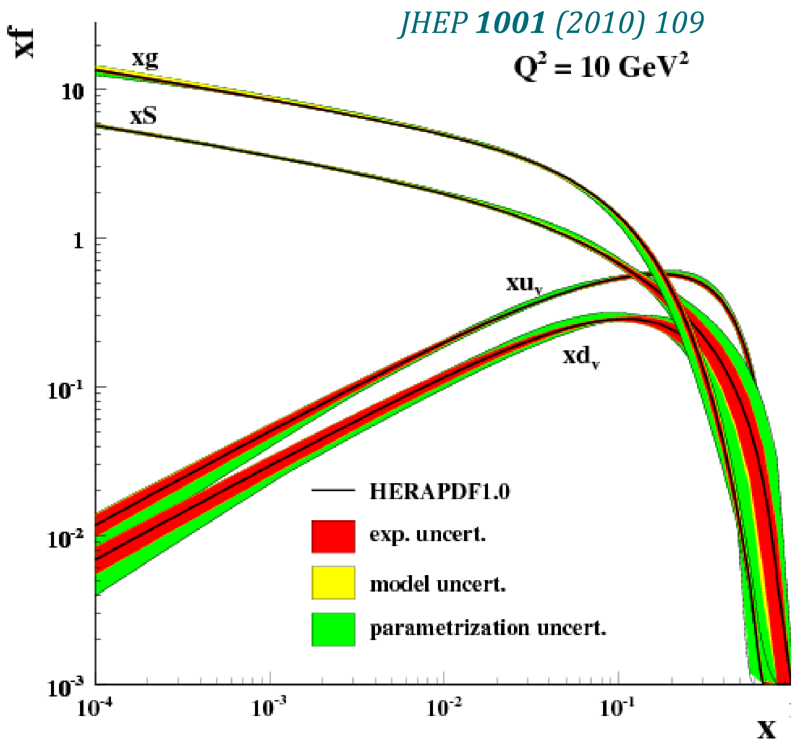


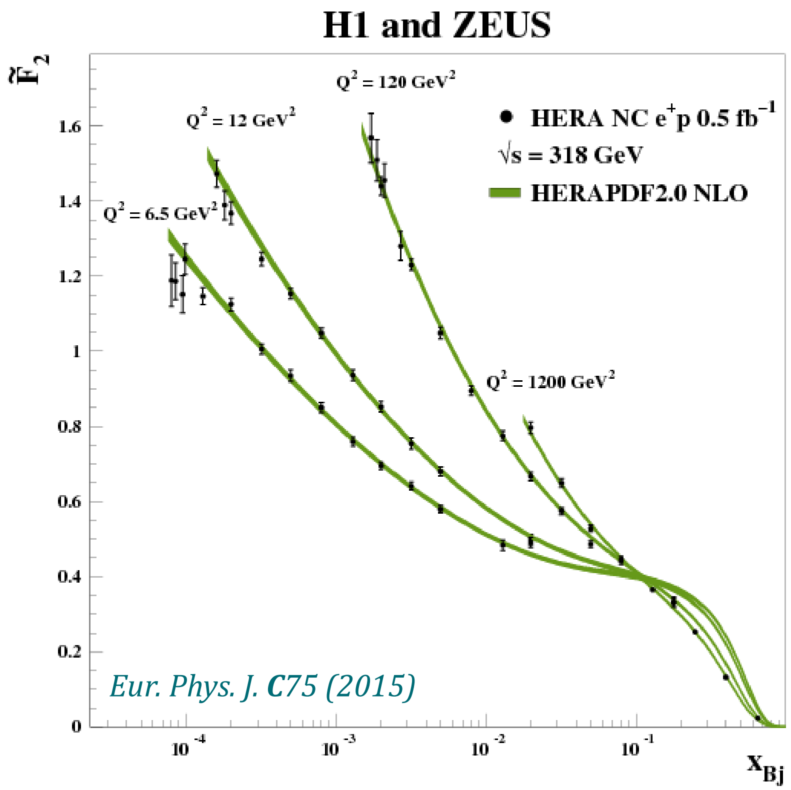
- Small $x \ll 1$
- Radiation-dominated
- Gluons and sea quarks

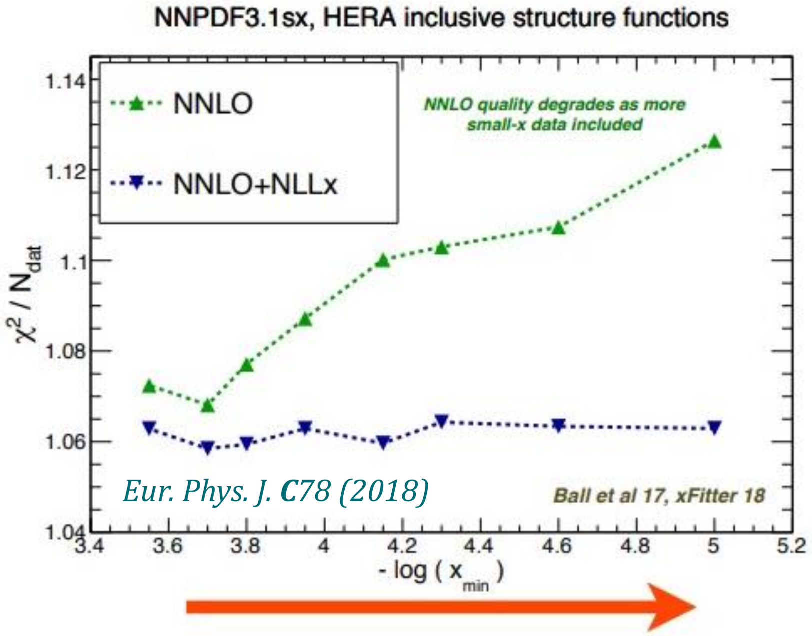




- Large $x \sim O(1)$
- Valence region







- The **dense proton** at **small x** reflects an **explosion of soft gluon** bremsstrahlung:

Systematic tension in the HERA fits at **small x**…

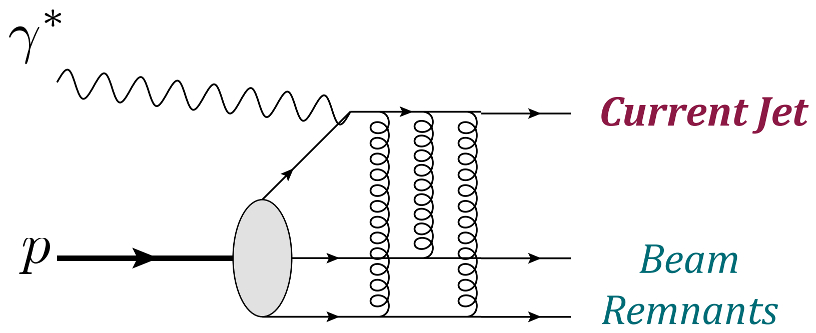
…reflect the **onset of new degrees of freedom**

DIS at Small x

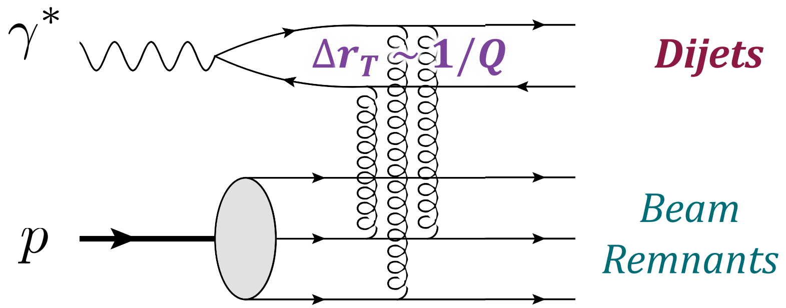
The Intersection of Structure and Hadronic Scattering

Onset of Small-x Degrees of Freedom in DIS

"Knockout" DIS at Large x:



"Dipole" DIS at Small x:

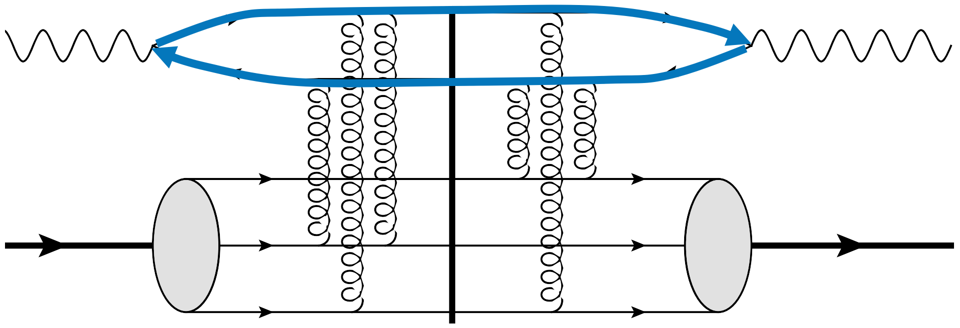


- At **small x**, DIS resembles a **hadronic scattering** process

("baby" pp collision)

  - γ^* fluctuates into a **$q\bar{q}$ dipole** with **variable size**

$$\langle \mathrm{tr}\left[V_{x_\perp} V^\dagger_{y_\perp}\right]\rangle$$

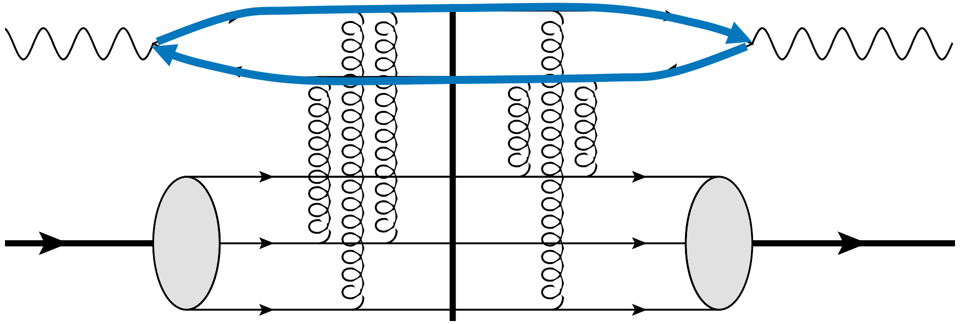


$$V_{x_\perp} = \mathcal{P}\exp\left[ig\int_{-\infty}^{\infty} dz^-\ A^{+a}(0^+, z^-, \vec{x}_\perp)t^a\right]$$

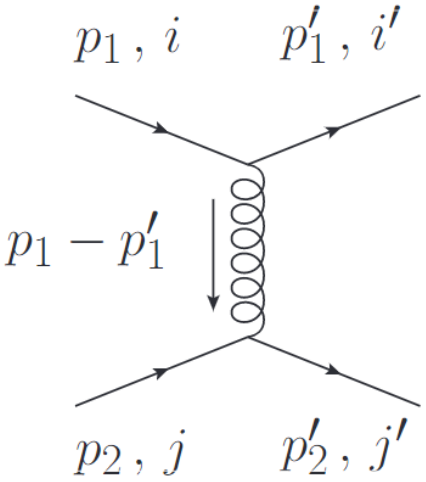
- New degrees of freedom: **Wilson lines**
  - **Boosted, coherent** scattering in a **background field**

Look Familiar?

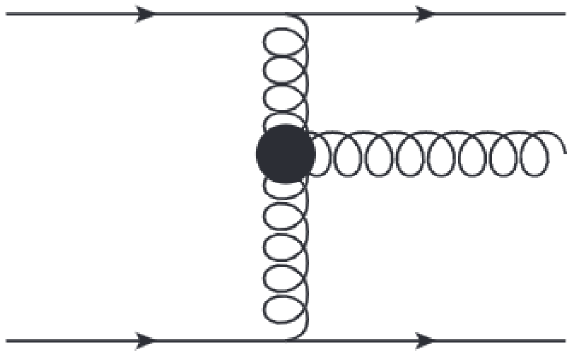
$$\left\langle \mathrm{tr}\left[V_{x_\perp} V^\dagger_{y_\perp}\right]\right\rangle$$



$$V_{x_\perp} = \mathcal{P}\exp\left[ig\int_{-\infty}^{\infty} dz^- \, A^{+a}(0^+, z^-, \vec{x}_\perp)t^a\right]$$



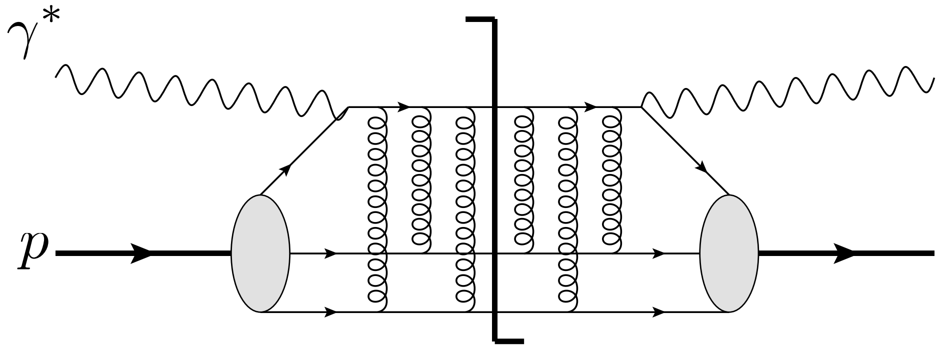
- ❖ Elastic q/q scattering (LO)
  - ➢ Unsuppressed
  - ➢ Wilson line!



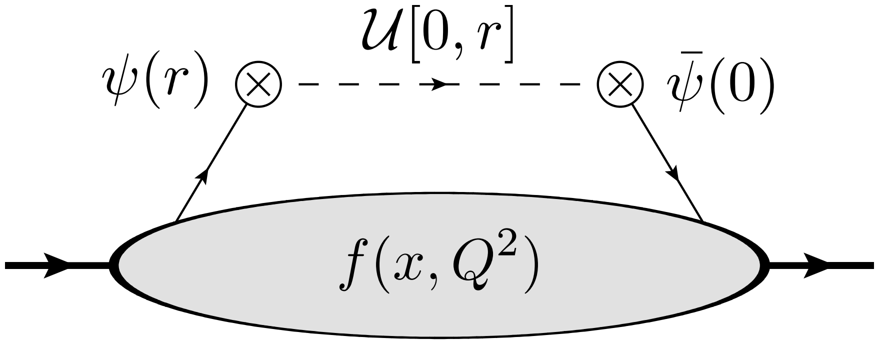
- ❖ Plus soft gluon radiation (NLO)
  - ➢ Large logs…
  - ➢ Resummation…?

- **Factorization** provides a one-to-one correspondence between the DIS **cross section** and hadronic structure: **PDFs**

$$\frac{Q^2}{4\pi^2\alpha_{EM}} \frac{d\sigma^{(\gamma^* p)}}{dx\, dQ^2} = F_2(x,Q^2) \overset{L.O.}{=} \sum_f e_f^2\, x q_f(x,Q^2)$$



$$q(x,Q^2) = \int \frac{dr^-}{2\pi} e^{ixp^+r^-} \langle p|\, \bar\psi(0)\, \mathcal{U}[0,r^-]\, \frac{\gamma^+}{2}\, \psi(r^-)\, |p\rangle$$



As Goes the Cross Section, So Go the PDFs

- At small x, the DIS cross section, and therefore the PDFs themselves, are expressed in terms of **dipole scattering amplitudes**

$$x q_f(x, Q^2) = \frac{Q^2 N_c}{4\pi^2 \alpha_{EM}} \int \frac{d^2 x_{10}\, dz}{4\pi z(1-z)} \sum_{L,T} \left|\Psi(x_{10}^2, z)\right|^2 \int d^2 b_{10} \left[2 - \frac{1}{N_c} \left\langle \mathrm{tr}\left[V_0 V_1^\dagger\right] \right\rangle_{(zs)} - \frac{1}{N_c} \left\langle \mathrm{tr}\left[V_1 V_0^\dagger\right] \right\rangle_{(zs)} \right]$$

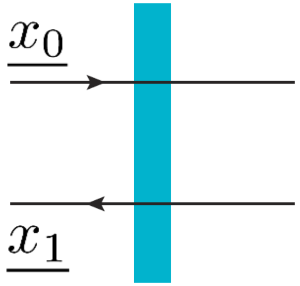
Photon splitting wave functions

Non-interacting terms

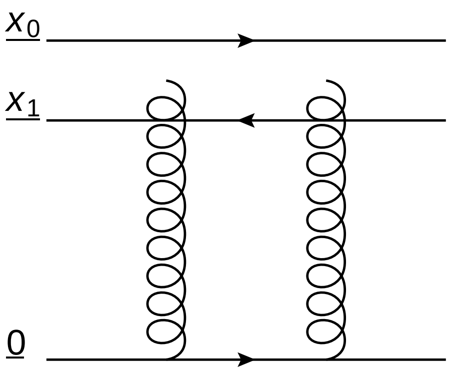
Dipole amplitudes

- The **x-dependence of the PDF** (TMD) is governed by the **energy dependence of the dipole amplitude**

$$\frac{1}{N_c}\left\langle \mathrm{tr}\left[V_0 V_1^\dagger\right]\right\rangle_{(zs)}$$



- Arises from the **phase-space enhanced quantum corrections** which are **resummed**



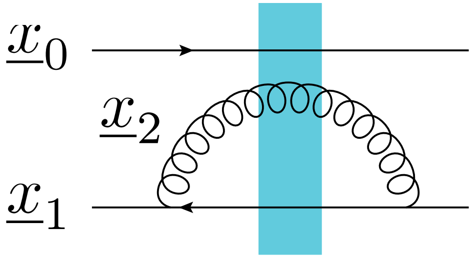
- The **initial conditions** can be taken from PDF fits at large x or, e.g.) the quark target model

$$\frac{1}{N_c}\left\langle \mathrm{tr}\left[V_0 V_1^\dagger\right]\right\rangle_{(zs)}^{(0)} = \frac{2\alpha_s^2 C_F}{N_c} \ln^2 \frac{x_{0T}}{x_{1T}}$$

- **“Real” gluon emissions** propagate through the **gauge field** of the proton

$$\frac{1}{N_c}\left\langle \mathrm{tr}\left[V_0 t^a V_1^\dagger t^b\right] U_2^{ba}\right\rangle_{(z' s)}$$

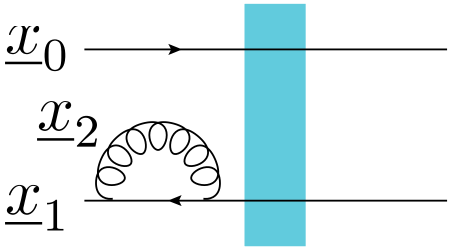
Adjoint Wilson line (gluon)



$$(x_2^-)_i < 0^- < (x_2^-)_f$$

- **“Virtual” gluon emissions** propagate through the **vacuum**, before or after hitting the proton.

$$-\frac{C_F}{N_c}\left\langle \mathrm{tr}\left[V_0 V_1^\dagger\right]\right\rangle_{(z' s)}$$



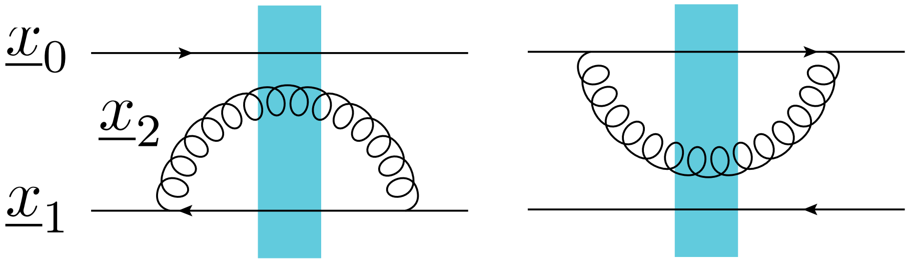
+ “c.c.”

$$(x_2^-)_i\,,\ (x_2^-)_f < 0^-$$

$$0^- < (x_2^-)_i\,,\ (x_2^-)_f$$

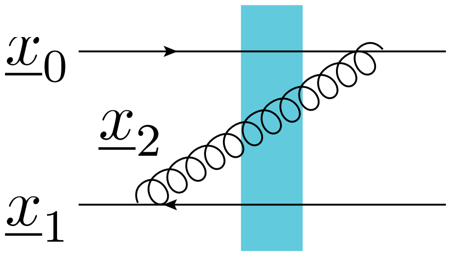
- **“Ladder” emissions** are emitted and absorbed by the same parton

$$\frac{\alpha_s}{\pi^2}\int\limits_{\Lambda^2/s}^{z}\frac{dz'}{z'}\int d^2x_2\left(\frac{1}{x_{21}^2}+\frac{1}{x_{20}^2}\right)\times\Big[\text{operator}\Big]$$

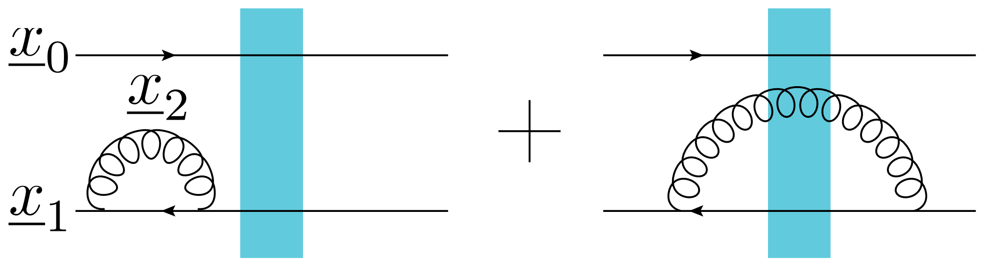


- **“Non-ladder” emissions** are emitted and absorbed by different partons

$$\frac{\alpha_s}{\pi^2}\int\limits_{\Lambda^2/s}^{z}\frac{dz'}{z'}\int d^2x_2\left(-2\,\frac{\underline{x}_{21}\cdot\underline{x}_{20}}{x_{21}^2\,x_{20}^2}\right)\times\Big[\text{operator}\Big]$$



What Happens to the Transverse Integral?



+ "c.c."

$$\frac{\alpha_s N_c}{2\pi^2} \int\limits_{\frac{\Lambda^2}{s}}^{z} \frac{dz'}{z'} \int \frac{d^2x_2}{x_{21}^2} \times \left[\frac{1}{N_c^2} \left\langle \mathrm{tr}\left[V_2 V_1^\dagger \right] \mathrm{tr}\left[V_0 V_2^\dagger \right] \right\rangle_{(z's)} - \frac{1}{N_c} \left\langle \mathrm{tr}\left[V_0 V_1^\dagger \right] \right\rangle_{(z's)} \right]$$

- "Ladder" emissions of **small-sized fluctuations are enhanced**
- **Transverse integral** appears divergent… a **second logarithm?**
- **No: Cancellation** of real + virtual diagrams due to **color transparency**

$$\frac{1}{N_c}\left\langle \mathrm{tr}\left[V_0 V_1^\dagger\right]\right\rangle_{(zs)} = \frac{1}{N_c}\left\langle \mathrm{tr}\left[V_0 V_1^\dagger\right]\right\rangle_{(zs)}^{(0)}$$

$$+ \frac{\alpha_s N_c}{2\pi^2} \int\limits_{\frac{\Lambda^2}{s}}^{z} \frac{dz'}{z'} \int d^2 x_2 \, \frac{x_{10}^2}{x_{20}^2 \, x_{21}^2} \left[\frac{1}{N_c^2} \left\langle \mathrm{tr}\left[V_2 V_1^\dagger\right] \mathrm{tr}\left[V_0 V_2^\dagger\right] \right\rangle_{(z's)} - \frac{1}{N_c} \left\langle \mathrm{tr}\left[V_0 V_1^\dagger\right]\right\rangle_{(z's)} \right]$$

Rapidity Logarithm

BFKL Kernel

New, more complex operator

- The dipole evolves into **increasingly complex operators**….

I. Balitsky, Nucl. Phys. ***B463*** *(1996) 99*
I. Balitsky, Phys. Rev. ***D60*** *(1999) 014020*

- Equivalent to a **functional differential equation**….

(JIMWLK)

Jalilian-Marian et al., Phys. Rev. ***D59*** *(1998) 014015*
Jalilian-Marian et al., Phys. Rev. ***D59*** *(1998) 014014*
Iancu et al., Phys. Lett. ***B510*** *(2001) 133*
Iancu et al., Nucl. Phys. ***A692*** *(2001) 583*

Linear BFKL Evolution at Small x

- The equations **linearize** in the **dilute limit (BFKL)** $1 - \frac{1}{N_c} \left\langle \mathrm{tr} \left[V_0 V_1^\dagger \right] \right\rangle \ll 1$

*Kuraev, et al., Sov. Phys. JETP **45** (1977) 199*
*Balitsky and Lipatov, Sov. J. Nucl. Phys. **28** (1978) 822*

$$\frac{1}{N_c} \left\langle \mathrm{tr} \left[V_0 V_1^\dagger \right] \right\rangle_{(zs)} = \frac{1}{N_c} \left\langle \mathrm{tr} \left[V_0 V_1^\dagger \right] \right\rangle_{(zs)}^{(0)}$$

$$+ \frac{\alpha_s N_c}{2\pi^2} \int\limits_{\frac{\Lambda^2}{s}}^{z} \frac{dz'}{z'} \int d^2 x_2 \, \frac{x_{10}^2}{x_{20}^2 \, x_{21}^2} \left\langle \frac{1}{N_c} \mathrm{tr} \left[V_2 V_1^\dagger \right] + \frac{1}{N_c} \mathrm{tr} \left[V_0 V_2^\dagger \right] - \frac{1}{N_c} \mathrm{tr} \left[V_0 V_1^\dagger \right] - 1 \right\rangle_{(z's)}$$

➢ Solution by **Laplace-Mellin transform** leads to **power-law growth** at small x

$$xq(x, Q^2) \sim xG(x, Q^2) \sim \left(\frac{1}{x} \right)^{\frac{4\alpha_s N_c}{\pi} \ln 2}$$

"Pomeron Intercept"

$$\alpha_P - 1 = \frac{4\alpha_s N_c}{\pi} \ln 2$$

Gluon Saturation

What It Is, and Why We Need to Find It

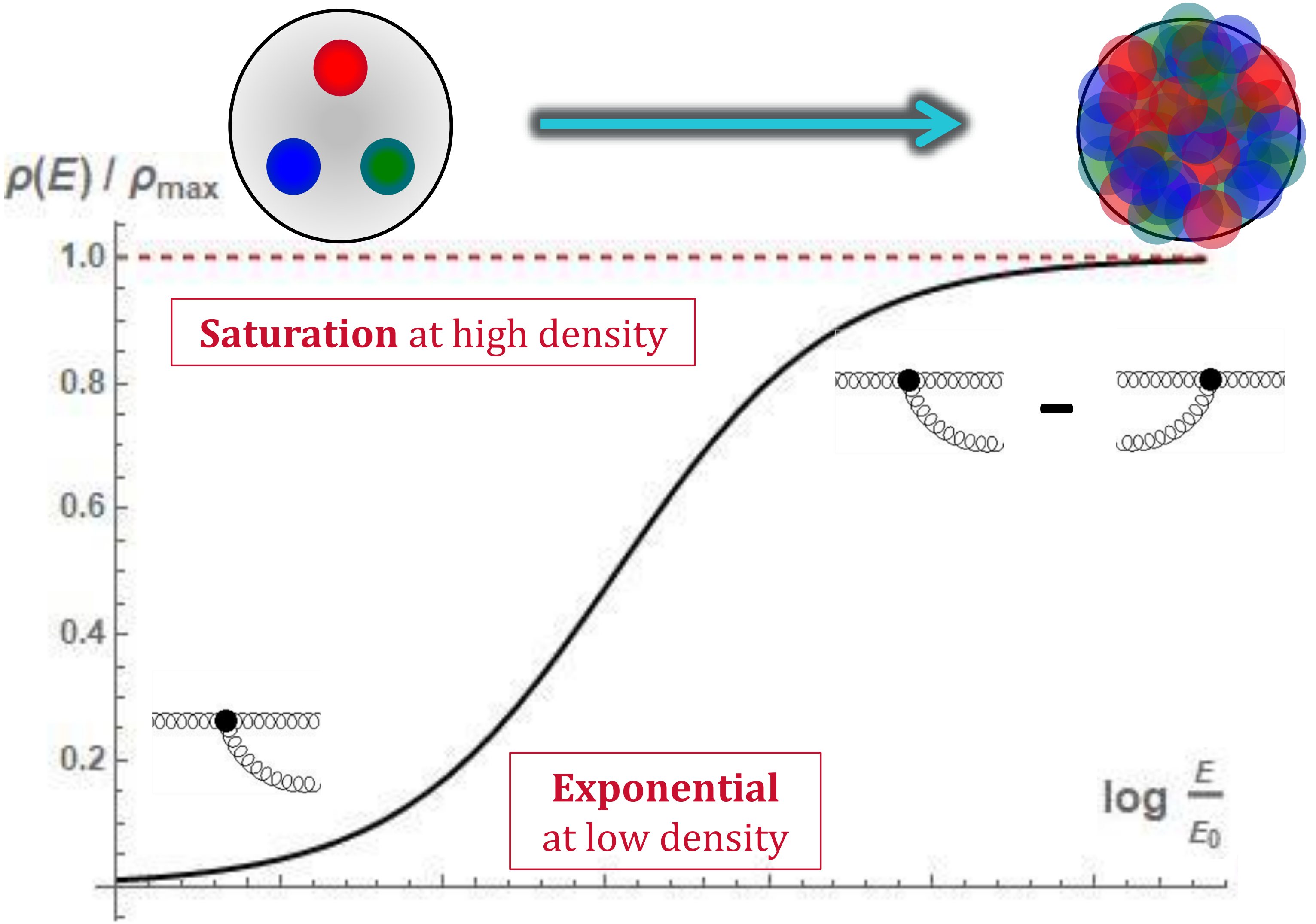
The Drive to Gluon Saturation

- QCD radiates an abundance of **soft gluons at small x**

➢ **Exponential growth** of charge density

- Also provides a mechanism to **regulate the growth**

➢ Gluon **recombination** can balance radiation, leading to a **saturation** of the density



Nonlinear BK Evolution at Large x

- The full nonlinear operator **hierarchy closes** under a **mean-field approximation**.

Balitsky, Nucl. Phys. ***B463*** *(1996) 99*
Balitsky, Phys. Rev. ***D60*** *(1999) 014020*
Kovchegov, Phys. Rev. ***D60*** *(1999) 034008*
Kovchegov, Phys. Rev. ***D61*** *(2000) 074018*

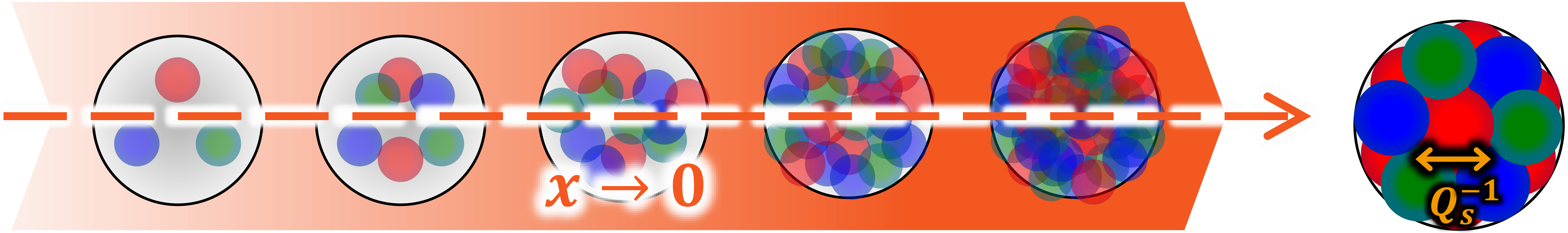
➢ **Large-Nc limit!**

$$\frac{1}{N_c}\left\langle \mathrm{tr}\left[V_0 V_1^\dagger\right]\right\rangle_{(zs)} = \frac{1}{N_c}\left\langle \mathrm{tr}\left[V_0 V_1^\dagger\right]\right\rangle_{(zs)}^{(0)}$$

$$+ \frac{\alpha_s N_c}{2\pi^2}\int\limits_{\frac{\Lambda^2}{s}}^{z}\frac{dz'}{z'}\int d^2x_2\,\frac{x_{10}^2}{x_{20}^2\,x_{21}^2}\left[\frac{1}{N_c^2}\left\langle \mathrm{tr}\left[V_2 V_1^\dagger\right]\right\rangle_{(z's)}\times\left\langle \mathrm{tr}\left[V_0 V_2^\dagger\right]\right\rangle_{(z's)} - \frac{1}{N_c}\left\langle \mathrm{tr}\left[V_0 V_1^\dagger\right]\right\rangle_{(z's)}\right]$$

❖ **Mean field** ❖ **Gluon-dominated** ❖ **Fully nonlinear** ❖ **Solvable!**

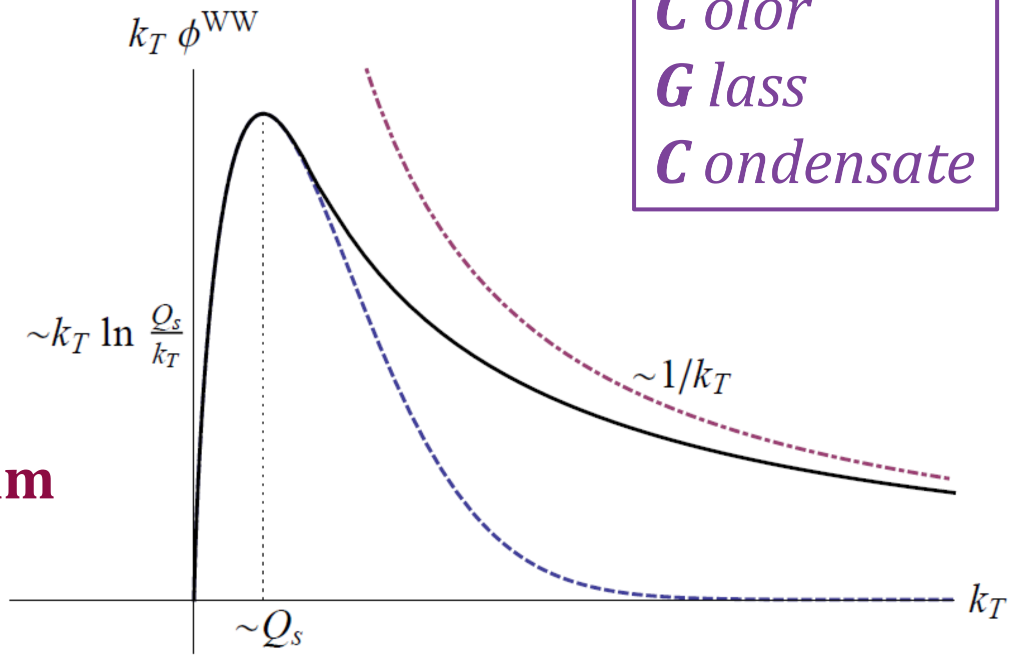
The "Color Glass Condensate" Effective Theory



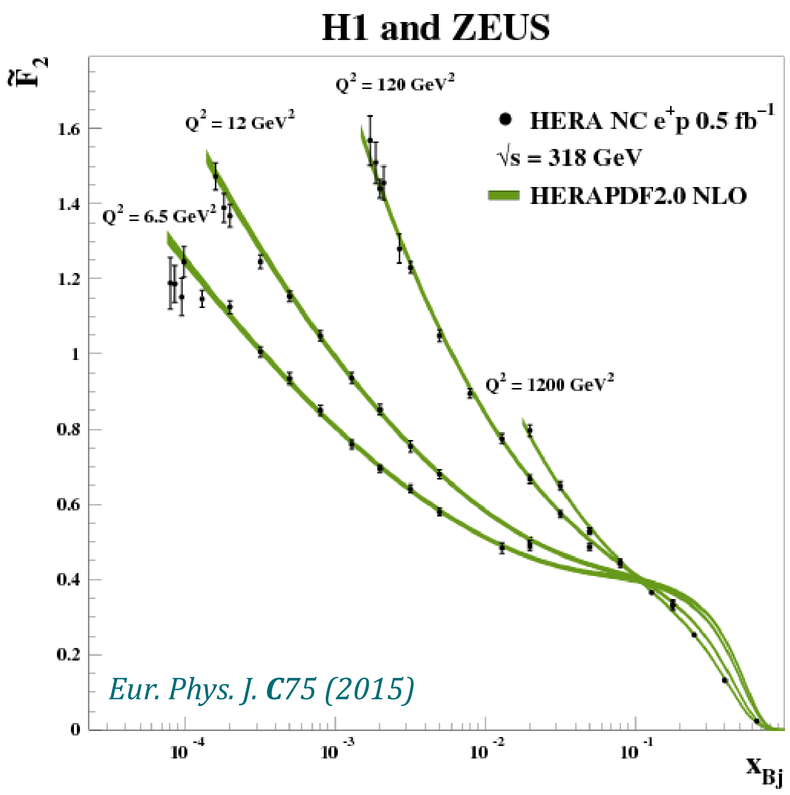
L. McLerran, R. Venugopalan, *Phys. Rev.* **D49** (1994), **D50** (1994)

- At small x, we approach an **emergent**, **universal**, **high-density state** of QCD characterized by:
  - Resummation of **coherent multiple scattering**
  - **Strong gluon fields**, semi-classical QCD
  - **Color domains** with emergent **saturation momentum**

$$Q_s^2(x) \approx 3.70\, \alpha_s(Q_s^2)\, \rho_G(x, Q_s^2)$$



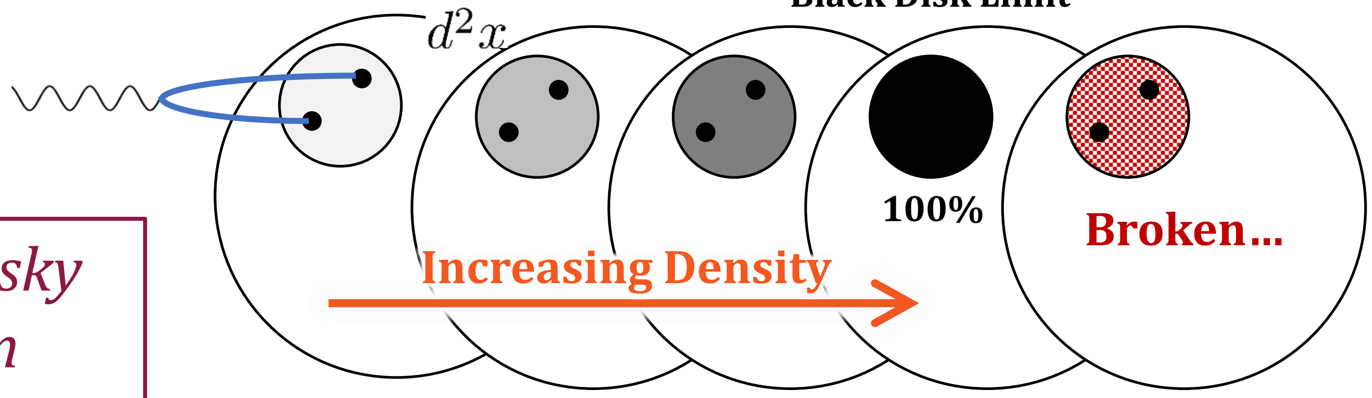
***C** olor*
***G** lass*
***C** ondensate*



Eur. Phys. J. ***C75*** *(2015)*

B *alitsky*
F *adin*
K *uraev*
L *ipatov*

"Little Unitarity Problem"



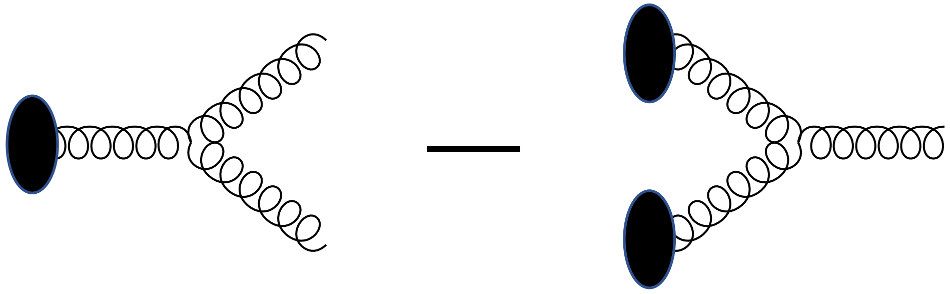
Kuraev, Lipatov, and Fadin, Sov. Phys. JETP ***45*** *(1977) 199*
Balitsky and Lipatov, Sov. J. Nucl. Phys. ***28*** *(1978) 822*

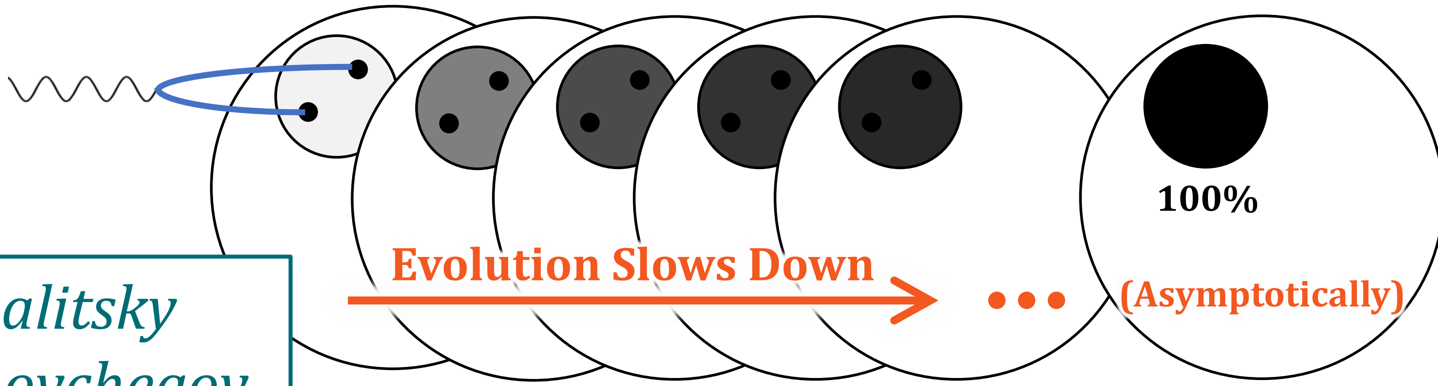
$$xG(x,Q^2) \sim \left(\frac{1}{x}\right)^{2.65\,\alpha_s(Q^2)}$$

$$\sigma^{q\bar{q}} \approx 3.29\frac{\alpha_s(Q^2)}{Q^2}\,xG(x,Q^2)$$

- **Resumming multi-emission** leads to a **power law growth** of the gluon density at small x.
  - ➢ **Breaks unitarity** locally: $\frac{dP}{d^2x} > 1$

Saturation: Necessary but not Sufficient for Unitarity





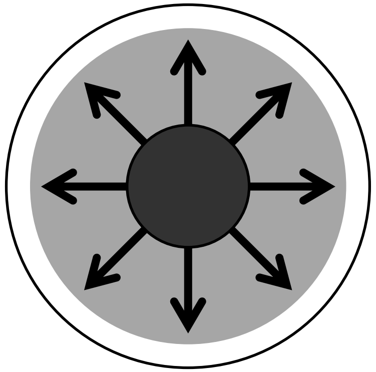
I. Balitsky, Nucl. Phys. **B463** (1996)
Y. Kovchegov, Phys. Rev. **D60** (1999)
J. Jalilian-Marian et al., Nucl. Phys. ***B504*** *(1997), Phys. Rev.* ***D59*** *(1998)*
A. Kovner, J. G. Milhano, H. Weigert, Phys. Rev. ***D62*** *(2000)*
E. Iancu, A. Leonidov, L. McLerran, Nucl. Phys. ***A692*** *(2001)*

***B** alitsky*
***K** ovchegov*

$$\rho_G \sim 1 - c\, e^{-0.556\alpha_s^2 Y^2}$$

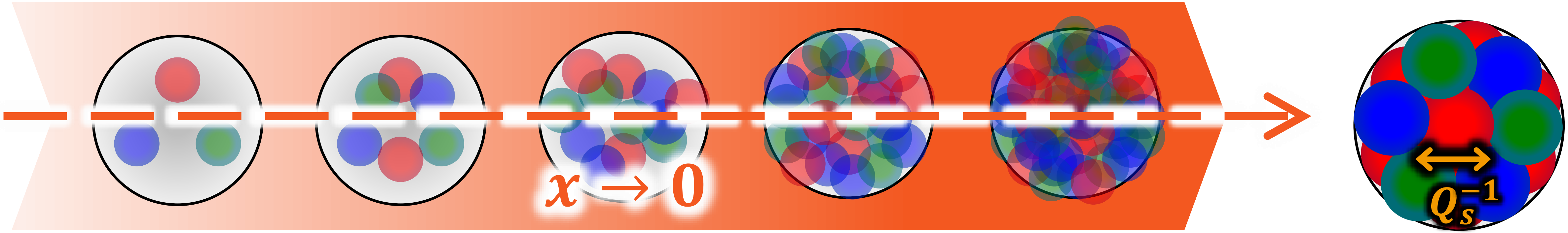
- At high density, **nonlinear gluon fusion slows down** the gluon cascade
  - Asymptotically **saturates** to the **black disk limit**
  - **Solves** the **"little unitarity problem"** (perturbatively!)
- But does **not** explicitly unitarize **total** cross sections due to the **diffusion** of the **black disk**. (nonperturbative)

$$R_{black} \sim e^Y$$



E. Ferreiro et al., Nucl. Phys. **A710** (2002)
A. Kovner, U. A. Wiedemann, Phys. Lett. **B551** (2003)

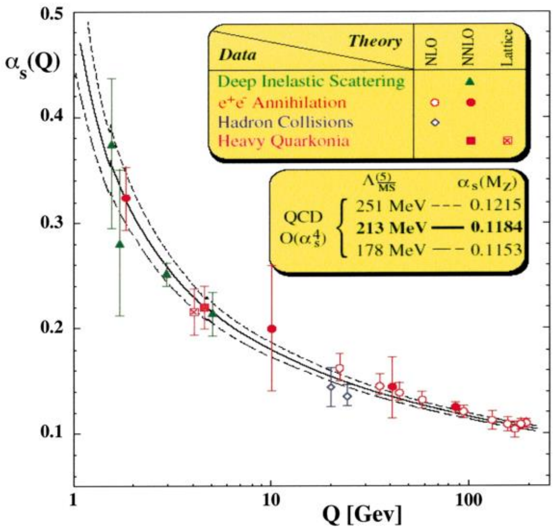
Why Should We Care About Saturation?



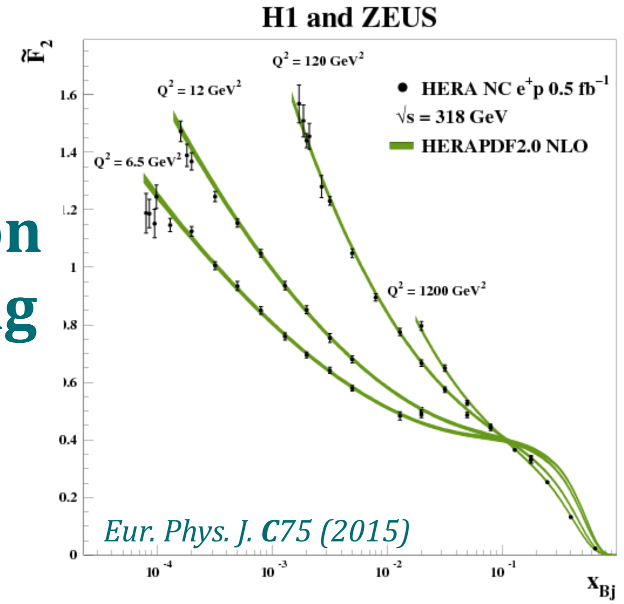
- Exotic state of nuclear matter characterized by (maximally!) intense gluon fields
- Initial conditions for heavy-ion collisions
- Perturbatively controllable regime of QCD with different dynamics (dense proton)
- ...Yes, and...

Why Should We Care About Saturation?

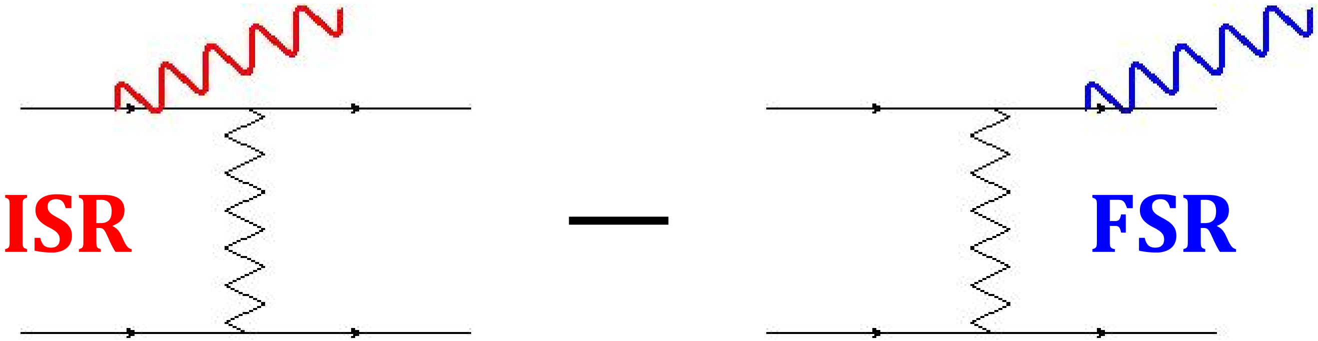
Any theory which is **asymptotically free** like **QCD**…



…will have an **explosion** of **soft bremsstrahlung** at **small x**.



- Radiation patterns at high energy $E_{cm} \to \infty \quad (x \to 0)$

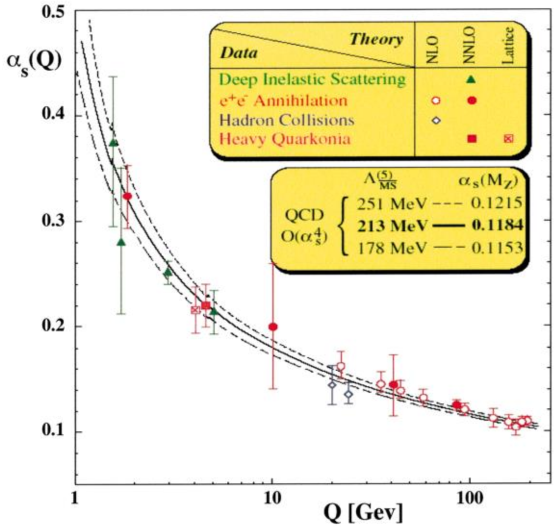


$$\frac{dN}{dy} = 0$$ **Abelian (QED)**

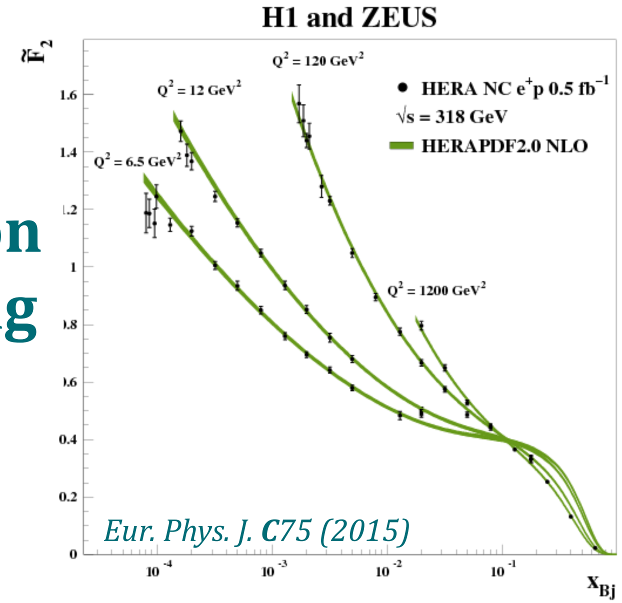
$$\frac{dN}{dy} = \alpha_s \times (\text{const})$$ **Non-Abelian (QCD)**

Why Should We Care About Saturation?

Any theory which is **asymptotically free** like **QCD**…

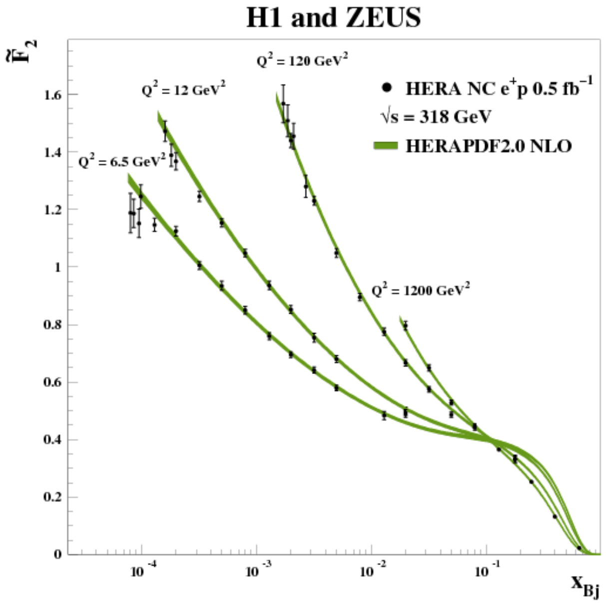


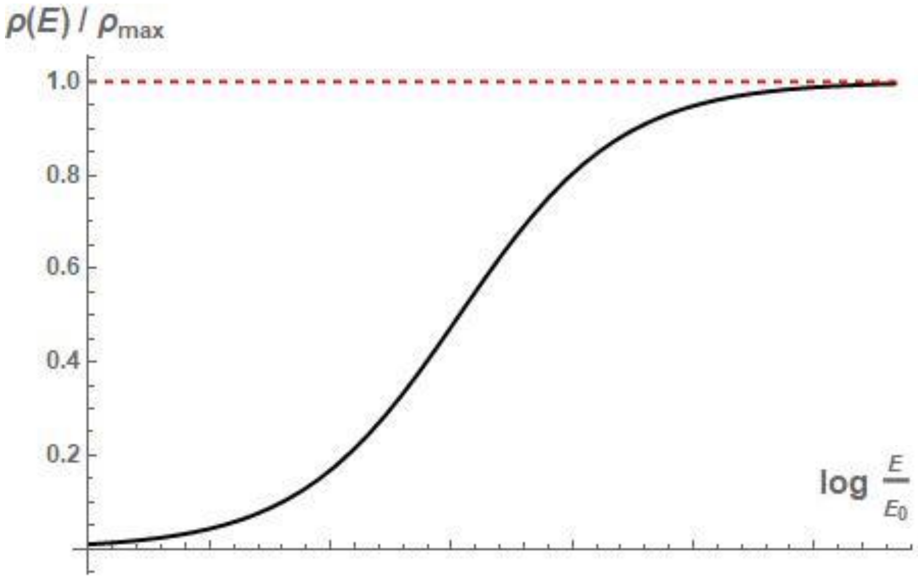
…will have an **explosion** of **soft bremsstrahlung** at **small x**.

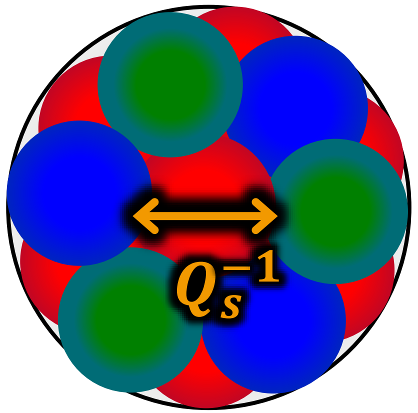


- ❖ The very feature that **makes QCD work** as an asymptotically free, UV-complete theory **threatens to break it** at small x.
  - ➢ *"Saturation is a non-negotiable consequence of QCD."* *- N. Armesto*
- ❖ If saturation **does not exist**, something is **irredeemably broken**
  - ➢ Not just for QCD, but for **any non-Abelian quantum field theory**

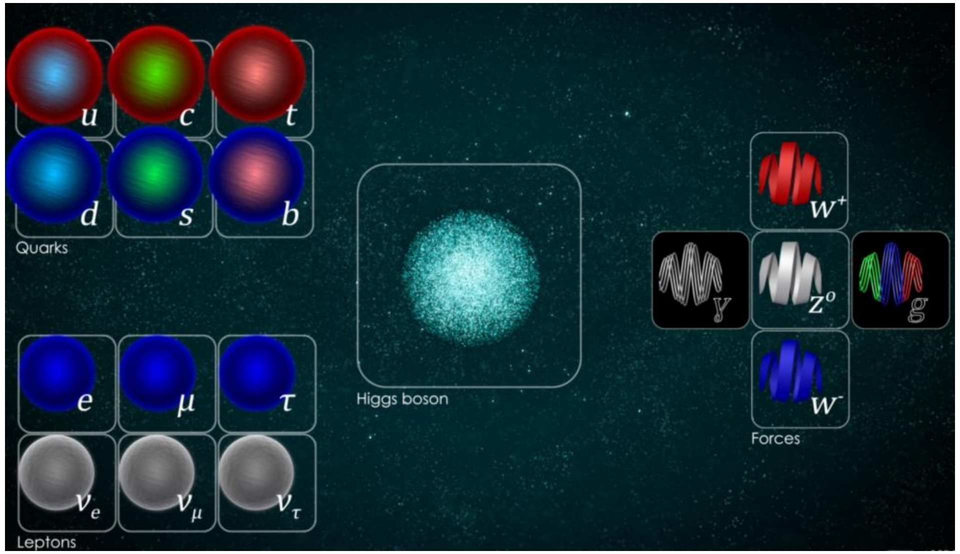
My Claim: Saturation is as Essential as the Higgs Boson







- ❖ Saturation, like the Higgs, is necessary for the **self-consistency of the QFT**
- ❖ Nothing guarantees we could find it…
- ❖ …But if it **does not exist**, we **need to know!**



Thank You!

Enjoy the Rest of the Summer School!