CHAPTER 1

HUGS 2022 Lecture 1: Construction of QCD (Part 1)

1.1 Overview: The Theoretical Framework

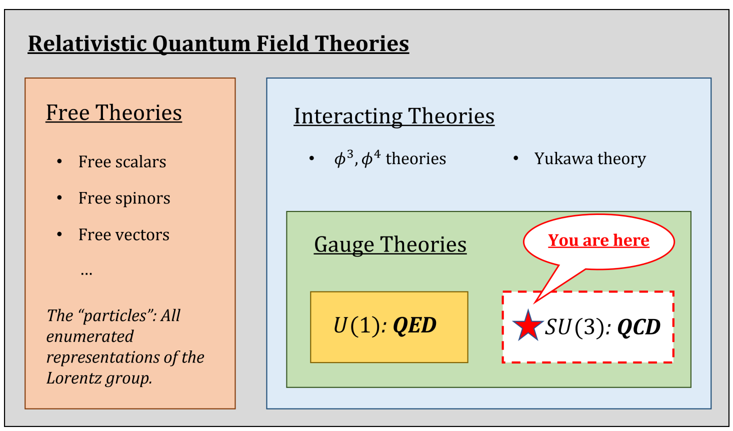

Figure 1.1: Illustration of QCD's place in the landscape of relativistic quantum field theories.

Quantum Field Theory

By way of introduction, let us begin by orienting ourselves with respect to the theoretical language and calculculational approach we will employ to describe QCD, as visualized in Fig. 1.1. The language of relativistic quantum mechanics is *quantum field theory* (QFT). QFT expresses the physics of multiparticle creation and annihilation in a manifestly Lorentz-covariant formalism, which elegantly solves the problem of causality for a relativistic quantum theory.

The simplest quantum theories are "free" theories – Lagrangians whose equations of motion are *linear*, such that the solutions obey a noninteracting superposition principle. These free theories define the "particles," with one Lorentz-invariant Lagrangian that can be enumerated for each representation of the Lorentz group $SO^{+}(3,1)$. More complex Lagrangians with nonlinear equations of motion are interpreted as *interactions* among the particles. The "traditional" approach to solving interacting theories is *perturbation theory*: a Taylor series expansion of physical observables in terms of a small coupling constant, such as the fine structure constant $\alpha_{EM} \approx 1/137$ in quantum electrodynamics (QED). This allows for a systematic order-by-order computation of observables with controllable error in QFT.

Perturbative and Nonperturbative Effects

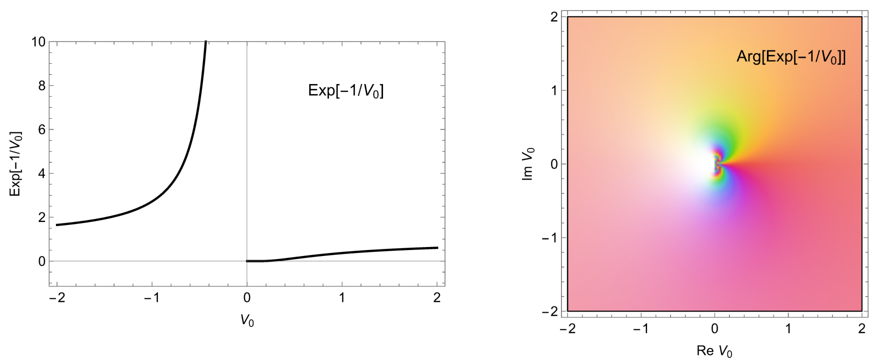

Figure 1.2: Behavior of the essential singularity $\exp[-1/V_0]$ in the complex plane.

Perturbation theory is a powerful, systematic approach to studying interacting theories, but it is not complete. The fundamental philosophy is an expansion of observables in a smooth Taylor series at small values of the coupling constant. The assumed smoothness of this expansion can fail in a number of important ways. First, there can be fundamental physics effects which are essentially *nonperturbative* in nature. Nonperturbative effects are often associated with a radical realignment of the degrees of freedom in a system which causes perturbation theory to break down. One famous example is nonperturbative transition from degrees of freedom resembling electrons and holes in a normal metal to Cooper pairs in a superconductor. As computed in the BCS theory of superconductivity, the binding energy E_b of a Cooper pair is

$$E_b = 2\omega_D \, e^{-2/V_0\rho(\varepsilon_F)} \,, \tag{1.1}$$

where V_0 is the strength of the effective potential between two electrons, ω_D is the Debye frequency and $\rho(\varepsilon_F)$ is the density of states at the Fermi energy[1]. Rather

[1] For more information, see these nice lecture notes.

than having a well-behaved Taylor series at weak coupling $V_0 \to 0$, the binding energy (1.1) instead has an essential singularity of the form $f(V_0) = e^{-1/V_0}$ shown in Fig. 1.2. This form, which often occurs in nonperturbative mechanisms, *has no Taylor series*:

$$\begin{aligned} f(V_0) &\overset{?}{=} f(V_0 = 0) + V_0 \left. \frac{df}{dV_0} \right|_{V_0=0} + \frac{1}{2} V_0^2 \left. \frac{d^2 f}{dV_0^2} \right|_{V_0=0} + \cdots \\ &\overset{!}{=} 0 + 0 + 0 + \cdots . \end{aligned} \tag{1.2}$$

A perturbative calculation of the binding energy (1.1) would yield *zero, to all orders in perturbation theory* and could never discover the superconducting phase transition. Likewise, one can construct a "proof" that $E_b = 0$ to all orders in perturbation theory, but this still misses the possibility of nonperturbative contributions.

This example illustrates that while perturbation theory can capture the smooth evolution of the degrees of freedom in a quantum field theory, by construction it cannot capture a severe rupture in those degrees of freedom such as during a phase transition. Exactly the same kind of mechanism (1.1) which leads to the formation of a Cooper-pair condensate and the superconducting phase transition in BCS theory is also responsible for the spontaneous breaking of chiral symmetry in QCD, which is associated with quark confinement and the emergence of the proton mass.

Asymptotic Series in Perturbation Theory

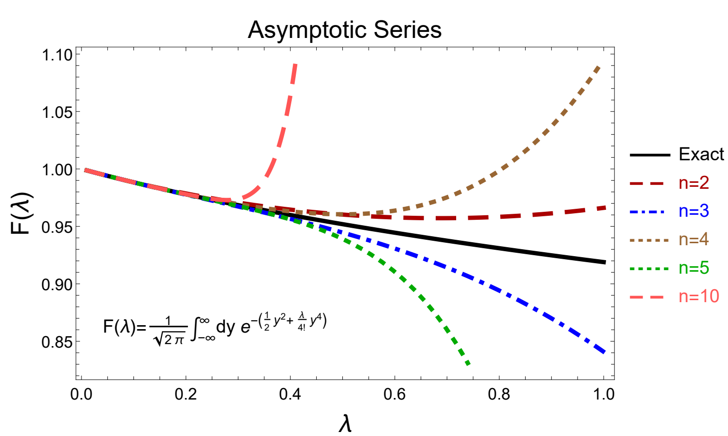

Figure 1.3: One example of the breakdown of convergence in an asymptotic series.

Another important caveat to the applicability of perturbation theory is the fact that the *perturbation series does not converge* in general. Rather than producing an absolutely-convergent series, the perturbation expansion is an asymptotic series. This means that, while a fixed-order calculation may provide a good estimate of the true solution, the accuracy of that estimate does not necessarily increase as one goes to higher orders. An example of an asymptotic series is shown in Fig. 1.3. Note that the fixed-order approximation gets better as one increases the accuracy from LO, to NLO ($n = 2$), to NNLO ($n = 3$). But as one pushes the perturbative expansion to higher and higher orders, the *error starts to increase* rather than decrease. The $n = 5$ curve is clearly worse than the $n = 3$ curve, and the $n = 10$ curve is a terrible approximation to the exact solution. This illustrates that, sometimes, "working harder" (computing to higher accuracy in the perturbation series) doesn't always pay off.

Resummation in Perturbation Theory

Finally, there can be a more gradual evolution of the degrees of freedom in a quantum system which **is** capturable in perturbation theory. This often occurs when there is a systematic enhancement of certain amplitudes due to a logarithmically large phase space, for instance in integrals of the form

$$\alpha \int_{\Lambda^2}^{E^2} \frac{dk_\perp^2}{k_\perp^2} = \alpha \ln \frac{E^2}{\Lambda^2} \text{ , or} \tag{1.3a}$$

$$\alpha \int_{\Lambda/E}^{1} \frac{dz}{z} = \alpha \ln \frac{E}{\Lambda} \,. \tag{1.3b}$$

This creates an interesting tension where the weak-coupling approximation $\alpha \ll 1$ may still be valid, but certain amplitudes are systematically enhanced by a large logarithm $\ln E/\Lambda$ coming from the limits of the phase space. If that logarithm becomes large enough, it can begin to compete with the smallness of the coupling α. In the limit

$$\alpha \ll 1 \qquad , \qquad \ln \frac{E}{\Lambda} \gg 1 \qquad , \qquad \alpha \ln \frac{E}{\Lambda} \sim \mathcal{O}(1) \ , \tag{1.4}$$

these systematically enhanced diagrams are not "small" at all, and we must *re-sum them all*, since

$$1 \sim \alpha \ln \frac{E}{\lambda} \sim \left(\alpha \ln \frac{E}{\lambda}\right)^2 \sim \left(\alpha \ln \frac{E}{\lambda}\right)^3 \sim \cdots \sim \left(\alpha \ln \frac{E}{\lambda}\right)^n \ . \tag{1.5}$$

Resumming these large logarithmic corrections can be accomplished by expressing them as a differential equation; if that equation can be solved, then its solution encodes the iteration of these enhanced corrections to all orders in perturbation theory. This procedure re-orders the perturbation series, giving the *leading-logarithmic approximation* (LLA) to the resummation. This description of a gradual transformation of degrees of freedom in QFT is often referred to as *quantum evolution*. It occurs prominently in QCD in several forms, including the DGLAP evolution with Q^2 and BFKL evolution with x_B of parton distribution functions.

1.2 Gauge Theory of $U(1)$: Quantum Electrodynamics

The Golden Archetype of Gauge Symmetry

Quantum Electrodynamics (QED) is defined by the Lagrangian

$$\mathcal{L}_{QED} = \bar{\psi}(i\not{\partial} - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - e\bar{\psi}\gamma_\mu\psi\, A^\mu \,, \tag{1.6}$$

which describes the interactions of charged fermions ("electrons") with vector bosons ("photons"). QED is a *gauge theory*, meaning that the particular *form of the photon/electron interaction vertex is uniquely dictated* by a symmetry transformation (called a "gauge symmetry") of the Lagrangian (1.6). The gauge symmetry is not a quirk of QED; it is an essential feature necessary to even define an interacting vector boson.

Global Symmetry and Conserved Current

The QED Lagrangian (1.6) is uniquely obtained from the free Dirac Lagrangian

$$\mathcal{L}_{Dirac} = \bar{\psi}(i\not{\partial} - m)\psi \tag{1.7}$$

through the process of *minimal coupling*. The free Dirac Lagrangian (1.7) is invariant under the global symmetry transformation

$$\psi'(x) = e^{i\phi}\psi(x) \qquad \text{or} \qquad \begin{bmatrix} \mathrm{Re}\,\psi'(x) \\ \mathrm{Im}\,\psi'(x) \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \mathrm{Re}\,\psi(x) \\ \mathrm{Im}\,\psi(x) \end{bmatrix} \,, \tag{1.8}$$

where ϕ is an arbitrary constant. This symmetry transformation is just a complex phase rotation, which may be regarded as a "1×1 unitary matrix". This describes the Lie group $U(1)$, which is a global symmetry of the free Dirac Lagrangian (1.7). Since charge conjugation $\psi \leftrightarrow \bar{\psi}$ changes particles into antiparticles, this $U(1)$ rotation may be regarded as a continuous rotation which redefines the particles and antiparticles.

By Noether's theorem, the invariance of the Lagrangian (1.7) under the continuous symmetry transformation (1.8) implies the existence of a conserved current:

$$\begin{aligned} \delta\mathcal{L} = 0 &= \frac{\delta\mathcal{L}}{\delta\psi}\delta\psi + \delta\bar{\psi}\frac{\delta\mathcal{L}}{\delta\bar{\psi}} + \frac{\delta\mathcal{L}}{\delta(\partial_\mu\psi)}\delta(\partial_\mu\psi) \\ &= \left(\partial_\mu \frac{\delta\mathcal{L}}{\delta(\partial_\mu\psi)}\right)\delta\psi + \frac{\delta\mathcal{L}}{\delta(\partial_\mu\psi)}\partial_\mu(\delta\psi) \\ &= \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\psi)}\delta\psi\right) \\ 0 &= \partial_\mu\left(\bar{\psi}\gamma^\mu\psi\right) \end{aligned} \tag{1.9}$$

where we have used the equations of motion. The *net particle number current* is conserved:

$$j^\mu = \bar{\psi}\gamma^\mu\psi \qquad , \qquad \partial_\mu j^\mu = 0\,, \tag{1.10}$$

reflecting the conservation of electric charge. The electromagnetic current $J^\mu_{EM} = e\bar{\psi}\gamma^\mu\psi$ is just the particle number current, weighted by the charge.

The Problem with Vector Bosons: Scalar Polarization Mode

The form of the QED interaction vertex (1.6) is of the form

$$\mathcal{L}_I = -J^\mu_{EM}A_\mu = -e\bar{\psi}\gamma^\mu\psi\, A_\mu, \tag{1.11}$$

which introduces the photon field A^μ as being *created by the conserved current* j^μ. This choice of vertex directly links the properties of the conserved current j^μ produced by the $U(1)$ global symmetry and the structure of the vector field A^μ. This is not just a curiosity; it is an essential feature necessary for the photon field A^μ to be well-defined at all. The reason is that, in 4-dimensional spacetime, there are potentially four independent polarization modes of A^μ, including the "timelike" or "scalar polarization" mode, which can be written as the gradient of a scalar field:

$$A^\mu_{(scalar)}(x) = \partial^\mu\phi(x)\,. \tag{1.12}$$

The timelike polarization is "unphysical" – if quantized, it would lead to states of negative norm, which are incompatible with a Hilbert space of quantum states. Getting rid of the sick scalar polarization is essential for any self-consistent quantum theory of vector bosons.

An interaction vertex of the form (1.11) which couples the vector boson to a *conserved* current arising from a global symmetry eliminates the scalar-polarized modes (1.12) in an elegant way: by reducing them to a *symmetry transformation* on the Lagrangian. If we shift A^μ by the addition of a scalar-polarized mode,

$$A'^{\,\mu}(x) = A^\mu(x) + \partial^\mu\phi(x) \tag{1.13a}$$

$$\mathcal{L}'_I = \mathcal{L}_I - ej^\mu\partial_\mu\phi = \mathcal{L}_I - \partial_\mu(ej^\mu\phi) \tag{1.13b}$$

the interaction term is *invariant* (up to an irrelevant total derivative). Thus, with a special interaction vertex of the form (1.11), the scalar modes are removed as "redundant, unphysical degrees of freedom" which have no consequence on observables.

CHAPTER 2

HUGS 2022 Lecture 2: Construction of QCD (Part 2)

2.1 Previously

- Taxonomy of quantum field theories: free theories, interacting theories, gauge theories
- Perturbation theory and its limitations
- Global symmetry and current conservation
- Problematic scalar-polarized mode of the vector field A^μ

Minimal Coupling: Gauging the QED Lagrangian

The extension of the free Dirac Lagrangian (1.7) to include the *minimal coupling* to the vector field A^μ through a term of the form $-j_\mu A^\mu$ can be compactly expressed using the *gauge-covariant derivative*:

$$\begin{aligned}\mathcal{L}_{gauged} &= \bar{\psi}(i\gamma_\mu\partial^\mu - m)\psi - e\,\bar{\psi}\gamma_\mu\psi\,A^\mu \\ &= \bar{\psi}[i\gamma_\mu(\partial^\mu + ie\,A^\mu) - m]\psi \\ &\equiv \bar{\psi}(i\not{D} - m)\psi\,. \end{aligned} \tag{2.1}$$

The remarkable physics of extending the local $U(1)$ symmetry to form the basis of a local $U(1)$ gauge theory is encoded in the simple replacement of the partial derivative with the covariant derivative:

$$\partial_\mu \to D_\mu \equiv \partial_\mu + ie\,A_\mu\,. \tag{2.2}$$

The covariant derivative expresses the fact that the shift (1.13) of A^μ by the addition of a scalar mode is now interconnected with the $U(1)$ symmetry of the Dirac Lagrangian (1.7). Under the more general version of the transformation (1.8) in which the rotation phase $\phi(x)$ can vary as a function of spacetime,

$$\psi'(x) = e^{i\phi(x)}\psi(x)\,, \tag{2.3}$$

the Lagrangian (2.1) transforms as

$$\begin{aligned}\mathcal{L}'(x) &= \bar{\psi}'(x)(i\gamma_\mu\partial^\mu - m)\psi'(x) - e\,\bar{\psi}'(x)\gamma_\mu\psi'(x)\,A^{\mu\,\prime}(x)\\ &= \bar{\psi}(x)\,e^{-i\phi(x)}(i\gamma_\mu\partial^\mu - m)e^{i\phi(x)}\,\psi(x) - e\,\bar{\psi}(x)\,e^{-i\phi(x)}\gamma_\mu e^{i\phi(x)}\,\psi(x)\,A^{\mu\,\prime}(x)\\ &= \bar{\psi}(x)(i\gamma_\mu\partial^\mu - m)\psi(x) - \bar{\psi}(x)\gamma_\mu\psi(x)\,\partial^\mu\phi(x) - e\,\bar{\psi}(x)\gamma_\mu\psi(x)\,A^{\mu\,\prime}(x)\\ &= \bar{\psi}(x)(i\gamma_\mu\partial^\mu - m)\psi(x) - e\,\bar{\psi}(x)\gamma_\mu\psi(x)\left[A^{\mu\,\prime}(x) + \frac{1}{e}\partial^\mu\phi(x)\right].\end{aligned} \tag{2.4}$$

The effect of the *local* transformation (2.3) can be interpreted as *shifting* A^μ *by a scalar-polarized mode*, exactly as we did in Eq. (1.13). Since the minimal coupling of A^μ to the conserved current j_mu guarantees that a shift by a scalar mode is an unphysical symmetry transformation, the two pieces (global $U(1)$ symmetry and scalar polarizations of A_mu) work in tandem as part of a single composite symmetry operation known as *gauge symmetry*.

Choosing a special interaction vertex of the form (1.11) has united the elimination of the scalar-polarized mode with the global $U(1)$ symmetry responsible for the conserved current j^μ. This choice effectively *enlarges the* $U(1)$ *symmetry group* from a global symmetry to a local symmetry and uses it to *define the photon* field A^μ. Without a gauge-invariant coupling of this form, the photon field *could not exist at all*, since it would be polluted with unworkable scalar-polarized modes.

Geometric Interpretation

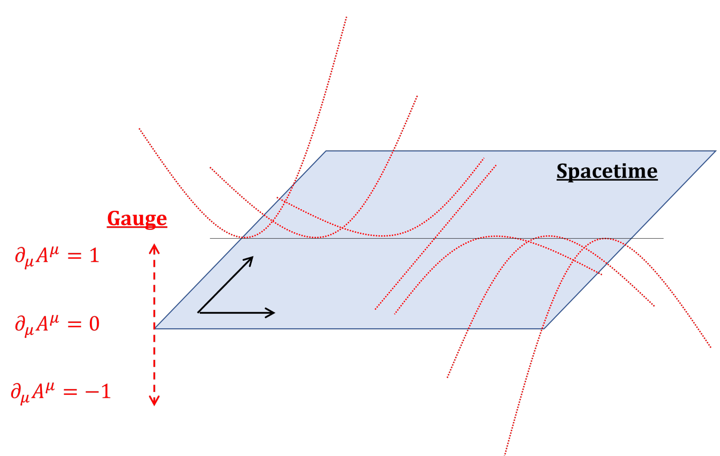

Figure 2.1: Sketch of the combined spacetime + gauge manifold.

From the point of view of the local gauge transformation (2.3), the covariant derivative (2.2) has a natural geometric interpretation. The ordinary partial derivative ∂_μ is not invariant under the local $U(1)$ transformation (2.3), because different points x^μ of spacetime transform differently, so the partial derivative makes an unequal comparison between two adjacent points. The covariant derivative (2.2) *compensates* for the different transformations of $\psi(x)$ at different points, permitting a simple meaningful (gauge-invariant) comparison between two points.

Under the combined transformation

$$\psi'(x) = e^{i\phi(x)}\psi(x) \,, \tag{2.5a}$$

$$A^{\mu\,\prime}(x) = A^{\mu}(x) - \frac{1}{e}\partial^{\mu}\phi(x) \,, \tag{2.5b}$$

the covariant derivative transforms as

$$\begin{aligned}\big(D_\mu\psi(x)\big)' &\equiv (\partial_\mu + ie\,A'_\mu)\,\psi'(x)\\ &= (\partial_\mu + ie\,A_\mu - i\partial_\mu\phi)\,e^{i\phi(x)}\psi(x)\\ &= e^{i\phi(x)}(\partial_\mu + ie\,A_\mu)\psi(x)\\ &= e^{i\phi(x)}\,D_\mu\psi(x) \,.\end{aligned} \tag{2.6}$$

That is: *the covariant derivative of a field has the same gauge transformation as the field itself.* This is the precise mathematical statement that the covariant derivative *cancels the nonlocal differences in the gauge transformation*, providing a meaningful way to compare two different points in spacetime in a gauge-invariant way. For instance, it is clear that $\bar{\psi}D_\mu\psi$ is gauge invariant.

We have found that, by minimally coupling the photon field to the conserved electric current, we have constructed a Lagrangian (2.1) which is invariant under a *mix* of Lorentz transformations and shifting A^μ by scalar modes. Even though spacetime itself is "flat" (in this discussion), the different transformations of neighboring points under a local $U(1)$ transformation act as if the *combined* spacetime + gauge manifold is *curved*. From this point of view, the problem of how to compute meaningful derivatives on a curved manifold is a standard one in differential geometry. The covariant derivative (2.2) precisely introduces the photon field A_μ as a metric connection which compensates for the curvature of the gauge dimensions along the physical spacetime dimensions.

Field-Strength Tensor as a Generalized Curl

The last piece of the QED Lagrangian (1.6) is the kinetic term associated with the free photon field. The free-field Lagrangian for A^μ must be quadratic in A, coupled to derivatives to contain momenta, and a Lorentz scalar. It must be massless, as an essential requirement of gauge symmetry[1]. From these considerations, the kinetic term can be deduced to be

$$\mathcal{L}_{kinetic} = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} \,, \tag{2.7}$$

[1] Massive vector bosons have a different relation with their scalar-polarized modes, whose removal is generally enforced by constraint.

where the antisymmetric field-strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is a four-dimensional generalization of the curl, transforming as an antisymmetric rank-2 tensor under Lorentz tranformations.

Based on the geometric interpretation of gauge transformations discussed previously, we expect that this generalized curl should also have a sensible interpretation under gauge transformations. Indeed, the field-strength tensor $F^{\mu\nu}$ has a particularly simple expression in terms of the covariant derivative (2.2):

$$\begin{aligned} \Big[D_\mu \, , \, D_\nu \Big] &= ie(\partial_\mu A_\nu - \partial_\nu A_\mu) = ie\, F_{\mu\nu} \\ \therefore \qquad F_{\mu\nu} &= \frac{-i}{e} \Big[D_\mu \, , \, D_\nu \Big] \, . \end{aligned} \tag{2.8}$$

This expression, as a commutator of covariant derivatives, describes the generalized curl *including the curvature along the gauge direction.* Since D_μ is itself a gauge-covariant quantity (transforming locally under gauge transformations), so is $F_{\mu\nu}$. In the case of QED, $F_{\mu\nu}$ is simply gauge-invariant, but in the more general case, $F_{\mu\nu}$ may transform under gauge transformations, and its transformation properties are dictated by those of the covariant derivative D_μ.

Next Steps

Taken together, the QED Lagrangian

$$\mathcal{L}_{QED} = \bar{\psi}(i\not{\partial} - m)\psi - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - e\bar{\psi}\gamma_\mu \psi \, A^\mu \, , \tag{2.9}$$

and all the physics of electrodynamics are consequences of a single unifying principle: $U(1)$ gauge symmetry. Physically, this gauge group encodes the statement that *electric charge is a scalar quantity.* Electric charge may be positive or negative (electrons and positrons), but it has no "direction" associated with it. This is reflected in the fact that $U(1)$ corresponds to rotation by a complex phase, without any matrix dimension to it. In a different gauge group, such as $SU(2)$ (the Pauli matrices), the gauge transformation could employ a nontrivial matrix structure. This single difference is responsible for the enormous complexity of QCD.

2.2 Quantum Chromodynamics: Gauge Theory of $SU(3)$

Given the context developed thus far for what a gauge theory looks like in the case of the *Abelian* (commutative) gauge group $U(1)$, we can now state concisely what the governing principle of Quantum Chromodynamics (QCD) is. QCD is the $SU(3)$ gauge theory describing interactions of fermions called "quarks" with vector gauge bosons called "gluons."

Physically, the statement that the gauge group is $SU(3)$ implies that the quarks possess a new kind of charge quantum number, termed "color charge,"

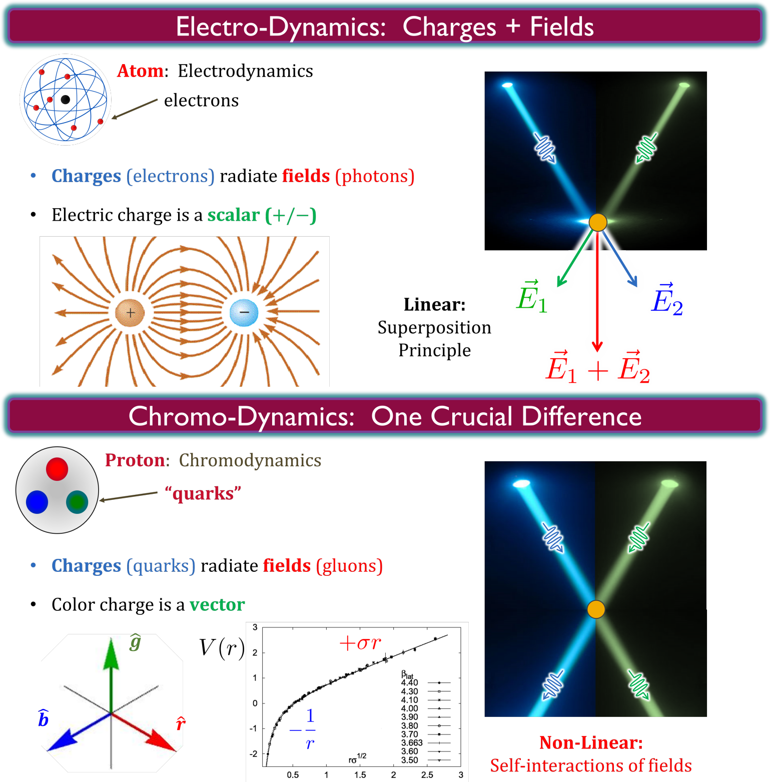

Figure 2.2: Comparison of the role of "charge" in QED versus QCD.

which come in three different varieties, referred to as "red, blue, and green." The fact that the color quantum number comes in multiple independent types, or means that unlike electric charge, color charge carries a particular "direction" to it. Color charge is a *vector*[2]. This structural change in the gauge theory leads to profound differences between QED and QCD, as illustrated in Fig. **??**.

As with QED, we construct the QCD Lagrangian by starting with the free Dirac Lagrangian. Now we posit that there are three independent varieties of noninteracting fermions (quarks), corresponding to the three color states:

$$\mathcal{L}_{free} = \bar{\psi}_i (i\partial\!\!\!/ - m)\psi_i \qquad , \qquad i = 1, 2, 3\,. \tag{2.10}$$

Here we use indices like i, j, k to denote quark colors, referred to as the "fundamental representation" of $SU(3)$. The multicomponent Lagrangian (2.10) contains a much larger global symmetry than just the $U(1)$ symmetry of QED. It possesses a symmetry of *rotations among the 3 color states*. Just as the unitary rotation operator is given by $e^{i\vec{\phi}\cdot\vec{J}}$, with $\vec{\phi}$ the angle and axis of rotation

[2]under $SU(3)$ transformations

and $\vec{J}$ the generators of rotations (angular momenta), the $SU(3)$ color rotations take a similar form:

$$\psi_i'(x) = \exp\left[i\phi^a t^a\right]_{ij} \psi_j(x) \qquad , \qquad a = 1, 2, \cdots, 8\,. \tag{2.11}$$

As with the rotation matrix, there are various rotation angles ϕ^a and $SU(3)$ rotation generators t^a, for each of the possible "axes of rotation" which can mix between the various color states. There are many such "axes" – in fact, there are more axes than the number of quark colors themselves. To go from any of 3 colors initially to any of 3 final colors, there are in principle $3^2 = 9$ possible axes of rotation. The group $SU(3)$ excludes the contribution proportional to the identity matrix[3], reducing the number to $3^2 - 1 = 8$ types of $SU(3)$ color rotations.

Group Structure of $SU(3)$

The nomenclature $SU(3)$ for the QCD gauge group refers to the group of *special, unitary,* 3×3 *matrices* which perform the color rotations among the quark colors. The term "special" indicates that the matrices have *unit determinant*, which is the condition that excludes the unit matrix ("QCD photon").

The 8 generators of $SU(3)$ in QCD are written

$$t^a = \frac{1}{2}\lambda^a \tag{2.12}$$

where λ^a are the Gell-Mann matrices

$$\begin{aligned}
\lambda^1 &= \begin{bmatrix} & 1 & \\ 1 & & \\ & & \end{bmatrix} & \lambda^2 &= \begin{bmatrix} & -i & \\ i & & \\ & & \end{bmatrix} & \lambda^3 &= \begin{bmatrix} 1 & & \\ & -1 & \\ & & \end{bmatrix} \\
\lambda^4 &= \begin{bmatrix} & & 1 \\ & & \\ 1 & & \end{bmatrix} & \lambda^5 &= \begin{bmatrix} & & -i \\ & & \\ i & & \end{bmatrix} & \lambda^6 &= \begin{bmatrix} & & \\ & & 1 \\ & 1 & \end{bmatrix} \\
\lambda^7 &= \begin{bmatrix} & & \\ & & -i \\ & i & \end{bmatrix} & \lambda^8 &= \frac{1}{\sqrt{3}}\begin{bmatrix} 1 & & \\ & 1 & \\ & & -2 \end{bmatrix} & & .
\end{aligned} \tag{2.13}$$

In this fundamental representation of $SU(3)$, the Gell-Mann matrices are these explicit 3×3 matrices, with the particular components t^3, t^8 being diagonal. The essential property of $SU(3)$ is the Lie algebra of its generators, which can be expressed through the commutator relation

$$[t^a\,,\, t^b] = i f^{abc}\, t^c\,, \tag{2.14}$$

[3] This would resemble the interaction of the photon, since it would again be a $U(1)$ gauge subgroup.

where f^{abc} are the totally antisymmetric structure constants

$$\begin{array}{c|ccccccccc} f^{abc} & 1 & \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} & \frac{\sqrt{3}}{2} & \frac{\sqrt{3}}{2} \\ \hline \mathrm{a} & 1 & 1 & 1 & 2 & 2 & 3 & 3 & 4 & 6 \\ \mathrm{b} & 2 & 4 & 5 & 4 & 5 & 4 & 6 & 5 & 7 \\ \mathrm{c} & 3 & 7 & 6 & 6 & 7 & 5 & 7 & 8 & 8 \end{array} \tag{2.15}$$

with $f^{abc} = -f^{bac} = -f^{acb}$ and all other components of f^{abc} equal to zero.

The structure constants f^{abc} themselves provide the 8-dimensional "adjoint representation" of $SU(3)$ (which we denote here in capital letters),

$$(T^a)_{bc} \equiv -i\, f^{abc}\,, \tag{2.16}$$

which satisfies the Lie (commutator) algebra (2.14)

$$[T^a\,,\,T^b] = i f^{abc}\, T^c \tag{2.17}$$

through the use of the Jacobi identity

$$\big[T^a\,,\,[T^b,T^c]\big] + \big[T^b\,,\,[T^c,T^a]\big] + \big[T^c\,,\,[T^a,T^b]\big] = 0\,. \tag{2.18}$$

Just as the 3×3 fundamental representation of $SU(3)$ describes the color states of the quarks and how they transform, the 8×8 adjoint representation describes the color states and interactions of the *gluons*.

Comparison to $SU(2)$

The Gell-Mann matrices (2.13) bear a clear resemblance to the Pauli matrices – for good reason. The Pauli matrices

$$\sigma^1 = \begin{bmatrix} & 1 \\ 1 & \end{bmatrix} \qquad \sigma^2 = \begin{bmatrix} & -i \\ i & \end{bmatrix} \qquad \sigma^3 = \begin{bmatrix} 1 & \\ & -1 \end{bmatrix} \tag{2.19}$$

define the generators $t^a = \frac{1}{2}\sigma^a$ (with $a = 1,2,3$) for the sister group $SU(2)$. The Lie algebra of $SU(2)$ is

$$[t^a\,,\,t^b] = i\epsilon^{abc}\, t^c\,, \tag{2.20}$$

where the structure constants for $SU(2)$ are just the elements of the antisymmetric Levi-Civita symbol.

In QCD, the quantum number for quark colors comes in 3 distinct states, giving the 3×3 Gell-Mann matrices (2.13). For the sister group $SU(2)$, the quantum number comes in 2 distinct states (usually in the context of "spin-up" and "spin-down"), giving the 2×2 Pauli matrices (2.19). For $SU(2)$, there are $2^2 - 1 = 3$ Pauli matrices, which correspond to the 3 independent *axes of rotation* $\sigma_x, \sigma_y, \sigma_z$. In the same way, the $3^2 - 1 = 8$ Gell-Mann matrices describe the *axes of (color) rotation* in QCD. With this analogy, we may speak of a rotation of a quark from fundamental color state (say) "red" ($i = 1$) to "blue" ($j = 2$) by the emission/absorption of a gluon in the adjoint color state $a = 1$. This rotation would be described by the element $(\lambda^1)_{21} = 1$ of the corresponding Gell-Mann matrix. This is what is meant by the statement "charge is a vector" in QCD.

The t'Hooft Large-N_c Limit

While for true QCD the number of quark colors is 3, the general gauge structure of QCD is only minimally modified for the case of arbitrary number of quark colors N_c. We have already benefited from the comparison of QCD ($N_c = 3$) with the Pauli matrices of $N_c = 2$. In fact, the algebra of the general gauge group $SU(N_c)$ becomes significantly *simpler* with clever usage of the number of colors N_c. One particularly powerful usage is the **t'Hooft large-N_c limit**

$$\alpha_s \to 0\,, \tag{2.21a}$$
$$N_c \to \infty\,, \tag{2.21b}$$
$$\alpha_s N_c = \text{const} \ll 1\,. \tag{2.21c}$$

In this limit, the "S" of $SU(N_c)$ essentially becomes irrelevant (reducing $SU(N_c)$ to $U(N_c)$), since the one omitted generator is negligible compared to the $N_c^2 - 1 \approx N_c^2$ generators retained as $N_c \to \infty$. This can be clearly seen in the form of the Fierz identity for $SU(N_c)$

$$(t^a)^i{}_j\,(t^a)^k{}_\ell = \frac{1}{2}\delta^i{}_\ell\,\delta^k{}_j - \frac{1}{2N_c}\delta^i{}_j\,\delta^k{}_\ell \overset{N_c \gg 1}{\approx} \frac{1}{2}\delta^i{}_\ell\,\delta^k{}_j\,, \tag{2.22}$$

where the subtraction term enforcing $(t^a)^i{}_i = 0$ drops out.

In the large-N_c limit, the number of gluons $N_c^2 - 1$ far exceeds the number of quarks N_c, so this limit *simplifies QCD to effectively contain only gluons*. For gluon-dominated phenomena like small-x gluon saturation, this approximation is an especially powerful simplification. The simplified Fierz identity (2.22) allows the adjoint color flow of gluons to be replaced with an equivalent fundamental color flow, as if the gluon were being replaced by a quark-antiquark pair[4]. Moreover, the Feynman diagrams which dominate the large-N_c limit are always *planar*, meaning that (in a graph theory sense), all the vertices and propagators can be laid out flat on a plane, without any lines needing to cross "underneath" each other to construct the diagram. This can lead to a tremendous simplification of the color structure and associated operators for high-energy scattering in QCD, making the large-N_c limit highly advantageous in QCD. As an approximation to QCD, corrections to the large-N_c limit in real QCD often occurs at $\mathcal{O}\left(1/N_c^2\right)$ for physical observables. One would accordingly expect that the large-N_c limit is accurate at the level of $1/9 \sim 10\%$; however, for many observables, the large-N_c limit works even better in practice than this naive estimate.

2.3 Gauging the QCD Lagrangian

With the structure of $SU(N_c)$ in hand, we have all the ingredients we need to construct the QCD Lagrangian following the template of QED. Since the

[4]Caution: this statement applies only to the color representation, not to any other quantum numbers such as spin.

free multicomponent Dirac Lagrangian (2.10) is invariant under global $SU(N_c)$ color rotations (2.11), we have

$$\begin{aligned}
\delta\mathcal{L} = 0 &= \frac{\delta\mathcal{L}}{\delta\psi_i}\delta\psi_i + \delta\bar{\psi}_i\frac{\delta\mathcal{L}}{\delta\bar{\psi}_i} + \frac{\delta\mathcal{L}}{\delta(\partial_\mu\psi_i)}\delta(\partial_\mu\psi_i) \\
&= \left(\partial_\mu\frac{\delta\mathcal{L}}{\delta(\partial_\mu\psi_i)}\right)\delta\psi_i + \frac{\delta\mathcal{L}}{\delta(\partial_\mu\psi_i)}\partial_\mu(\delta\psi_i) \\
&= \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\psi_i)}\delta\psi_i\right) \\
&= \partial_\mu\left(-\bar{\psi}_i\gamma^\mu\phi^a(t^a)_{ij}\psi_j\right) \\
\therefore \qquad 0 &= \partial_\mu\left(\bar{\psi}_i\gamma^\mu(t^a)_{ij}\psi_j\right) \ ,
\end{aligned} \tag{2.23}$$

which gives a conserved color current

$$j^{\mu a} \equiv \bar{\psi}\gamma_\mu t^a\psi \tag{2.24}$$

for each of the $a = 1, \ldots, (N_c^2 - 1)$ possible “axes” of color rotation.

With the conserved currents, we can introduce the gluon field $A^{\mu a}$ by minimally coupling it to the free Lagrangian (2.10). Adding the term $g{j_\mu}^a A^{\mu a}$ with g the QCD coupling renders the scalar modes of $A^{\mu a}$ unphysical symmetry transfromations, constructing the Lagrangian

$$\begin{aligned}
\mathcal{L}_{gauged} &= \bar{\psi}(i\partial\!\!\!/ - m)\psi + g\bar{\psi}\gamma_\mu t^a\psi\, A^{\mu a} \\
&= \bar{\psi}\Big(i\gamma_\mu\left(\partial^\mu - igA^{\mu a}t^a\right) - m\Big)\psi \\
&= \bar{\psi}\Big(iD\!\!\!\!/ - m\Big)\psi \ ,
\end{aligned} \tag{2.25}$$

where we have defined the gauge-covariant derivative

$$D_\mu \equiv \partial_\mu - igA_\mu^a t^a \ . \tag{2.26}$$

For the Lagrangian (2.25) to be gauge invariant, the covariant derivative (2.26) must cancel the additional terms entering from a local gauge transformation. That is, the covariant derivative must satisfy

$$\begin{aligned}
\Big(D_\mu\psi\Big)' &\equiv e^{i\phi^a t^a}\Big(D_\mu\psi\Big) \\
\therefore \qquad (\partial_\mu - igA_\mu^{a\,\prime}t^a)e^{i\phi^a t^a}\psi &= e^{i\phi^a t^a}(\partial_\mu - igA_\mu^a t^a)\psi \ .
\end{aligned} \tag{2.27}$$

Writing the unitary gauge rotation for compactness as $\mathcal{U} \equiv e^{i\phi^a t^a}$, this gives the transformation law for the gauge field $A^{\mu a}$ as

$$A^{\mu a}t^a = A^{\mu a\,\prime}\left(\mathcal{U}^\dagger t^a\mathcal{U}\right) + \frac{i}{g}\mathcal{U}^\dagger(\partial_\mu\mathcal{U}) \ , \tag{2.28}$$

or, equivalently,

$$A^{\mu a\,\prime} = A^{\mu a} + \frac{1}{g}(\partial^\mu \phi^a) + f^{abc} A^{\mu b} \phi^c \,. \tag{2.29}$$

Interestingly, for a non-Abelian gauge theory like QCD, the gauge transformation includes both the shift of $A^{\mu a}$ by a scalar-polarized mode and the rotation of the gluon color from a to b (depending on the choice of rotation angles ϕ^c).

The last ingredient in the construction of QCD is the non-Abelian field-strength tensor, from which we can build the pure gluon term. From the commutator of the covariant derivatives, we have

$$\begin{aligned}
\left[D_\mu \,,\, D_\nu\right] &= \left[\partial_\mu - igA_\mu^a t^a \,,\, \partial_\nu - igA_\nu^b t^b\right] \\
&= -ig\partial_\mu A_\nu^b t^b + ig\partial_\nu A_\mu^a t^a - g^2 A_\mu^a A_\nu^b \left[t^a \,,\, t^b\right] \\
&= -ig\partial_\mu A_\nu^b t^b + ig\partial_\nu A_\mu^a t^a - ig^2 f^{abc} A_\mu^a A_\nu^b \, t^c \\
&= -ig\big(\partial_\mu A_\nu^c - \partial_\nu A_\mu^c + g f^{abc} A_\mu^a A_\nu^b\big)\, t^c \\
\left[D_\mu \,,\, D_\nu\right] &\equiv -ig\, F_{\mu\nu}^c \, t^c \,,
\end{aligned} \tag{2.30}$$

where we have defined the non-Abelian field-strength tensor

$$F_{\mu\nu}^c \equiv \partial_\mu A_\nu^c - \partial_\nu A_\mu^c + g f^{abc} A_\mu^a A_\nu^b \,. \tag{2.31}$$

By constructing $F_{\mu\nu}^a$ in a gauge-covariant way using (2.30), we have produced a field-strength tensor (2.31) which is more than just the free kinetic part which occurs in QCD. This is because, unlike in QED, now the free part $(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)$ is *not gauge invariant* (or even gauge covariant). Instead, the "chromo-electric" and "chromo-magnetic" fields are themselves not separately gauge invariant, transforming as

$$F_{\mu\nu}^{\prime\, a} = \mathcal{U} F_{\mu\nu}^{\prime\, a} \mathcal{U}^\dagger \,. \tag{2.32}$$

While the field-strength tensor (2.31) is not gauge invariant, its *square* still is. This allows us to immediately write down the QCD Lagrangian,

$$\begin{aligned}
\mathcal{L}_{QCD} &= \bar\psi(i\not{D} - m)\psi - \frac{1}{4} F_{\mu\nu}^a F^{\mu\nu\, a} \\
&= \bar\psi(i\not{\partial} - m)\psi - \frac{1}{4}(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a)(\partial^\mu A^{\nu\, a} - \partial^\nu A^{\mu\, a}) \\
&\qquad + g\bar\psi\gamma_\mu t^a \psi\, A^{\mu a} - g f^{abc}\, A_\mu^b A_\nu^c (\partial^\mu A^{\nu a}) \\
&\qquad - \frac{1}{4} g^2 f^{abc} f^{ab'c'}\, A_\mu^b A_\nu^c A^{\mu\, b'} A^{\nu\, c'} \,.
\end{aligned} \tag{2.33}$$

The pure glue part $-\frac{1}{4}F_{\mu\nu}^a F^{\mu\nu\, a}$ and the covariant quark part $\bar\psi(i\not{D} - m)\psi$ are *separately gauge invariant*, leading to the generation of not only the quark-gluon interaction vertex $g\bar\psi\gamma_\mu t^a \psi\, A^{\mu a}$, but also to interactions among the gluons themselves through the three-gluon vertex $g\bar\psi\gamma_\mu t^a \psi\, A^{\mu a}$ and four-gluon vertex $-g f^{abc}\, A_\mu^b A_\nu^c (\partial^\mu A^{\nu a})$.