CHAPTER 1

HUGS 2022 Lecture 1: Construction of QCD (Part 1)

1.1 Overview: The Theoretical Framework

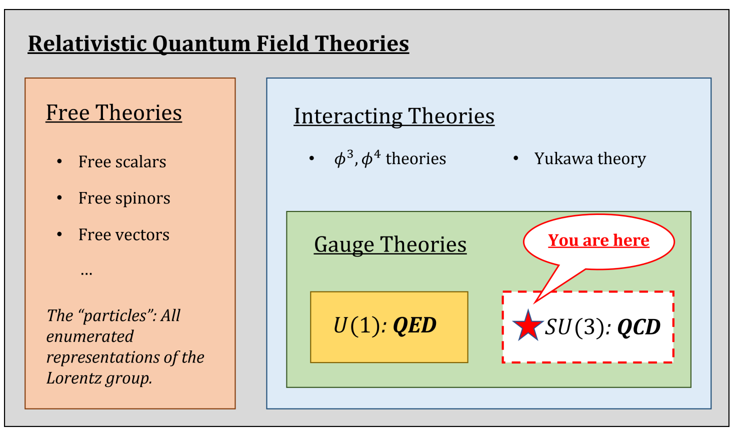

Figure 1.1: Illustration of QCD's place in the landscape of relativistic quantum field theories.

Quantum Field Theory

By way of introduction, let us begin by orienting ourselves with respect to the theoretical language and calculculational approach we will employ to describe QCD, as visualized in Fig. 1.1. The language of relativistic quantum mechanics is *quantum field theory* (QFT). QFT expresses the physics of multiparticle creation and annihilation in a manifestly Lorentz-covariant formalism, which elegantly solves the problem of causality for a relativistic quantum theory.

The simplest quantum theories are "free" theories – Lagrangians whose equations of motion are *linear*, such that the solutions obey a noninteracting superposition principle. These free theories define the "particles," with one Lorentz-invariant Lagrangian that can be enumerated for each representation of the Lorentz group $SO^+(3,1)$. More complex Lagrangians with nonlinear equations of motion are interpreted as *interactions* among the particles. The "traditional" approach to solving interacting theories is *perturbation theory*: a Taylor series expansion of physical observables in terms of a small coupling constant, such as the fine structure constant $\alpha_{EM} \approx 1/137$ in quantum electrodynamics (QED). This allows for a systematic order-by-order computation of observables with controllable error in QFT.

Perturbative and Nonperturbative Effects

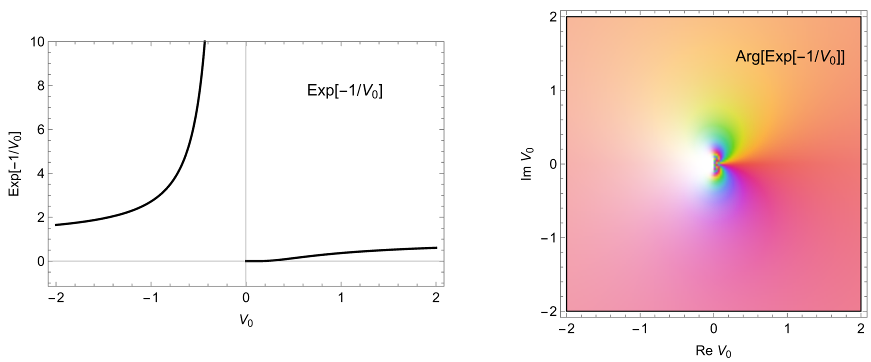

Figure 1.2: Behavior of the essential singularity $\exp[-1/V_0]$ in the complex plane.

Perturbation theory is a powerful, systematic approach to studying interacting theories, but it is not complete. The fundamental philosophy is an expansion of observables in a smooth Taylor series at small values of the coupling constant. The assumed smoothness of this expansion can fail in a number of important ways. First, there can be fundamental physics effects which are essentially *nonperturbative* in nature. Nonperturbative effects are often associated with a radical realignment of the degrees of freedom in a system which causes perturbation theory to break down. One famous example is nonperturbative transition from degrees of freedom resembling electrons and holes in a normal metal to Cooper pairs in a superconductor. As computed in the BCS theory of superconductivity, the binding energy E_b of a Cooper pair is

$$E_b = 2\omega_D \, e^{-2/V_0 \rho(\varepsilon_F)} \,, \tag{1.1}$$

where V_0 is the strength of the effective potential between two electrons, ω_D is the Debye frequency and $\rho(\varepsilon_F)$ is the density of states at the Fermi energy[1]. Rather

[1] For more information, see these nice lecture notes.

than having a well-behaved Taylor series at weak coupling $V_0 \to 0$, the binding energy (1.1) instead has an essential singularity of the form $f(V_0) = e^{-1/V_0}$ shown in Fig. 1.2. This form, which often occurs in nonperturbative mechanisms, *has no Taylor series*:

$$\begin{aligned} f(V_0) &\stackrel{?}{=} f(V_0 = 0) + V_0 \left. \frac{df}{dV_0} \right|_{V_0=0} + \frac{1}{2} V_0^2 \left. \frac{d^2 f}{dV_0^2} \right|_{V_0=0} + \cdots \\ &\stackrel{!}{=} 0 + 0 + 0 + \cdots . \end{aligned} \tag{1.2}$$

A perturbative calculation of the binding energy (1.1) would yield *zero, to all orders in perturbation theory* and could never discover the superconducting phase transition. Likewise, one can construct a "proof" that $E_b = 0$ to all orders in perturbation theory, but this still misses the possibility of nonperturbative contributions.

This example illustrates that while perturbation theory can capture the smooth evolution of the degrees of freedom in a quantum field theory, by construction it cannot capture a severe rupture in those degrees of freedom such as during a phase transition. Exactly the same kind of mechanism (1.1) which leads to the formation of a Cooper-pair condensate and the superconducting phase transition in BCS theory is also responsible for the spontaneous breaking of chiral symmetry in QCD, which is associated with quark confinement and the emergence of the proton mass.

Asymptotic Series in Perturbation Theory

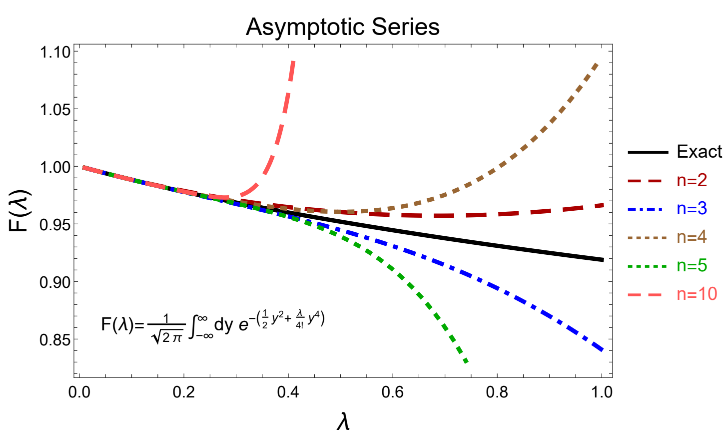

Figure 1.3: One example of the breakdown of convergence in an asymptotic series.

Another important caveat to the applicability of perturbation theory is the fact that the *perturbation series does not converge* in general. Rather than producing an absolutely-convergent series, the perturbation expansion is an asymptotic series. This means that, while a fixed-order calculation may provide a good estimate of the true solution, the accuracy of that estimate does not necessarily increase as one goes to higher orders. An example of an asymptotic series is shown in Fig. 1.3. Note that the fixed-order approximation gets better as one increases the accuracy from LO, to NLO ($n = 2$), to NNLO ($n = 3$). But as one pushes the perturbative expansion to higher and higher orders, the *error starts to increase* rather than decrease. The $n = 5$ curve is clearly worse than the $n = 3$ curve, and the $n = 10$ curve is a terrible approximation to the exact solution. This illustrates that, sometimes, "working harder" (computing to higher accuracy in the perturbation series) doesn't always pay off.

Resummation in Perturbation Theory

Finally, there can be a more gradual evolution of the degrees of freedom in a quantum system which **is** capturable in perturbation theory. This often occurs when there is a systematic enhancement of certain amplitudes due to a logarithmically large phase space, for instance in integrals of the form

$$\alpha \int_{\Lambda^2}^{E^2} \frac{dk_\perp^2}{k_\perp^2} = \alpha \ln \frac{E^2}{\Lambda^2} \text{ , or} \tag{1.3a}$$

$$\alpha \int_{\Lambda/E}^{1} \frac{dz}{z} = \alpha \ln \frac{E}{\Lambda} \, . \tag{1.3b}$$

This creates an interesting tension where the weak-coupling approximation $\alpha \ll 1$ may still be valid, but certain amplitudes are systematically enhanced by a large logarithm $\ln E/\Lambda$ coming from the limits of the phase space. If that logarithm becomes large enough, it can begin to compete with the smallness of the coupling α. In the limit

$$\alpha \ll 1 \qquad , \qquad \ln \frac{E}{\Lambda} \gg 1 \qquad , \qquad \alpha \ln \frac{E}{\Lambda} \sim \mathcal{O}(1) \; , \tag{1.4}$$

these systematically enhanced diagrams are not "small" at all, and we must *re-sum them all*, since

$$1 \sim \alpha \ln \frac{E}{\lambda} \sim \left(\alpha \ln \frac{E}{\lambda} \right)^2 \sim \left(\alpha \ln \frac{E}{\lambda} \right)^3 \sim \cdots \sim \left(\alpha \ln \frac{E}{\lambda} \right)^3 \, . \tag{1.5}$$

Resumming these large logarithmic corrections can be accomplished by expressing them as a differential equation; if that equation can be solved, then its solution encodes the iteration of these enhanced corrections to all orders in perturbation theory. This procedure re-orders the perturbation series, giving the *leading-logarithmic approximation* (LLA) to the resummation. This description of a gradual transformation of degrees of freedom in QFT is often referred to as *quantum evolution*. It occurs prominently in QCD in several forms, including the DGLAP evolution with Q^2 and BFKL evolution with x_B of parton distribution functions.

1.2 Gauge Theory of $U(1)$: Quantum Electrodynamics

The Golden Archetype of Gauge Symmetry

Quantum Electrodynamics (QED) is defined by the Lagrangian

$$\mathcal{L}_{QED} = \bar{\psi}(i\partial\!\!\!/ - m)\psi - \frac{1}{4}F_{\mu\nu}F^{\mu\nu} - e\bar{\psi}\gamma_\mu\psi\, A^\mu\,, \tag{1.6}$$

which describes the interactions of charged fermions ("electrons") with vector bosons ("photons"). QED is a *gauge theory*, meaning that the particular *form of the photon/electron interaction vertex is uniquely dictated* by a symmetry transformation (called a "gauge symmetry") of the Lagrangian (1.6). The gauge symmetry is not a quirk of QED; it is an essential feature necessary to even define an interacting vector boson.

Global Symmetry and Conserved Current

The QED Lagrangian (1.6) is uniquely obtained from the free Dirac Lagrangian

$$\mathcal{L}_{Dirac} = \bar{\psi}(i\partial\!\!\!/ - m)\psi \tag{1.7}$$

through the process of *minimal coupling*. The free Dirac Lagrangian (1.7) is invariant under the global symmetry transformation

$$\psi'(x) = e^{i\phi}\psi(x) \qquad \text{or} \qquad \begin{bmatrix} \mathrm{Re}\,\psi'(x) \\ \mathrm{Im}\,\psi'(x) \end{bmatrix} = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix} \begin{bmatrix} \mathrm{Re}\,\psi(x) \\ \mathrm{Im}\,\psi(x) \end{bmatrix}\,, \tag{1.8}$$

where ϕ is an arbitrary constant. This symmetry transformation is just a complex phase rotation, which may be regarded as a "1×1 unitary matrix". This describes the Lie group $U(1)$, which is a global symmetry of the free Dirac Lagrangian (1.7). Since charge conjugation $\psi \leftrightarrow \bar{\psi}$ changes particles into antiparticles, this $U(1)$ rotation may be regarded as a continuous rotation which redefines the particles and antiparticles.

By Noether's theorem, the invariance of the Lagrangian (1.7) under the continuous symmetry transformation (1.8) implies the existence of a conserved current:

$$\begin{aligned} \delta\mathcal{L} = 0 &= \frac{\delta\mathcal{L}}{\delta\psi}\delta\psi + \delta\bar{\psi}\frac{\delta\mathcal{L}}{\delta\bar{\psi}} + \frac{\delta\mathcal{L}}{\delta(\partial_\mu\psi)}\delta(\partial_\mu\psi) \\ &= \left(\partial_\mu \frac{\delta\mathcal{L}}{\delta(\partial_\mu\psi)}\right)\delta\psi + \frac{\delta\mathcal{L}}{\delta(\partial_\mu\psi)}\partial_\mu(\delta\psi) \\ &= \partial_\mu\left(\frac{\delta\mathcal{L}}{\delta(\partial_\mu\psi)}\delta\psi\right) \\ 0 &= \partial_\mu\left(\bar{\psi}\gamma^\mu\psi\right) \end{aligned} \tag{1.9}$$

where we have used the equations of motion. The *net particle number current* is conserved:

$$j^\mu = \bar{\psi}\gamma^\mu\psi \qquad , \qquad \partial_\mu j^\mu = 0\,, \tag{1.10}$$

reflecting the conservation of electric charge. The electromagnetic current $J^\mu_{EM} = e\bar{\psi}\gamma^\mu\psi$ is just the particle number current, weighted by the charge.

The Problem with Vector Bosons: Scalar Polarization Mode

The form of the QED interaction vertex (1.6) is of the form

$$\mathcal{L}_I = -J^\mu_{EM} A_\mu = -e\bar{\psi}\gamma^\mu\psi\, A_\mu, \tag{1.11}$$

which introduces the photon field A^μ as being *created by the conserved current* j^μ. This choice of vertex directly links the properties of the conserved current j^μ produced by the $U(1)$ global symmetry and the structure of the vector field A^μ. This is not just a curiosity; it is an essential feature necessary for the photon field A^μ to be well-defined at all. The reason is that, in 4-dimensional spacetime, there are potentially four independent polarization modes of A^μ, including the "timelike" or "scalar polarization" mode, which can be written as the gradient of a scalar field:

$$A^\mu_{(scalar)}(x) = \partial^\mu\phi(x)\,. \tag{1.12}$$

The timelike polarization is "unphysical" – if quantized, it would lead to states of negative norm, which are incompatible with a Hilbert space of quantum states. Getting rid of the sick scalar polarization is essential for any self-consistent quantum theory of vector bosons.

An interaction vertex of the form (1.11) which couples the vector boson to a *conserved* current arising from a global symmetry eliminates the scalar-polarized modes (1.12) in an elegant way: by reducing them to a *symmetry transformation* on the Lagrangian. If we shift A^μ by the addition of a scalar-polarized mode,

$$A'^\mu(x) = A^\mu(x) + \partial^\mu\phi(x) \tag{1.13a}$$

$$\mathcal{L}'_I = \mathcal{L}_I - ej^\mu\partial_\mu\phi = \mathcal{L}_I - \partial_\mu(ej^\mu\phi) \tag{1.13b}$$

the interaction term is *invariant* (up to an irrelevant total derivative). Thus, with a special interaction vertex of the form (1.11), the scalar modes are removed as "redundant, unphysical degrees of freedom" which have no consequence on observables.

Minimal Coupling: Gauging the QED Lagrangian

The extension of the free Dirac Lagrangian (1.7) to include the *minimal coupling* to the vector field A^μ through a term of the form $-j_\mu A^\mu$ can be compactly expressed using the *gauge-covariant derivative*:

$$\mathcal{L}_{gauged} = \bar{\psi}(i\gamma_\mu\partial^\mu - m)\psi - e\,\bar{\psi}\gamma_\mu\psi\, A^\mu$$

$$
\begin{aligned}
&= \bar{\psi}[i\gamma_\mu(\partial^\mu + ie\,A^\mu) - m]\psi \\
&\equiv \bar{\psi}(i\not{D} - m)\psi\,. \qquad (1.14)
\end{aligned}
$$

The remarkable physics of extending the local $U(1)$ symmetry to form the basis of a local $U(1)$ gauge theory is encoded in the simple replacement of the partial derivative with the covariant derivative:

$$\partial_\mu \to D_\mu \equiv \partial_\mu + ie\,A_\mu\,. \qquad (1.15)$$

The covariant derivative expresses the fact that the shift (1.13) of A^μ by the addition of a scalar mode is now interconnected with the $U(1)$ symmetry of the Dirac Lagrangian (1.7). Under the more general version of the transformation (1.8) in which the rotation phase $\phi(x)$ can vary as a function of spacetime,

$$\psi'(x) = e^{i\phi(x)}\psi(x)\,, \qquad (1.16)$$

the Lagrangian (1.14) transforms as

$$
\begin{aligned}
\mathcal{L}'(x) &= \bar{\psi}'(x)(i\gamma_\mu\partial^\mu - m)\psi'(x) - e\,\bar{\psi}'(x)\gamma_\mu\psi'(x)\,A^{\mu\,\prime}(x) \\
&= \bar{\psi}(x)\,e^{-i\phi(x)}(i\gamma_\mu\partial^\mu - m)e^{i\phi(x)}\,\psi(x) - e\,\bar{\psi}(x)\,e^{-i\phi(x)}\gamma_\mu e^{i\phi(x)}\,\psi(x)\,A^{\mu\,\prime}(x) \\
&= \bar{\psi}(x)(i\gamma_\mu\partial^\mu - m)\psi(x) - \bar{\psi}(x)\gamma_\mu\psi(x)\,\partial^\mu\phi(x) - e\,\bar{\psi}(x)\gamma_\mu\psi(x)\,A^{\mu\,\prime}(x) \\
&= \bar{\psi}(x)(i\gamma_\mu\partial^\mu - m)\psi(x) - e\,\bar{\psi}(x)\gamma_\mu\psi(x)\left[A^{\mu\,\prime}(x) + \frac{1}{e}\partial^\mu\phi(x)\right]. \qquad (1.17)
\end{aligned}
$$

The effect of the *local* transformation (1.16) can be interpreted as *shifting* A^μ *by a scalar-polarized mode*, exactly as we did in Eq. (1.13). Since the minimal coupling of A^μ to the conserved current $j_m u$ guarantees that a shift by a scalar mode is an unphysical symmetry transformation, the two pieces (global $U(1)$ symmetry and scalar polarizations of $A_m u$) work in tandem as part of a single composite symmetry operation known as *gauge symmetry*.

Choosing a special interaction vertex of the form (1.11) has united the elimination of the scalar-polarized mode with the global $U(1)$ symmetry responsible for the conserved current j^μ. This choice effectively *enlarges the* $U(1)$ *symmetry group* from a global symmetry to a local symmetry and uses it to *define the photon* field A^μ. Without a gauge-invariant coupling of this form, the photon field *could not exist at all*, since it would be polluted with unworkable scalar-polarized modes.

Geometric Interpretation

From the point of view of the local gauge transformation (1.16), the covariant derivative (1.15) has a natural geometric interpretation. The ordinary partial derivative ∂_μ is not invariant under the local $U(1)$ transformation (1.16), because different points x^μ of spacetime transform differently, so the partial derivative makes an unequal comparison between two adjacent points. The covariant derivative (1.15) *compensates* for the different transformations of $\psi(x)$ at

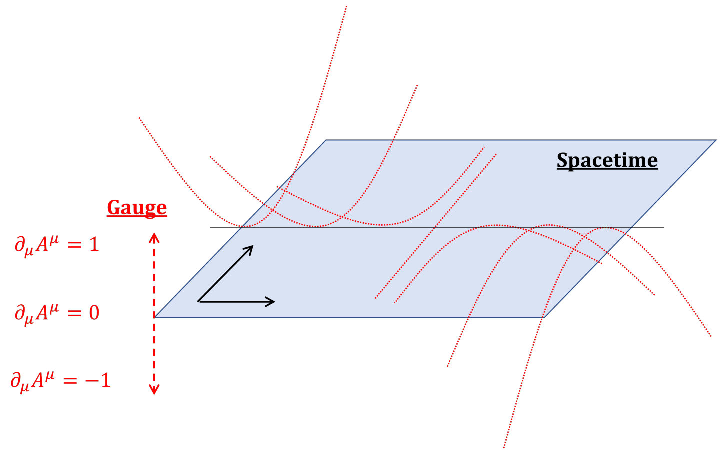

Figure 1.4: Sketch of the combined spacetime + gauge manifold.

different points, permitting a simple meaningful (gauge-invariant) comparison between two points.

Under the combined transformation

$$\psi'(x) = e^{i\phi(x)}\psi(x) \,, \tag{1.18a}$$

$$A^{\mu\,\prime}(x) = A^\mu(x) - \frac{1}{e}\partial^\mu\phi(x) \,, \tag{1.18b}$$

the covariant derivative transforms as

$$\begin{aligned}\big(D_\mu\psi(x)\big)' &\equiv (\partial_\mu + ie\,A'_\mu)\,\psi'(x) \\ &= (\partial_\mu + ie\,A_\mu - i\partial_\mu\phi)\,e^{i\phi(x)}\psi(x) \\ &= e^{i\phi(x)}(\partial_\mu + ie\,A_\mu)\psi(x) \\ &= e^{i\phi(x)}\,D_\mu\psi(x) \,. \end{aligned} \tag{1.19}$$

That is: *the covariant derivative of a field has the same gauge transformation as the field itself.* This is the precise mathematical statement that the covariant derivative *cancels the nonlocal differences in the gauge transformation*, providing a meaningful way to compare two different points in spacetime in a gauge-invariant way. For instance, it is clear that $\bar{\psi}D_\mu\psi$ is gauge invariant.

We have found that, by minimally coupling the photon field to the conserved electric current, we have constructed a Lagrangian (1.14) which is invariant under a *mix* of Lorentz transformations and shifting A^μ by scalar modes. Even though spacetime itself is "flat" (in this discussion), the different transformations of neighboring points under a local $U(1)$ transformation act as if the *combined* spacetime + gauge manifold is *curved*. From this point of view, the problem of how to compute meaningful derivatives on a curved manifold is a standard one in differential geometry. The covariant derivative (1.15) precisely introduces the photon field A_μ as a metric connection which compensates for the curvature of the gauge dimensions along the physical spacetime dimensions.

Field-Strength Tensor as a Generalized Curl

The last piece of the QED Lagrangian (1.6) is the kinetic term associated with the free photon field. The free-field Lagrangian for A^μ must be quadratic in A, coupled to derivatives to contain momenta, and a Lorentz scalar. It must be massless, as an essential requirement of gauge symmetry[2]. From these considerations, the kinetic term can be deduced to be

$$\mathcal{L}_{kinetic} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \,, \tag{1.20}$$

where the antisymmetric field-strength tensor $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is a four-dimensional generalization of the curl, transforming as an antisymmetric rank-2 tensor under Lorentz tranformations.

Based on the geometric interpretation of gauge transformations discussed previously, we expect that this generalized curl should also have a sensible interpretation under gauge transformations. Indeed, the field-strength tensor $F^{\mu\nu}$ has a particularly simple expression in terms of the covariant derivative (1.15):

$$\begin{aligned} \Big[D_\mu \,,\, D_\nu \Big] &= ie(\partial_\mu A_\nu - \partial_\nu A_\mu) = ie\, F_{\mu\nu} \\ \therefore \qquad F_{\mu\nu} &= \frac{-i}{e} \Big[D_\mu \,,\, D_\nu \Big] \,. \end{aligned} \tag{1.21}$$

This expression, as a commutator of covariant derivatives, describes the generalized curl *including the curvature along the gauge direction*. Since D_μ is itself a gauge-covariant quantity (transforming locally under gauge transformations), so is $F_{\mu\nu}$. In the case of QED, F_μ

$issimplygauge - invariant, butinthemoregeneralcase,F_{\mu\nu}$ may transform under gauge transformations, and its transformation properties are dictated by those of the covariant derivative D_μ.

Next Steps

Taken together, the QED Lagrangian (1.6) and all the physics of electrodynamics are consequences of a single unifying principle: $U(1)$ gauge symmetry. Physically, this gauge group encodes the statement that *electric charge is a scalar quantity*. Electric charge may be positive or negative (electrons and positrons), but it has no "direction" associated with it. This is reflected in the fact that $U(1)$ corresponds to rotation by a complex phase, without any matrix dimension to it. In a different gauge group, such as $SU(2)$ (the Pauli matrices), the gauge transformation could employ a nontrivial matrix structure. This single difference is responsible for the enormous complexity of QCD.

[2] Massive vector bosons have a different relation with their scalar-polarized modes, whose removal is generally enforced by constraint.

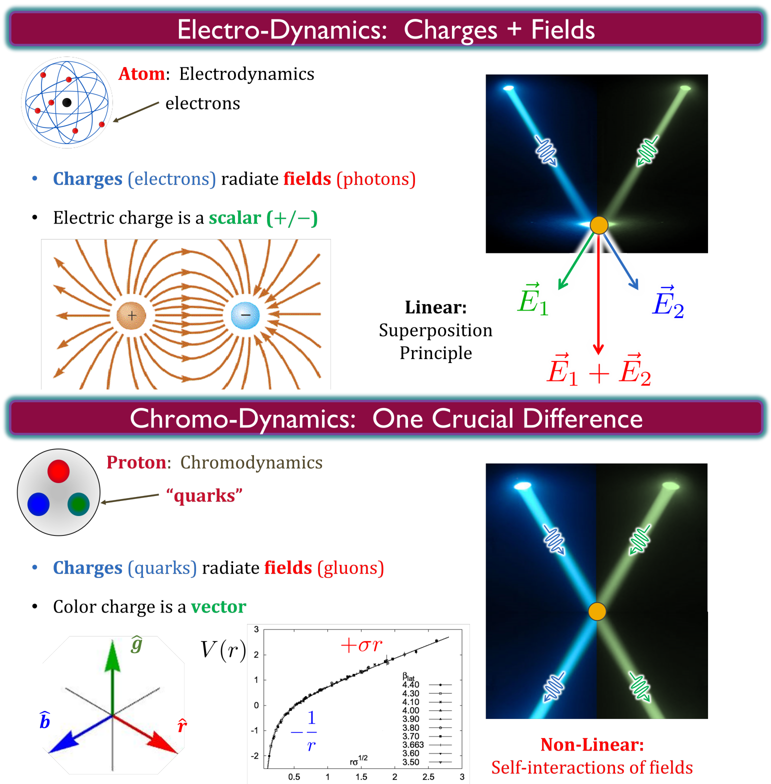

Figure 1.5: Comparison of the role of "charge" in QED versus QCD.