Connecting Lattice and Continuum TMDs with Factorization

Iain Stewart

MIT

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Massachusetts Institute of Technology

U.S. DEPARTMENT OF ENERGY | Office of Science

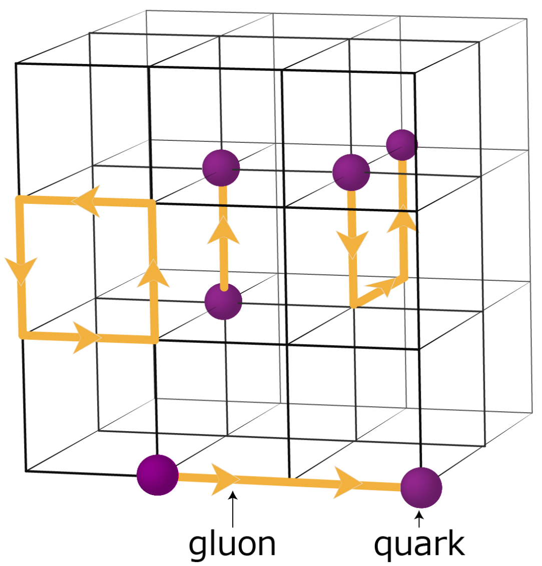
TMD Collaboration

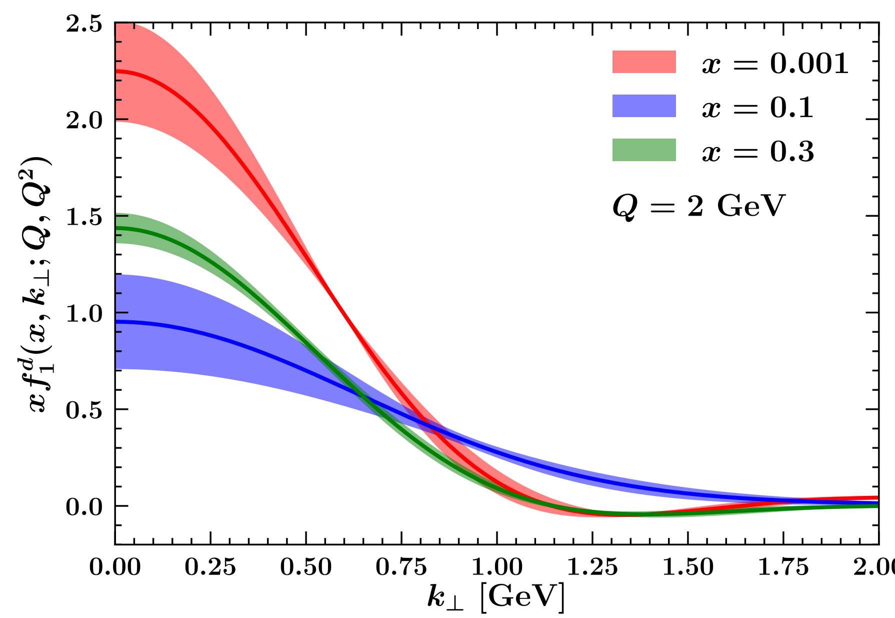
Goal

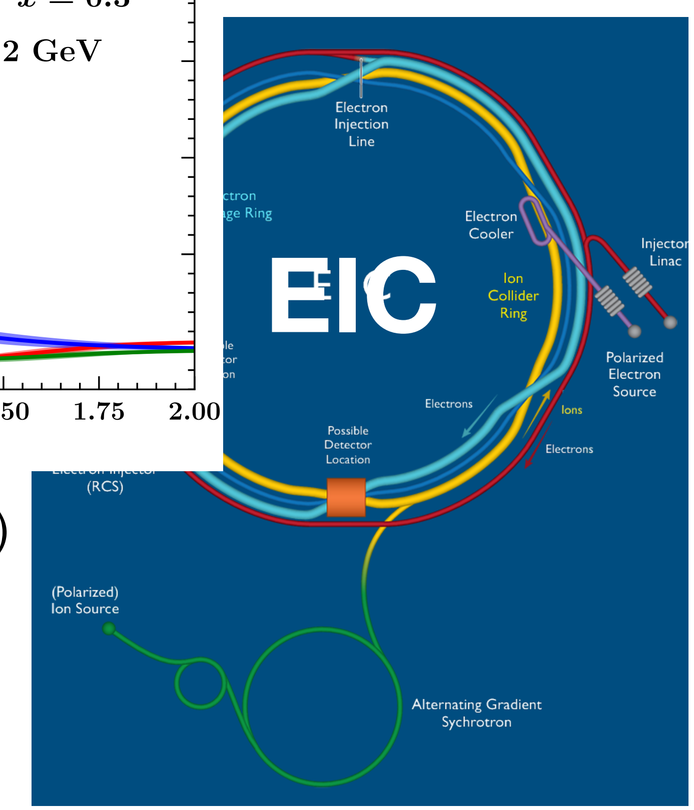
Ebert, Schindler, IS, Zhao (arXiv:2201.08401)

Proof of Factorization connecting
Quasi-TMDs (Lattice) and Collins-TMDs (Continuum)

$$\lim_{\tilde\eta\to\infty} \tilde f_{i/h}(x, \vec b_T, \mu, \zeta, x\tilde P^z, \tilde\eta) = C_i\big(x\tilde P^z, \mu\big)\, f^C_{i/h}(x, \vec b_T, \mu, \zeta) + \ldots$$







Bachetta et.al. (1912.07550)

Ebert, Schindler, IS, Zhao (arXiv:2201.08401)

Goal

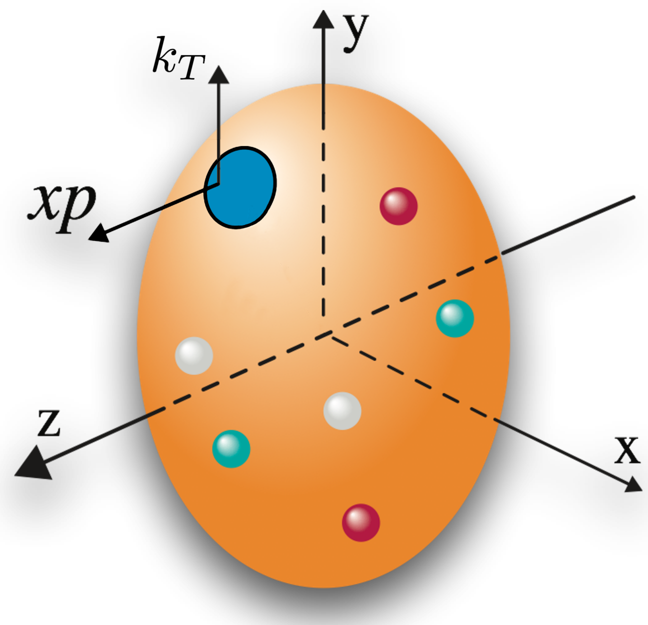
Proof of Factorization connecting
Quasi-TMDs (Lattice) and Collins-TMDs (Continuum)

$$\lim_{\tilde{\eta}\to\infty} \tilde{f}_{i/h}(x, \vec{b}_T, \mu, \zeta, x\tilde{P}^z, \tilde{\eta}) = C_i\big(x\tilde{P}^z, \mu\big)\, f^C_{i/h}(x, \vec{b}_T, \mu, \zeta) + \ldots$$

Outline

- Introduction
- Setup a General Framework
- Proof
- Implications

TMDs

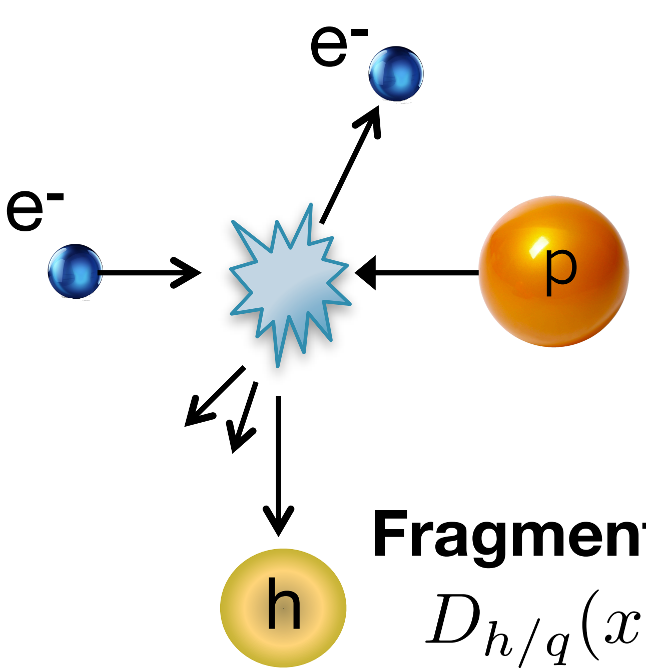


$$f_{q/P}(x, k_T, \mu, \zeta)$$

longitudinal & Transverse

Semi-Inclusive DIS

$$\sigma \sim f_{q/P}(x, k_T) D_{h/q}(x, k_T)$$

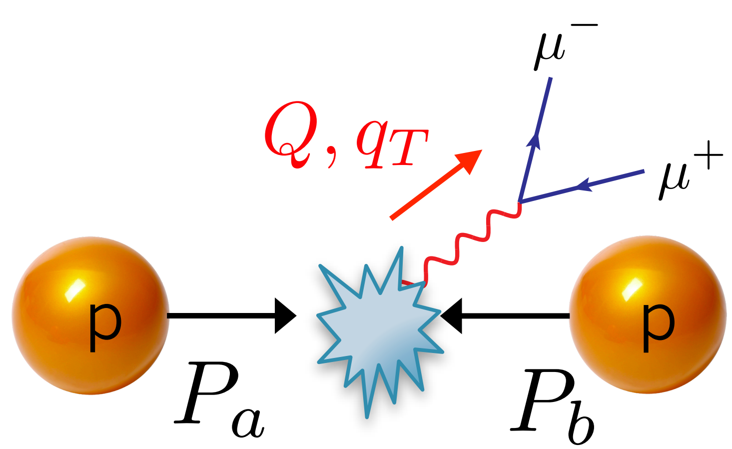


Fragmentation

$$D_{h/q}(x, k_T)$$

Drell-Yan

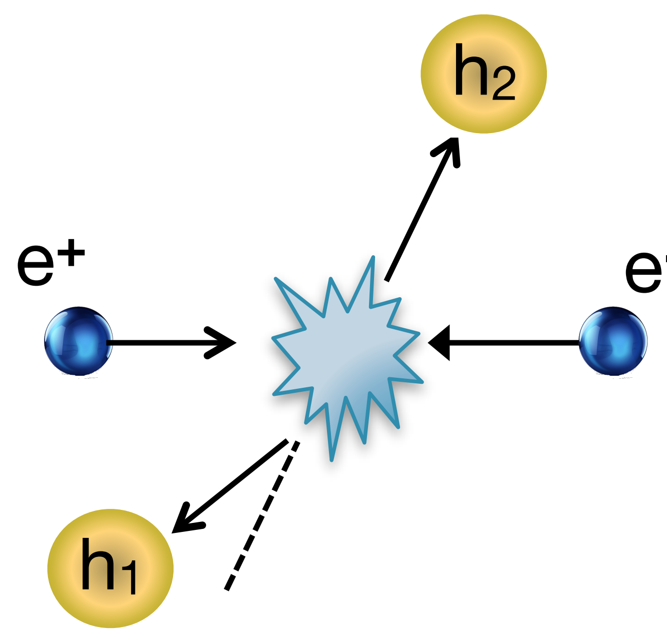
$$\sigma \sim f_{q/P}(x, k_T) f_{q/P}(x, k_T)$$



$$q_T \ll Q$$

Dihadron in e+e-

$$\sigma \sim D_{h_1/q}(x, k_T) D_{h_2/q}(x, k_T)$$



TMD Factorization (Drell Yan)

CSS (Collins, Soper, Sterman)

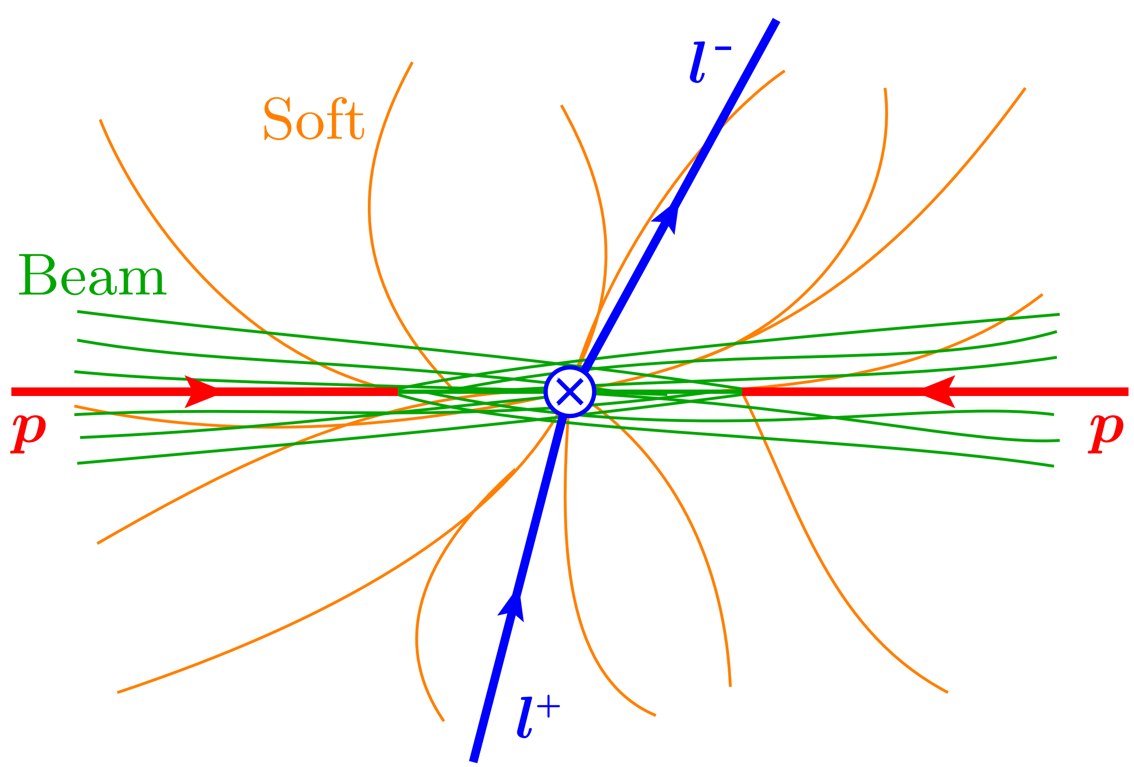
SCET (Soft Collinear Effective Theory)

$$\frac{d\sigma}{dQ\,dY\,dq_T^2} = H(Q,\mu) \int d^2\vec{b}_T\, e^{i\vec{q}_T\cdot\vec{b}_T}\, f_q(x_a,\vec{b}_T,\mu,\zeta_a)\, f_q(x_b,\vec{b}_T,\mu,\zeta_b) \left[1 + \mathcal{O}\Big(\frac{q_T^2}{Q^2}\Big)\right]$$

Hard virtual corrections

TMDs

$$f_q(x,\vec{b}_T,\mu,\zeta) \sim Z_{\rm uv} B_q / \sqrt{S_q}$$



TMD Definitions

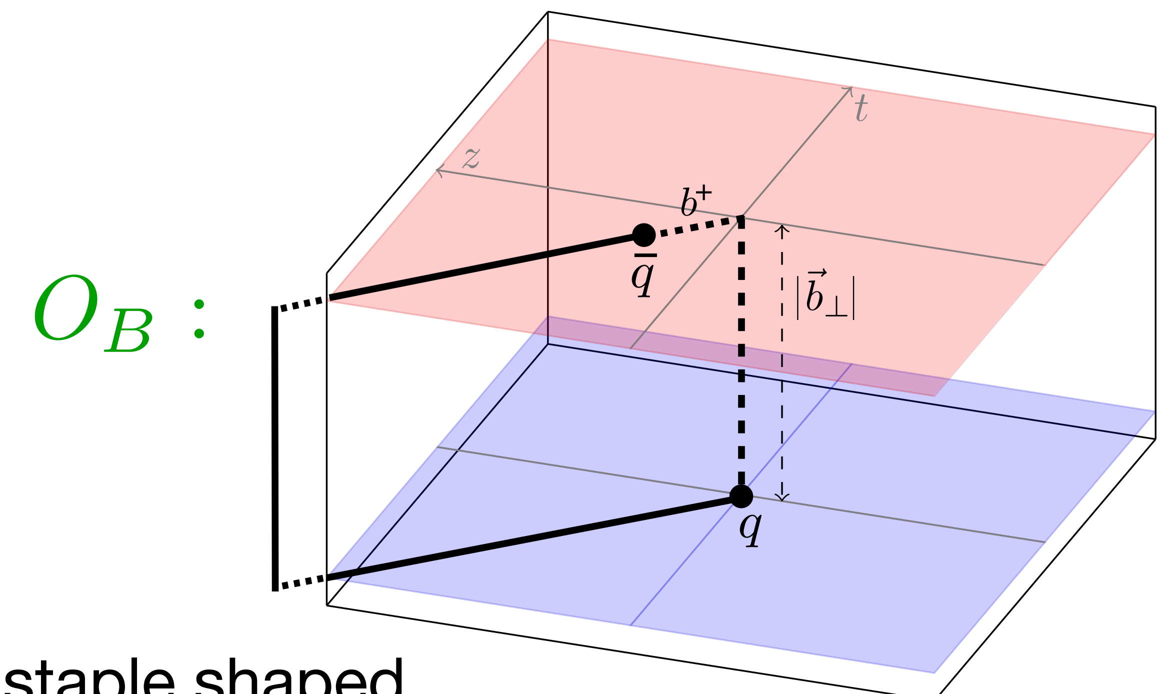
Beam Function **Soft factor**

$$f_q(x,\vec{b}_T,\mu,\zeta) = \lim_{\epsilon\to 0} Z_{\rm uv}(\epsilon,\mu,\zeta) \lim_{\tau\to 0} B_q(x,\vec{b}_T,\epsilon,\tau,\zeta) \;/ \sqrt{S_q(b_T,\epsilon,\tau)}$$

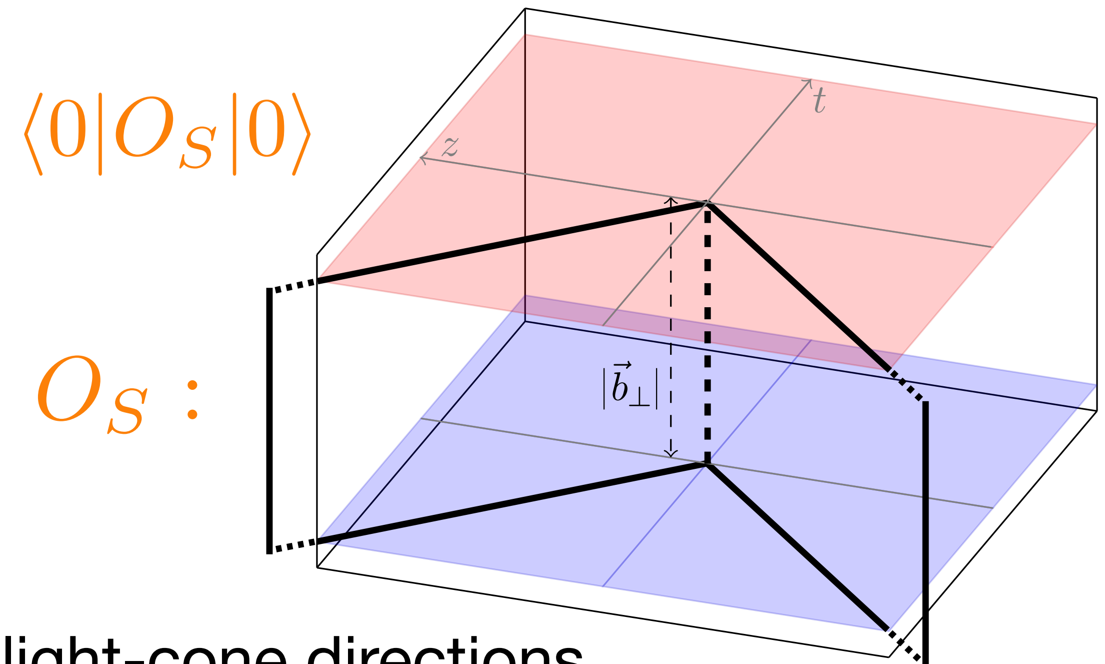
rapidity limit

$$B_q = \mathrm{FT}_{b^+} \langle p|O_B|p\rangle$$

$$S_q = \langle 0|O_S|0\rangle$$



staple shaped Wilson lines



two light-cone directions

Lattice calculations must overcome light-cone nature of TMD definitions.

Evolution

$$f_q(x, \vec{b}_T, \mu, \zeta)$$

μ = renormalization scale

ζ = Collins-Soper parameter

Sum large logarithms: $\ln(Q^2 b_T^2) \sim \ln \frac{Q^2}{q_T^2}$

$$\zeta = 2\big(xP^+ e^{-y_n}\big)^2$$

<u>Solution:</u>

$$f_q(x, \vec{b}_T, \mu, \zeta) = \exp\Big[\int_{\mu_0}^{\mu} \frac{d\mu'}{\mu'} \gamma_\mu^q(\mu', \zeta_0)\Big] \exp\Big[\frac{1}{2}\gamma_\zeta^q(\mu, b_T) \ln \frac{\zeta}{\zeta_0}\Big] \times f_q(x, \vec{b}_T, \mu_0, \zeta_0)$$

CS kernel

Connect Lattice calc. or model: $\mu_0, \sqrt{\zeta_0} \sim \mathrm{GeV}$

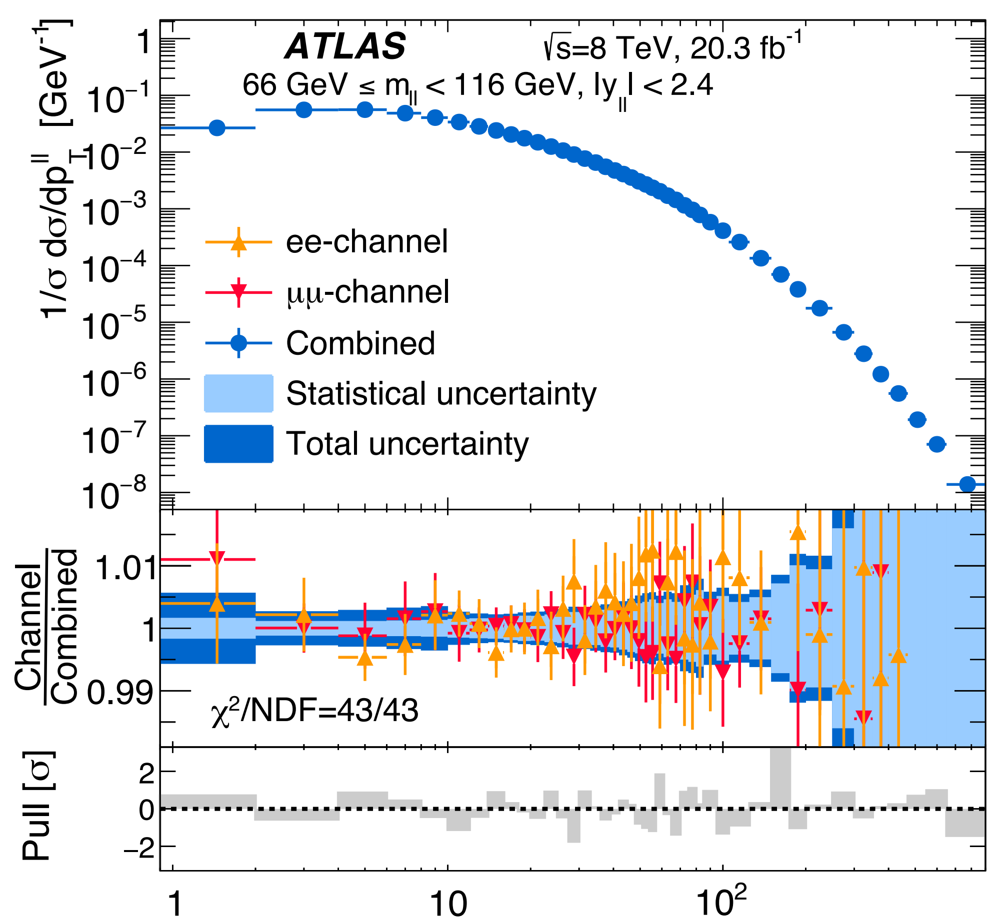
with scales needed for σ : $\mu, \sqrt{\zeta} \sim Q$

Nonperturbative contributions in both

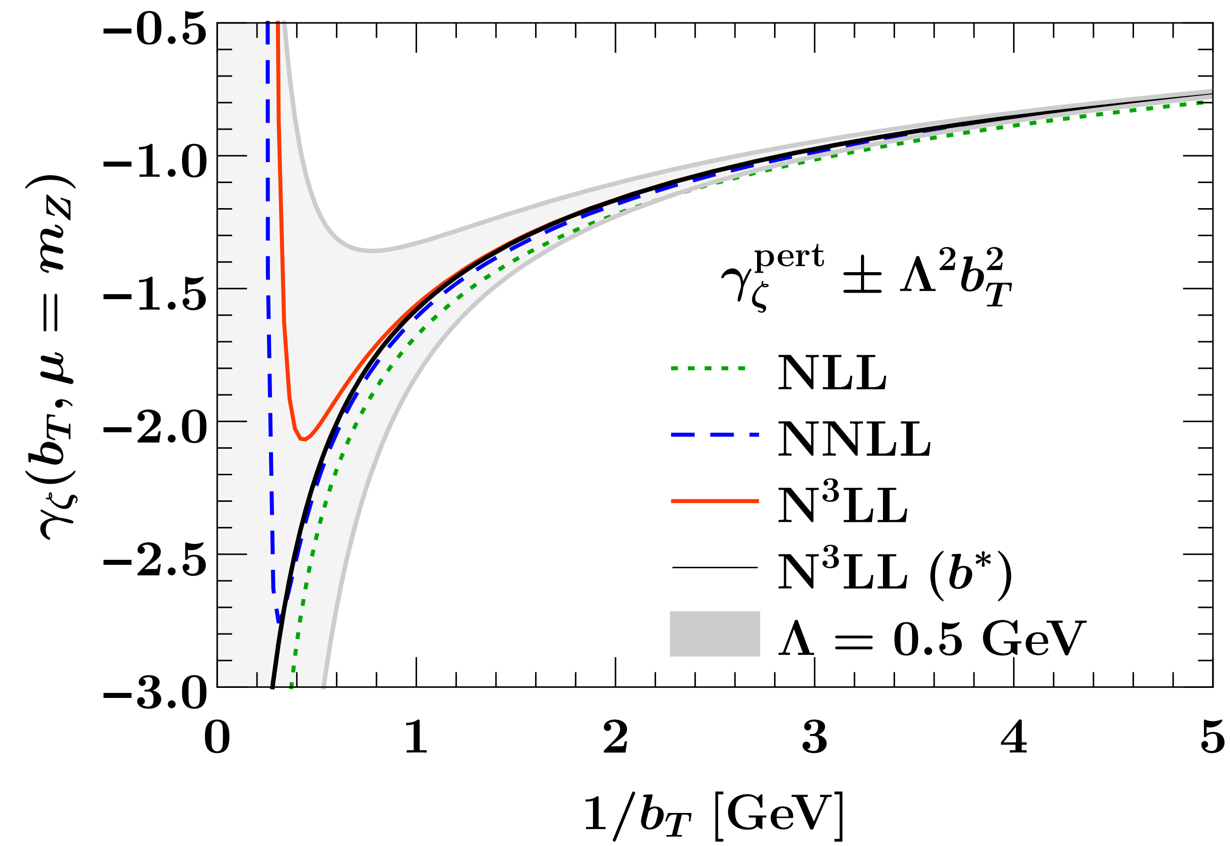
$$\underbrace{f_q(x, b_T, \mu_0, \zeta_0), \quad \gamma_\zeta^q(\mu, b_T)}_{\text{Targets for Lattice Calculations}} \qquad b_T^{-1} \sim \Lambda_{\mathrm{QCD}}$$

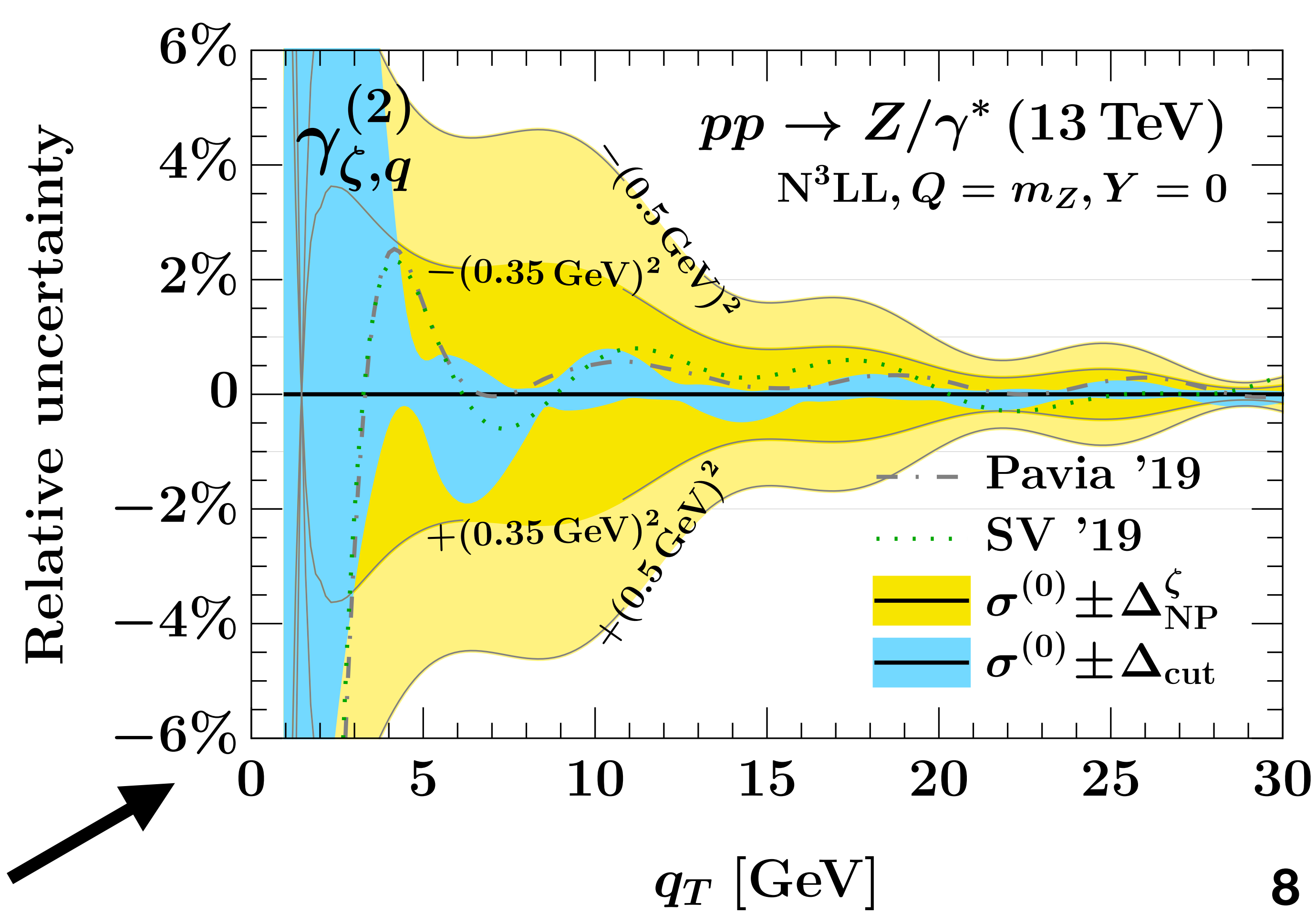
Nonperturbative Contributions to DY Cross Sections

Collins-Soper kernel



[ATLAS '15]

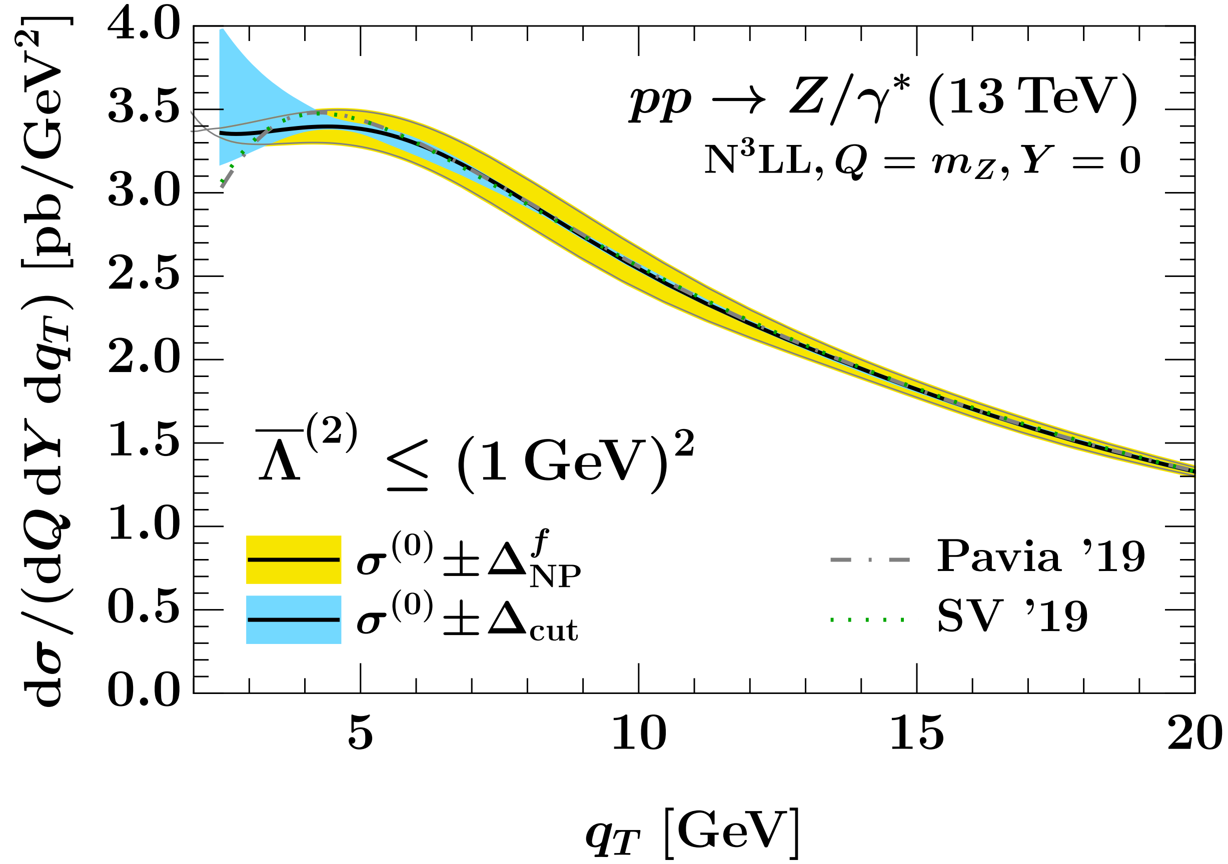


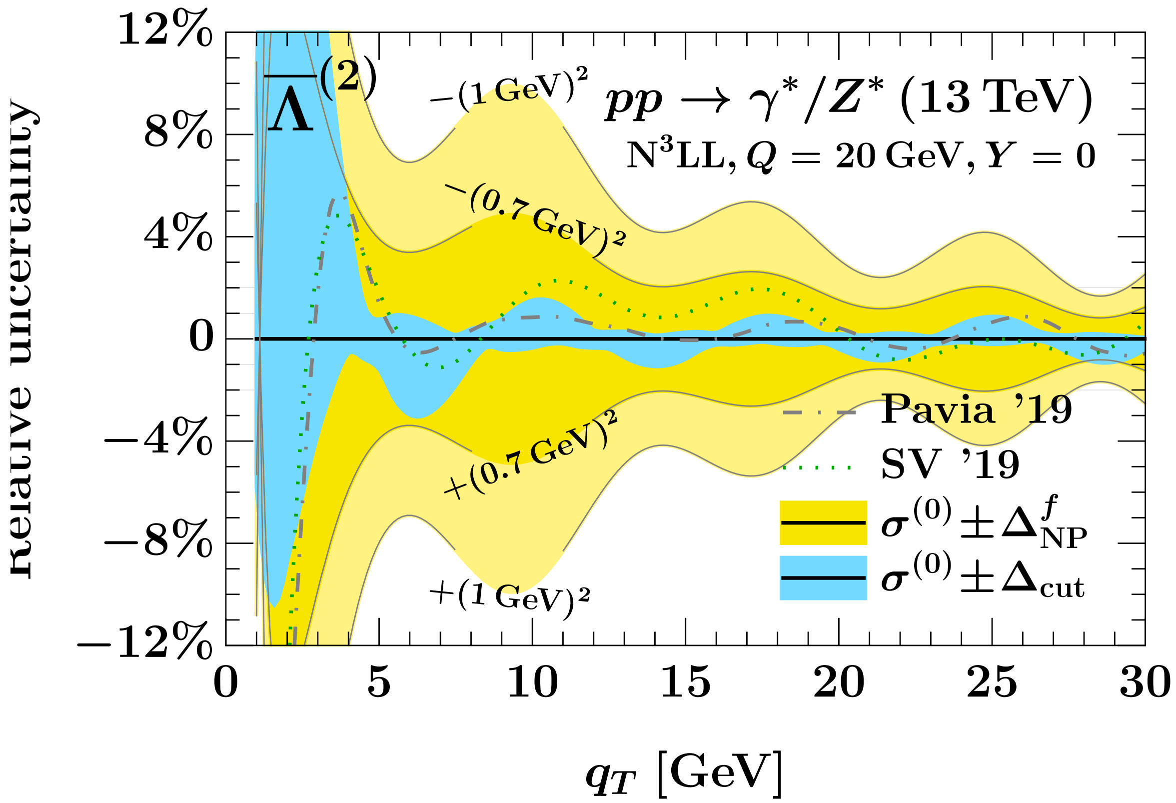


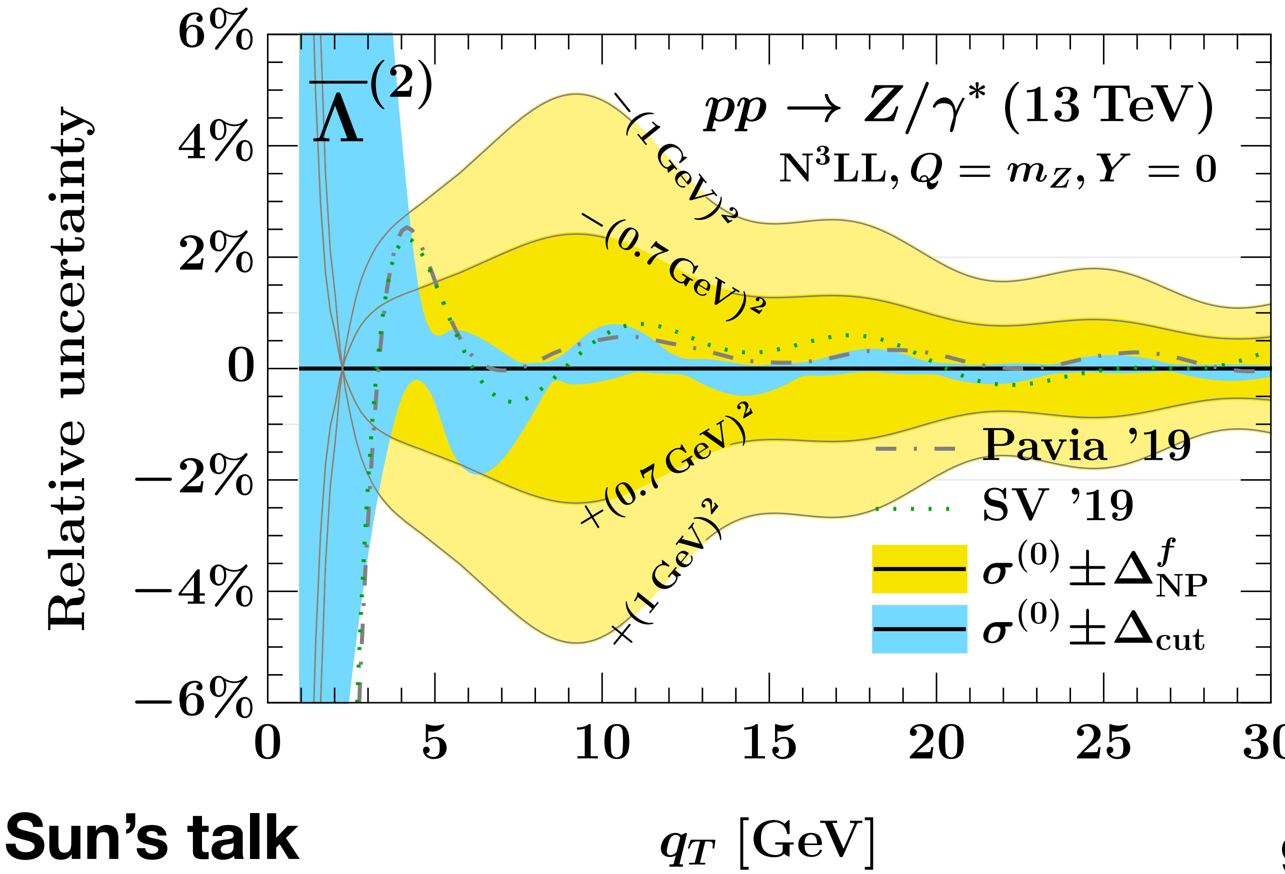
See Zhiquan Sun's talk

Nonperturbative Contributions to DY Cross Sections

Boundary TMD PDF







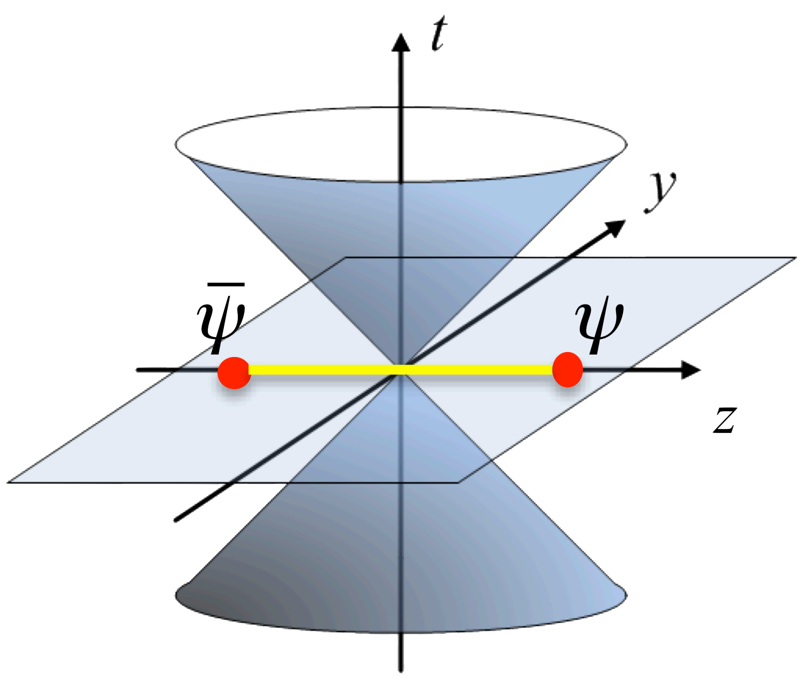
See Zhiquan Sun's talk

Large Momentum EFT: Quasi-PDFs

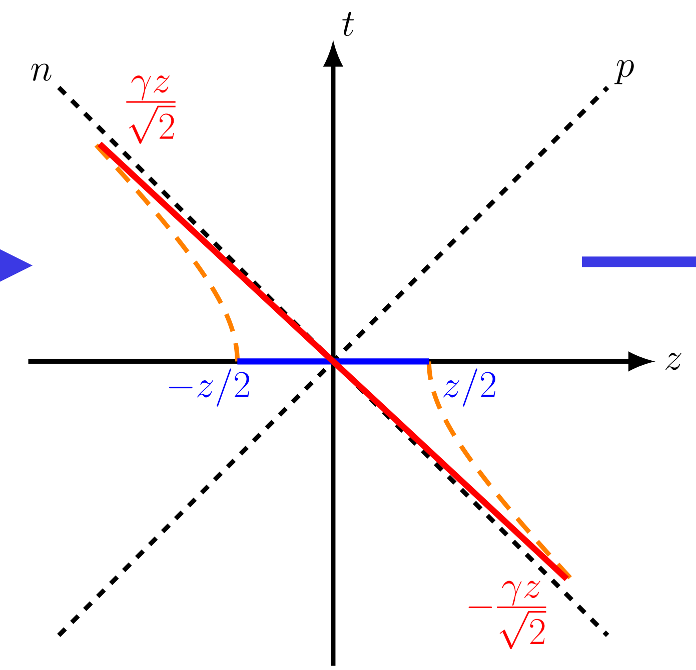
Xiangdong Ji
2013

$$\Lambda_{\rm QCD} \ll P^z \quad (\text{finite large } P^z)$$

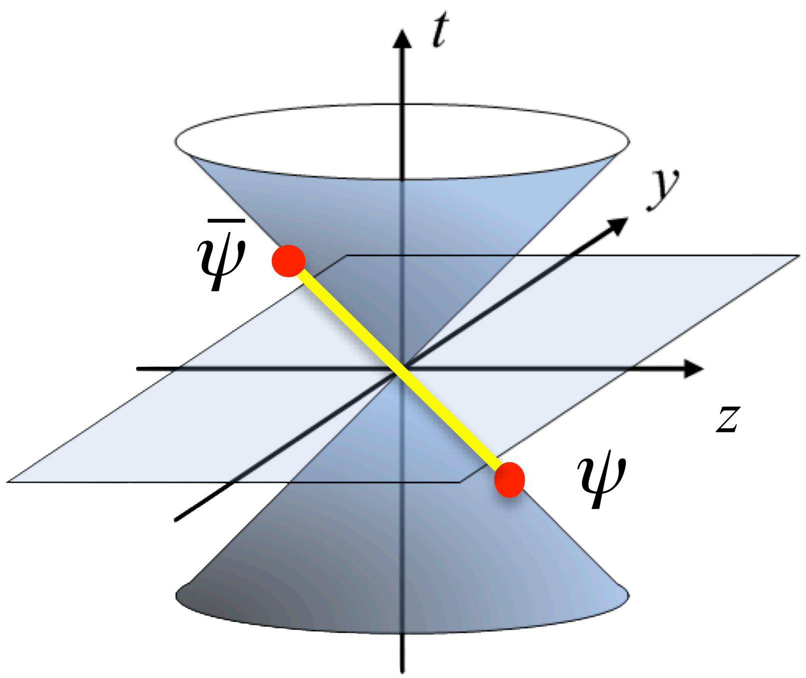
$t = 0, \;\; z \neq 0$



Related by Lorentz boost



$z + ct = 0, \;\; z - ct \neq 0$



quasi-PDF computable with Lattice QCD

Perturbative matching coefficient

PDF

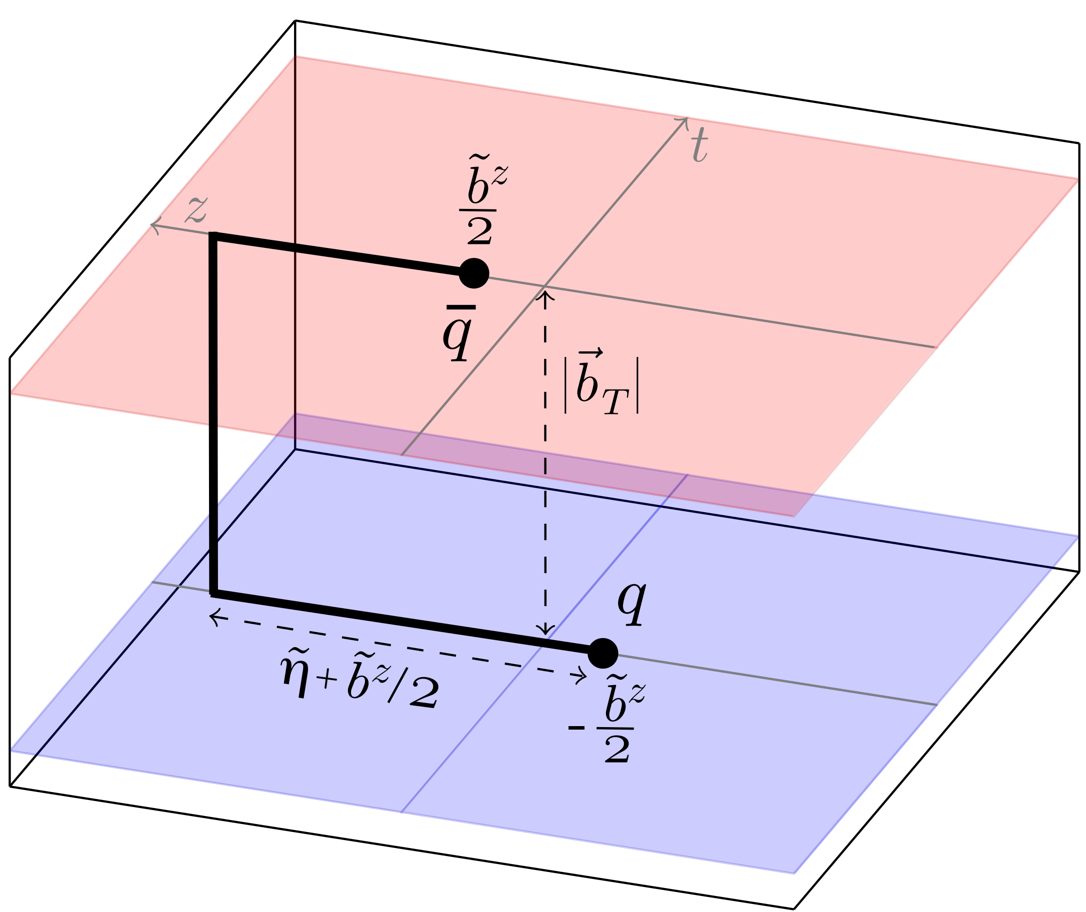
$$\tilde{f}_i(x, P^z, \tilde{\mu}) = \int_{-1}^{1} \frac{dy}{|y|}\, C_{ij}\left(\frac{x}{y}, \frac{\tilde{\mu}}{P^z}, \frac{\mu}{yP^z}\right) f_j(y, \mu) \; + \; \mathcal{O}\left(\frac{M^2}{P_z^2}, \frac{\Lambda_{\rm QCD}^2}{x^2 P_z^2}\right)$$

Power corrections

quasi-PDF and PDF: same IR physics

Requirements for a useful Lattice-TMD

1. Tractable on Lattice: Equal time correlators $\tilde{B}$
2. Same IR physics $(b_T \,\&\, \Lambda_{\rm QCD})$ as continuum TMDs
3. Finite length Wilson lines (staples) $\tilde{\eta}$ = finite
4. Cancel linear divergences due to self-energies (requires a soft factor $\tilde{S}$)
5. Relation to continuum TMDs should exist: $\tilde{f} \to f$



$$\tilde{f} \sim Z_{\rm UV}\, \tilde{B}/\sqrt{\tilde{S}}$$

Two approaches:

- Lorentz Invariant method (MHENS TMDs)
- quasi-TMDs

Literature:

- MHENS:Musch, Hägler, Engelhardt, Negele, Schäfer ('10, '11, '15)

  Pioneered Lattice studies of TMDs, exploit Lorentz Invariance ratios to cancel soft, focus on moments ($b^z \to 0$)

- Ji, Sun, Xiong, Yuan ('14); Ji, Link, Yuan, Zhang, Zhao ('18)

  Quasi TMDs, propose factorization ($\eta = \infty$), calculate C

- Ebert, Stewart, Zhao ('18, '19, '19)

  Propose factorization (finite η) and CS kernel method,
  IR tests, calculate C, lattice renormalization

- Ji, Liu, Liu ('19, '19)

  Proposal for diagrammatic proof of factorization
  & lattice method for required quasi-soft factor

- Vladimirov, Schäfer ('20)

  Factorization analysis

- Many other Lattice studies … (see Yong Zhao's talk next)

Unifying Correlators

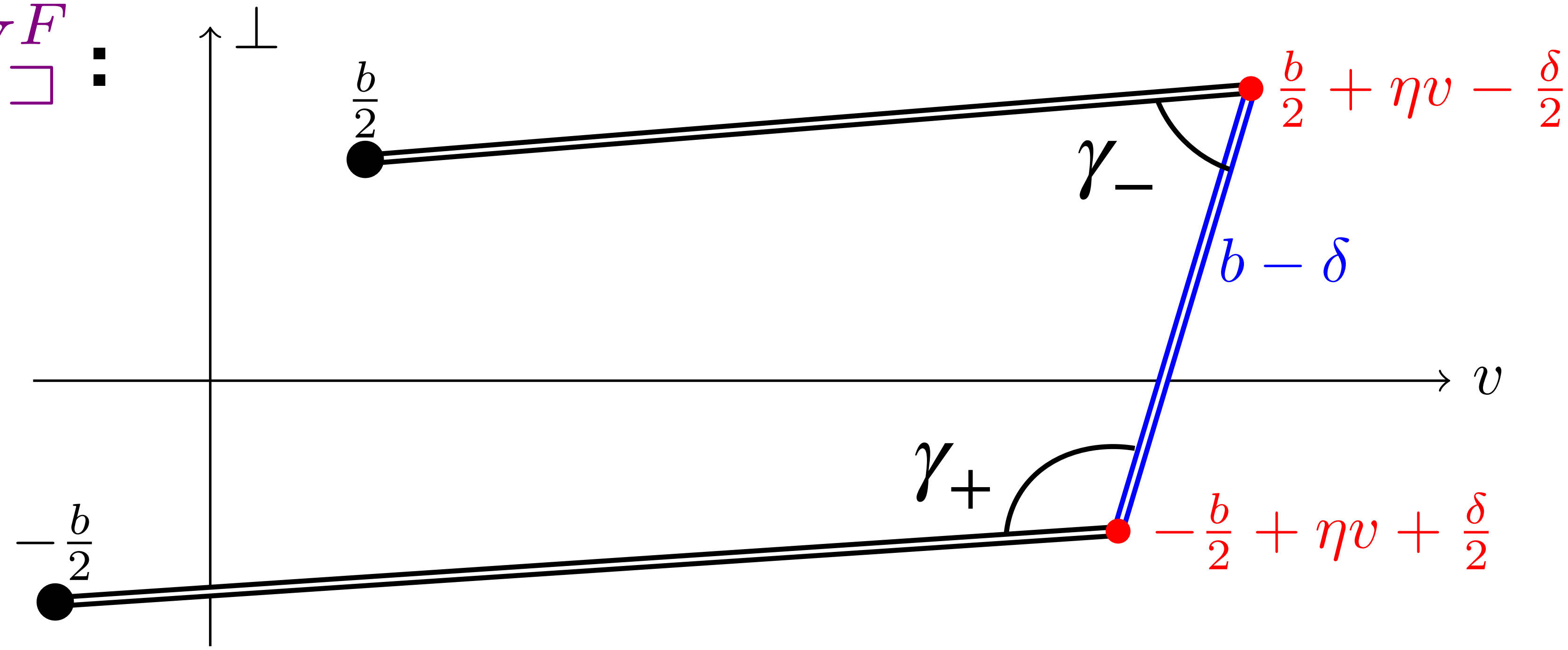
Ebert, Schindler, IS, Zhao (arXiv:2201.08401)

MHENS and Quasi Lattice TMDs use different proton matrix elements
A priori, relation to continuum (eg. Collins-TMD) is unclear

To clarify differences, we introduce a universal Beam Function:

$$\Omega_{q/h}^{[\Gamma]}(b, P, \epsilon, \eta v, \delta) = \Big\langle h(P) \Big| \bar{q}_i\Big(\frac{b}{2}\Big)\frac{\Gamma}{2} W_{\sqsupset}^{F}(b, \eta v, \delta) q_i\Big(-\frac{b}{2}\Big)\Big| h(P)\Big\rangle$$

$W_{\sqsupset}^{F}$:



MHENS: $\delta = 0$

Quasi: $b - \delta = b_T$

Collins: also works

Path Length:

$$L_{\sqsupset} = |\eta v - \delta/2| + |\eta v + \delta/2| + |b - \delta|$$

Cusp angles:

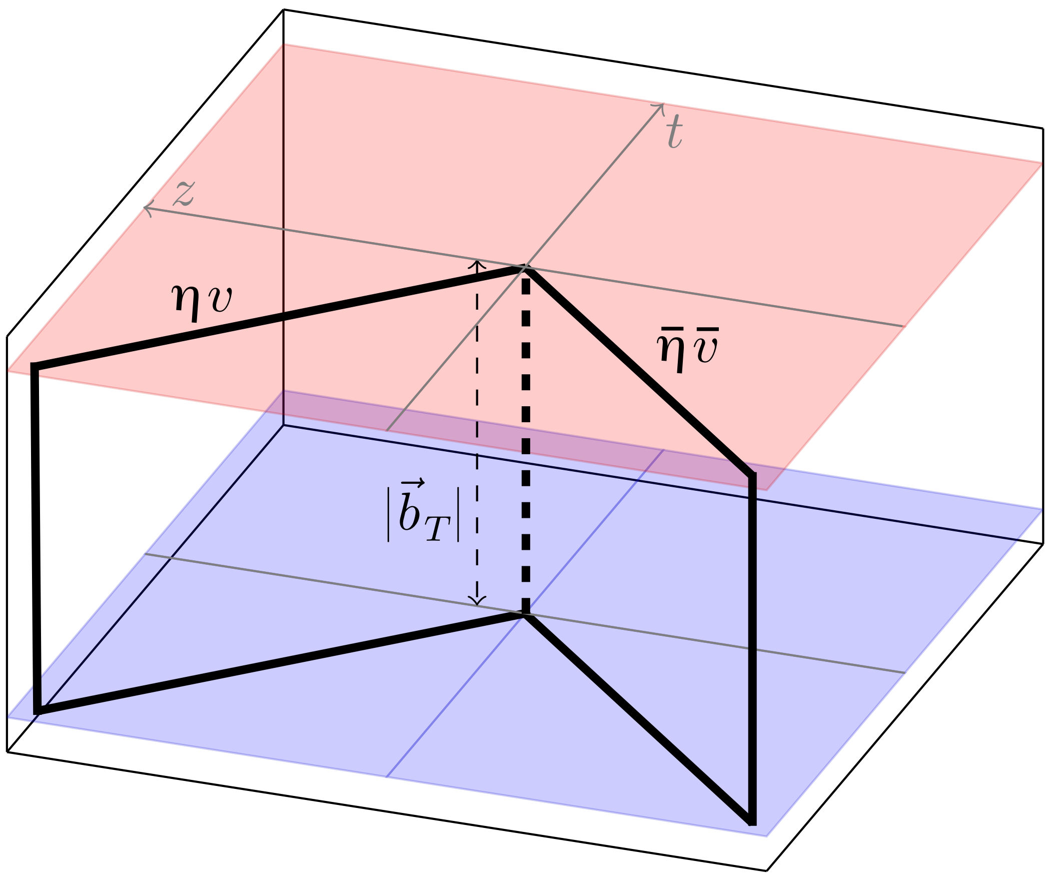
$$\cosh \gamma_{\pm} = \frac{(\eta v \pm \delta/2)\cdot(b-\delta)}{|\eta v \pm \delta/2||b-\delta|}$$

These matter for Renormalization

General Soft Function (finite η)

$$S^R(b,\epsilon,\eta v,\bar\eta\bar v) = \frac{1}{d_R}\Big\langle 0\Big|\mathrm{Tr}\Big[S^R_{\gg}(b,\eta v,\bar\eta\bar v)\Big]\Big|0\Big\rangle$$

$S^R_{\gg}$:



Path Length:

$$L_{\gg} = 2|\bar\eta\bar v| + 2|\eta v| + 2|b|$$

- Needed to match IR structure (Ebert, IS, Zhao '19)
- Needed to cancel η linear div. in Ω

- **No direct method to calculate on the Lattice.**
- **Indirect method exists (Ji, Liu, Liu '19) [See Yong Zhao's talk]**

Can parameterize Ω with 10 Lorentz Invariants:

$$P^2\,,\quad b^2\,,\quad \eta^2 v^2\,,\quad P\cdot b\,,\quad \frac{P\cdot(\eta v)}{\sqrt{P^2|(\eta v)^2|}}\,,\quad \frac{b\cdot(\eta v)}{\sqrt{|b^2(\eta v)^2|}}\,,$$

$$\frac{\delta^2}{b^2}\,,\quad \frac{b\cdot\delta}{b^2}\,,\quad \frac{P\cdot\delta}{P\cdot b}\,,\quad \frac{\delta\cdot(\eta v)}{b\cdot(\eta v)}\,.$$

6 like Musch et.al.

+4 that fix scheme category

Choices for various TMDs:

Continuum TMDs **Lattice TMDs**

	Collins / LR	JMY	Quasi	MHENS
b^μ	$(0, b^-, b_\perp)$	$(0, b^-, b_\perp)$	$(0, b_T^x, b_T^y, \tilde{b}^z)$	$(0, b_T^x, b_T^y, \tilde{b}^z)$
v^μ	$(-e^{2y_B}, 1, 0_\perp)$	$(v^- e^{2y'_B}, v^-, 0_\perp)$	$(0, 0, 0, -1)$	$(0, v^x, v^y, v^z)$
δ^μ	$(0, b^-, 0_\perp)$	$(0, b^-, 0_\perp)$	$(0, 0, 0, \tilde{b}^z)$	$(0, 0, 0_\perp)$
P^μ	$\frac{m_h}{\sqrt{2}}(e^{y_P}, e^{-y_P}, 0_\perp)$	$\frac{m_h}{\sqrt{2}}(e^{y_P}, e^{-y_P}, 0_\perp)$	$m_h(\cosh y_{\tilde{P}}, 0, 0, \sinh y_{\tilde{P}})$	$m_h\left(\cosh y_P, \frac{P^x}{m_h}, \frac{P^y}{m_h}, \sinh y_P\right)$

TMDs differ by how 3 limits (UV, large rapidity, large η) are taken:

$$\Omega^{[\Gamma]}_{q/h}(b, P, \epsilon, \eta v, \delta) \qquad S^R(b, \epsilon, \eta v, \bar{\eta}\bar{v})$$

	TMD	Beam function	Soft function
Collins	$\lim\limits_{\epsilon\to 0} Z^R_{\mathrm{UV}} \lim\limits_{y_B\to-\infty} \frac{\Omega_{i/h}}{\sqrt{S^R}}$	$\Omega^{[\gamma^+]}_{q/h}\left[b, P, \epsilon, -\infty n_B(y_B), b^- n_b\right]$	$S^R\left[b_\perp, \epsilon, -\infty n_A(y_A), -\infty n_B(y_B)\right]$
LR	$\lim\limits_{-y_B\gg 1} \lim\limits_{\epsilon\to 0} Z^R_{\mathrm{UV}} \frac{\Omega_{i/h}}{\sqrt{S^R}}$	$\Omega^{[\gamma^+]}_{q/h}\left[b, P, \epsilon, -\infty n_B(y_B), b^- n_b\right]$	$S^R\left[b_\perp, \epsilon, -\infty n_A(y_A), -\infty n_B(y_B)\right]$
JMY	$\lim\limits_{\frac{v^-}{v^+}\gg 1} \lim\limits_{\epsilon\to 0} Z^R_{\mathrm{UV}} \frac{\Omega_{i/h}}{\sqrt{S^R}}$	$\Omega^{[\gamma^+]}_{q/h}\left[b, P, \mu, -\infty v, b^- n_b\right]$	$S^R\left[b_\perp, \mu, -\infty v, -\infty \tilde{v}\right]$
Quasi	$\lim\limits_{a\to 0} Z_{\mathrm{UV}} \frac{B_{i/h}}{\sqrt{\tilde{S}^R}}$	$\Omega^{[\gamma^{0,z}]}_{q/h}(\tilde{b}, \tilde{P}, a, \tilde{\eta}\hat{z}, \tilde{b}^z\hat{z})$	$S^R\left[b_\perp, a, -\tilde{\eta}\frac{n_A(y_A)}{\lvert n_A(y_A)\rvert}, -\tilde{\eta}\frac{n_A(y_A)}{\lvert n_A(y_A)\rvert}\right]$
MHENS		$\Omega^{[\Gamma]}_{q/h}(b, P, a, \eta v, 0)$	

Also differ by how 3 limits (UV, large rapidity, large η) are taken:

$$\Omega^{[\Gamma]}_{q/h}(b, P, \epsilon, \eta v, \delta) \qquad S^R(b, \epsilon, \eta v, \bar{\eta}\bar{v})$$

	TMD	Beam function	Soft function
Collins	$\lim\limits_{\epsilon\to 0} Z^R_{\mathrm{UV}} \lim\limits_{y_B\to-\infty} \frac{\Omega_{i/h}}{\sqrt{S^R}}$	$\Omega^{[\gamma^+]}_{q/h}\left[b, P, \epsilon, -\infty n_B(y_B), b^- n_b\right]$	$S^R\left[b_\perp, \epsilon, -\infty n_A(y_A), -\infty n_B(y_B)\right]$
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JMY	$\lim\limits_{\frac{v^-}{v^+}\gg 1} \lim\limits_{\epsilon\to 0} Z^R_{\mathrm{UV}} \frac{\Omega_{i/h}}{\sqrt{S^R}}$	$\Omega^{[\gamma^+]}_{q/h}\left[b, P, \mu, -\infty v, b^- n_b\right]$	$S^R\left[b_\perp, \mu, -\infty v, -\infty \tilde{v}\right]$
Quasi	$\lim\limits_{a\to 0} Z_{\mathrm{UV}} \frac{B_{i/h}}{\sqrt{\tilde{S}^R}}$	$\Omega^{[\gamma^{0,z}]}_{q/h}(\tilde{b}, \tilde{P}, a, \tilde{\eta}\hat{z}, \tilde{b}^z\hat{z})$	$S^R\left[b_\perp, a, -\tilde{\eta}\frac{n_A(y_A)}{\lvert n_A(y_A)\rvert}, -\tilde{\eta}\frac{n_A(y_A)}{\lvert n_A(y_A)\rvert}\right]$
MHENS		$\Omega^{[\Gamma]}_{q/h}(b, P, a, \eta v, 0)$	

Collins:

$$f^C_{i/h}(x, \vec{b}_T, \mu, \zeta) = \lim_{\epsilon\to 0} Z^R_{\mathrm{uv}}(\epsilon, \mu, \zeta) \lim_{y_B\to-\infty} \frac{B^C_{i/h}(x, \vec{b}_T, \epsilon, y_P - y_B)}{\sqrt{S^R_C(b_T, \epsilon, 2y_n, 2y_B)}}$$

$$B^C_{q_i/h}(x, \vec{b}_T, \epsilon, y_P - y_B) = \int \frac{db^-}{2\pi} e^{-ib^-(xP^+)} \, \Omega^{[\gamma^+]}_{q_i/h}\left[b, P, \epsilon, -\infty n_B(y_B), b^- n_b\right]$$

$$n^\mu_B(y_B) = (-e^{2y_B}, 1, 0_\perp)$$

Also differ by how 3 limits (UV, large rapidity, large η) are taken:

$$\Omega^{[\Gamma]}_{q/h}(b, P, \epsilon, \eta v, \delta) \qquad S^R(b, \epsilon, \eta v, \bar{\eta}\bar{v})$$

	TMD	Beam function	Soft function
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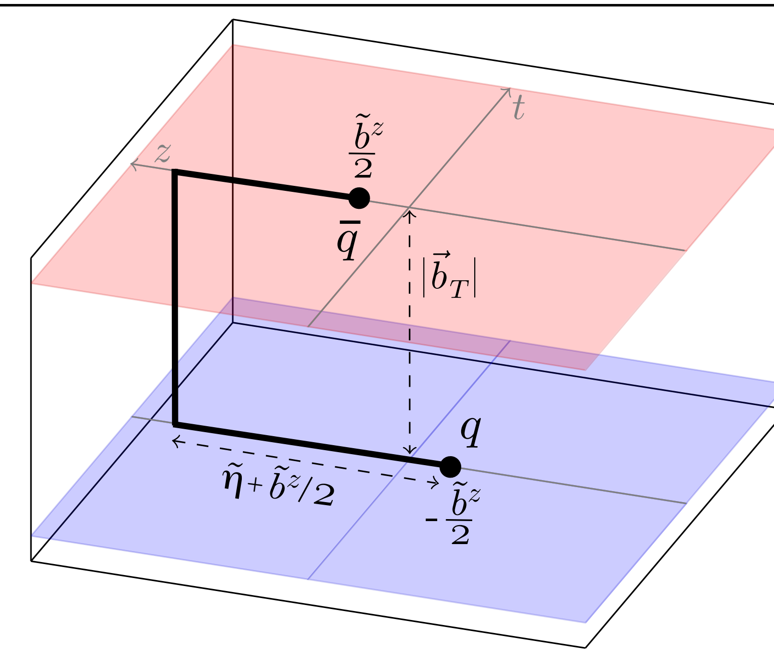
Quasi	$\lim\limits_{a\to 0} Z_{\rm UV} \frac{B_{i/h}}{\sqrt{\tilde{S}^R}}$	$\Omega^{[\gamma^{0,z}]}_{q/h}(\tilde{b}, \tilde{P}, a, \tilde{\eta}\hat{z}, \tilde{b}^z\hat{z})$	$S^R\left[b_\perp, a, -\tilde{\eta}\frac{n_A(y_A)}{\lvert n_A(y_A)\rvert}, -\tilde{\eta}\frac{n_A(y_A)}{\lvert n_A(y_A)\rvert}\right]$
MHENS		$\Omega^{[\Gamma]}_{q/h}(b, P, a, \eta v, 0)$	

Quasi Beam:

$$\tilde{B}^{[\tilde{\Gamma}]}_{i/h}(x, \vec{b}_T, a, \tilde{\eta}, x\tilde{P}^z)$$

$$= \int \frac{d\tilde{b}^z}{2\pi} e^{i\tilde{b}^z(x\tilde{P}^z)} \Omega^{[\tilde{\Gamma}]}_{q_i/h}(\tilde{b}, \tilde{P}, a, \tilde{\eta}\hat{z}, \tilde{b}^z\hat{z})$$

equal time:



a = lattice spacing (UV regulator)

Also differ by how 3 limits (UV, large rapidity, large η) are taken:

$$\Omega^{[\Gamma]}_{q/h}(b, P, \epsilon, \eta v, \delta) \qquad S^R(b, \epsilon, \eta v, \bar{\eta}\bar{v})$$

	TMD	Beam function	Soft function
Collins	$\lim\limits_{\epsilon\to 0} Z^R_{\mathrm{UV}} \lim\limits_{y_B\to-\infty} \dfrac{\Omega_{i/h}}{\sqrt{S^R}}$	$\Omega^{[\gamma^+]}_{q/h}\left[b, P, \epsilon, -\infty n_B(y_B), b^- n_b\right]$	$S^R\left[b_\perp, \epsilon, -\infty n_A(y_A), -\infty n_B(y_B)\right]$
LR	$\lim\limits_{-y_B\gg 1} \lim\limits_{\epsilon\to 0} Z^R_{\mathrm{UV}} \dfrac{\Omega_{i/h}}{\sqrt{S^R}}$	$\Omega^{[\gamma^+]}_{q/h}\left[b, P, \epsilon, -\infty n_B(y_B), b^- n_b\right]$	$S^R\left[b_\perp, \epsilon, -\infty n_A(y_A), -\infty n_B(y_B)\right]$
JMY	$\lim\limits_{\frac{v^-}{v^+}\gg 1} \lim\limits_{\epsilon\to 0} Z^R_{\mathrm{UV}} \dfrac{\Omega_{i/h}}{\sqrt{S^R}}$	$\Omega^{[\gamma^+]}_{q/h}\left[b, P, \mu, -\infty v, b^- n_b\right]$	$S^R\left[b_\perp, \mu, -\infty v, -\infty \tilde{v}\right]$
Quasi	$\lim\limits_{a\to 0} Z_{\mathrm{UV}} \dfrac{B_{i/h}}{\sqrt{\tilde{S}^R}}$	$\Omega^{[\gamma^{0,z}]}_{q/h}(\tilde{b}, \tilde{P}, a, \tilde{\eta}\hat{z}, \tilde{b}^z\hat{z})$	$S^R\left[b_\perp, a, -\tilde{\eta}\dfrac{n_A(y_A)}{\lvert n_A(y_A)\rvert}, -\tilde{\eta}\dfrac{n_A(y_A)}{\lvert n_A(y_A)\rvert}\right]$
MHENS		$\Omega^{[\Gamma]}_{q/h}(b, P, a, \eta v, 0)$	

Quasi TMD:

$$\tilde{f}^{[\tilde{\Gamma}]}_{i/h}(x, \vec{b}_T, \mu, \tilde{\zeta}, x\tilde{P}^z) = \lim_{\substack{\tilde{\eta}\to\infty \\ a\to 0}} Z'_{\mathrm{uv}}(\mu, \tilde{\mu}) Z_{\mathrm{uv}}(a, \tilde{\mu}, y_n - y_B) \frac{\tilde{B}^{[\tilde{\Gamma}]}_{i/h}(x, \vec{b}_T, a, \tilde{\eta}, x\tilde{P}^z)}{\sqrt{S^R\left[b_\perp, a, -\tilde{\eta}\frac{n_A(y_A)}{\lvert n_A(y_A)\rvert}, -\tilde{\eta}\frac{n_B(y_B)}{\lvert n_B(y_B)\rvert}\right]}}$$

Finite η Collins soft function

(In ratio: limit $\tilde{\eta} \to \infty$ exists)

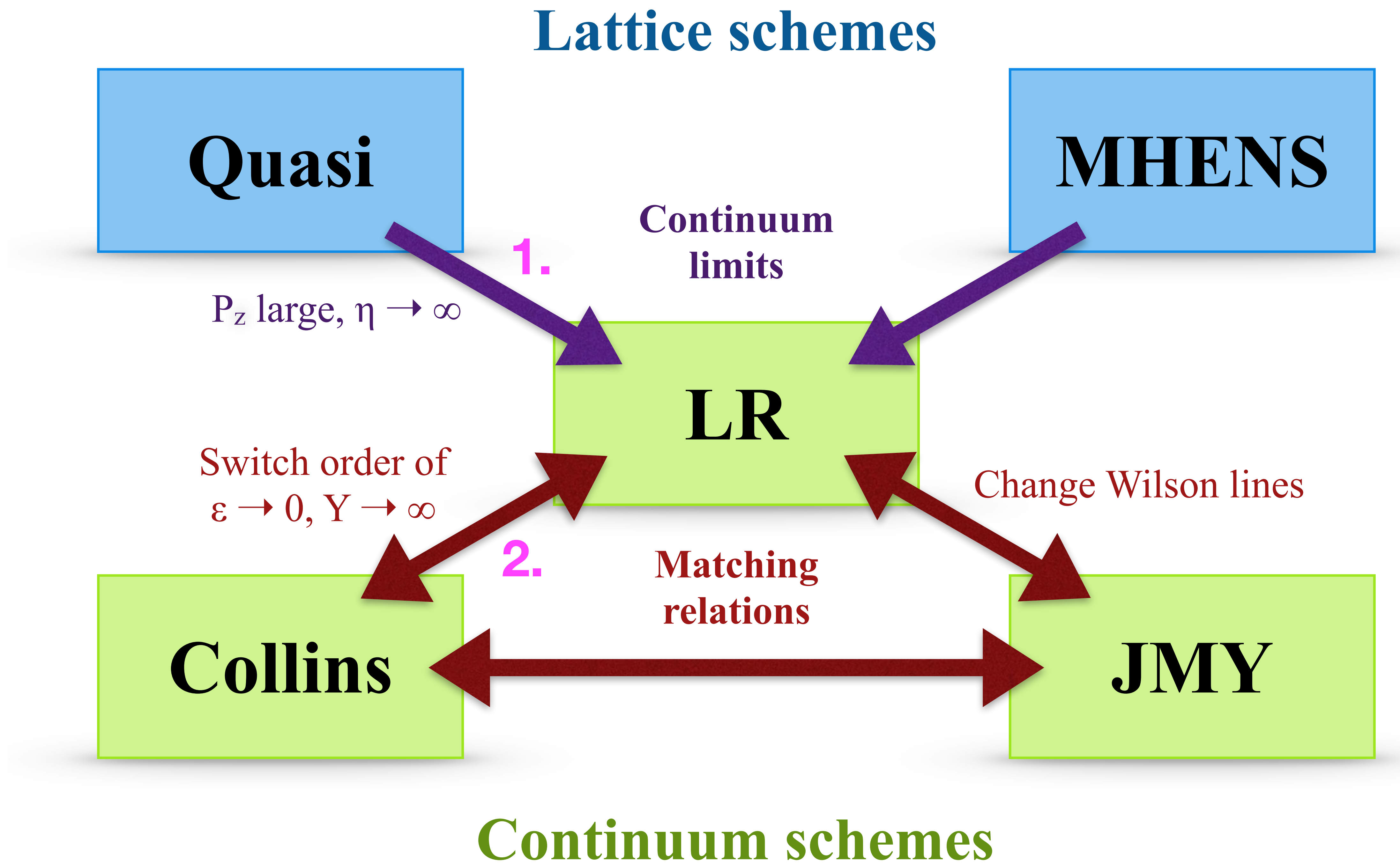
Also differ by how 3 limits (UV, large rapidity, large η) are taken:

$$\Omega^{[\Gamma]}_{q/h}(b, P, \epsilon, \eta v, \delta) \qquad S^R(b, \epsilon, \eta v, \bar{\eta}\bar{v})$$

	TMD	Beam function	Soft function
Collins	$\lim\limits_{\epsilon\to 0} Z^R_{\rm UV} \lim\limits_{y_B\to-\infty} \frac{\Omega_{i/h}}{\sqrt{S^R}}$	$\Omega^{[\gamma^+]}_{q/h}\left[b, P, \epsilon, -\infty n_B(y_B), b^- n_b\right]$	$S^R\left[b_\perp, \epsilon, -\infty n_A(y_A), -\infty n_B(y_B)\right]$
LR	$\lim\limits_{-y_B\gg 1} \lim\limits_{\epsilon\to 0} Z^R_{\rm UV} \frac{\Omega_{i/h}}{\sqrt{S^R}}$	$\Omega^{[\gamma^+]}_{q/h}\left[b, P, \epsilon, -\infty n_B(y_B), b^- n_b\right]$	$S^R\left[b_\perp, \epsilon, -\infty n_A(y_A), -\infty n_B(y_B)\right]$
JMY	$\lim\limits_{\frac{v^-}{v^+}\gg 1} \lim\limits_{\epsilon\to 0} Z^R_{\rm UV} \frac{\Omega_{i/h}}{\sqrt{S^R}}$	$\Omega^{[\gamma^+]}_{q/h}\left[b, P, \mu, -\infty v, b^- n_b\right]$	$S^R\left[b_\perp, \mu, -\infty v, -\infty \tilde{v}\right]$
Quasi	$\lim\limits_{a\to 0} Z_{\rm UV} \frac{B_{i/h}}{\sqrt{\tilde{S}^R}}$	$\Omega^{[\gamma^{0,z}]}_{q/h}(\tilde{b}, \tilde{P}, a, \tilde{\eta}\hat{z}, \tilde{b}^z\hat{z})$	$S^R\left[b_\perp, a, -\tilde{\eta}\frac{n_A(y_A)}{\lvert n_A(y_A)\rvert}, -\tilde{\eta}\frac{n_A(y_A)}{\lvert n_A(y_A)\rvert}\right]$
MHENS		$\Omega^{[\Gamma]}_{q/h}(b, P, a, \eta v, 0)$	

LR scheme: **new,**
differs from Collins only by order of (UV & rapidity) limits,
useful for our proof

Proof



Steps:

1. Quasi → LR: related by large rapidity $(P^z \gg \Lambda_{\rm QCD})$
   IF we properly map variables,
   take $|\eta| \to \infty$

2. LR → Collins: UV ren. & non-trivial Matching coefficient

Step 1

	Collins / LR	Quasi	MHENS
b^2	$-b_T^2$	$-b_T^2 - (\tilde{b}^z)^2$	$-b_T^2 - (\tilde{b}^z)^2$
$(\eta v)^2$	$-2\eta^2 e^{2y_B}$	$-\tilde{\eta}^2$	$-\eta^2 \vec{v}^{\,2}$
$P \cdot b$	$\frac{m_h}{\sqrt{2}} b^- e^{y_P}$	$-m_h \tilde{b}^z \sinh y_{\tilde{P}}$	$m_h \sinh y_P \tilde{b}^z + P^x b_T^x + P^y b_T^y$
$\frac{b \cdot (\eta v)}{\sqrt{\lvert(\eta v)^2 b^2\rvert}}$	$-\frac{b^- e^{y_B}}{\sqrt{2}\, b_T} \operatorname{sgn}(\eta)$	$\frac{\tilde{b}^z}{\sqrt{(\tilde{b}^z)^2 + b_T^2}} \operatorname{sgn}(\eta)$	$\frac{b_T^x v^x + b_T^y v^y + \tilde{b}^z v^z}{\sqrt{v_T^2 + (v^z)^2}\sqrt{b_T^2 + (\tilde{b}^z)^2}}$
$\frac{P \cdot (\eta v)}{\sqrt{P^2 \lvert\eta v\rvert^2}}$	$\sinh(y_P - y_B) \operatorname{sgn}(\eta)$	$\sinh y_{\tilde{P}} \operatorname{sgn}(\eta)$	$\frac{P^x v^x + P^y v^y + m_h v^z \sinh y_P}{\sqrt{v_T^2 + (v^z)^2}\sqrt{m_h^2 + P_x^2 + P_y^2}}$
$\frac{\delta^2}{b^2}$	0	$\frac{(\tilde{b}^z)^2}{b_T^2 + (\tilde{b}^z)^2}$	0
$\frac{b \cdot \delta}{b^2}$	0	$\frac{(\tilde{b}^z)^2}{b_T^2 + (\tilde{b}^z)^2}$	0
$\frac{P \cdot \delta}{P \cdot b}$	1	1	0
$\frac{\delta \cdot (\eta v)}{b \cdot (\eta v)}$	1	1	0
P^2	m_h^2	m_h^2	m_h^2

Step 1

	Collins / LR	Quasi	MHENS
b^2	$-b_T^2$	$-b_T^2 - (\tilde{b}^z)^2$	$-b_T^2 - (\tilde{b}^z)^2$
$(\eta v)^2$	$-2\eta^2 e^{2y_B}$	$-\tilde{\eta}^2$	$-\eta^2 \vec{v}^{\,2}$
$P \cdot b$	$\frac{m_h}{\sqrt{2}} b^- e^{y_P}$	$-m_h \tilde{b}^z \sinh y_{\tilde{P}}$	$m_h \sinh y_P \tilde{b}^z + P^x b_T^x + P^y b_T^y$
$\frac{b \cdot (\eta v)}{\sqrt{\lvert(\eta v)^2 b^2\rvert}}$	$-\frac{b^- e^{y_B}}{\sqrt{2}\, b_T} \operatorname{sgn}(\eta)$	$\frac{\tilde{b}^z}{\sqrt{(\tilde{b}^z)^2 + b_T^2}} \operatorname{sgn}(\eta)$	$\frac{b_T^x v^x + b_T^y v^y + \tilde{b}^z v^z}{\sqrt{v_T^2 + (v^z)^2}\sqrt{b_T^2 + (\tilde{b}^z)^2}}$
$\frac{P \cdot (\eta v)}{\sqrt{P^2 \lvert\eta v\rvert^2}}$	$\sinh(y_P - y_B) \operatorname{sgn}(\eta)$	$\sinh y_{\tilde{P}} \operatorname{sgn}(\eta)$	$\frac{P^x v^x + P^y v^y + m_h v^z \sinh y_P}{\sqrt{v_T^2 + (v^z)^2}\sqrt{m_h^2 + P_x^2 + P_y^2}}$
[illegible]	[illegible]	$\frac{(\tilde{b}^z)^2}{b_T^2 + (\tilde{b}^z)^2}$	0
[illegible]	[illegible]	$\frac{(\tilde{b}^z)^2}{b_T^2 + (\tilde{b}^z)^2}$	0
$\frac{P \cdot \delta}{P \cdot b}$	1	1	0
$\frac{\delta \cdot (\eta v)}{b \cdot (\eta v)}$	1	1	0
P^2	m_h^2	m_h^2	m_h^2

$$\sinh(y_P - y_B) = \sinh y_{\tilde{P}}$$

$$\Rightarrow y_{\tilde{P}} = y_P - y_B$$

Step 1

	Collins / LR	Quasi	MHENS
b^2	$-b_T^2$	$-b_T^2 - (\tilde{b}^z)^2$	$-b_T^2 - (\tilde{b}^z)^2$
$(\eta v)^2$	$-2\eta^2 e^{2y_B}$	$-\tilde{\eta}^2$	$-\eta^2 \vec{v}^{\,2}$
$P \cdot b$	$\boxed{\frac{m_h}{\sqrt{2}} b^- e^{y_P}}$	$\boxed{-m_h \tilde{b}^z \sinh y_{\tilde{P}}}$	$m_h \sinh y_P \tilde{b}^z + P^x b_T^x + P^y b_T^y$
			$\cdots b_T^y v^y + \tilde{b}^z v^z$ / $\cdots \sqrt{b_T^2 + (\tilde{b}^z)^2}$
			$\cdots + m_h v^z \sinh y_P$ / $\cdots \sqrt{m_h^2 + P_x^2 + P_y^2}$
			0
$\dfrac{b \cdot \delta}{b^2}$	0	$\dfrac{(\tilde{b}^z)^2}{b_T^2 + (\tilde{b}^z)^2}$	0
$\dfrac{P \cdot \delta}{P \cdot b}$	1	1	0
$\dfrac{\delta \cdot (\eta v)}{b \cdot (\eta v)}$	1	1	0
P^2	m_h^2	m_h^2	m_h^2

$y_P,\, b^-$ finite **Boost quasi by $y_B = y_P - y_{\tilde{P}}$**

$$\tilde{P}^z = m_h \sinh y_{\tilde{P}} \gg \Lambda_{\rm QCD} \qquad (y_{\tilde{P}} \to \infty, y_B \to -\infty)$$

$$-m_h \tilde{b}^z \sinh y_{\tilde{P}} = m_h \sqrt{2} e^{y_B} b^- \sinh(y_P - \tilde{y}_B) \xrightarrow{y_B \to -\infty} \frac{m_h}{\sqrt{2}} b^- e^{y_P}$$

Step 1

	Collins / LR	Quasi	MHENS
b^2	$-b_T^2$	$-b_T^2 - (\tilde{b}^z)^2$	$-b_T^2 - (\tilde{b}^z)^2$
$(\eta v)^2$	$-2\eta^2 e^{2y_B}$	$-\tilde{\eta}^2$	[illegible]
$P \cdot b$	$\frac{m_h}{\sqrt{2}} b^- e^{y_P}$	$-m_h \tilde{b}^z \sinh y_{\tilde{P}}$	$m_h \sinh y_P \tilde{b}^z + P^x b_T^x + P^y b_T^y$
$\frac{b \cdot (\eta v)}{\sqrt{\lvert(\eta v)^2 b^2\rvert}}$	$-\frac{b^- e^{y_B}}{\sqrt{2}\, b_T} \operatorname{sgn}(\eta)$	$\frac{\tilde{b}^z}{\sqrt{(\tilde{b}^z)^2 + b_T^2}} \operatorname{sgn}(\eta)$	[illegible]
$\frac{P \cdot (\eta v)}{\sqrt{P^2 \lvert\eta v\rvert^2}}$	$\sinh(y_P - y_B) \operatorname{sgn}(\eta)$	$\sinh y_{\tilde{P}} \operatorname{sgn}(\eta)$	[illegible]
$\frac{\delta^2}{b^2}$	0	$\frac{(\tilde{b}^z)^2}{b_T^2 + (\tilde{b}^z)^2}$	0
$\frac{b \cdot \delta}{b^2}$	0	$\frac{(\tilde{b}^z)^2}{b_T^2 + (\tilde{b}^z)^2}$	0
$\frac{P \cdot \delta}{P \cdot b}$	1	1	0
$\frac{\delta \cdot (\eta v)}{b \cdot (\eta v)}$	1	1	0
P^2	m_h^2	m_h^2	m_h^2

$$\tilde{\eta} = \sqrt{2} e^{y_B} \eta$$

$$\frac{\tilde{b}^z}{b_T} = \frac{\sqrt{2} b^- e^{y_B}}{b_T} \xrightarrow{y_B \to -\infty} 0$$

Also $\tilde{\zeta} = \zeta$

Step 1 Quasi $\to$ LR

$$\tilde{f}_{i/h}^{[\tilde{\Gamma}]}(x, \vec{b}_T, \mu, \tilde{\zeta}, x\tilde{P}^z) = \lim_{\substack{\tilde{\eta}\to\infty \\ a\to 0}} Z'_{\rm uv}(\mu, \tilde{\mu}) Z_{\rm uv}(a, \tilde{\mu}, y_n - y_B) \frac{\tilde{B}_{i/h}^{[\tilde{\Gamma}]}(x, \vec{b}_T, a, \tilde{\eta}, x\tilde{P}^z)}{\sqrt{S^R\left[b_\perp, a, -\tilde{\eta}\frac{n_A(y_A)}{|n_A(y_A)|}, -\tilde{\eta}\frac{n_B(y_B)}{|n_B(y_B)|}\right]}}$$

Quasi and LR have same UV renormalization

Quasi and LR have same $\tilde{\eta} \to \infty$ limit

Thus Quasi = LR after expansion

Step 2 Quasi=LR → Collins

LR and Collins differ by order of $y_B \to -\infty$ and $\epsilon \to 0$ limits

LaMET: this induces a matching coefficient

$$\tilde{f}_{q_i/h}(x, \vec{b}_T, \mu, \tilde{\zeta}, x\tilde{P}^z)$$

$$= C_q(x\tilde{P}^z, \mu) f_{q_i/h}(x, \vec{b}_T, \mu, \tilde{\zeta}) + \mathcal{O}(\tilde{y}_P^k e^{-\tilde{y}_P})$$

$$= C_q(x\tilde{P}^z, \mu) \exp\left[\frac{1}{2}\gamma_\zeta^q(\mu, b_T) \ln \frac{\tilde{\zeta}}{\zeta}\right] f_{q_i/h}(x, \vec{b}_T, \mu, \zeta) + \mathcal{O}(\tilde{y}_P^k e^{-\tilde{y}_P})$$

standard CS evolution

Result

Ebert, Schindler, IS, Zhao (arXiv:2201.08401)

Same steps work for any spin structure & for gluon TMDs

$$\tilde{f}_{i/h}(x, \vec{b}_T, \mu, \tilde{\zeta}, x\tilde{P}^z, \tilde{\eta}) = C_i\big(x\tilde{P}^z, \mu\big) \exp\left[\frac{1}{2}\gamma_\zeta^i(\mu, b_T) \ln\frac{\tilde{\zeta}}{\zeta}\right] f_{i/h}^C(x, \vec{b}_T, \mu, \zeta)$$

$$\times\left\{1 + \mathcal{O}\left[\frac{b_T}{\tilde{\eta}}, \frac{1}{\tilde{x}P^z\tilde{\eta}}, \frac{1}{(x\tilde{P}^z b_T)^2}, \frac{\Lambda_{\rm QCD}^2}{(x\tilde{P}^z)^2}\right]\right\}$$

✔

Also cross-checked all properties at 1-loop ✔

Implications

Quasi → Collins

$$\tilde{f}_{i/h}(x, \vec{b}_T, \mu, \tilde{\zeta}, x\tilde{P}^z, \tilde{\eta}) = C_i\big(x\tilde{P}^z, \mu\big) \exp\left[\frac{1}{2}\gamma_\zeta^q(\mu, b_T) \ln\frac{\tilde{\zeta}}{\zeta}\right] f^C_{i/h}(x, \vec{b}_T, \mu, \zeta)$$

- **Extract CS Kernel from ratios of quasi-TMDs** (Ebert, IS, Zhao '18)
- **No mixing of flavors, quarks and gluons, or spin structures (except perhaps by lattice-fermion discretization)**

Ebert, Schindler, IS, Zhao '20
Vladimirov, Schafer '20
Ji, Liu, Schaefer, Yuan '20

- **Ratios can be calculated in x-space**

$$\lim_{\tilde{\eta}\to\infty} \frac{\tilde{B}^{[\tilde{\Gamma}_1]}_{q_i/h}(x, \vec{b}_T, \mu, \tilde{\eta}, x\tilde{P}^z)}{\tilde{B}^{[\tilde{\Gamma}_2]}_{q_j/h'}(x, \vec{b}_T, \mu, \tilde{\eta}, x\tilde{P}^z)} = \frac{f^{[\Gamma_1]}_{q_i/h}(x, \vec{b}_T, \mu, \zeta)}{f^{[\Gamma_2]}_{q_j/h'}(x, \vec{b}_T, \mu, \zeta)}$$

- **NLL calculation** $\quad C_q\big(x\tilde{P}^z, \mu\big)^{\mathrm{NLL}} = \exp\Big[-2K_\Gamma^q\big(2x\tilde{P}^z, \mu\big) - K_\gamma^q\big(2x\tilde{P}^z, \mu\big)\Big]$

$$K_\Gamma^q(\mu_0, \mu) = -\frac{\Gamma_0^q}{4\beta_0^2}\left\{\frac{4\pi}{\alpha_s(\mu_0)}\left(1 - \frac{1}{r} - \ln r\right) + \left(\frac{\Gamma_1^q}{\Gamma_0^q} - \frac{\beta_1}{\beta_0}\right)(1 - r + \ln r) + \frac{\beta_1}{2\beta_0}\ln^2 r\right\}$$

$$K_\gamma^q(\mu_0, \mu) = -\frac{\gamma_{C0}^q}{2\beta_0}\ln r$$

MHENS → Collins

Ebert, Schindler, IS, Zhao (arXiv:2201.08401)

	Collins / LR	Quasi	MHENS
b^2	$-b_T^2$	$-b_T^2 - (\tilde{b}^z)^2$	$-b_T^2 - (\tilde{b}^z)^2$
$(\eta v)^2$	$-2\eta^2 e^{2y_B}$	$-\tilde{\eta}^2$	$-\eta^2 \vec{v}^{\,2}$
$P \cdot b$	$\frac{m_h}{\sqrt{2}} b^- e^{y_P}$	$-m_h \tilde{b}^z \sinh y_{\tilde{P}}$	$m_h \sinh y_P \tilde{b}^z + P^x b_T^x + P^y b_T^y$
$\frac{b \cdot (\eta v)}{\sqrt{\|(\eta v)^2 b^2\|}}$	$-\frac{b^- e^{y_B}}{\sqrt{2}\, b_T} \operatorname{sgn}(\eta)$	$\frac{\tilde{b}^z}{\sqrt{(\tilde{b}^z)^2 + b_T^2}} \operatorname{sgn}(\eta)$	$\frac{b_T^x v^x + b_T^y v^y + \tilde{b}^z v^z}{\sqrt{v_T^2 + (v^z)^2}\sqrt{b_T^2 + (\tilde{b}^z)^2}}$
$\frac{P \cdot (\eta v)}{\sqrt{P^2 \|\eta v\|^2}}$	$\sinh(y_P - y_B) \operatorname{sgn}(\eta)$	$\sinh y_{\tilde{P}} \operatorname{sgn}(\eta)$	$\frac{P^x v^x + P^y v^y + m_h v^z \sinh y_P}{\sqrt{v_T^2 + (v^z)^2}\sqrt{m_h^2 + P_x^2 + P_y^2}}$
$\frac{\delta^2}{b^2}$	0	$\frac{(\tilde{b}^z)^2}{b_T^2 + (\tilde{b}^z)^2}$	0
$\frac{b \cdot \delta}{b^2}$	0	$\frac{(\tilde{b}^z)^2}{b_T^2 + (\tilde{b}^z)^2}$	0
$\frac{P \cdot \delta}{P \cdot b}$	1	1	0
$\frac{\delta \cdot (\eta v)}{b \cdot (\eta v)}$	1	1	0
P^2	m_h^2	m_h^2	m_h^2

?

MHENS → Collins

Ebert, Schindler, IS, Zhao (arXiv:2201.08401)

- $P \cdot b = 0$ **case, our proof applies**

  MHENS equivalent to Quasi (same soft fn, renormalization, …)

$$\int \mathrm{d}x \; \tilde{f}^{[\Gamma]}_{q_i/h}(x, \vec{b}_T, \mu, \tilde{\zeta}, x\tilde{P}^z, \tilde{\eta}) = f^{[\Gamma]\mathrm{MHENS}}_{q_i/h}(b^z = 0, \vec{b}_T, \mu, \tilde{P}^z, y_n - y_B, \tilde{\eta})$$

This case was focus of Musch, Hägler, Engelhardt, Negele, Schäfer

- $P \cdot b \neq 0$ **case (** x **dependence)**

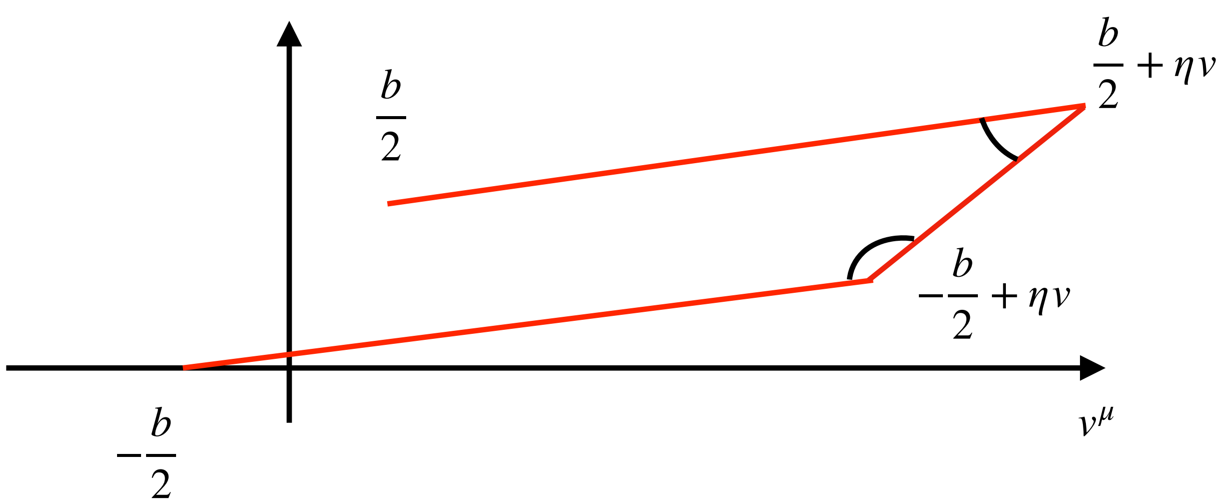
Additional challenges

  - b^z - dependent renormalization

    Linear: $\propto \left(2|\eta v| + \sqrt{\tilde{b}_z^2 + b_T^2}\right)/a$

    Cusp: $\propto \left[3 - \frac{2\tilde{b}^z}{b_T} \tan^{-1}\frac{b_T}{\tilde{b}^z}\right] \; \ln(a)$

  - b^z - dependent soft function?



With proper lattice renormalization, Lorentz Inv. compensation, and construction of a suitable soft function, could connect MHENS to LR scheme (thus to Collins).

Conclusion

Quasi-TMD → Collins-TMD
Our proof enables rigorous lattice studies

$$\tilde{f}_{i/h}(x,\vec{b}_T,\mu,\tilde{\zeta},x\tilde{P}^z,\tilde{\eta}) = C_i\big(x\tilde{P}^z,\mu\big)\ \exp\left[\frac{1}{2}\gamma_\zeta^q(\mu,b_T)\ln\frac{\tilde{\zeta}}{\zeta}\right] f_{i/h}^C(x,\vec{b}_T,\mu,\zeta)$$

Lattice Targets:

- Non-perturbative CS Kernel
- Info on Spin-dependent TMDPDFs (in ratios)
- Info about 3D structure, x and b_T (in ratios)
- proton vs. pion TMD PDFs (in ratios)
- flavor dependence of TMD PDFs (in ratios)
- soft function for TMDs
- TMD PDF with x and b_T (normalization)
- Gluon TMD PDFs

Current Status of Lattice Calculations: Yong Zhao's talk next