Observables for scattering on targets with any spin

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Motivation

Current matrix elements for composite particles with arbitrary spin

- Decompose matrix element in independent non-perturbative objects

$$\langle p', s' | j^\mu | p, s \rangle = \left(G_1\left(Q^2\right) \left[\varepsilon'^* \cdot \varepsilon\right] + G_3\left(Q^2\right) \frac{\left(q \cdot \varepsilon'^*\right)\left(q \cdot \varepsilon\right)}{2m^2} \right) \left(p + p'\right)^\mu + G_M\left(Q^2\right) \left(\left(q \cdot \varepsilon'^*\right) \varepsilon^\mu - \left(q \cdot \varepsilon\right) \left(\varepsilon'^*\right)^\mu \right)$$

- Spin-j fields embedded in objects with $> 2j + 1$ components
  - Polarization four vector (ε) for spin 1 $\rightarrow$ $p_\mu \epsilon^\mu(p, s) = 0$
  - Rarita Schwinger for spin 3/2 $\rightarrow$ $\gamma^\mu \psi_\mu(p, s) = 0$

  (need for constraints, subsidiary conditions)

- Use $(2j + 1)$-component spinors
  - Via SL(2,C) fundamental rep tensor products [Zwanziger 60s, Polyzou '18]
  - Weinberg's construction [64-65] (not yet applied in this context)

Advantages of Weinberg's construction

- Use only exact degrees of freedom (chiral reps), no need for constraints
- No kinematic singularities (improved analyticity properties of operators)
- Physical interpretation becomes more straightforward (amplitude matrix elements)
- "Basic" in construction and implementation of su(2) algebra
- For parity conserving interactions a generalized Dirac algebra is obtained
- Easy to switch between forms of dynamics (instant form, light front)

Weinberg's "Feynman rules for Any Spin" [1964]

- Algebra for Generators of the Lorentz group

$$[\mathbb{J}_l, \mathbb{J}_m] = i\epsilon_{lmn}\mathbb{J}_n \ , \quad [\mathbb{J}_l, \mathbb{K}_m] = i\epsilon_{lmn}\mathbb{K}_n \ , \quad [\mathbb{K}_l, \mathbb{K}_m] = -i\epsilon_{lmn}\mathbb{J}_n$$

- Two independent su(2) subalgebras → irreps (j_A, j_B)

$$\mathbb{A}_m = \frac{1}{2}(\mathbb{J}_m + i\mathbb{K}_m) \quad , \quad \mathbb{B}_m = \frac{1}{2}(\mathbb{J}_m - i\mathbb{K}_m)$$

$$[\mathbb{A}_l, \mathbb{A}_m] = i\epsilon_{lmn}\mathbb{A}_n \ , \quad [\mathbb{B}_l, \mathbb{B}_m] = i\epsilon_{lmn}\mathbb{B}_n \ , \quad [\mathbb{A}_l, \mathbb{B}_m] = 0$$

- Simplest irreps that contain spin-j → $(2j + 1$ components$)$
  - Right-handed $(j, 0)$: $\mathbb{K}_m \to -i\mathbb{J}_m$
  - Left-handed $(0, j)$: $\mathbb{K}_m \to +i\mathbb{J}_m$

[Wigner(1939)]

Some Representations constructed out of the Chiral ones

- $(0,0) \rightarrow$ Scalar
- $(1/2,0) \rightarrow$ Right Weyl spinors & $(0,1/2) \rightarrow$ Left Weyl spinors
- $(1/2,0) \bigoplus (0,1/2) \rightarrow$ Dirac (spin 1/2) spinors (direct sum)
- $(1/2,1/2) \rightarrow$ Vector (Defining representation)
- $(1,0) \rightarrow$ Right Chiral (spin 1) spinors & $(0,1) \rightarrow$ Left Chiral (spin 1) spinors
- $(1,0) \bigoplus (0,1) \rightarrow$ Dirac (spin 1) spinors (direct sum)
- $(1,1) \rightarrow$ Tensor

Canonical Space-Time Parameterization

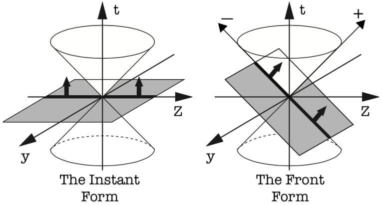
Parameterizations (Foliations) of space-time → Specify equal time surfaces

Canonical or Instant time

- Defined by rotationless boosts from rest: $\overset{\circ}{p}{}^{\mu} = (m, 0, 0, 0)$

  to final momentum: $p^{\mu} = (E_p, \vec{p}) = (\sqrt{m^2 + \vec{p}^2}, \vec{p})$

$$\Lambda^{\text{IF}} = \exp\left(i\vec{\mathbb{K}} \cdot \vec{\phi}\right) = \exp\left(i\phi\vec{\mathbb{K}} \cdot \hat{\phi}\right)$$



- Then, $p^{\mu} = (E, \vec{p}) = (\Lambda^{\text{IF}})^{\mu}{}_{\nu} \overset{\circ}{p}{}^{\nu}$

  implies, $\cosh(\phi) = \frac{E}{m}$, $\hat{\phi}_j \sinh(\phi) = \frac{p_j}{m}$

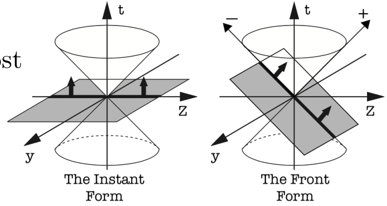
Leading to the well known result: $(\Lambda^{\text{IF}})^{\mu}{}_{\nu} = \begin{pmatrix} \frac{E}{m} & \frac{\vec{p}}{m} \\ \frac{\vec{p}}{m} & \delta_{ij} + \frac{p_i p_j}{(E+m)m} \end{pmatrix}$

Light-Front Space-Time Parameterization

Light Front

Dirac(1949)

$p^+ = E_p + p_z \ , \ \ p^- = E_p - p_z$



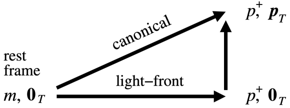
- Defined by a longitudinal boost followed by a transverse boost

$$\Lambda^{\rm LF}_{\rm def.} = \exp\left[i\vec{\mathbb{G}}\cdot\vec{\rm v}_{\rm T}\right]\cdot\exp\left[i\mathbb{K}_3\eta\right]$$

- LF Boost Generators (light front along $z-$axis),

$$\mathbb{G}_1 = \mathbb{G}_x = \mathbb{K}_x - \mathbb{J}_y \ , \quad \mathbb{G}_2 = \mathbb{G}_y = \mathbb{K}_y + \mathbb{J}_x \ , \quad \mathbb{K}_3 = \mathbb{K}_z$$

- Comparing the action of both boosts on the same rest momentum we find the LF boost parameters



$$e^{\eta} = \frac{p^+}{m} \ , \quad \vec{\rm v}_T = \frac{\vec{p}_T}{p^+} \to \Lambda^{\rm LF} = \exp\left[i\frac{\eta}{p^+ - m}\vec{p}_T\cdot\vec{\mathbb{G}} + i\eta\mathbb{K}_3\right]$$

Propagator of chiral fields

- Numerator (invariant)

$$\Pi^{(j)}_{\sigma\sigma'}(\vec{p},\omega) = m^{2j} D^{(j)}_{\sigma\sigma'}[L(\vec{p})] \left(D^{(j)}_{\sigma'\sigma''}[L(\vec{p})]\right)^{\dagger} = m^{2j}\left(e^{-2\hat{p}\cdot\vec{J}^{(j)}\theta}\right)_{\sigma\sigma'}$$

$$\bar{\Pi}^{(j)}_{\sigma\sigma'}(\vec{p},\omega) = m^{2j} \bar{D}^{(j)}_{\sigma\sigma'}[L(\vec{p})] \left(\bar{D}^{(j)}_{\sigma'\sigma''}[L(\vec{p})]\right)^{\dagger} = m^{2j}\left(e^{2\hat{p}\cdot\vec{J}^{(j)}\theta}\right)_{\sigma\sigma'}$$

- Introduction of the t-tensors

$$\Pi^{(j)}_{\sigma\sigma'}(\vec{p},\omega) = t^{\mu_1\mu_2\dots\mu_{2j}}_{\sigma\sigma'} p_{\mu_1}p_{\mu_2}\dots p_{\mu_{2j}}$$

$$\bar{\Pi}^{(j)}_{\sigma\sigma'}(\vec{p},\omega) = \bar{t}^{\mu_1\mu_2\dots\mu_{2j}}_{\sigma\sigma'} p_{\mu_1}p_{\mu_2}\dots p_{\mu_{2j}}$$

- These can also be used to write the boosts/spinors

$$D^{(j)}_{[L(p)]} = t^{\mu_1\mu_2\dots\mu_{2j}}\tilde{p}_{\mu_1}\tilde{p}_{\mu_2}\dots\tilde{p}_{\mu_{2j}}$$

$$\bar{D}^{(j)}_{[L(p)]} = \bar{t}^{\mu_1\mu_2\dots\mu_{2j}}\tilde{p}_{\mu_1}\tilde{p}_{\mu_2}\dots\tilde{p}_{\mu_{2j}}$$

Instant form (Canonical)

$$\tilde{p}^{\mu}_{\mathrm{C}} = \sqrt{\frac{m}{2(m+p^0)}}(p^0+m,\vec{p}\,)$$

Light-Front

$$\tilde{p}^{\mu}_{\mathrm{LF}} = \sqrt{\frac{m}{4p^+}}(p^+ + m, p_\ell, ip_\ell, p^+ - m)$$

Properties of the t-tensors

- Symmetric and (covariantly) traceless $g_{\mu_k\mu_l} t_{\sigma\sigma'}^{\mu_1\ldots\mu_k\ldots\mu_l\ldots\mu_{2j}} = 0$

- Transform covariantly $\left(D_{[\Lambda]}^{(j)}\right)_{\sigma\delta} t_{\delta\delta'}^{\mu_1\ldots\mu_{2j}} \left(D^{(j)}{}_{[\Lambda]}^{\dagger}\right)_{\delta'\sigma'} = \Lambda_{\nu_1}{}^{\mu_1} \ldots \Lambda_{\nu_{2j}}{}^{\mu_{2j}}\; t_{\sigma\sigma'}^{\nu_1\ldots\nu_{2j}}$

- Right chiral (t) and left chiral ($\bar{t}$) are related by charge conjugation (+ for even (− for odd) spacelike indices) $\bar{t}_{\sigma\sigma'}^{\mu_1\mu_2\ldots\mu_{2j}} = (\pm) t_{\sigma\sigma'}^{\mu_1'\mu_2'\ldots\mu_{2j}'}$

Construction for t-tensor more insightful than Weinberg's expressions

- The 0-th degree polynomial in the J's is always $\quad t^{0\ldots0} = \mathbf{1}$

- The linear polynomials
  are the Rotation Group Generators $\quad t^{0\ldots i\ldots0} = \frac{2}{2j} J_i = \frac{1}{j} J_i$

- From pairwise symmetrizations of the rotation generators

$$t^{0\ldots m\ldots0\ldots n\ldots0} = t^{mn0\ldots0} = \frac{1}{\frac{(2j)!}{2!(2j-2)!}} \left(\{J_m, J_n\} - \frac{1}{3}\delta_{mn} \sum_{r=1}^{3} \{J_r, J_r\} \right) + \frac{1}{3} t^{0\ldots0} \delta_{mn}$$

$$= \frac{j}{(2j-1)} \left(\{t^{m0\ldots0}, t^{n0\ldots0}\} - \frac{1}{j} \delta_{mn} t^{0\ldots0} \right)$$

Algorithm for construction of t-tensors

- Continues for higher orders
  - Matrices have more and more off-diagonal elements

$$t^{lmn0\dots0} = t^{0\dots0l0\dots0m0\dots0n0\dots0} = \frac{j}{(2j-2)}\frac{1}{3}\Big(\{t^{l0\dots0}, t^{mn0\dots0}\} + \{t^{m0\dots0}, t^{nl0\dots0}\} + \{t^{n0\dots0}, t^{lm0\dots0}\} - \frac{2}{j}\{\delta_{lm}t^{n0\dots0} + \delta_{ln}t^{m0\dots0} + \delta_{mn}t^{l0\dots0}\}\Big)$$

- Construction stops after j steps (Cayley-Hamilton) $(J-s)(J-s-1)...(J+s)=0$
- t-tensors contain an independent basis for the su(2j+1) algebra
- A basis to decompose operators with physical interpretation for each term (multipole expansion → mono-, di-, quadrupole, ...)

$$\hat{O} = Tr\,[O]\,\mathbf{1} + Tr\,[OJ_i]\,J_i + Tr\,[OJ_{ij}]\,J_{ij} + \cdots = \langle O\rangle\,\mathbf{1} + O_iJ_i + O_{ij}J_{ij} + \cdots$$

spin 1 example

Left Chiral Rep

- $t^{00} = \mathbf{1}$, $t^{0i} = t^{i0} = J_i^{(1)}$, $t^{ij} = \{J_1^{(1)}, J_1^{(1)}\} - \mathbf{1}\delta_{ij}$
- $t^{\mu\nu}$ Transform covariantly $D_{[\Lambda]}^{(1)} t^{\mu\nu} D^{(1)}{}_{[\Lambda]}^{\dagger} = \Lambda_\rho{}^\mu \Lambda_\sigma{}^\nu t^{\rho\sigma}$
- Propagator $(p_\mu = (E_p, \vec{p}))$: $\Pi^{(1)}(p) = t^{\mu\nu} p_\mu p_\nu = \begin{pmatrix} (p^-)^2 & -\sqrt{2}p_\ell p^- & p_\ell^2 \\ \sqrt{2}p_r p^+ & p^+p^- + p_T^2 & \sqrt{2}p_\ell p^- \\ p_r^2 & \sqrt{2}p_r p^- & (p^+)^2 \end{pmatrix}$
- Boost/spinors $(t^{\mu\nu}\tilde{p}_\mu \tilde{p}_\nu)$

  Canonical: $D_{\mathrm{IF}}^{(1)} = \dfrac{1}{2m\,(m+p_0)} \begin{pmatrix} (m+p^-)^2 & -\sqrt{2}p_\ell(m+p^-) & p_\ell^2 \\ -\sqrt{2}p_r(m+p^-) & 2(m^2+mp_0+p_T^2) & -\sqrt{2}p_\ell(m+p^+) \\ p_r^2 & -\sqrt{2}p_r(m+p^+) & (m+p^+)^2 \end{pmatrix}$

  $$\tilde{p}_{\mathrm{C}}^\mu = \sqrt{\frac{m}{2(m+p^0)}}(p^0 + m, \vec{p}\,)$$

Similarly for the Right Chiral Rep, only change is: $J_i^{(1)} \to \bar{J}^\mu = (1, -\vec{J}^{(1)})$

Dirac (bispinors)

Generalized Dirac algebra

- Parity conserving reactions are simpler in the direct sum of both chiral representations (like the spin 1/2 case)
- This leads to generalized Gamma matrices $\rightarrow$ $\Gamma^{\mu_1\cdots\mu_{2j}} = \begin{pmatrix} 0 & t^{\mu_1\cdots\mu_{2j}} \\ \bar{t}^{\mu_1\cdots\mu_{2j}} & 0 \end{pmatrix}$
- Dirac basis for spin-1 $\rightarrow$ $\mathbf{1},\ \Gamma_5,\ (9)\Gamma^{\mu\nu},\ (9)\Gamma^{\mu\nu}\Gamma_5,\ (6)[\Gamma^{\mu_1\mu_2},\Gamma^{\mu_3\mu_4}],\ (10)\{\Gamma^{\mu_1\mu_2},\Gamma^{\mu_3\mu_4}\}$
- Amplitudes can be evaluated by
    - Constructing expressions for the generalized bilinears
    - Using trace algebra
- Similarly expressions for covariant density matrices can be constructed

Generalized Dirac and Gordon identities

- Dirac Equation (constraint on bispinors)

$$\left(\gamma^{\mu_1\ldots\mu_{2j}} p_{\mu_1\ldots\mu_{2j}} - m^{2j}\right) u_p^s = (\not{p}^{(j)} - m^{2j}) u_p^s = 0$$

- Gordon identity separates general bilinears into convection and magnetization currents (spin 1/2 Lorce-2017)

$$u_{p'}^{s'} \left(\Gamma\right) u_p^s = \frac{1}{2\bar{m}^{2j}} u_{p'}^{s'} \left(\left\{ \not{P}^{(j)}, \Gamma \right\} + \frac{1}{2} \left[\not{\Delta}^{(j)}, \Gamma \right] \right) u_p^s$$

$$0 = u_{p'}^{s'} \left(\frac{1}{2} \left\{ \not{\Delta}^{(j)}, \Gamma \right\} + \left[\not{P}^{(j)}, \Gamma \right] \right) u_p^s$$

with,
$$\not{P}^{(j)} = \gamma^{\mu_1\ldots\mu_{2j}} P_{\mu_1\ldots\mu_{2j}}$$
$$\not{\Delta}^{(j)} = \gamma^{\mu_1\ldots\mu_{2j}} \Delta_{\mu_1\ldots\mu_{2j}}$$

EM Current Parameterized by Sachs Form Factors

From spinor Representation:

$$\langle p', s' | j^\mu(0) | p, s \rangle = 2P^\mu \left(\mathbf{1} G_C(Q^2) - \frac{\Delta^\rho \Delta^\sigma \left(t_{\rho\sigma} - \frac{1}{3} g_{\rho\sigma} \mathbf{1} \right)}{2M^2} \frac{P^2}{M^2} G_Q(Q^2) \right)_{s's} - i\epsilon^{\mu\rho\sigma\lambda} \left(\frac{\Delta_\rho P_\sigma \left(t_{\lambda\nu} - \frac{1}{3} g_{\lambda\nu} \mathbf{1} \right) n_t^\nu}{\sqrt{P^2}} G_M(Q^2) \right)_{s's}$$

$$P = \tfrac{1}{2}(p' + p)$$
$$\Delta = p' - p \quad (\Delta^2 = -Q^2)$$
$$n_t^\nu = (1, 0, 0, 0)$$

Using polarization vectors [Wang & Lorcé (2022)]

$$\Gamma^{\mu\alpha\beta} = 2P^\mu \left(\Pi^{\alpha\beta} G_C(Q^2) - \frac{\Delta^\rho \Delta^\sigma \left(\Sigma_{\rho\sigma} \right)^{\alpha\beta}}{2M^2} \frac{P^2}{M^2} G_Q(Q^2) \right)_{s's} - i\epsilon^{\mu\rho\sigma\lambda} \left(\frac{\Delta_\rho P_\sigma \left(\Sigma_\lambda \right)^{\alpha\beta}}{\sqrt{P^2}} G_M(Q^2) \right)_{s's}$$

Current conservation is guaranteed: $\Gamma^\mu \Delta_\mu = 0$

(on-shell condition $\rightarrow P^\mu \Delta_\mu = 0$)

EM Current

From Spinor Representation (Parameterized by Sachs Form Factors):

$$j^{\mu}_{\mathrm{Ch}}(P,\Delta) = 2P^{\mu}\left(\mathbf{1}G_C - \frac{\Delta^{\rho}\Delta^{\sigma}\left(t_{\rho\sigma} - \frac{1}{3}g_{\rho\sigma}\mathbf{1}\right)}{2M^2}\frac{P^2}{M^2}G_Q\right) - \frac{i\epsilon^{\mu\rho\sigma\lambda}\Delta_{\rho}P_{\sigma}\left(t_{\lambda\nu} - \frac{1}{3}g_{\lambda\nu}\mathbf{1}\right)n^{\nu}_{t}}{\sqrt{P^2}}G_M$$

Textbook Representation (using Polarization vectors)

$$\langle p', s' \,| j^{\mu}(0) |\, p, s\rangle = \varepsilon^{*\,\alpha}_{s'}(p')\, j^{\mu}_{\alpha\beta}(P,\Delta)\varepsilon^{\beta}_{s}(p)$$

Parameterized by Covariant Form Factors

$$j^{\mu}_{\alpha\beta}(P,\Delta) = 2P^{\mu}\left[g_{\alpha\beta}G_1\left(Q^2\right) - \frac{\Delta_{\alpha}\Delta_{\beta}}{M^2}G_2\left(Q^2\right)\right] + \left[\Delta^{\alpha}g^{\mu}_{\beta} - \Delta^{\beta}g^{\mu}_{\alpha}\right]G_3\left(Q^2\right)$$

$$\varepsilon^{\mu}_{s}(p) = \left(\frac{\boldsymbol{p}\cdot\boldsymbol{\epsilon}_s}{M}, \boldsymbol{\epsilon}_s + \frac{\boldsymbol{p}(\boldsymbol{p}\cdot\boldsymbol{\epsilon}_s)}{M(p^0+M)}\right)$$

$$\boldsymbol{\epsilon}_{\pm} = \frac{1}{\sqrt{2}}(\mp 1, -i, 0), \;\; \boldsymbol{\epsilon}_0 = (0,0,1)$$

The two sets of form factors are related by:

$$\tau = Q^2/(4M^2)$$

$$G_C\left(Q^2\right) = G_1\left(Q^2\right) + \frac{2}{3}\tau G_Q\left(Q^2\right)$$

$$G_M\left(Q^2\right) = G_2\left(Q^2\right)$$

$$G_Q\left(Q^2\right) = G_1\left(Q^2\right) - G_2\left(Q^2\right) + (1+\tau)G_3\left(Q^2\right)$$

EM Current - Example: Breit Frame

In the Breit Frame $(\vec{P}=0) \to p_B^{\mu} = \left(P^0, -\Delta/2\right)$ and $p_B^{\mu} = \left(P^0, \Delta/2\right)$ with $P_B^0 = \sqrt{M^2 + \frac{\Delta^2}{4}}$

Covariant Chiral Representation:

$$\langle p', s' \left| j^{\mu}(0) \right| p, s\rangle = \Gamma^{\mu}(P,\Delta) = 2P^{\mu}\left(\mathbf{1}G_C\left(Q^2\right) - \frac{\Delta^{\rho}\Delta^{\sigma}\left(t_{\rho\sigma} - \frac{1}{3}g_{\rho\sigma}\mathbf{1}\right)}{2M^2}\frac{P^2}{M^2}G_Q\left(Q^2\right)\right)_{s's}$$

$$-i\epsilon^{\mu\rho\sigma\lambda}\left(\frac{\Delta_{\rho}P_{\sigma}\left(t_{\lambda\nu} - \frac{1}{3}g_{\lambda\nu}\mathbf{1}\right)n_t^{\nu}}{\sqrt{P^2}}G_M\left(Q^2\right)\right)_{s's}$$

Textbook Representation in terms of Polarization vectors:

$$\left\langle p_B', s' \left| j^0(0) \right| p_B, s\right\rangle = 2P_B^0\left[\left(\boldsymbol{\epsilon}_{s'}^{*}\cdot\boldsymbol{\epsilon}_s\right)G_C\left(Q^2\right) + \left(\left(\boldsymbol{\Delta}\cdot\boldsymbol{\epsilon}_{s'}^{*}\right)\left(\boldsymbol{\Delta}\cdot\boldsymbol{\epsilon}_s\right) - \frac{1}{3}\Delta^2\left(\boldsymbol{\epsilon}_{s'}^{*}\cdot\boldsymbol{\epsilon}_s\right)\right)\frac{G_Q\left(Q^2\right)}{2M^2}\right]$$

$$\langle p_B', s' | \boldsymbol{j}(0) | p_B, s\rangle = 2P_B^0\left[\left(\boldsymbol{\Delta}\cdot\boldsymbol{\epsilon}_{s'}^{*}\right)\boldsymbol{\epsilon}_s - \left(\boldsymbol{\Delta}\cdot\boldsymbol{\epsilon}_s\right)\boldsymbol{\epsilon}_{s'}^{*}\right]\frac{G_M\left(Q^2\right)}{2M}$$

$$\boldsymbol{\epsilon}_{\pm} = \frac{1}{\sqrt{2}}(\mp 1, -i, 0), \quad \boldsymbol{\epsilon}_0 = (0,0,1), \quad p_{\mu}\varepsilon^{\mu}(p,s) = 0$$

Summary

- Weinberg's construction allows for an efficient and manifestly covariant calculation of currents for any spin
- Central (and multifaceted) role for the covariant t-tensors
- Simple algorithm. Only need to know the matrices for the Generators of rotations in the representation of interest.
- Many applications and extensions possible (Parameterization for SIDIS and DVCS)