Introducing Jets for the EIC

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CFNS

G. Sterman, SBU

A. Quick history of jets in experiment: partons and inclusive cross sections.

B. A little quantum field theory: the role of soft and collinear radiation.

C. Why energy flow is a guide to calculable cross sections: infrared safety.

D. Briefly: infrared safety and beyond at the EIC.

A. Quick history of jets

- A "jet" is an set of particles whose total invariant mass is much smaller than its total energy. It is intrinsically relativistic.

- The first observations of particle "jets" was in cosmic ray detection.

  Particle jets in cosmic rays . . .

  "The average transverse momentum resulting from our measurements is p_T=0.5 BeV/c for pions . . . a summary of jet events observed to date . . ." (B. Edwards et al, Phil. Mag. 3, 237 (1957))

- The late 1960s and the dawning era of high energy accelerators:

  The parton picture for deep-inelastic scattering (Feynman, Bjorken)

$$\sigma^{\rm incl}_{e\,\rm proton}\left(Q, x=\frac{Q^2}{2p\cdot q}\right) \rightarrow \sigma^{\rm excl}_{e\,\rm parton}(Q) \times F_{\rm proton}(x)\,,$$

- the question arose: what happens to partons in the final state?
  (Feynman, Bjorken & Paschos, Drell, Levy & Yan, 1969)
  Do "the hadrons 'remember' the directions along which the bare constituents were emitted? . . . "the observation of such 'jets' in colliding beam processes would be most spectacular." (Bjorken & Brodsky, 1969) Or does confinement forbid a it?

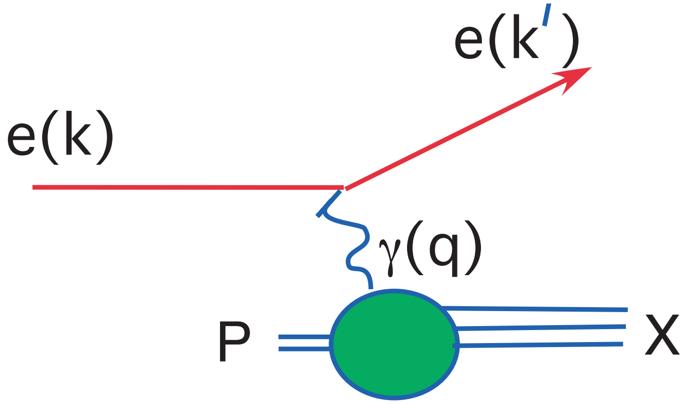
- The inclusive DIS cross section is described by exclusive partonic scattering. Could something similar happen in a less inclusive observable? The answer was yes . . .

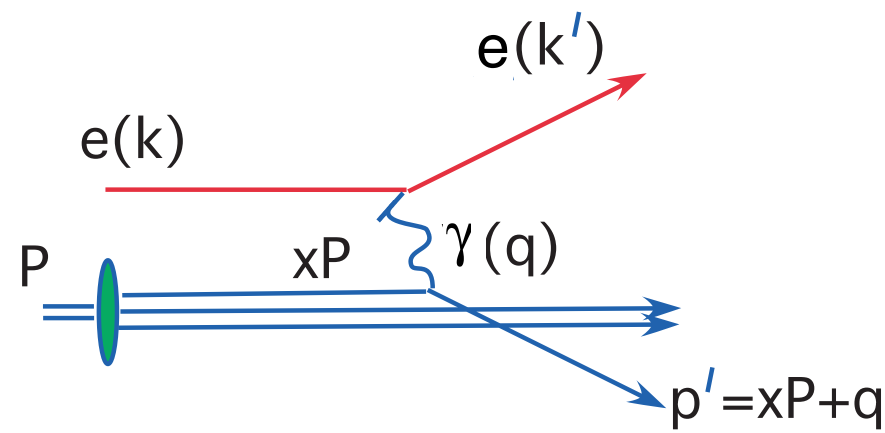
- To make this long story short: Quantum Chromodynamics (QCD) reconciled the irreconcilable. Here was the problem.

  1. Quarks and gluons explain spectroscopy, but aren't seen directly – confinement.

  2. In highly ("deep") inelastic, electron-proton scattering, the inclusive cross section was found to well-approximated by lowest-order elastic scattering of point-like (spin-1/2) particles (="partons" = quarks here) a result called "scaling":

$$\frac{d\sigma_{e+p}(Q, p\cdot q)}{dQ^2}\Big|_{\text{inclusive}} \quad \propto \quad F\left(x = \frac{Q^2}{2p\cdot q}\right) \frac{d\sigma^{\text{free}}_{e+\text{spin}\,\frac{1}{2}}}{dQ^2}\Big|_{\text{elastic}}$$





- If the "spin-$\frac{1}{2}$ is a quark, how can a confined quark scatter freely?

- This paradoxical combination of confined bound states at long distances and nearly free behavior at short distances was explained by asymptotic freedom: In QCD, the force between quarks behaves at short distances like

$$f(r) \sim \frac{\alpha_s(r)}{r^2}, \qquad \alpha_s(r^2) = \frac{4\pi}{\ln\left(\frac{1}{r^2\Lambda^2}\right)}$$

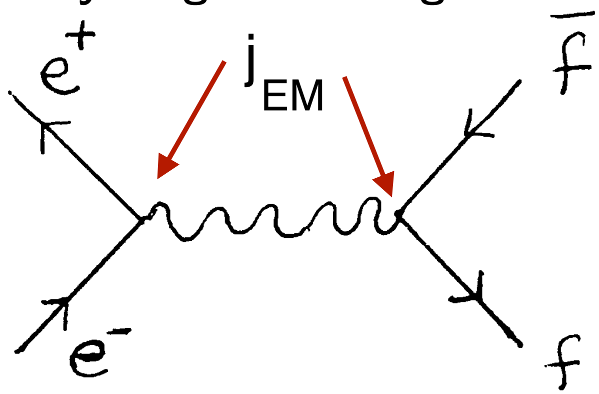
where $\Lambda \sim 0.2$ GeV. For distances much less than $1/(0.2GeV) \sim 10^{-13}$cm the force weakens. These are distances that began to be probed in deep inelastic scattering experiments at SLAC in the 1970s.

- The short explanation of DIS: Over the times $ct \leq \hbar/GeV$ it takes the electron to scatter from a quark-parton, the quark really does seem free. Later, the quark is eventually confined, but by then it's too late to change the probability for an event that has already happened.

- The function $F(x)$ is interpreted as the probability to find quark of momentum xP in a target of total momentum P – a parton distribution.

- To explore further, SLAC used the quantum mechanical credo: anything that can happen, will happen.

- Quarks have electric charge, so if they are there to be produced, they will be. This can happen when colliding electron-positron pairs annihilate to a virtual photon, which (ungratefully) decays to just anything with charge.



- Of course because of confinement it's not that. But more generally, we believe that a virtual photon decays at a point through a local operator: $j_{\rm em}(x)$.

- This enables translating measurements into correlation functions . . . In fact, the cross section for electron-positron annihilation probes the vacuum with an electromagnetic current.

- On the one hand, all final states are familiar hadrons, with nothing special about them to tell the tale of QCD, $|N\rangle = |\text{pions, protons}\ldots\rangle$,

$$\sigma_{\mathrm{e^+e^-\to\ hadrons}}(Q) \;\propto\; \sum_N |\langle 0|j^\mu_{\mathrm{em}}(0)|N\rangle|^2\, \delta^4(Q-p_N)$$

- On the other hand, $\sum_N |N\rangle\langle N| = 1$, and using translation invariance this gives

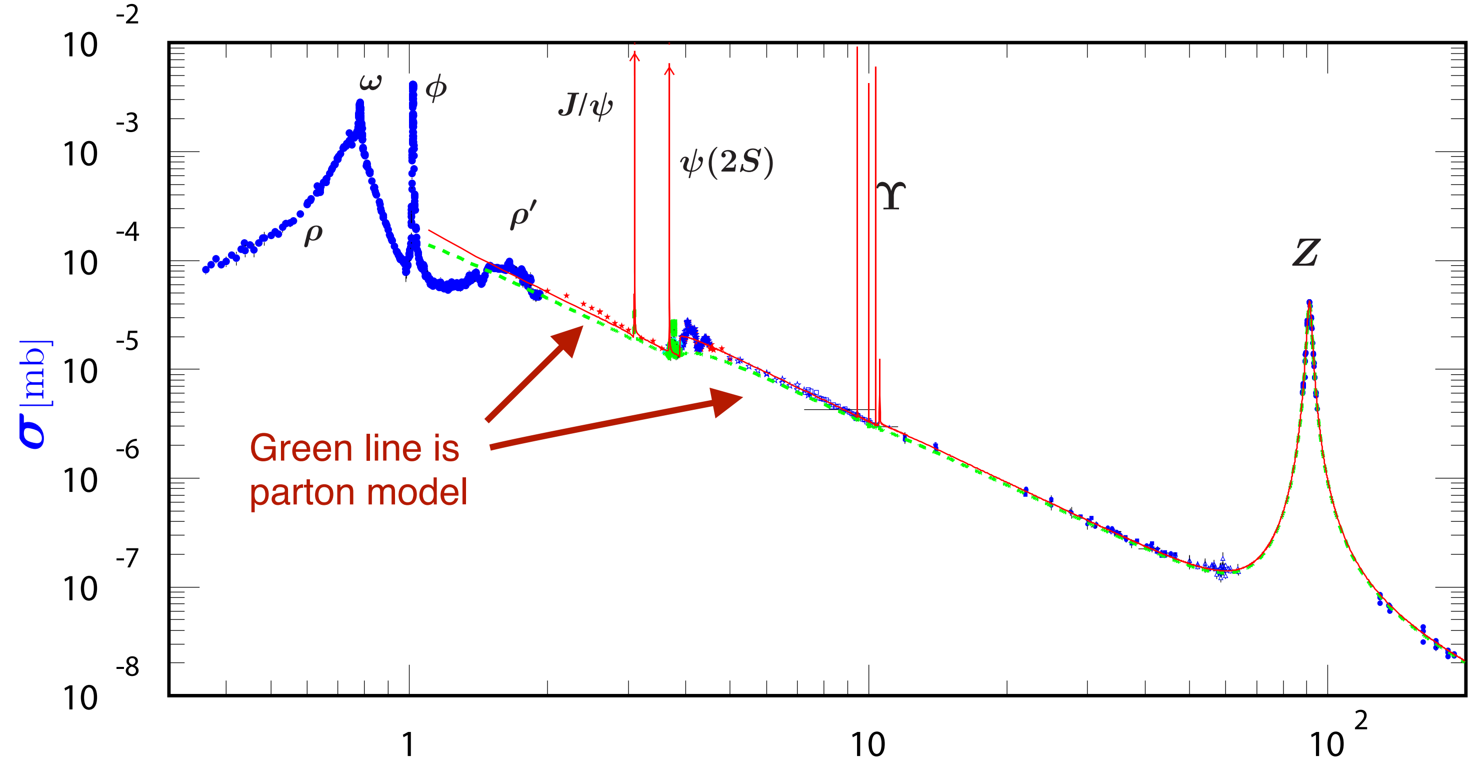
$$\sigma_{\mathrm{e^+e^-\to\ hadrons}}(Q) \;\propto\; \int d^4x\, e^{-iQ\cdot x}\; \langle 0|j^\mu_{\mathrm{em}}(0)\; j^\mu_{\mathrm{em}}(x)|0\rangle$$

- We are probing the vacuum at short distances, imposed by the Fourier transform as $Q \to \infty$. The currents are only a distance $1/Q$ apart.

- Asymptotic freedom suggests a "free" result: QCD at lowest order ("quark-parton model") at cm. energy Q and angle θ

$$\sigma^{tot}_{e^+e^-\to\mathrm{hadrons}} \;=\; \sum_{\mathrm{quarks}\ a} e_a^2 N_c\, \frac{4\pi\alpha^2_{\mathrm{EM}}}{3Q^2}$$

- **This works for σ_{tot} to quite a good approximation (with calculable corrections)**



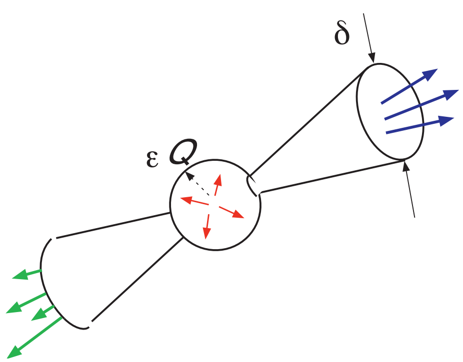
- **So the "free" theory again describes the inclusive sum over confined (nonperturbative) bound states – another "paradox".**

- **Is there an imprint on these states of their origin? Yes. What to look for? The spin of the quarks is imprinted in their angular distribution:**

$$\frac{d\sigma(Q)}{d\cos\theta} = \frac{\pi\alpha_{\rm EM}^2}{2Q^2}\left(1+\cos^2\theta\right)$$

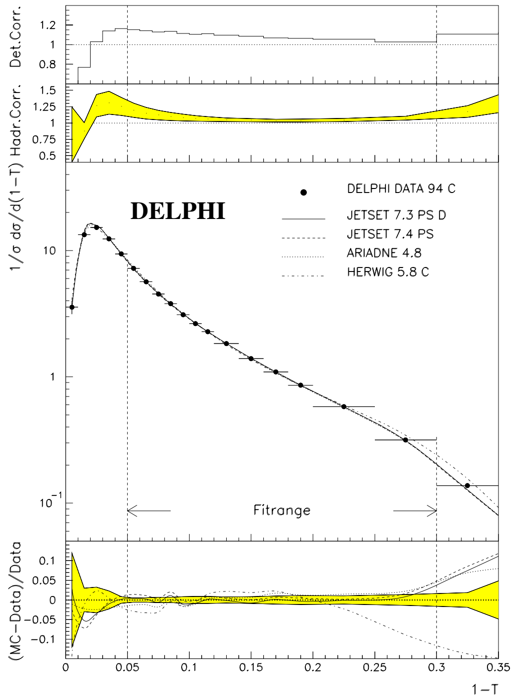
- **It's not quarks, but can look for a back to back flow of energy by finding an axis that maximizes the projection of particle momenta ("thrust") measuring a "jet-like" structure**

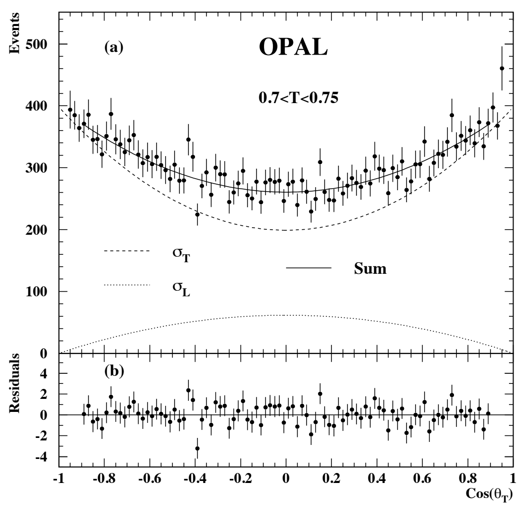
$$\frac{d\sigma_{e^+e^-\to\ \rm hadrons}(Q)}{dT} \propto \sum_N |\langle 0|j^\mu_{\rm em}(0)|N\rangle|^2\,\delta^4(Q-p_N)\;\delta\left(T-\frac{1}{Q}\max_{\hat{n}}\sum_{i\in N}|\vec{p}_i\cdot\hat{n}|\right)$$

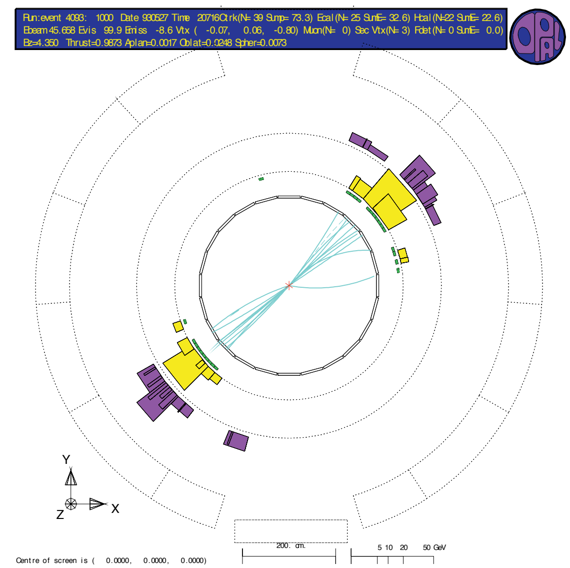


- **When the particles all line up $T \to 1$ (neglecting masses). So what happens?**

- **Here's what was found (from a little later, at LEP):**

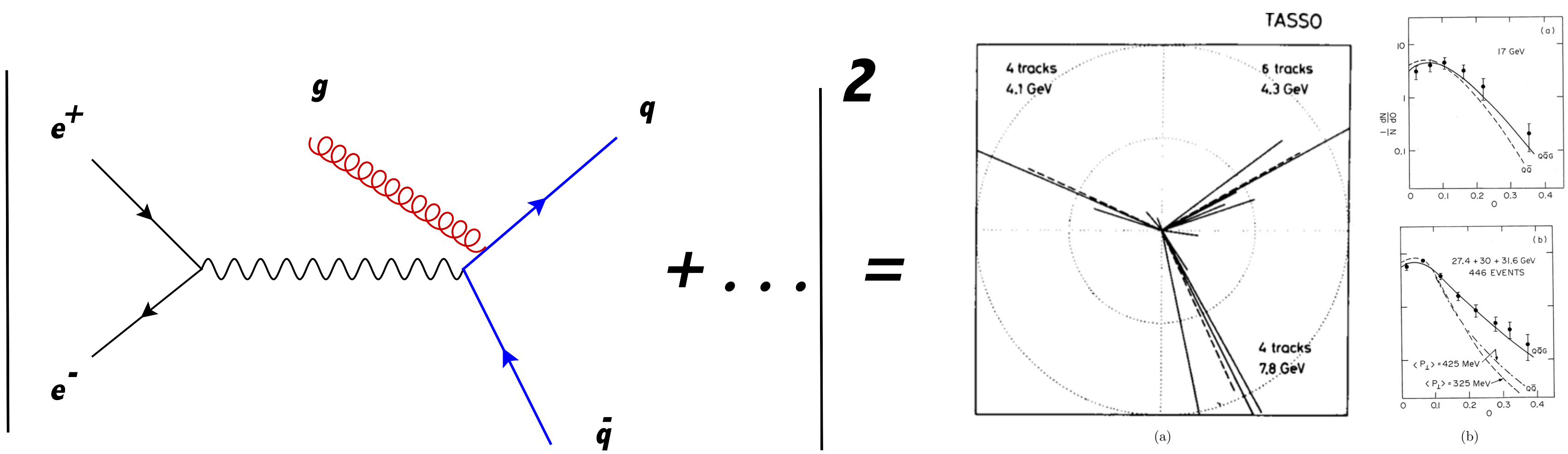






- **Thrust is peaked near unity and follows the $1 + \cos^2\theta$ distribution – reflecting the production of spin $\frac{1}{2}$ particles – back-to-back. All this despite confinement. Quarks have been replaced by "jets" of hadrons. What could be better? But what's going on? How can we understand persistence of short-distance structure into the final state, evolving over many many orders of magnitude in time?**

- Back to the Timeline . . . 1975 -1980: the first quark and gluon jets
- As we've seen: in electron-positron annihilation to hadrons, the angular distribution for energy flow follows the lowest-order ("Born") cross section for the creation of spin-1/2 pairs of quarks and antiquarks (As first seen by Hanson *et al*, at SLAC in 1975)
- Jets are "rare" because the high momentum transfer scattering of partons is rare (but calculable), but in e^+e^- annihilation to hadrons the "rarity" is in the likelihood of annihilation. Once that takes places, jets are nearly always produced.
- And then (Ellis, Gaillard, Ross (1976) Ellis, Karliner (1979)): hints of three gluons in Upsilon decay, and then unequivocal gluon jets at Petra (1979) (S.L. Wu (1984))

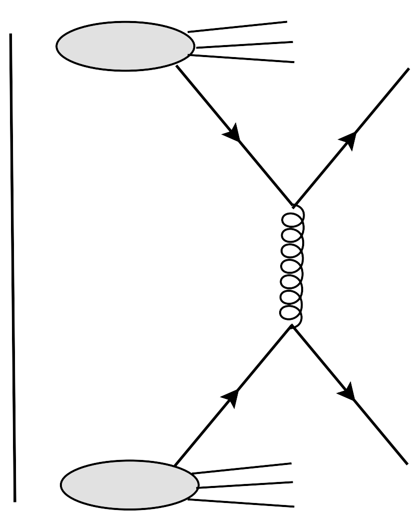


(On the right, O is oblateness, which measures the spread of energy in a plane.)

- confirmed color as a dynamical variable.

- **Jets at hadron colliders . . .**

- **80's: direct and indirect 'sightings' of scattered parton jets at Fermilab and the ISR at CERN, often in the context of single-particle spectra. Overall, however, an unsettled period until the SPS large angular coverage makes possible (UA2) 'lego plots' in terms of energy flow, and leads to the unequivocal observation of high-p_T jet pairs that represent scattered partons.**



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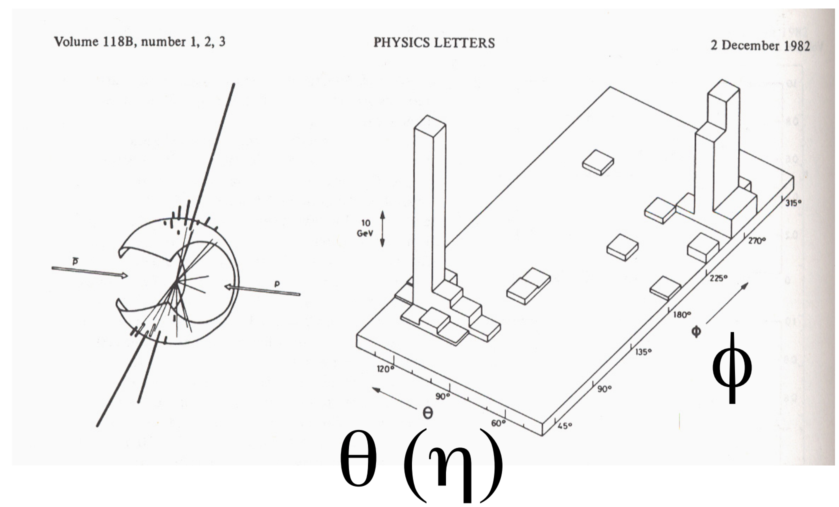
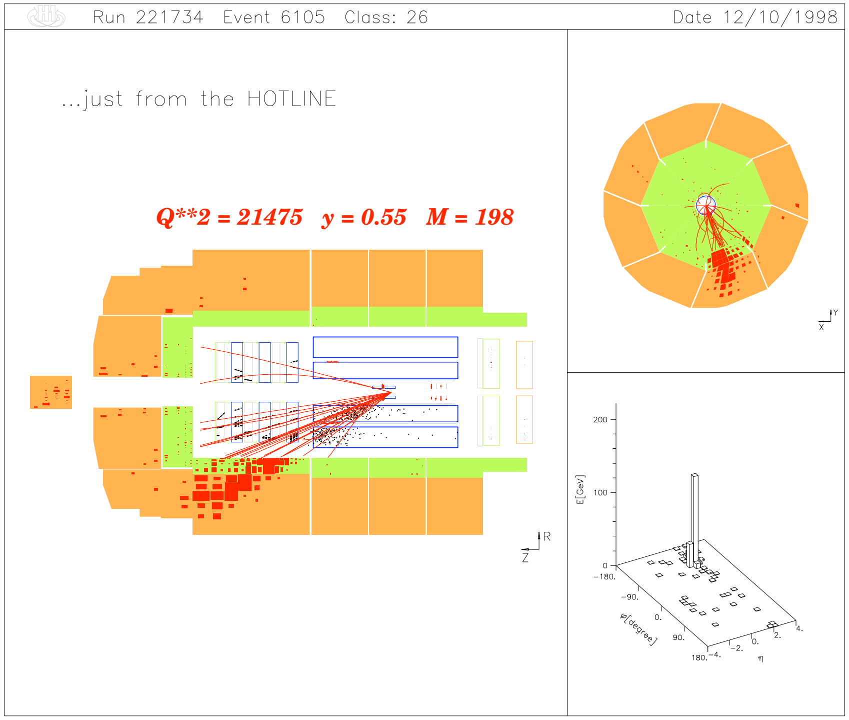
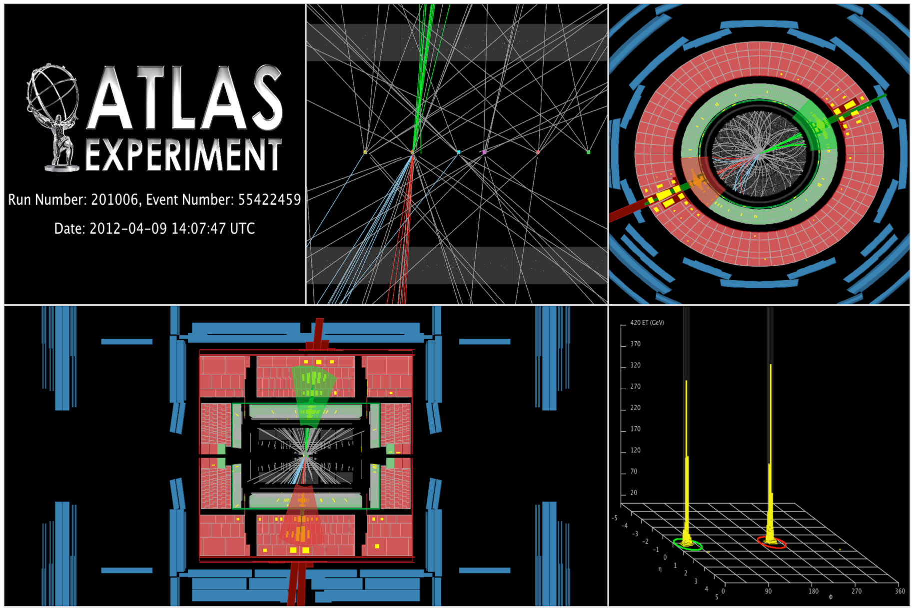


- **1990's – 2005: The great Standard Model machines: HERA, the Tevatron Run I, and LEP I and II provided jet cross sections over multiple orders of magnitude. The scattered quark appears.**



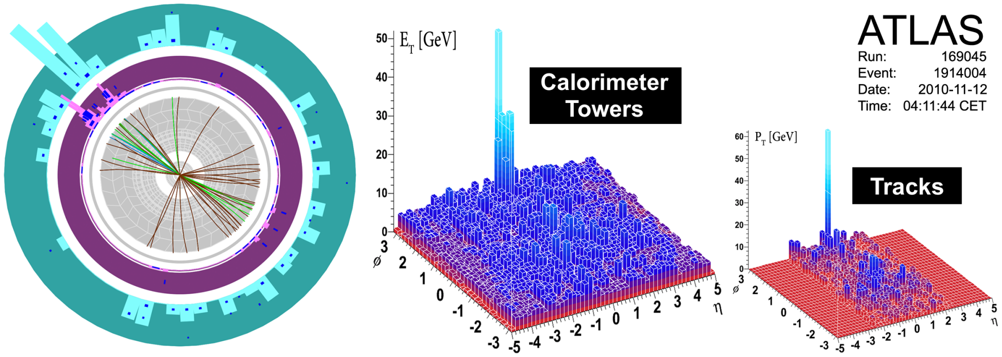
- **And now . . . the era of jets at the anticipated limits of the SM, ushered in by Tevatron Run II, on to the LHC: 2 $\rightarrow$ 7 $\rightarrow$ 8 $\rightarrow$ 13 TeV .**

- **Events at the scale** $\delta x \sim \frac{\hbar}{1\ \mathrm{TeV}} \sim 2 \times 10^{-19}$ **meters . . . observed about 10 meters away.**



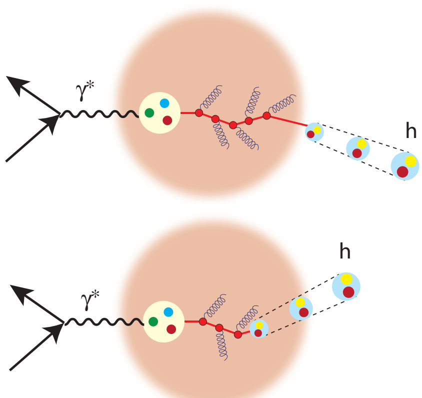
- And at the EIC: unprecedented access to the fate of the scattered quark. Radiation and hadronization, and ...

- shining from the inside, jets are probe of new phases of strongly-interacting matter in nuclear collisions at RHIC and the LHC, (Bjorken (1983) ...)



(From 1011.6182)

- And of "cold nuclei" in electron-ion collisions, through radiation & hadronization
(A. Arccadi *et al.*, Electron-ion Collider White Paper (1212.1701))



B: A little quantum field theory: the role of soft and collinear radiation.

To find the quantum mechanics in all this . . . how can we make jets quantitative?

- At lowest order, $\mathrm{e^+e^-} \to \mathrm{q\bar{q}}$ is easy to calculate, but what can we do with $\mathrm{e^+e^-} \to \mathrm{q\bar{q}g}$? It is (IR) divergent!

- And what to do about the running of the asymptotically free QCD coupling? Doesn't the coupling blow up, making the entire process nonperturbative?

- The glorious example of QED: At lowest order, electron-electron scattering is finite, but at next to leading order it is IR divergent for both virtual corrections and photon emission. But in a partially inclusive sum over soft photon emission only, the divergences cancel, and we derive a finite cross section.

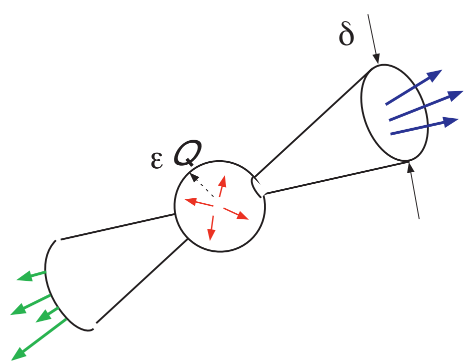
- How? We introduce an "energy resolution", $\epsilon E_{\mathrm{c.m}}$, below which we count all photons. Then divergences are replaced by factors $\alpha \ln(E_{\mathrm{c.n.}}/\epsilon E_{\mathrm{c.m.}})$, and this "inclusive" cross section is well-approximated by the lowest order (again).

- For $|\ln \epsilon| \ll 137$, the sum over orders n is very close to the Born cross section. All the higher order singularities cancel The paradoxical lesson: "the more inclusive, the closer to the lowest order."

- **Once QCD was invented, QED served an inspiration for energies and momentum transfers much larger than masses.**

- **At very high energy we had to introduce an energy resolution and another, "angular" resolution.**

- **From now on, all our particles will be massless. The picture ($E_{\mathrm{c.m.}} = Q$):**



- **With ϵQ the energy resolution, an δ an angular resolution. Defines a "cone jet".**

- **Looks promising, but how does it work? First, we have to isolate the problem, then show how the jet approach solves it.**

- **Let's think of what we'd like to calculate. A "transition probability", or cross section, summed over final states "f":**

$$
\begin{aligned}
P[S] &= \sum_f S[f] \, |\langle m_f | m_0 \rangle|^2 \\
&= \sum_f S[f] \sum_{n',n} \langle m_0 | m_f \rangle^{(n')} \langle m_f | m_0 \rangle^{(n)}
\end{aligned}
$$

The "measurement function" $S[f]$ defines the cross section. It can be unity for some states, zero for others, or in between. We'll assume it's a smooth function.

- **To calculate $P[S]$, we'll start with the amplitude $\langle m_f | m_0 \rangle^{(n)}$ at fixed perturbative order (n) in QCD or some other theory. This is "just" a bunch of Feynman diagrams, but we'll consider a variation of this route.**

Perturbation theory "from the beginning"

- **It really just follows from Schrödinger equation for mixing of free particle states $|m\rangle$,**

$$i\hbar \frac{\partial}{\partial t}|\psi(t)> = \left(H^{(0)} + V\right)|\psi(t)>$$

Usually with free-state "IN" boundary condition :

$$|\psi(t=-\infty)> = |m_0> = |p_1^{\mathrm{IN}}, p_2^{\mathrm{IN}}\rangle$$

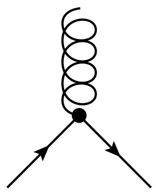
- **Notation : $V_{ji} = \langle m_j|V|m_i\rangle$ (vertices)**

- **Theories differ in their list of particles and their (hermitian) Vs.**

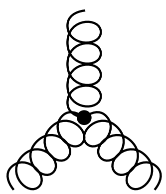
For QCD, the Lagrange density

$$
\begin{aligned}
\mathcal{L}_{\mathrm{QCD}} &= \overline{\psi}_i (i\gamma^\mu \partial_\mu - m)\psi_i - \tfrac{1}{4} F_a^{\mu\nu} F_{\mu\nu}^a - g_{\mathrm{s}}\, \overline{\psi}_i \lambda_{ij}^a \psi_j\, \gamma^\mu A_\mu^a \\
F_a^{\mu\nu} &= \partial^\mu A_a^\nu - \partial^\nu A_a^\mu - 2 g_{\mathrm{s}}\, f_{abc} A_b^\mu A_c^\nu
\end{aligned}
$$

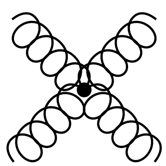
And vertices



$g_{\mathrm{s}}\, \bar{\psi}_i \lambda_{ij}^a \psi_j \gamma^\mu A_\mu^a$ quark-gluon vertex



$g_{\mathrm{s}}\, (\partial^\mu A_a^\nu - \partial^\nu A_a^\mu) f_{abc} A_\mu^b A_\nu^c$ 3-gluon vertex



$g_{\mathrm{s}}^2\, f_{abc} A_b^\mu A_c^\nu\, f_{ade} A_\mu^d A_\nu^e$ 4-gluon vertex

- **Solutions to the Schrödinger equation are sums of ordered time integrals. "Old-fashioned perturbation theory."** **(For *any* theory.)**

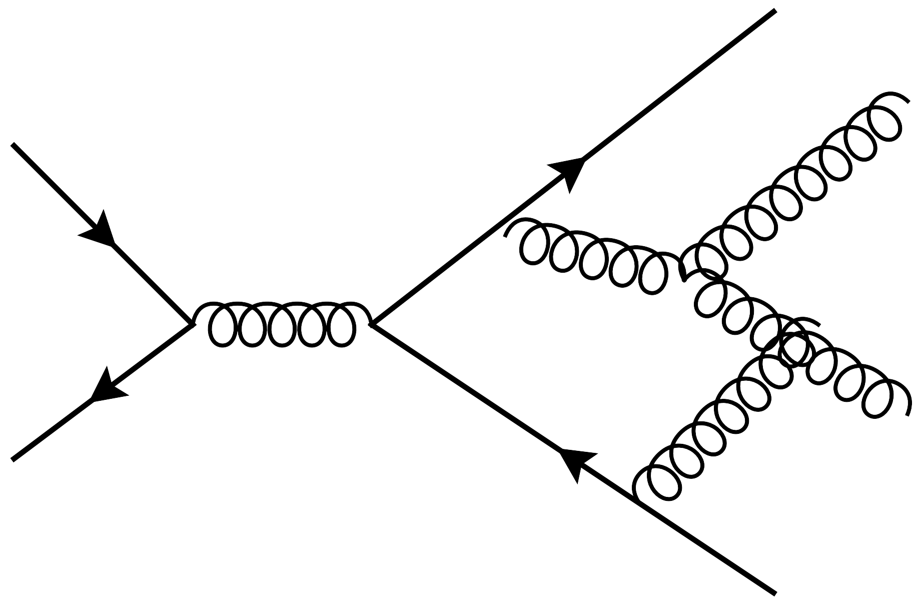
$$\langle m_F | m_0 \rangle^{(n)} = \sum_{\tau\ \mathrm{orders}} \int_{-\infty}^{\infty} d\tau_n \ldots \int_{-\infty}^{\tau_2} d\tau_1$$

$$\times \prod_{\mathrm{loops}\ i} \int \frac{d^3\ell_i}{(2\pi)^3} \prod_{\mathrm{lines}\ j} \frac{1}{2E_j} \times \prod_{\mathrm{vertices}\ a=1}^{n} iV_{a-1\to a}$$

$$\times \exp\left[i \sum_{\mathrm{states}\ m=1}^{n-1} \left(\sum_{j\ \mathrm{in}\ m} E(\vec{p}_j) \right) (\tau_m - \tau_{m+1}) - iE_0\tau_1 \right]$$

- **Perturbative QFT in a nutshell: integrals are divergent in QFT from:**
- $\tau_i \to \tau_j$ **(UV) and** $\tau_i \to \infty$ **(IR).**
- **Renormalization takes care of coinciding times. We'll just assume this is done.**

Each term in this expansion corresponds to a "time-ordered" diagram



Here the vertices are ordered at different times. Sums of orderings give (topologically equivalent) "Feynman diagrams", which exhibit the Lorentz invariance manifestly.

The integrals over loop momenta are exactly the sums over all virtual states.

- **Once renormalized, infinities only come from large times in . . . (same formula)**

$$\langle m_n | m_0 \rangle = \sum_{\tau \text{ orders}} \int_{-\infty}^{\infty} d\tau_n \ldots \int_{-\infty}^{\tau_2} d\tau_1$$
$$\times \prod_{\text{loops } i} \int \frac{d^3\ell_i}{(2\pi)^3} \prod_{\text{lines } j} \frac{1}{2E_j} \times \prod_{\text{vertices } a=1}^{n} iV_{a-1\to a}$$
$$\times \exp\left[i \sum_{\text{states } m=1}^{n-1} \left(\sum_{j \text{ in } m} E(\vec{p}_j) \right) (\tau_m - \tau_{m+1}) - iE_0\tau_1 \right]$$

- **Divergences from $\tau_i \to \infty$ are "Infrared=IR". In some sense, their "solution" is jets,**

- **because – it's not as bad as it looks. Time integrals extend to infinity, but usually oscillations damp them and answers are finite. Long-time, "infrared" divergences (logs) come about when phases vanish and the time integrals diverge.**

- **When does this happen? Here's the phase:**

$$\exp\left[\, i \sum_{\text{states}\, m=1}^{n-1} \left(\sum_{j \text{ in } m} E(\vec{p}_j) \right) (\tau_m - \tau_{m+1}) \right] =$$

$$\exp\left[\, i \sum_{\text{vertices}\, m=1}^{n} \left(\sum_{j \text{ in } m} E(\vec{p}_j) \; - \sum_{j \text{ in } m-1} E(\vec{p}_j) \right) \; \tau_m \right]$$

- **Divergences for $\tau_i \to \infty$ requires two things:**

i) (RHS) the phase must vanish $\leftrightarrow$ "degenerate states"

$$\sum_{j \in m} E(\vec{p}_j) = \sum_{j \in m+1} E(\vec{p}_j) \,, \quad \text{and}$$

ii) (LHS) the phase must be stationary in loop momenta (sums over states):

$$\frac{\partial}{\partial \ell_{i\mu}} \; [\,\text{phase}\,] = \sum_{\text{states}\, m} \; \sum_{j \text{ in } m} (\pm \beta_j^\mu)(\tau_m - \tau_{m-1}) = 0$$

where the β_js are normal 4-velocities:

$$\beta_j = \pm \partial E_j / \partial \ell_i \,.$$

- Condition of stationary phase:

$$\sum_{\text{states } m} \sum_{j \text{ in } m} (\pm\beta_j^\mu)(\tau_m - \tau_{m-1}) = 0$$

- $\beta^\mu \Delta\tau = x^\mu$ is a classical translation. For IR divergences, there must be free, classical propagation as $t \to \infty$. Easy to satisfy if all the β_j's are equal.

- Whenever fast partons (quarks or gluons) emerge from the same point in space-time,

  they will rescatter for long times only with collinear partons.

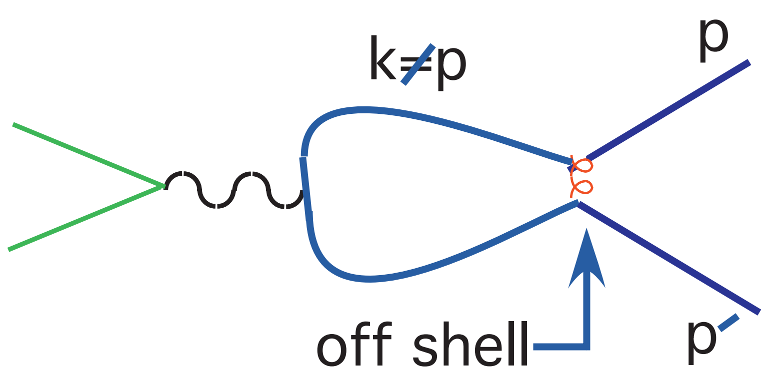
  Of course, radiating or absorbing zero momentum particles also don't affect the phase, but adds a time integral.

  Note, all the states we can reach by rescattering or zero momentum interactions describe the same energy flow.

  These are the sources of "soft" and "collinear" IR divergences.

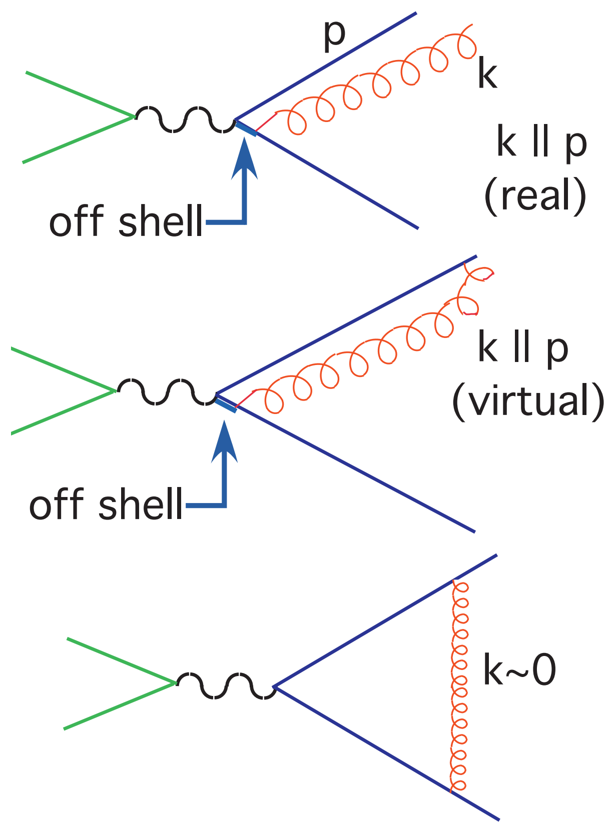
- **Let's illustrate the role of classical propagation.**

- **Example: degenerate states that cannot give long-time divergences:**



- **This makes identifying enhancements a lot simpler!**

- RESULT: For particles emerging from a local scattering, (only) collinear or soft lines can give long-time behavior and enhancement. These are jets, evolving in time **Examples:**



- **This generalizes to any order, and any field theory, but gauge theories alone have soft ($k \to 0$) divergences.**

- We can calculate if we regulate IR divergences, but these aren't physical. **Let's find out what we can compute that is physical.**

C. Why energy flow is a guide to calculable cross sections: infrared safety

- **Rewrite our general amplitude:**

$$\langle m_F | m_0 \rangle^{(n)} = \sum_{\text{orders } m_1 \ldots m_n} \prod_{\text{loops } i} \int \frac{d^3\ell_i}{(2\pi)^3} \prod_{\text{lines } j} \frac{1}{2E_j}$$
$$\times \prod_{\text{vertices } a=1}^{n} \int_{-\infty}^{\tau_{a+1}} iV_{a-1\to a} \exp\left[i \left(\sum_{j \text{ in } a-1} E(\vec{p}_j) \right) (\tau_{a-1} - \tau_a) - iE_0\tau_1 \right]$$

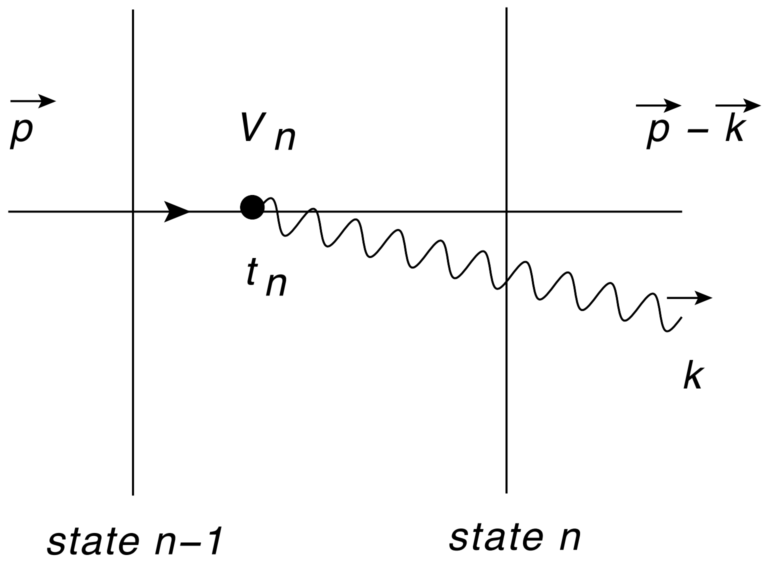
$$= \sum_{\text{orders } m_1 \ldots m_n} \prod_{\text{loops } i} \int \frac{d^3\ell_i}{(2\pi)^3} \prod_{\text{lines } j} \frac{1}{2E_j}$$
$$\times \prod_{\text{vertices } a=1}^{n} \int_{-\infty}^{\tau_{a+1}} iV_{a-1\to a} \exp\left[i \left(\sum_{j \text{ in } a} E(\vec{p}_j) - \sum_{j \text{ in } a-1} E(\vec{p}_j) \right) \tau_a \right]$$

With $\tau_{n+1} = \infty$.

- **IR divergences are controlled by the τ_n integral ("n=F"): the "largest time".**

$$\int_{\tau_{n-1}}^{\infty} iV_{n-1\to n} \exp\left[i \left(\sum_{j \text{ in } F} E(\vec{p}_j) - \sum_{j \text{ in } n-1} E(\vec{p}_j) \right) \tau_n \right]$$

Say the final interaction is the splitting of one particle into two, all treated as massless:



Here state n = the final state F
All the other energies cancel, and the largest time integral is

$$\int_{\tau_{n-1}}^{\infty} d\tau_n i V_{n-1\to F} e^{i\left(\sum_{j \,\mathrm{in}\, n} E(\vec{p}_j) \;-\; \sum_{j \,\mathrm{in}\, n-1} E(\vec{p}_j)\right)\tau_n}$$

$$= \int_{\tau_{n-1}}^{\infty} d\tau_n i V_{n-1\to F} e^{i\Delta_n \tau_n}$$

Relabel: $p \to k_1,\ k \to k_2$**:**

$$\Delta_n \;=\; E(\vec{k}_1 - \vec{k}_2) + E(\vec{k}_2) - \; E(\vec{k}_1)$$

Can use the $i\epsilon$ prescription $\Delta \to \Delta + i\epsilon$ to make the τ_n integral converge. Or, we can observe that most of this integral cancels out "oscillation by oscillation": (For simplicity, take $\tau_{n-1} \to 0$)

$$
\begin{aligned}
\int_0^\infty d\tau_n e^{i\Delta_n \tau_n} &= \frac{1}{\Delta_n}\int_0^\infty dx\,[\cos x + i\sin x] \\
&= \frac{1}{\Delta_n}\int_0^\infty dx \frac{d}{dx}[\sin x - i\cos x] \\
&= -\frac{1}{\Delta_n}[\sin 0 - i\cos 0] \\
&= \frac{i}{\Delta_n}\int_0^{\pi/2} dx\,\sin x
\end{aligned}
$$

– Only times smaller than $\pi/2\Delta_n$ really contribute in the amplitude.

– $1/\Delta_n$ is called the "formation time" of state n.

What is Δ_n and when does it vanish? When it does, we're going to get an IR divergence.

$$\Delta_n = E(\vec{k}_1 - \vec{k}_2) + E(\vec{k}_2) - E(\vec{k}_1)$$

– Kinematics

$$\vec{k}_1 = (P, \vec{0}_T)\,,\ \vec{k}_2 = (zP, \vec{k}_T)\,,\ k_T \leq zP \ll P$$

– Then

$$\Delta_n = \frac{k_T^2}{2zP} \Leftrightarrow \frac{1}{\Delta_n} = \frac{2zP}{k_T^2}$$

– Formation time and the τ_n integral diverge for $k_T \to 0$ at fixed z (collinear radiation) and when $z \to 0$ (with $k_T \leq zP$) (soft radiation).

– In terms of the angle: $k_T = zP \sin\theta$, for small θ,

$$\frac{1}{\Delta_n} \sim \frac{1}{\theta^2 zP} \sim \frac{1}{\theta k_T}$$

– The time integral diverges whenever we find a $\Delta_n = 0$.

– Now we can motivate the construction of IR finite cross sections.

Aside . . .

– **The point $\Delta_n = 0$ is exactly a point of stationary phase in k_T.**

$$\int d^2k_T \int^{\infty} d\tau_n e^{i\Delta_n \tau_n} = \int d^2k_T \int^{\infty} d\tau_n e^{i\tau_n k_T^2/2zzP}$$
$$\sim 2\pi z P \int^{\infty} \frac{d\tau_n}{\tau_n}$$

Finite-time cross sections and what they represent. Consider the probability for a sum over states f, each weighted by $S[f]$,

$$P[S] \;=\; \sum_f S[f] \sum_{n',n} \langle m_0|m_f\rangle^{(n')} \langle m_f|m_0\rangle^{(n)}$$

- **Each matrix element and complex conjugate is a sum of ordered time integrals**
- **In any term of $P[S]$, as we integrate over times, there is a largest time.**
- **The largest time may be in the amplitude, or in the complex conjugate. We combine these two possibilities. Inside the sum over states, we find**

$$\ldots \times \int_{\tau'_{n-2}}^{\tau'_n} e^{i\Delta_{n-1}\tau_{n-1}} (-iV'_{f-2\to f-1}) e^{-i\Delta_{n-1}\tau'_{n-1}} \quad \Leftarrow \text{in } \langle m_0|m_f\rangle$$

$$\times \int_{\tau_{n-1}}^{\infty} d\tau_n V_{n-1\to n} \left\{ i e^{i\Delta_n \tau_n} S[n] - i e^{-i(-\Delta_n)\tau_n} S[n-1] \right\}$$

$$\text{in } \langle m_f|m_0\rangle \Rightarrow \quad \times \int_{\tau_{n-2}}^{\tau_n} e^{i\Delta_{n-1}\tau_{n-1}} i V_{f-2\to f-1} e^{i\Delta_{n-1}\tau_{n-1}} \times \ldots$$

- **When $S[n] = S[n-1]$ this vanishes! This is called the "largest time equation". It is an expression of unitarity – the sum of all probabilities has to be one.**
- **All that matters is the difference due to the last interaction: $n-1 \to n$. When this produces a difference in $S[f]$, the result is nonzero.**

- **General formulation.** We define a set of smooth (symmetric) functions that depend only on the flow of energy, and not particle content:

$$S_{N+1}(p_1 \ldots (1-z)p_N, zp_N) \;=\; S_{N+1}(p_1 \ldots p_N)$$

Whenever $\Delta_n \to 0$, we only need

$$S_{N+1}[n] - S_N[n-1] \;\sim c\, k_\perp^b$$

for some constant c with $b > 0$. Then

$$\int d\tau_n \; e^{i\Delta_n \tau_n} \left(S_{N+1}[n] - S_N[n-1]\right) \;\to\; c \int d\tau_n \; k_\perp^b e^{i\Delta_n \tau_n}$$

- There is now suppression for large times:

$$c \int d^2k_T \, k_\perp^b \int^\infty d\tau_n e^{i\Delta_n \tau_n} \;\; = \; \pi c\, \Gamma(1+b) \int^\infty \frac{d\tau_n}{\tau_n^{1+b/2}}$$

- The largest time integral now converges, and so must the smaller ones, Our calculations now give predictions, rather than infinities. This is infrared safety.

- Every calculable jet cross section is based on such a weight function.

- In summary, For any $S[f]$ that respects energy flow, we compute the cross section

$$\boldsymbol{P}[\boldsymbol{S}] \;=\; \sum_{\boldsymbol{f}} \boldsymbol{S}[\boldsymbol{f}]\; |\langle \boldsymbol{m}_{\boldsymbol{f}} | \boldsymbol{m}_0 \rangle|^2$$

- The same applies to jet cross sections themselves if they are designed to respect the flow of energy. Here, S[f] is chosen to be unity for states that obey certain conditions in jet finding algorithms – which depend only on energy flows,

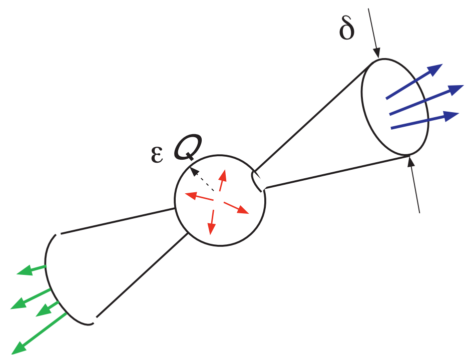
$$\boldsymbol{\sigma}[\boldsymbol{S}_{\mathrm{n-jet}}] \;=\; \sum_{\boldsymbol{f}} \boldsymbol{\theta}(\boldsymbol{S}_{\mathrm{n-jet}}[\boldsymbol{f}])\; |\langle \boldsymbol{m}_{\boldsymbol{f}} | \boldsymbol{m}_0 \rangle|^2$$

- Once we have identified a set of jets, we can then explore their properties by using weight functions $w_{\mathrm{n-jet}}[f]$ that reveal their structure,

$$\langle \boldsymbol{w}_{\mathrm{n-jet}} \rangle \;=\; \frac{\sum_{\boldsymbol{f}} \boldsymbol{w}_{\mathrm{n-jet}}[\boldsymbol{f}]\, \boldsymbol{\theta}(\boldsymbol{S}_{\mathrm{n-jet}}[\boldsymbol{f}])\; |\langle \boldsymbol{m}_{\boldsymbol{f}} | \boldsymbol{m}_0 \rangle|^2}{\sum_{\boldsymbol{f}} \boldsymbol{\theta}(\boldsymbol{S}_{\mathrm{n-jet}}[\boldsymbol{f}])\; |\langle \boldsymbol{m}_{\boldsymbol{f}} | \boldsymbol{m}_0 \rangle|^2}$$

- These are what we can compute.

- **An example is the cross section for a cone jet with a given energies,**



- **The smaller (larger) the "resolutions" ϵ and δ, the more (less) sensitivity to long times. We follow the story only to times like $1/Q\delta$.**

  Other fundamental choices: radiation pattern and and energy-energy correlation

$$\begin{aligned} S_{\mathrm{rad}}[\hat{n}] &= \sum_i E_i\, \delta^2 \left(\hat{n} - \hat{n}(\vec{k}_i)\right) \\ S_{\mathrm{EEC}}(\hat{n}_1, \hat{n}_2) &= \sum_{i,j} E_i\, E_j\, \delta^2 \left(\hat{n}_1 - \hat{n}(\vec{k}_i)\right) \delta^2 \left(\hat{n}_2 - \hat{n}(\vec{k}_i)\right) . \end{aligned}$$

  Perhaps surprisingly, we can treat the delta functions as if they were smooth, and if we integrate over $\hat{n}_1 \ldots$, with these we can generate any weight function.

D. Briefly: Infrared safety and beyond at the EIC.

- Jet cross sections at the EIC are factorized products of parton distributions and IR safe sums over final states:

$$\sigma(p,q,k_{\rm jet}) \; = \; \sum_{{\rm partons} a} \int_x^1 d\xi \, f_{a/p}(\xi,\mu) \, \omega_{e+a\to {\rm jet}+X} \, (\xi p, q, k_{\rm jet})$$

- $\int dx$ cand PDF $f_{a/p}$ an be generalized to TMDs for appropriate observables.

- These and more exclusive cross sections involving jets can probe the "initial state" structure of nucleons and nuclei, precisely because the jets reflect energy flow established at short distances.

- Jet substructure encodes both perturbative and nonperturbative dynamics of how the energy of a scattered quark emerges as hadrons.

- Planned particle identification and momentum capabilities may make possible tests of theories of hadronization in vacuum and in nuclei.

- **With a firm basis of IR safety in sufficiently inclusive cross sections, the way will be open to study how perturbative probability distributions are "redistributed" at long times by nonperturbative processes.**

- **Very likely advanced data analysis, and in time quantum simulations will play a role in the decoding of particle jets.**