Medusa

Multithread 4-body decay fitting and simulation software

Alessandro Maria Ricci[1],

A. A. Alves Junior[2], D. Brundu[3], A. Contu[3], F. Dordei[3], P. Muzzetto[3]

[1] University of Pisa and INFN Section of Pisa,

[2] Institute for Astroparticle Physics – Karlsruhe Institute of Technology (IAP/KIT),

[3] INFN Section of Cagliari

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Contact: alessandro.ricci@df.unipi.it

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The scientific case

- Among the biggest computational challenges for High Energy Physics (HEP) experiments, there are the increasingly larger datasets that are being collected, which often require correspondingly complex data analyses.
- In particular, the PDFs used for data modelling become more and more complicated, with hundred of free parameters.
- The optimization of such models involves a significant computational effort and a considerable amount of time, of the order of days, before reaching a result.

- The current strategy to increase overall performance is to parallelize the software in order to benefit from the large performance gains that can be achieved with multithreading in CPU and/or GPU.
- Despite ongoing modernization efforts, a large fraction of the software used in HEP is inherited. It consists of libraries of single threaded, mono-platform routines.

Medusa

- Medusa is a C++ 14 compliant application to perform physics data analyses of generic 4-body decays in massively parallel platforms on Linux systems.

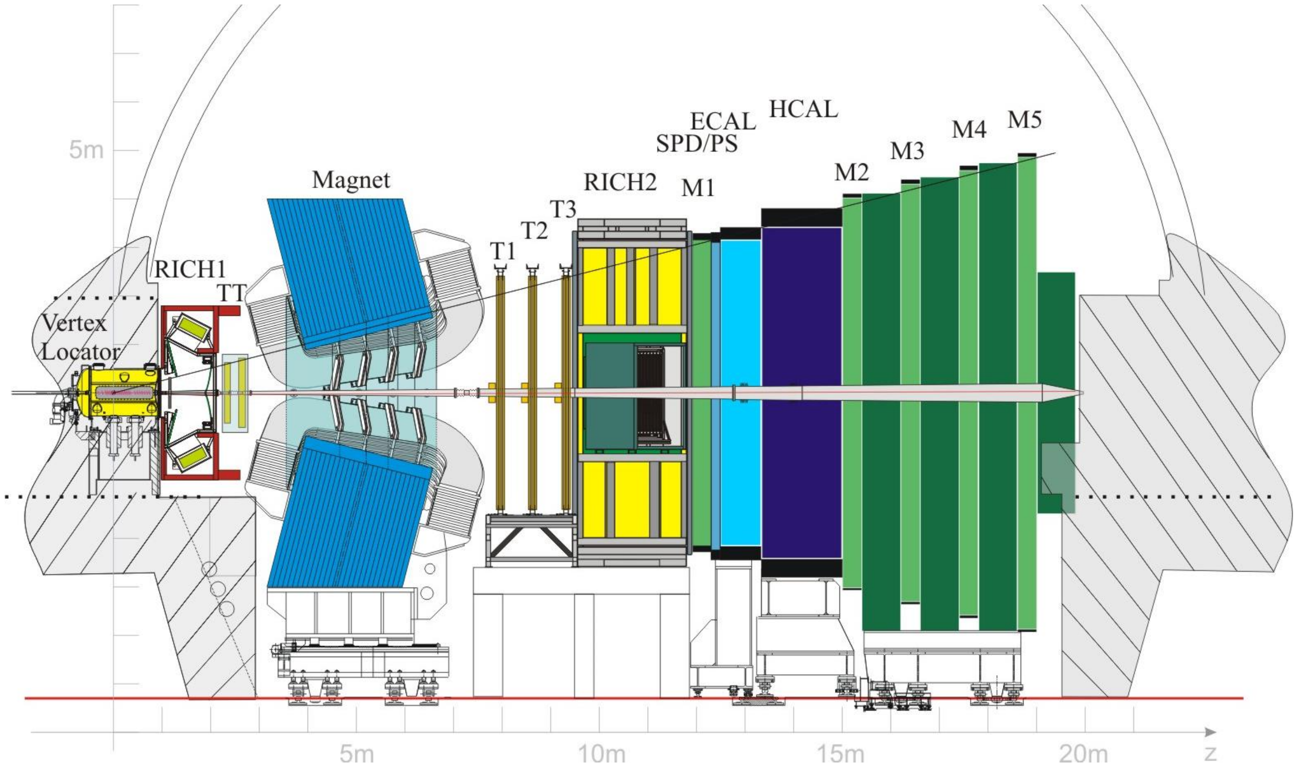
- Medusa is highly based on Hydra v3.0 (link), a C++ 14 compliant and header only library that hides most of complexities of writing parallel code for different architectures.

- Hydra provides a collection of containers and algorithms commonly used in HEP data analysis, which can transparently exploit enabled devices for OpenMP, CUDA, and TBB, allowing the user to re-use the same code across a large range of available multi-core CPU and GPU.

- Medusa wants to be a set of ready-made models to perform data analysis, as the CP-violating phase model in $B_s^0(\bar{B}_s^0)$ decay, and ready-made multithread functions, as the Fadeeva functions, to accelerate the develop of new models.

Use case: $B_s^0(\bar{B}_s^0)$ decay at LHCb

- Medusa has been tested through the measurement of the CP-violating phase ϕ_s in $B_s^0(\bar{B}_s^0) \to J/\psi\ \phi\ \to \mu^+\mu^-K^+K^-$ decay, one of the golden channels for this type of research at LHCb.
- Between 2015 and 2016, LHCb Collaboration collected a sample of about 209000 events.
- For extracting the phase ϕ_s is necessary to perform a maximum-likelihood fit with 32 free parameters, by using a model which includes both the signal and the background.
- This model must include the modeling of the distribution of the B_s^0-decay times, of the decay angles, of the so-called flavour tagging to distinguish between B_s^0 and $\bar{B}_s^0$ mesons and other experimental artifacts.



Signal-only model

- The full time- and angle-dependent decay rate is (Eur. Phys. J. C (2019) 79, 106 , Eur. Phys. J. C (2020) 80, 601 and Stemmle, Ph.D. Thesis, Heidelberg, Germany, 2019):

$$\frac{\mathrm{d}^4\Gamma}{dt\, d\theta_\mu\, d\theta_K\, d\phi} \propto \sum_{k=1}^{10} A_k h_{k,q}(t) f_k(\theta_\mu, \theta_K, \phi)$$

Signal-only model

$$h_{k,1}(t) = \frac{3}{4\pi} e^{-\Gamma t}\left(a_k \cosh\frac{\Delta\Gamma\, t}{2} + b_k \sinh\frac{\Delta\Gamma\, t}{2} + c_k \cos(\Delta m\, t) + d_k \sin(\Delta m\, t)\right) \; for \;\; B_s^0$$

$$h_{k,-1}(t) = \frac{3}{4\pi} e^{-\Gamma t}\left(a_k \cosh\frac{\Delta\Gamma\, t}{2} + b_k \sinh\frac{\Delta\Gamma\, t}{2} - c_k \cos(\Delta m\, t) - d_k \sin(\Delta m\, t)\right) \; for \;\; \bar{B}_s^0$$

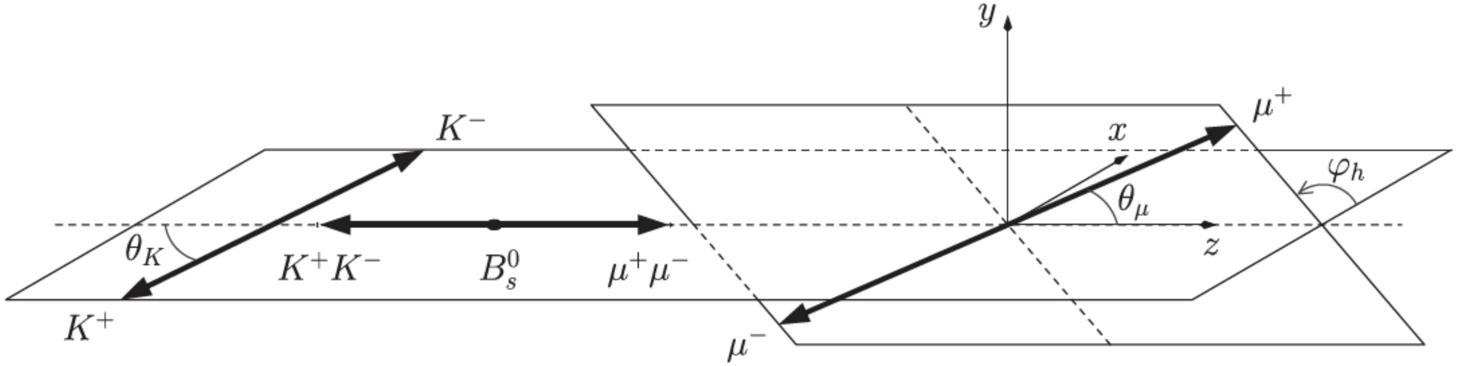
- Free parameters: $\Gamma = \frac{\Gamma_H + \Gamma_L}{2}$, $\Delta\Gamma = \Gamma_\mathrm{L} - \Gamma_\mathrm{H}$, $\Delta m = M_H - M_L$.

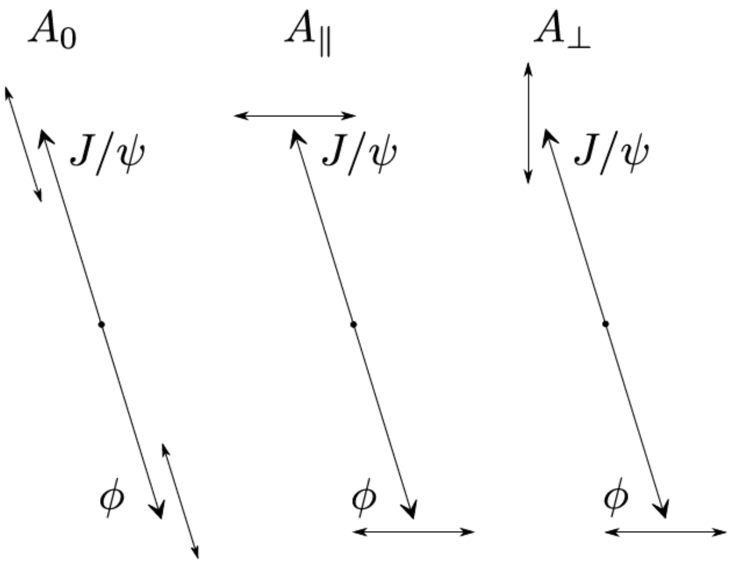
Coefficients in the time functions

- Coefficients in the time functions $h_{k,1}(t)$ and $h_{k,-1}(t)$ with the polarization dependent CP violation.
- Free parameters: $\lambda_0, \frac{\lambda_\parallel}{\lambda_0}, \frac{\lambda_\perp}{\lambda_0}, \frac{\lambda_S}{\lambda_0}, \phi_0, (\phi_\parallel - \phi_0), (\phi_\perp - \phi_0)$, $(\phi_S - \phi_0), \delta_0, (\delta_\parallel - \delta_0), (\delta_\perp - \delta_0)$, $(\delta_S - \delta_\perp)$.

k	a_k	b_k	c_k	d_k
1	$\frac{1}{2}(1+\lvert\lambda_0\rvert^2)$	$-\lvert\lambda_0\rvert\cos(\phi_0)$	$\frac{1}{2}(1-\lvert\lambda_0\rvert^2)$	$\lvert\lambda_0\rvert\sin(\phi_0)$
2	$\frac{1}{2}(1+\lvert\lambda_\parallel\rvert^2)$	$-\lvert\lambda_\parallel\rvert\cos(\phi_\parallel)$	$\frac{1}{2}(1-\lvert\lambda_\parallel\rvert^2)$	$\lvert\lambda_\parallel\rvert\sin(\phi_\parallel)$
3	$\frac{1}{2}(1+\lvert\lambda_\perp\rvert^2)$	$\lvert\lambda_\perp\rvert\cos(\phi_\perp)$	$\frac{1}{2}(1-\lvert\lambda_\perp\rvert^2)$	$-\lvert\lambda_\perp\rvert\sin(\phi_\perp)$
4	$\frac{1}{2}\Big[\sin(\delta_\perp-\delta_\parallel)-\lvert\lambda_\perp\lambda_\parallel\rvert \sin(\delta_\perp-\delta_\parallel-\phi_\perp+\phi_\parallel)\Big]$	$\frac{1}{2}\Big[\lvert\lambda_\perp\rvert\sin(\delta_\perp-\delta_\parallel-\phi_\perp) +\lvert\lambda_\parallel\rvert\sin(\delta_\parallel-\delta_\perp-\phi_\parallel)\Big]$	$\frac{1}{2}\Big[\sin(\delta_\perp-\delta_\parallel)+\lvert\lambda_\perp\lambda_\parallel\rvert \sin(\delta_\perp-\delta_\parallel-\phi_\perp+\phi_\parallel)\Big]$	$-\frac{1}{2}\Big[\lvert\lambda_\perp\rvert\cos(\delta_\perp-\delta_\parallel-\phi_\perp) +\lvert\lambda_\parallel\rvert\cos(\delta_\parallel-\delta_\perp-\phi_\parallel)\Big]$
5	$\frac{1}{2}\Big[\cos(\delta_0-\delta_\parallel)+\lvert\lambda_0\lambda_\parallel\rvert \cos(\delta_0-\delta_\parallel-\phi_0+\phi_\parallel)\Big]$	$-\frac{1}{2}\Big[\lvert\lambda_0\rvert\cos(\delta_0-\delta_\parallel-\phi_0) +\lvert\lambda_\parallel\rvert\cos(\delta_\parallel-\delta_0-\phi_\parallel)\Big]$	$\frac{1}{2}\Big[\cos(\delta_0-\delta_\parallel)-\lvert\lambda_0\lambda_\parallel\rvert \cos(\delta_0-\delta_\parallel-\phi_0+\phi_\parallel)\Big]$	$-\frac{1}{2}\Big[\lvert\lambda_0\rvert\sin(\delta_0-\delta_\parallel-\phi_0) +\lvert\lambda_\parallel\rvert\sin(\delta_\parallel-\delta_0-\phi_\parallel)\Big]$
6	$-\frac{1}{2}\Big[\sin(\delta_0-\delta_\perp)-\lvert\lambda_0\lambda_\perp\rvert \sin(\delta_0-\delta_\perp-\phi_0+\phi_\perp)\Big]$	$\frac{1}{2}\Big[\lvert\lambda_0\rvert\sin(\delta_0-\delta_\perp-\phi_0) +\lvert\lambda_\perp\rvert\sin(\delta_\perp-\delta_0-\phi_\perp)\Big]$	$-\frac{1}{2}\Big[\sin(\delta_0-\delta_\perp)+\lvert\lambda_0\lambda_\perp\rvert \sin(\delta_0-\delta_\perp-\phi_0+\phi_\perp)\Big]$	$-\frac{1}{2}\Big[\lvert\lambda_0\rvert\cos(\delta_0-\delta_\perp-\phi_0) +\lvert\lambda_\perp\rvert\cos(\delta_\perp-\delta_0-\phi_\perp)\Big]$
7	$\frac{1}{2}(1+\lvert\lambda_S\rvert^2)$	$\lvert\lambda_S\rvert\cos(\phi_S)$	$\frac{1}{2}(1-\lvert\lambda_S\rvert^2)$	$-\lvert\lambda_S\rvert\sin(\phi_S)$
8	$\frac{1}{2}\Big[\cos(\delta_S-\delta_\parallel)-\lvert\lambda_S\lambda_\parallel\rvert \cos(\delta_S-\delta_\parallel-\phi_S+\phi_\parallel)\Big]$	$\frac{1}{2}\Big[\lvert\lambda_S\rvert\cos(\delta_S-\delta_\parallel-\phi_S) -\lvert\lambda_\parallel\rvert\cos(\delta_\parallel-\delta_S-\phi_\parallel)\Big]$	$\frac{1}{2}\Big[\cos(\delta_S-\delta_\parallel)+\lvert\lambda_S\lambda_\parallel\rvert \cos(\delta_S-\delta_\parallel-\phi_S+\phi_\parallel)\Big]$	$\frac{1}{2}\Big[\lvert\lambda_S\rvert\sin(\delta_S-\delta_\parallel-\phi_S) -\lvert\lambda_\parallel\rvert\sin(\delta_\parallel-\delta_S-\phi_\parallel)\Big]$
9	$-\frac{1}{2}\Big[\sin(\delta_S-\delta_\perp)+\lvert\lambda_S\lambda_\perp\rvert \sin(\delta_S-\delta_\perp-\phi_S+\phi_\perp)\Big]$	$-\frac{1}{2}\Big[\lvert\lambda_S\rvert\sin(\delta_S-\delta_\perp-\phi_S) -\lvert\lambda_\perp\rvert\sin(\delta_\perp-\delta_S-\phi_\perp)\Big]$	$-\frac{1}{2}\Big[\sin(\delta_S-\delta_\perp)-\lvert\lambda_S\lambda_\perp\rvert \sin(\delta_S-\delta_\perp-\phi_S+\phi_\perp)\Big]$	$-\frac{1}{2}\Big[-\lvert\lambda_S\rvert\cos(\delta_S-\delta_\perp-\phi_S) +\lvert\lambda_\perp\rvert\cos(\delta_\perp-\delta_S-\phi_\perp)\Big]$
10	$\frac{1}{2}\Big[\cos(\delta_S-\delta_0)-\lvert\lambda_S\lambda_0\rvert \cos(\delta_S-\delta_0-\phi_S+\phi_0)\Big]$	$\frac{1}{2}\Big[\lvert\lambda_S\rvert\cos(\delta_S-\delta_0-\phi_S) -\lvert\lambda_0\rvert\cos(\delta_0-\delta_S-\phi_0)\Big]$	$\frac{1}{2}\Big[\cos(\delta_S-\delta_0)+\lvert\lambda_S\lambda_0\rvert \cos(\delta_S-\delta_0-\phi_S+\phi_0)\Big]$	$\frac{1}{2}\Big[\lvert\lambda_S\rvert\sin(\delta_S-\delta_0-\phi_S) -\lvert\lambda_0\rvert\sin(\delta_0-\delta_S-\phi_0)\Big]$

Amplitudes and angular distributions

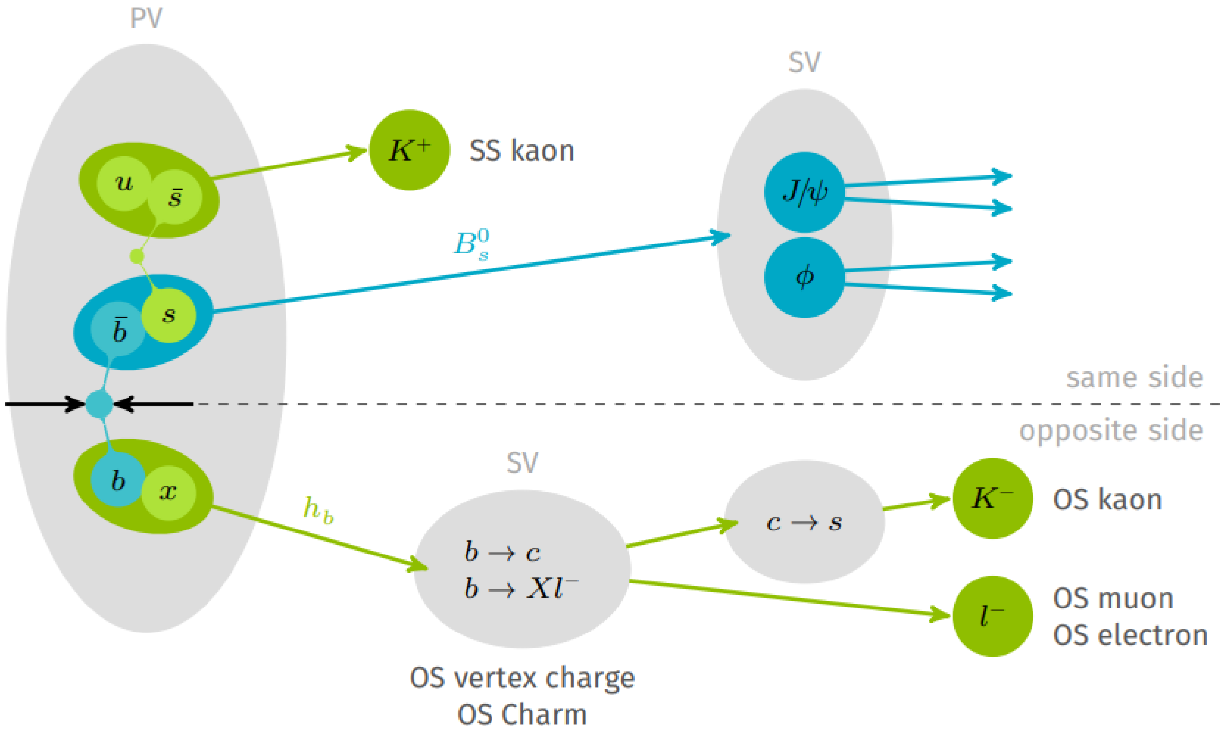




k	A_k	$f_k(\theta_\mu, \theta_K, \varphi_h)$
1	$\|A_0\|^2$	$2\cos^2\theta_K \sin^2\theta_\mu$
2	$\|A_\parallel\|^2$	$\sin^2\theta_k(1-\sin^2\theta_\mu\cos^2\varphi_h)$
3	$\|A_\perp\|^2$	$\sin^2\theta_k(1-\sin^2\theta_\mu\sin^2\varphi_h)$
4	$\|A_\parallel A_\perp\|$	$\sin^2\theta_k\sin^2\theta_\mu\sin 2\varphi_h$
5	$\|A_0 A_\parallel\|$	$\frac{1}{2}\sqrt{2}\sin 2\theta_k\sin 2\theta_\mu\cos\varphi_h$
6	$\|A_0 A_\perp\|$	$-\frac{1}{2}\sqrt{2}\sin 2\theta_k\sin 2\theta_\mu\sin\varphi_h$
7	$\|A_S\|^2$	$\frac{2}{3}\sin^2\theta_\mu$
8	$\|A_S A_\parallel\|$	$\frac{1}{3}\sqrt{6}\sin\theta_k\sin 2\theta_\mu\cos\varphi_h$
9	$\|A_S A_\perp\|$	$-\frac{1}{3}\sqrt{6}\sin\theta_k\sin 2\theta_\mu\sin\varphi_h$
10	$\|A_S A_0\|$	$\frac{4}{3}\sqrt{3}\cos\theta_K\sin^2\theta_\mu$

- Figure on the left: angular distribution in the helicity basis describing the decay geometry.
- Figure on the right: polarization amplitudes of the system $J/\psi\phi$. The short arrows indicate the spin orientation of the two vector mesons.
- Table: definition of the polarization amplitudes A_k and the angular functions $f_k(\theta_\mu, \theta_K, \phi)$.
- Free parameters: $A_0^2, A_\perp^2, A_S^2$.

Flavour tagging



- In LHCb, the tagging algorithms (taggers) can be divided into two classes.
- The opposite-side (OS) taggers rely on the fact that b quarks are predominantly produced in $b\bar{b}$ pairs and try to infer the initial flavour from the decay chain of the respective other b quark.
- The same-side (SS) taggers exploit the charge of particles that are created in association with the fragmentation of the signal b quark.
- Each tagger returns 2 values:

$$q = 1\ (B_s^0), 0\ (\text{no tagged}), -1\ (\bar{B}_s^0)$$

$$\eta = \text{mistag probability}$$

- $p_0, \Delta p_0, p_1, \Delta p1$ are free parameters.

$$\omega(\eta) = \left(p_0 + \frac{\Delta p_0}{2}\right) + \left(p_1 + \frac{\Delta p1}{2}\right)(\eta - \langle\eta\rangle) \quad for\ B_s^0$$

$$\bar{\omega}(\eta) = \left(p_0 - \frac{\Delta p_0}{2}\right) + \left(p_1 - \frac{\Delta p1}{2}\right)(\eta - \langle\eta\rangle) \quad for\ \bar{B}_s^0$$

Experimental artifacts and simultaneous fit

- The signal-only model has 18 free parameters. Flavour tagging adds 4 free parameters.
- We need to consider other experimental artifacts: decay-time resolution, decay-time and angular acceptances, S-wave.
- Decay-time resolution, decay-time and angular acceptances increase the model complexity without raising the number of free parameters.
- The S-wave is composed of K^+K^--couples, which originate from the direct decay $B_s^0(\bar{B}_s^0) \rightarrow J/\psi\ K^+K^-$. This contribution has a different angular distribution and then must be separated from the signal-only model.
- The S-wave is split in the 6 m_{KK}-bins: $[990 - 1008\ ], [1008 - 1016], [1016 - 1020], [1020 - 1024], [1024 - 1032], [1032 - 1050]\,\frac{Mev}{c^2}$.
- The 2 S-wave associated parameters, A_{S}^2 and $\delta_S - \delta_\perp$, are left free to vary in each bin.
- This brings to a simultaneous fit on 6 m_{KK}-bins with 20 free parameters in common between the bins and 2 specific for each bin. Hence, totally we have 32 free parameters.
- Finally, the background is considered by associating a weight to each event, which indicates the probability that the event is a signal or background.

Probability Density Function (PDF)

$$PDF^{j}_{y,c}(t,\Omega|q^{OS},q^{SS},\eta^{OS},\eta^{SS},\delta_t) =$$

m_{KK}-bins for S-wave

$$\frac{1}{N^{\eta^{OS},\eta^{SS},\delta_t,j}_{q^{OS},q^{SS},y,c}}\sum_{k=1}^{10}\tilde{A}^{j}_{k}f_k(\Omega)\varepsilon_{y,c}(t)$$

Cubic Spline for time acceptance

Effective Gaussian for time resolution

$$\Big\{\Big[\Big(1+q^{OS}\big(1-2\omega^{OS}(\eta_{OS})\big)\Big)\Big(1+q^{SS}\big(1-2\omega^{SS}(\eta_{SS})\big)\Big)\cdot h_{k,1}(t)$$

$$+\Big(1-q^{OS}\big(1-2\bar{\omega}^{OS}(\eta_{OS})\big)\Big)\Big(1-q^{SS}\big(1-2\bar{\omega}^{SS}(\eta_{SS})\big)\Big)\cdot h_{k,-1}(t)\Big]\otimes G\big(t|\sigma_{eff}(\delta_t)\big)\Big\}$$

Tagging coefficients

Normalization Factor

m_{KK}-bins for S-wave

Cubic Spline for time acceptance

Coefficients for angular acceptance

Tagging coefficients

Effective Gaussian for time resolution

$$N_{q^{OS},q^{SS},y,c}^{\eta^{OS},\eta^{SS},\delta_t,j} =$$

$$\int_{t=0.3\,ps}^{15\,ps} \sum_{k=1}^{10} \tilde{A}_k^{i} \varepsilon_{y,c}(t) \omega_{y,c}^{k}$$

$$\left\{\left[\left(1+q^{OS}\left(1-2\omega^{OS}(\eta_{OS})\right)\right)\left(1+q^{SS}\left(1-2\omega^{SS}(\eta_{SS})\right)\right)\cdot h_{k,1}(t)\right.\right.$$

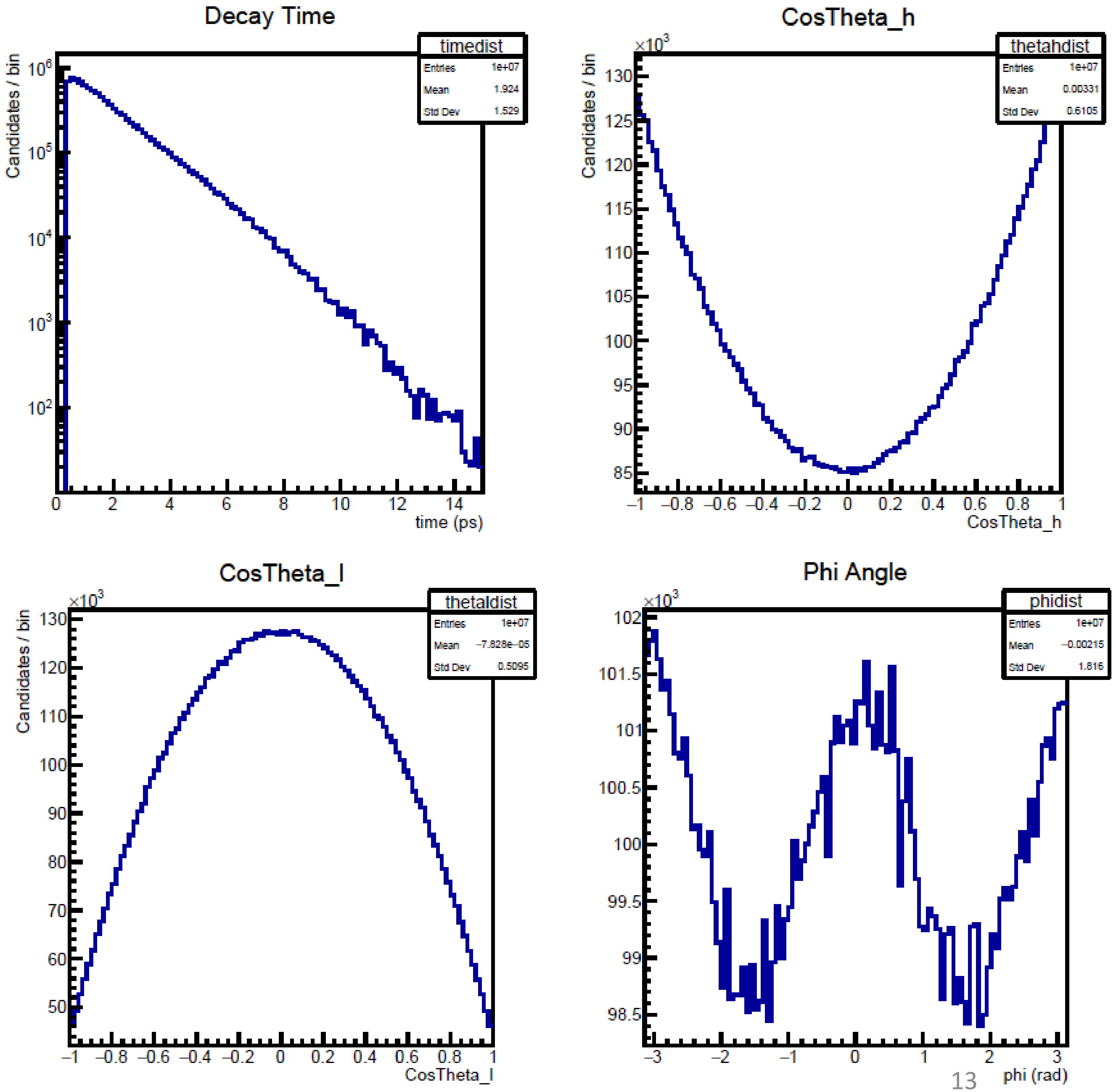
$$\left.\left.+\left(1-q^{OS}\left(1-2\overline{\omega}^{OS}(\eta_{OS})\right)\right)\left(1-q^{SS}\left(1-2\overline{\omega}^{SS}(\eta_{SS})\right)\right)\cdot h_{k,-1}(t)\right]\otimes G\left(t|\sigma_{eff}(\delta_t)\right)\right\} dt$$

- The normalization factor can be analytically computed, but the computation is expensive.

Validation

Difference between LHCb and Medusa	Statistical uncertainties in LHCb
$\phi_s^{LHCb} - \phi_s^M = 0.0009$	$\delta\phi_s^{LHCb} = 0.043$
$\lambda_0^{LHCb} - \lambda_0^M = -0.013$	$\delta\lambda_0^{LHCb} = 0.045$
$\Gamma^{LHCb} - \Gamma^M = 0.00055$	$\delta\Gamma^{LHCb} = 0.0024$
$\Delta\Gamma^{LHCb} - \Delta\Gamma^M = -0.0007$	$\delta\Delta\Gamma^{LHCb} = 0.0077$
$\Delta m^{LHCb} - \Delta m^M = 0.029$	$\delta\Delta m = 0.060$
$A_\perp^{2,LHCb} - A_\perp^{2,M} = -0.0035$	$\delta A_\perp^{2,LHCb} = 0.025$
$A_0^{2,LHCb} - A_0^{2,M} = 0.0011$	$\delta A_0^{2,LHCb} = 0.018$
$(\delta_\perp^{LHCb} - \delta_0^{LHCb}) - (\delta_\perp^M - \delta_0^M) = 0.031$	$\delta(\delta_\perp^{LHCb} - \delta_0^{LHCb}) = 0.15$
$(\delta_\parallel^{LHCb} - \delta_0^{LHCb}) - (\delta_\parallel^M - \delta_0^M) = 0.029$	$\delta(\delta_\parallel^{LHCb} - \delta_0^{LHCb}) = 0.083$

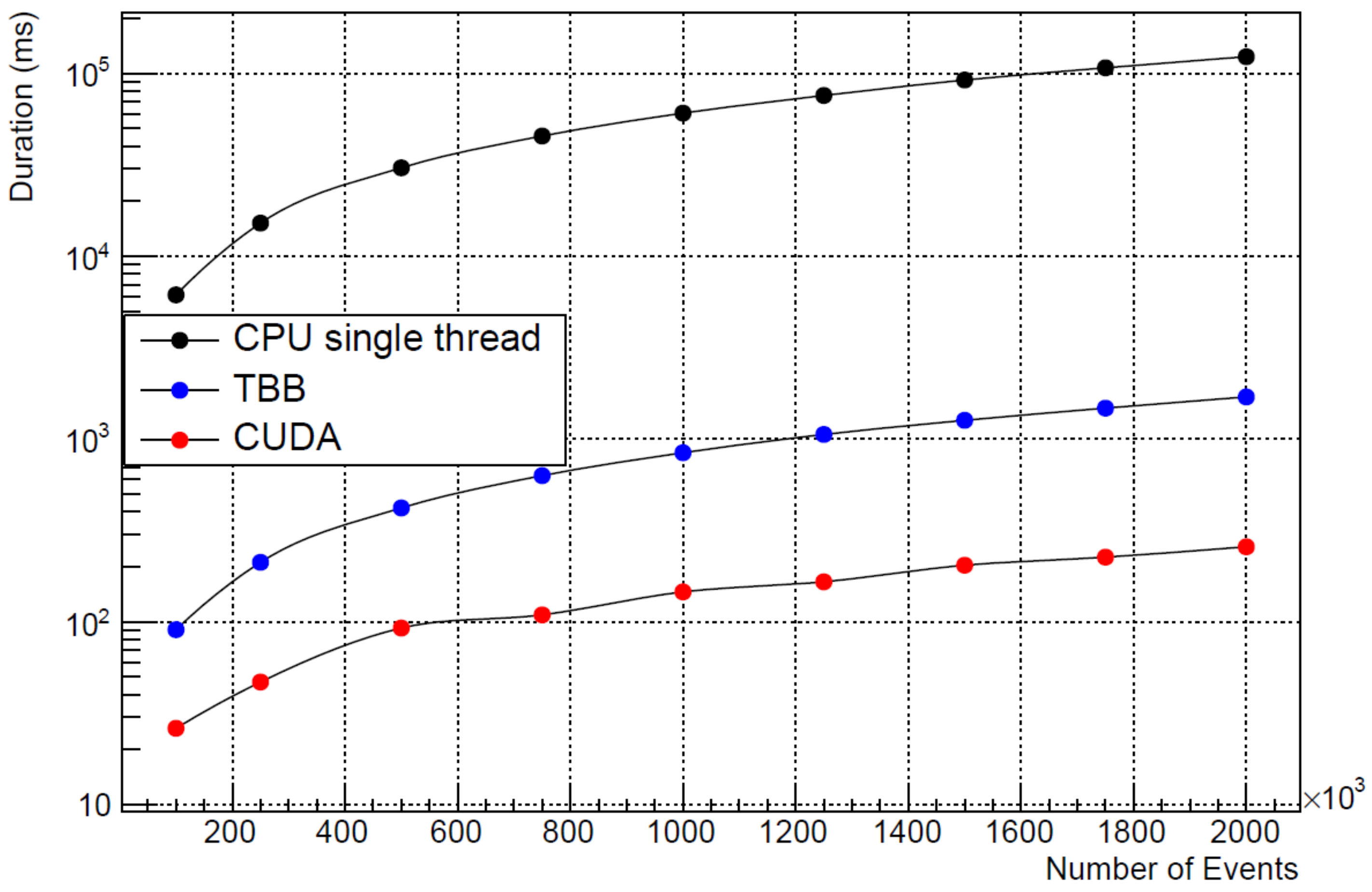
- The comparison has been done with the results reported in LHCb-ANA-2017-028.



Performance

System	Time/call (ms)
AMD EPYC 7452 @ 1.50 GHz (1 Threads)	30424
AMD EPYC 7452 @ 1.50 GHz (128 Threads)	419
NVIDIA A100	92

- The table summarizes the time spent to perform the objective function (FCN) evaluation with 500k events.



Conclusions

- HEP experiments collect ever-larger datasets and their analyses get more and more complex. Moreover, the PDFs used for data modelling become more and more complicated, with hundred of free parameters.
- Not rarely, a computation spends hours to reach a result, which very often needs to be re-tuned.
- Medusa is a multithread 4-body decay fitting and simulation software created to speed up the physics data analysis.
- As a use case, we used the measurement of CP-violating ϕ_s-phase in $B_s^0(\bar{B}_s^0) \to J/\psi\ \phi \ \to \mu^+\mu^-K^+K^-$ decay, one of the golden channel for this type of research at LHCb.
- The evaluation time of the objective function (FCN) with 500000 events is about 92 ms on CUDA, which is about 330 times faster than a non-parallelized software.
- The compilation times also have been optimized. The GCC compiler spends about 1 minute to create the executable for TBB and OpenMP and NVCC about 4 minutes for CUDA.

Distribution

- Medusa will be released on GitHub as open-source software under GPL v.3.0 license soon.

- If you are interested in the software, you can contact:
  1. andrea.contu@cern.ch
  2. francesca.dordei@cern.ch
  3. alessandro.ricci@df.unipi.it, alessandro.ricci@pi.infn.it
  4. davide.brundu@cern.ch

Thank you for the attention!

Backup

CP-violating phase ϕ_s

- Typically, the phase of the transition amplitude A can be split into a strong phase δ, which does not change sign, and a weak phase ϕ, which changes sign under CP transformation:

$$\boldsymbol{CP}A = \boldsymbol{CP}|A|e^{i(\phi+\delta)} = |A|e^{i(-\phi+\delta)}.$$

- The heavy and light eigenstates are:

$$\left|B^0_{s,H}\right\rangle = p|B^0_s\rangle + q|\bar{B}^0_s\rangle, \qquad \left|B^0_{s,L}\right\rangle = p|B^0_s\rangle - q|\bar{B}^0_s\rangle$$

- Given the transition amplitudes $A_{J/\psi\phi} = \langle J/\psi\phi|\mathrm{H}\big|B^0_s\big\rangle$ and $\bar{A}_{J/\psi\phi} = \langle J/\psi\phi|\mathrm{H}\big|\bar{B}^0_s\big\rangle$, we have

$$\lambda_{J/\psi\phi} = \frac{q}{p}\frac{\bar{A}_{J/\psi\phi}}{A_{J/\psi\phi}} = \left|\lambda_{J/\psi\phi}\right|e^{i\phi_s}$$

- The CP is violated if $\phi_s \neq 0$.

Normalization factor

$$\int_{t=0.3\ ps}^{15\ ps} \sum_{k=1}^{10} \tilde{A}_k^j \varepsilon_{y,c}(t) \omega_{y,c}^k * rPDF(t)\, dt \rightarrow \sum_{k=1}^{10} \tilde{A}_k^j \omega_{y,c}^k \int_{t=0.3\ ps}^{15\ ps} \varepsilon_{y,c}(t) * rPDF(t)\, dt$$

$$\sum_{k=1}^{10} \tilde{A}_k^j \omega_{y,c}^k \int_{t=0.3\ ps}^{15\ ps} (a_i t^3 + b_i t^2 + c_i t + d_i) * rPDF(t)\, dt \rightarrow$$

$$\sum_{k=1}^{10} \tilde{A}_k^j \omega_{y,c}^k \left[\sum_{i=1}^{7} \left(a_i \int_{t_i}^{t_{i+1}} t^3 * rPDF(t)\, dt \right) \sum_{i=1}^{7} \left(b_i \int_{t_i}^{t_{i+1}} t^2 * rPDF(t)\, dt \right) + \sum_{i=1}^{7} \left(c_i \int_{t_i}^{t_{i+1}} t * rPDF(t)\, dt \right) + \sum_{i=1}^{7} \left(d_i \int_{t_i}^{t_{i+1}} rPDF(t)\, dt \right) \right]$$

Normalization Factor

$$a_i \int_{t_i}^{t_{i+1}} t^p * rPDF(t)\, dt \rightarrow a_i * C_{tag} * \int_{t_i}^{t_{i+1}} t^p * h_{k,q}(t) \otimes G\left(t|\sigma_{eff}(\delta_t)\right) dt \rightarrow$$

$$\frac{3C_{tag}\sigma_{eff}a_i p!}{8\pi\sqrt{2}} \frac{\left(\sqrt{2}\sigma_{eff}\right)^p}{2^p} \sum_{j=0}^{p} \frac{1}{j!\,(p-j)!} \Bigg[a_k \left(K_j(z_1)M_{p-j}(x_1, x_2; z_1) + K_j(z_2)M_{p-j}(x_1, x_2; z_2)\right)$$
$$+ b_k \left(K_j(z_1)M_{p-j}(x_1, x_2; z_1) - K_j(z_2)M_{p-j}(x_1, x_2; z_2)\right)$$
$$+ qc_k \left(K_j(z_3)M_{p-j}(x_1, x_2; z_3) + K_j(z_4)M_{p-j}(x_1, x_2; z_4)\right)$$
$$+ \frac{qd_k}{i} \left(K_j(z_3)M_{p-j}(x_1, x_2; z_3) - K_j(z_4)M_{p-j}(x_1, x_2; z_4)\right) \Bigg]$$