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meson spectroscopy *"illustrating the problem"*

resonances, scattering, elastic phase-shifts

lattice QCD *"introducing the tool"*

discrete spectrum, finite volume, computing the spectrum

elastic scattering *"solving the simplest problem"*

lattice QCD phase-shift results

coupled-channel scattering *"a more realistic situation"*

mapping the discrete spectrum to the t-matrix

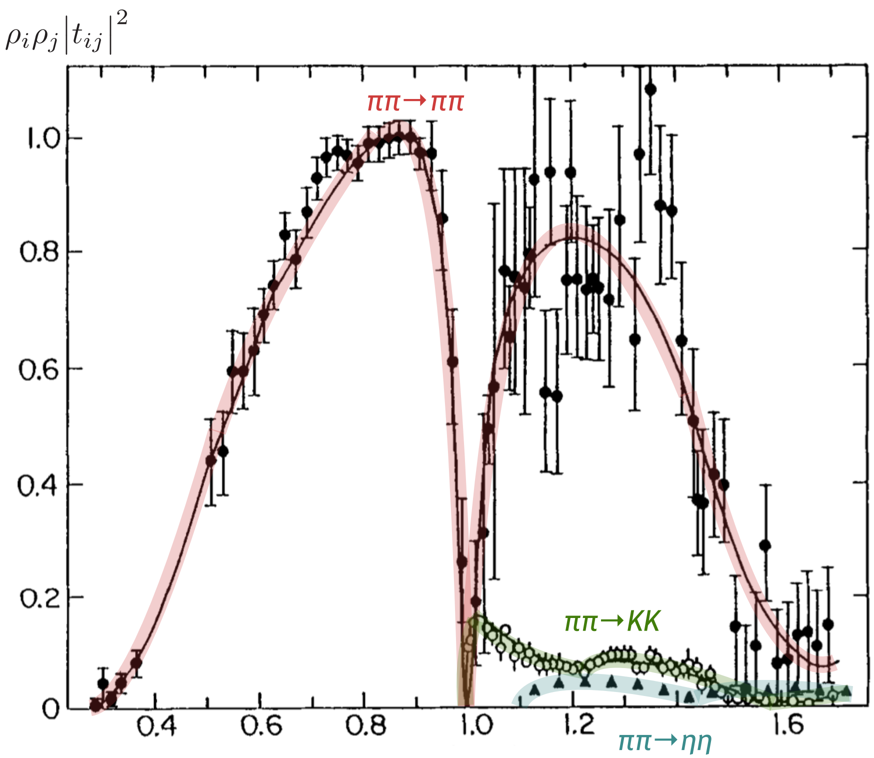
lattice QCD calculation results

the complex energy plane *"well-defined quantities"*

rigorously determining resonances

singularities in the complex energy plane

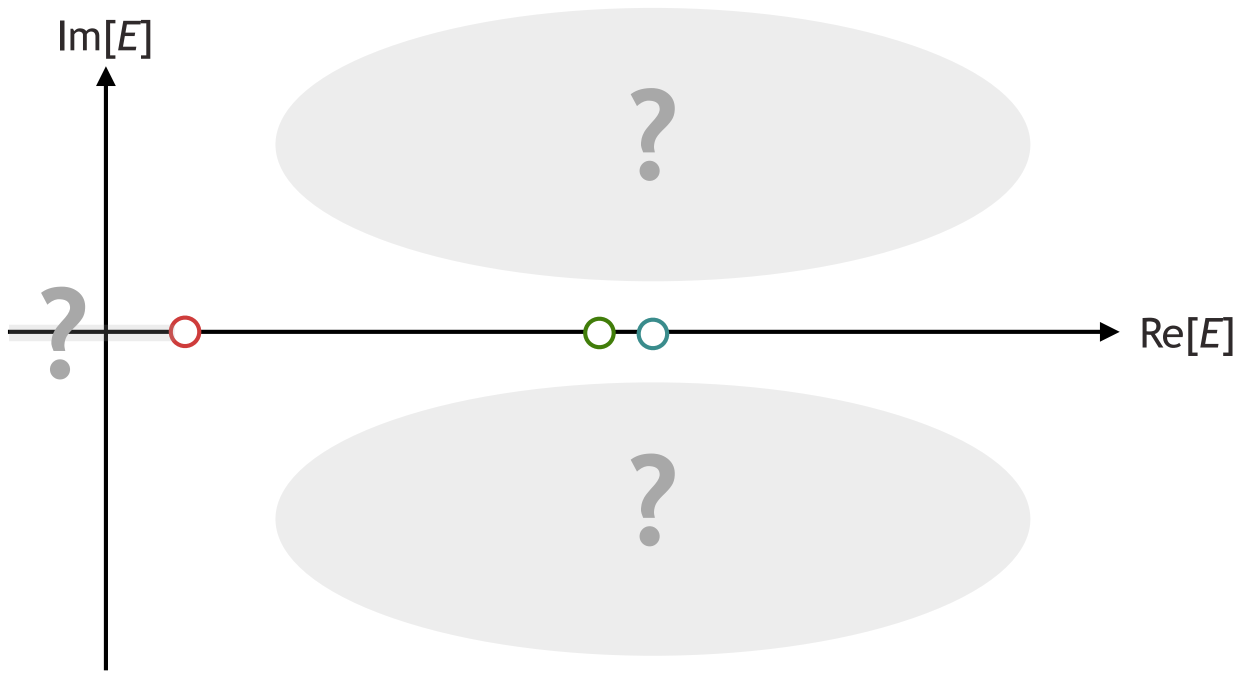
scattering amplitudes are measured for real energies above threshold



and we've seen that lattice calculations can lead to something similar

singularities in the complex energy plane

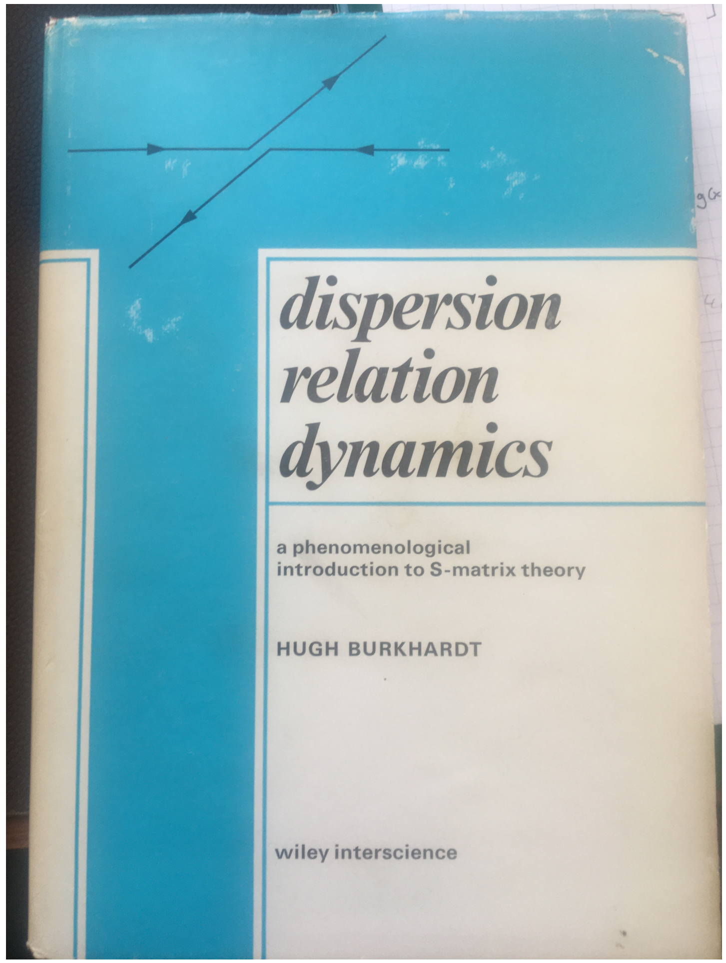
does it make sense to consider how the amplitude behaves 'elsewhere'
— below threshold ?
— for complex values of E ?



complex variable theory tells us that **singularities** (**poles**, **cuts**) 'control' the behaviour of functions

— what singularities can our amplitudes have ?

there’s a rather nice (old) book that gives a gentle introduction to this topic



This book is a lowbrow exposition of S-matrix theory, which is the approach to the dynamics of the strong interactions of elementary particles that uses dispersion relations and unitarity as its main tools. This inductive approach has the important advantage that the calculations can be made accessible to experimenters and others who would not wish to follow the more sophisticated derivations. A good deal of attention has been paid to explaining the grammar of the language of analytic functions.

The reasons for choosing an inductive approach lie in the present state of the theory. S-matrix dynamics has proved reasonably successful as a model for correlating and predicting experimental results, but it is still far from being a complete predictive theory. It is important that those methods which are fairly well established and useful in calculations should be widely understood.

the unitarity cut

unitarity gives us one guaranteed singularity — **a branch cut starting at threshold**

e.g. elastic partial-wave case: $\mathrm{Im}\, t_\ell(s) = \rho(s) \big| t_\ell(s) \big|^2 \, \Theta(s - 4m^2)$

the unitarity cut

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e.g. elastic partial-wave case: $\mathrm{Im}\, t_\ell(s) = \rho(s) \big| t_\ell(s) \big|^2 \, \Theta(s - 4m^2)$

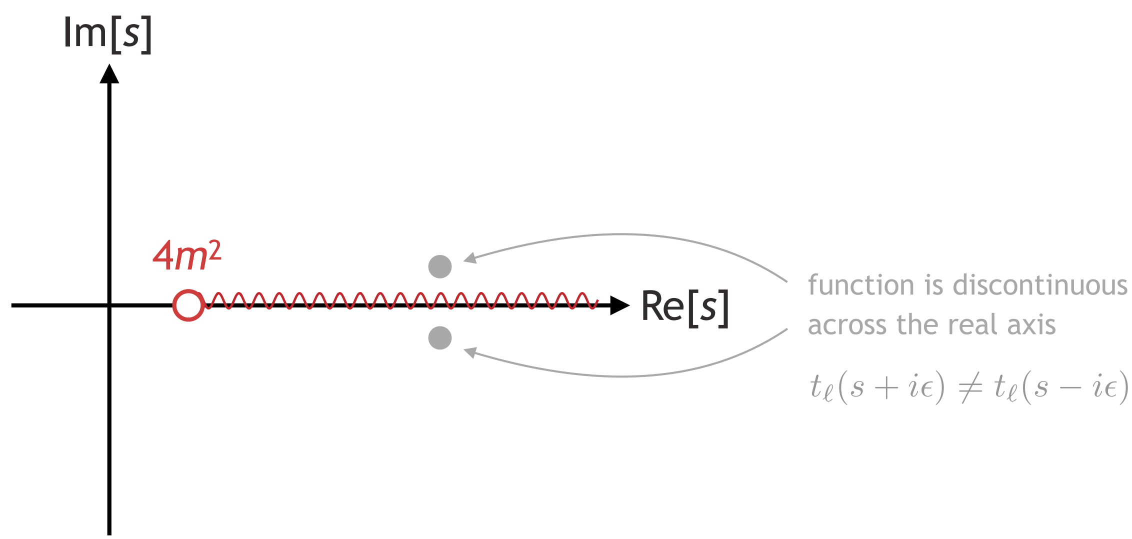
$$\rho(s) = \frac{2k(s)}{\sqrt{s}} = \frac{\boxed{\sqrt{s - 4m^2}}}{\sqrt{s}}$$

square root branch cut

unitarity gives us one guaranteed singularity — **a branch cut starting at threshold**

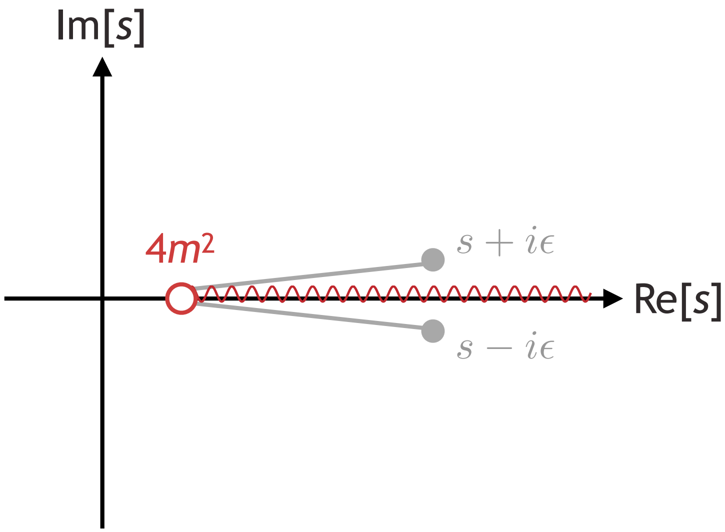
e.g. elastic partial-wave case: $\mathrm{Im}\, t_\ell(s) = \rho(s) \left|t_\ell(s)\right|^2 \Theta(s-4m^2)$

$$\rho(s) = \frac{2k(s)}{\sqrt{s}} = \frac{\boxed{\sqrt{s-4m^2}}}{\sqrt{s}}$$ square root branch cut



has an immediate consequence
— the complex plane must be **multi-sheeted**

unitarity cut $\sqrt{s-4m^2}$



just above the cut

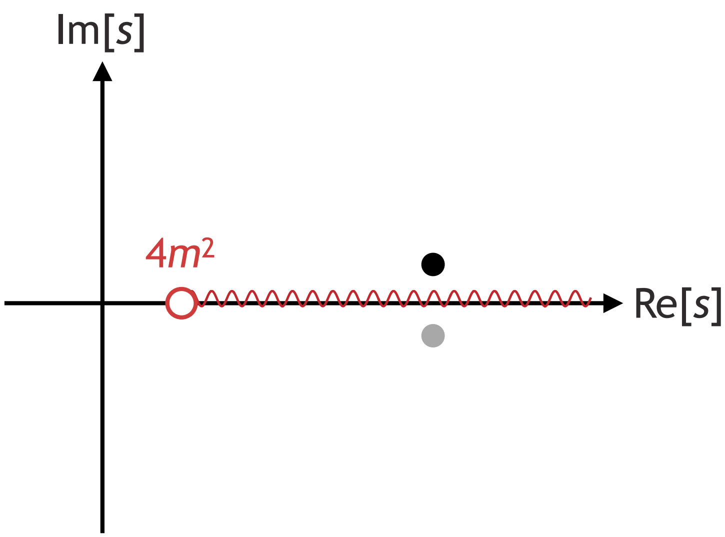
$$(s+i\epsilon)-4m^2 = \left|s-4m^2\right| e^{i\tilde{\epsilon}} \quad \Rightarrow \sqrt{s-4m^2} = \left|s-4m^2\right|^{1/2}$$

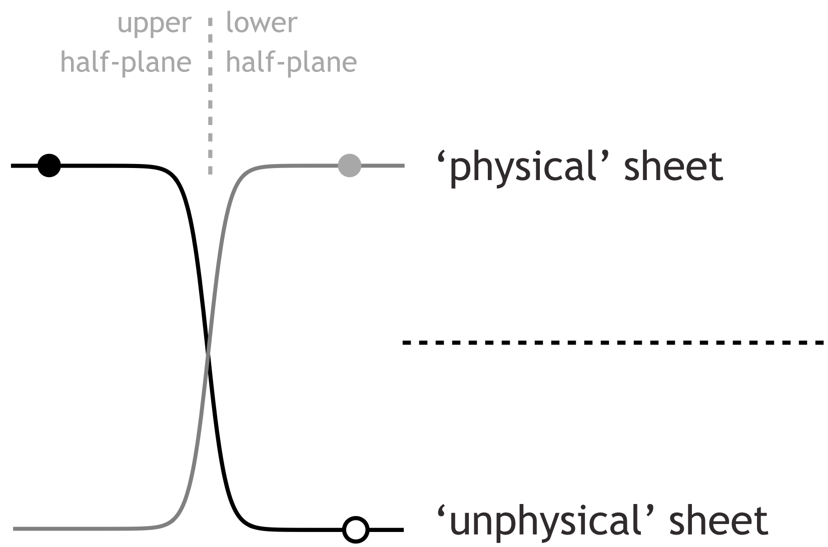
just below the cut

$$(s-i\epsilon)-4m^2 = \left|s-4m^2\right| e^{i(2\pi-\tilde{\epsilon})} \Rightarrow \sqrt{s-4m^2} = -\left|s-4m^2\right|^{1/2}$$

discontinuity across the cut !

Riemann sheet structure



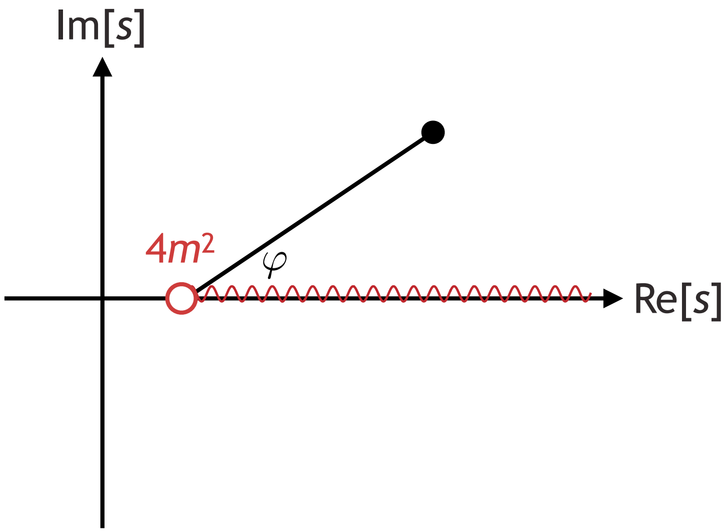


sheets can be characterised by the sign of Im[k]

physical sheet = sheet I = Im[k] > 0

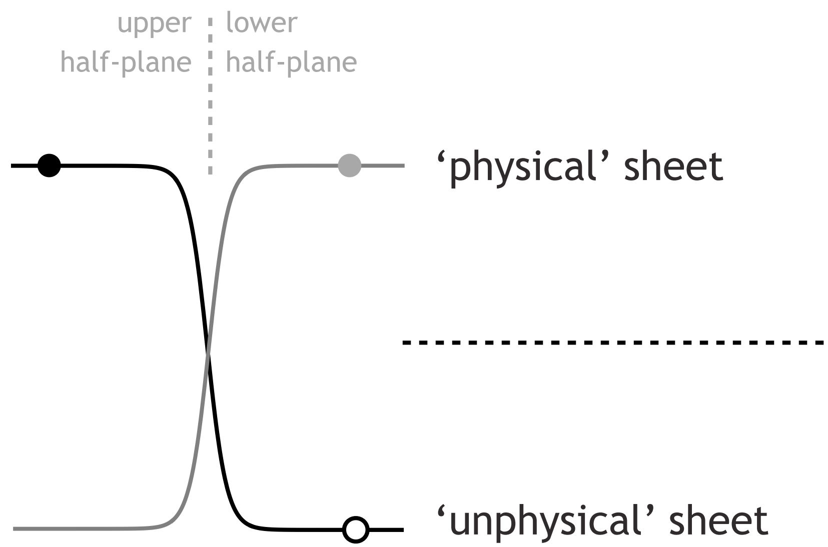
unphysical sheet = sheet II = Im[k] < 0

Riemann sheet structure



physical sheet = sheet I : $0 < \varphi < 2\pi$

unphysical sheet = sheet II : $2\pi < \varphi < 4\pi$



$\mathrm{Im}\, k \propto \sin(\varphi/2)$ $\Rightarrow$ sheets can be characterised by the sign of Im[*k*]

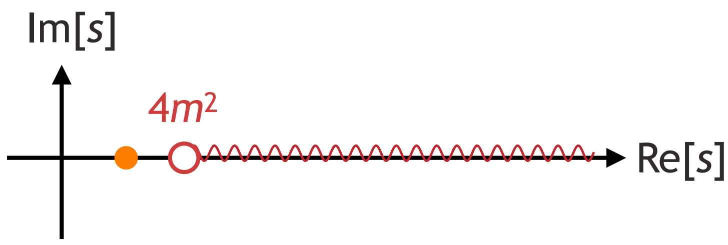
physical sheet = sheet I = Im[*k*] > 0

unphysical sheet = sheet II = Im[*k*] < 0

pole singularities ?

scattering amplitudes can have pole singularities only in certain locations

real energy axis, below threshold on **physical sheet**



corresponds to a **stable bound-state**

pole singularities ?

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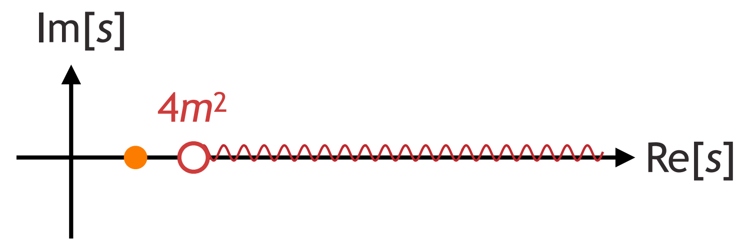
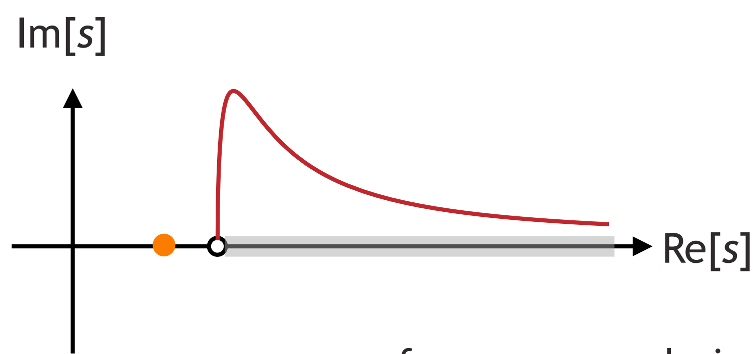
real energy axis, below threshold on **physical sheet**



corresponds to a **stable bound-state**

a stable bound-state
will strongly enhance scattering at threshold



famous example is the
deuteron at *NN* threshold

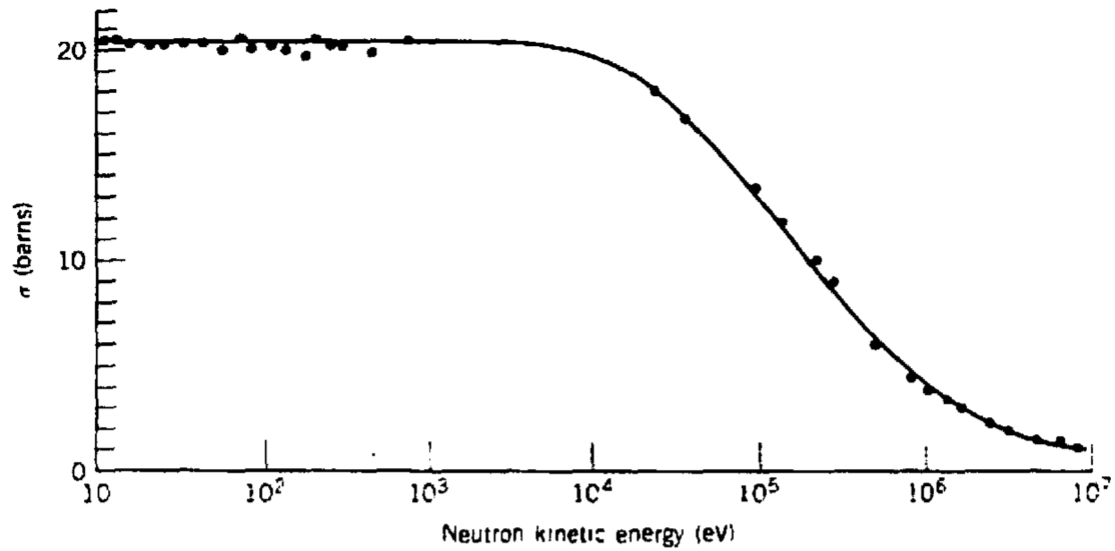


Figure 4.6 The neutron–proton scattering cross section at low energy. Data taken from a review by R. K. Adair, *Rev. Mod. Phys.* **22**, 249 (1950), with additional recent results from T. L. Houk, *Phys. Rev. C* **3**, 1886 (1970).

pole singularities ?

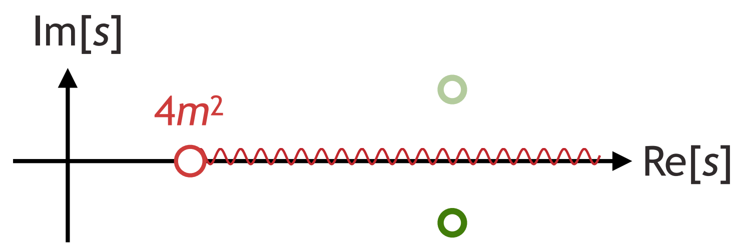
scattering amplitudes can have pole singularities only in certain locations

real energy axis, below threshold on **physical sheet**

corresponds to a **stable bound-state**

off the real axis, on the **unphysical sheet**

(in complex conjugate pairs)

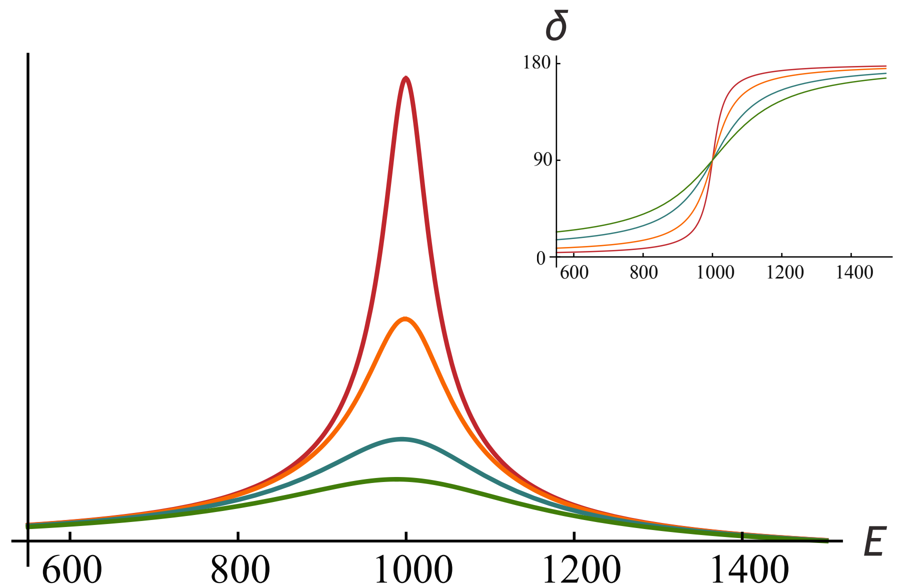


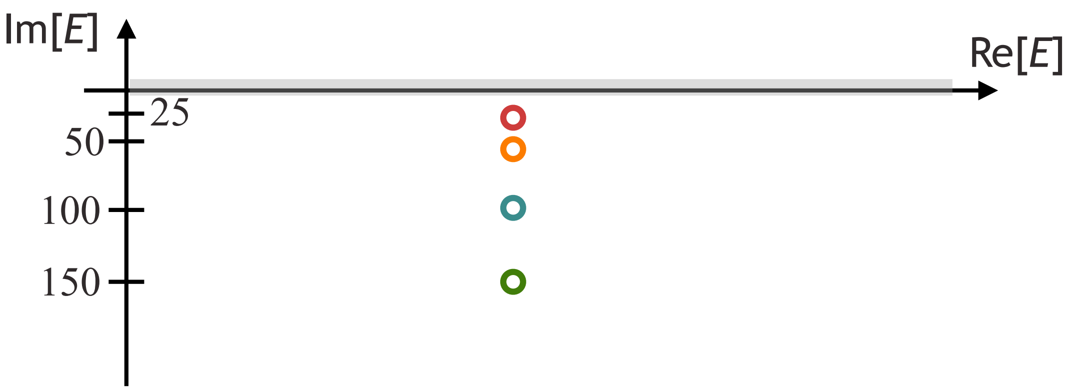
corresponds to a **resonance**

pole on the unphysical sheet

an **isolated** pole on the unphysical sheet will produce a bump on the real axis

– the classic resonance signature





close to the pole

$$t_\ell(s) \sim \frac{1}{s_0 - s}$$

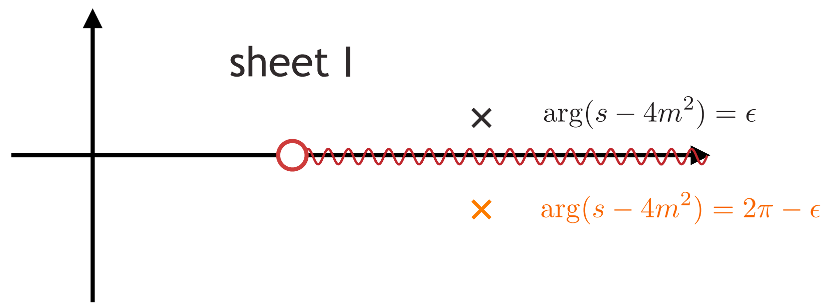
$$s_0 = \left(m - i\tfrac{1}{2}\Gamma\right)^2$$

Breit-Wigner poles on the unphysical sheet

X

$$\frac{1}{t(s)} \propto M^2 - s - i g^2 \rho(s) \qquad \rho(s) = \frac{\sqrt{s-4m^2}}{\sqrt{s}}$$

$$\propto -s + \left(M^2 - i g^2 \rho(s)\right) \quad \text{[D]}$$



$\rho = \frac{\sqrt{|s+4m^2|}}{\sqrt{s}}\, e^{i\epsilon/2}$

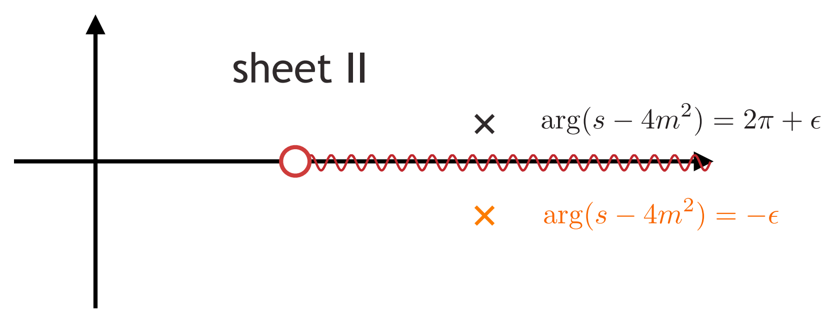
$\mathrm{Re}\,\rho > 0$

so no way for [D] to be zero

$\rho = \frac{\sqrt{|s+4m^2|}}{\sqrt{s}}\, e^{i\pi}\, e^{i\epsilon/2}$

$\mathrm{Re}\,\rho < 0$

so no way for [D] to be zero



$\rho = \frac{\sqrt{|s+4m^2|}}{\sqrt{s}}\, e^{i\pi}\, e^{i\epsilon/2}$

$\mathrm{Re}\,\rho < 0$

so [D] can be zero

$\rho = \frac{\sqrt{|s+4m^2|}}{\sqrt{s}}\, e^{-i\epsilon/2}$

$\mathrm{Re}\,\rho > 0$

so [D] can be zero

Breit-Wigner poles on the unphysical sheet

$$\frac{1}{t(s)} \propto M^2 - s - ig^2\rho(s) \qquad \rho(s) = \frac{\sqrt{s-4m^2}}{\sqrt{s}}$$

$$\propto -s + \left(M^2 - ig^2\rho(s)\right) \quad \text{[D]}$$

$$s = 4m^2 + 4k^2$$
$$\rho = \frac{2k}{\sqrt{s}} \approx \frac{2k}{M}$$

[D]=0→ $0 = k^2 + i\left(\frac{g^2}{2M}\right)k - k_M^2$

$$\Rightarrow \quad k = \frac{1}{2}\left[-i\left(\frac{g^2}{2M}\right) \pm \sqrt{-\left(\frac{g^2}{2M}\right)^2 + 4k_M^2}\right]$$

Im[k] < 0

unphysical sheet

e.g. for narrow resonance $k \approx \pm k_M - i\frac{g^2}{4M}$

pole singularities ?

scattering amplitudes can have pole singularities only in certain locations

real energy axis, below threshold on **physical sheet**

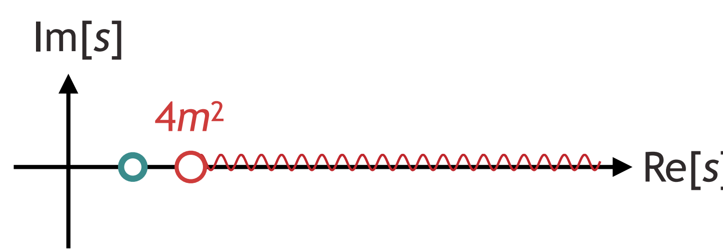
corresponds to a **stable bound-state**

off the real axis, on the **unphysical sheet**

(in complex conjugate pairs)

corresponds to a **resonance**

real energy axis, below threshold on **unphysical sheet**



corresponds to a **virtual bound-state**

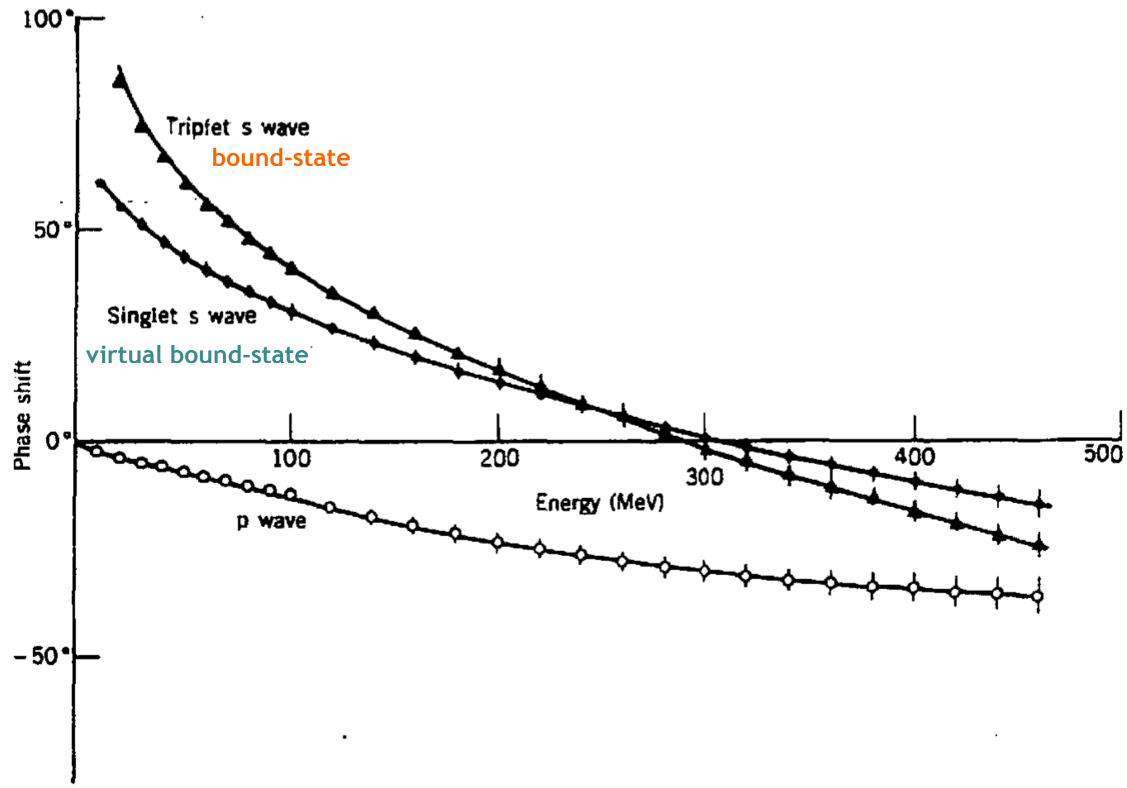
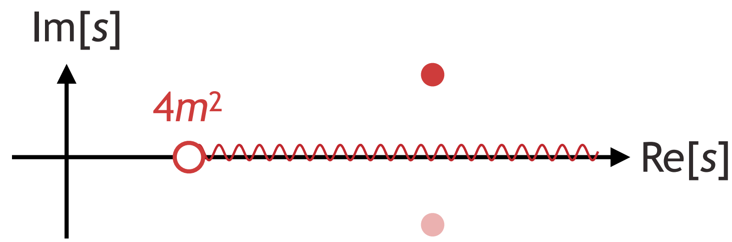


Figure 4.12 The phase shifts from neutron–proton scattering at medium energies. The change in the s-wave phase shift from positive to negative at about 300 MeV shows that at these energies the incident nucleon is probing a repulsive core in the nucleon–nucleon interaction. ▲, $^3S_1$; ●, $^1S_0$; O, $^1P_1$. Data from M. MacGregor et al., *Phys. Rev.* **182**, 1714 (1969).

pole singularities ?

scattering amplitudes can have pole singularities only in certain locations

not allowed: poles off the real axis
on the **physical sheet**



would violate **causality**

poor-man's causality → analyticity argument ...

the response of a physical system, ***G(t)*** to an input ***g(t)*** can be described in general by a **convolution**

$$G(t) = \int_{-\infty}^{\infty} dt'\, f(t-t')\, g(t')$$

where ***f(t′-t)*** describes the effect time *t′* has on time *t*

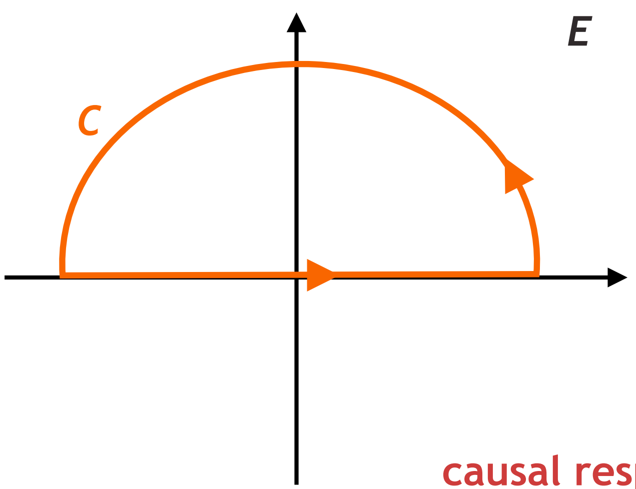
in order for the response to be **causal**, we require

$$f(\tau) = 0 \;\; \text{for} \;\; \tau < 0$$

consider the Fourier transform of $f(\tau)$

$$\tilde{f}(E) \equiv \frac{1}{2\pi} \int_{-\infty}^{\infty} d\tau\, e^{iE\tau}\, f(\tau) \qquad \& \qquad f(\tau) = \int_{-\infty}^{\infty} dE\, e^{-iE\tau}\, \tilde{f}(E)$$

in order to be causal, $\tilde{f}(E)$, must have no poles in the upper half-plane



$$\begin{aligned} 0 &= \oint_C dE\, e^{-iE\tau}\, \tilde{f}(E) \\ &= f(\tau) + \big(\text{semicircle at } \infty\big) \end{aligned}$$

$e^{-iE\tau} \propto e^{\tau \cdot \mathrm{Im}\, E}$

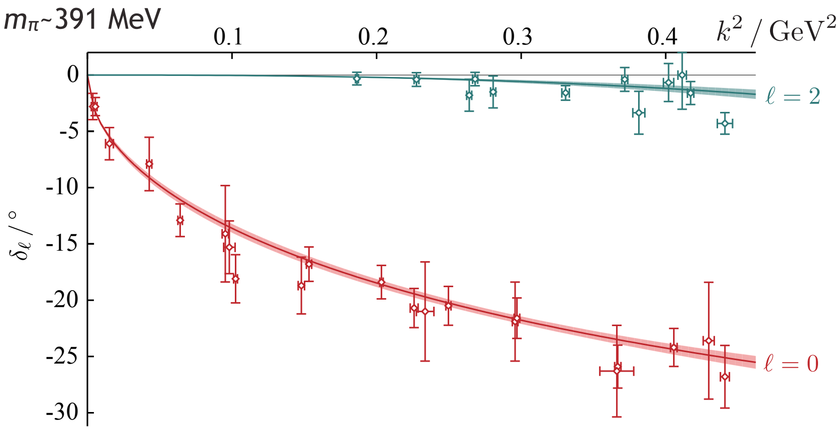
upper half-plane ⇒ Im(*E*) > 0

vanishes for $\tau<0$

causal response functions are analytic in the upper-half plane of complex energy

singularity structure from lattice calculations — elastic

$\pi\pi$ isospin=2



PRD86 034031 (2012)

no nearby poles

weak and repulsive interaction

$$k \cot\delta_0 = \frac{1}{a_0} + \ldots$$

[first term in the 'effective range expansion' $k \cot\delta_0 = \frac{1}{a_0} + \frac{1}{2} r_0 k^2 + \ldots$]

$$m_\pi a_0 = -0.285(6)$$

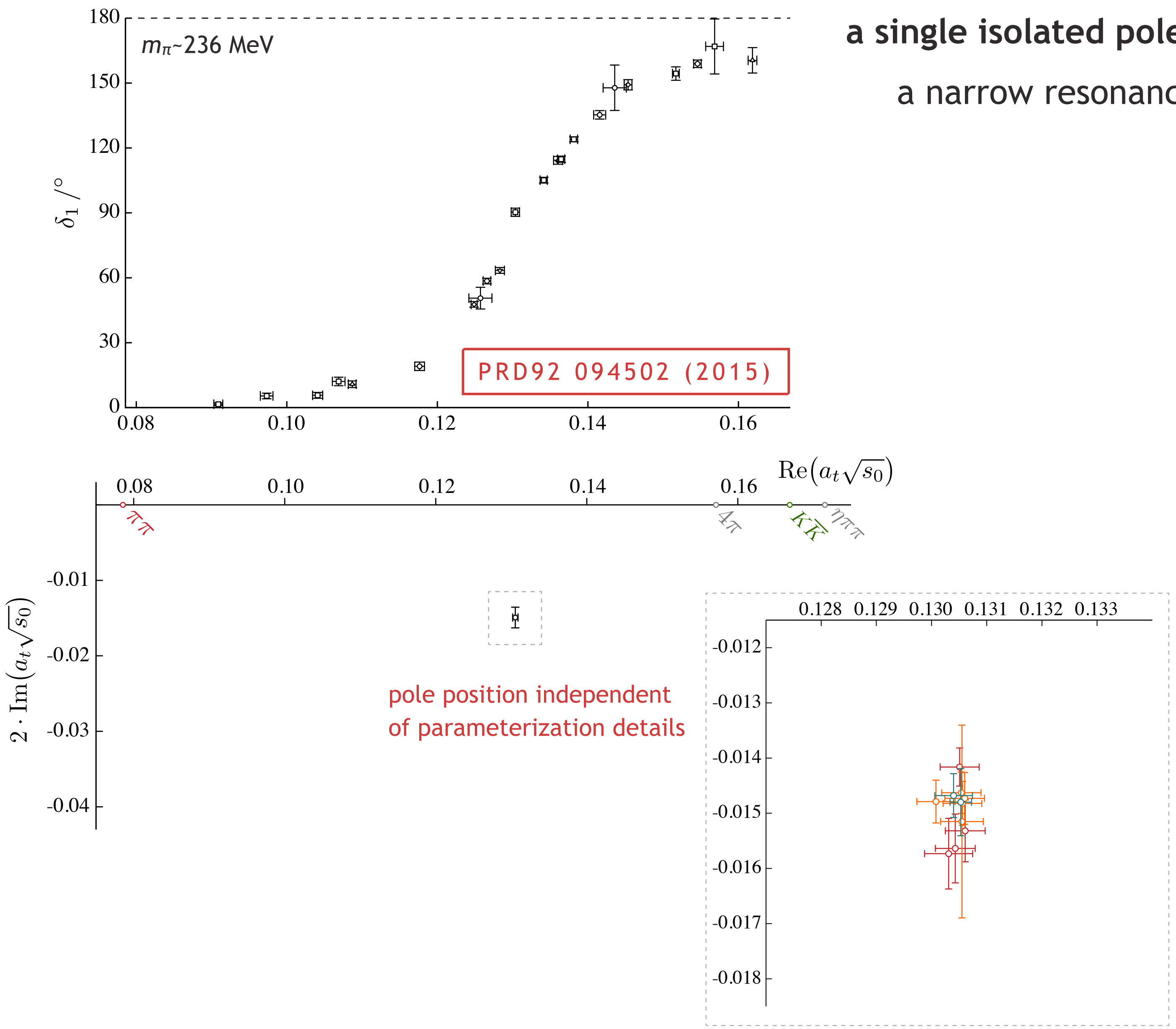
$$t_{\ell=0} = \frac{\sqrt{s}}{2} \frac{1}{k \cot\delta_0 - ik}$$

$$s_0 \approx -45\, m_\pi^2$$

a pole, but very far away

singularity structure from lattice calculations — elastic

$\pi\pi$ isospin=1



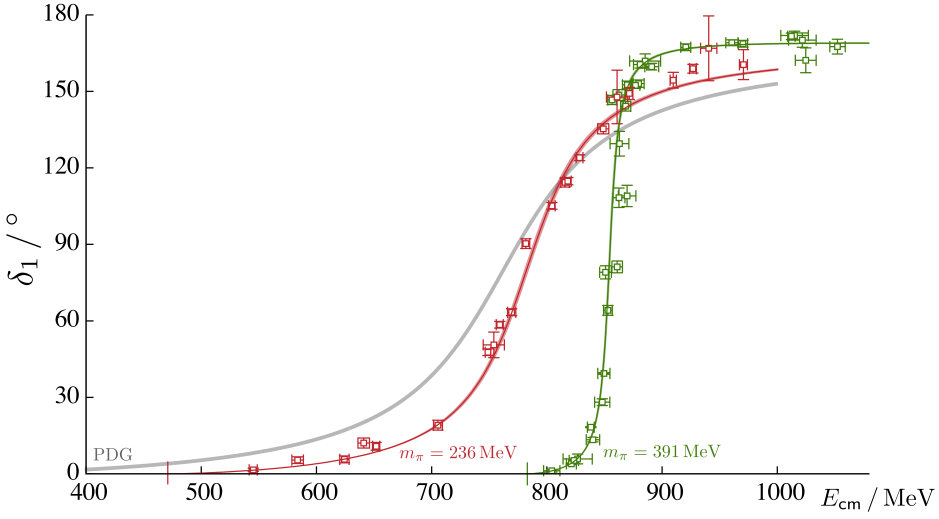
a single isolated pole

a narrow resonance

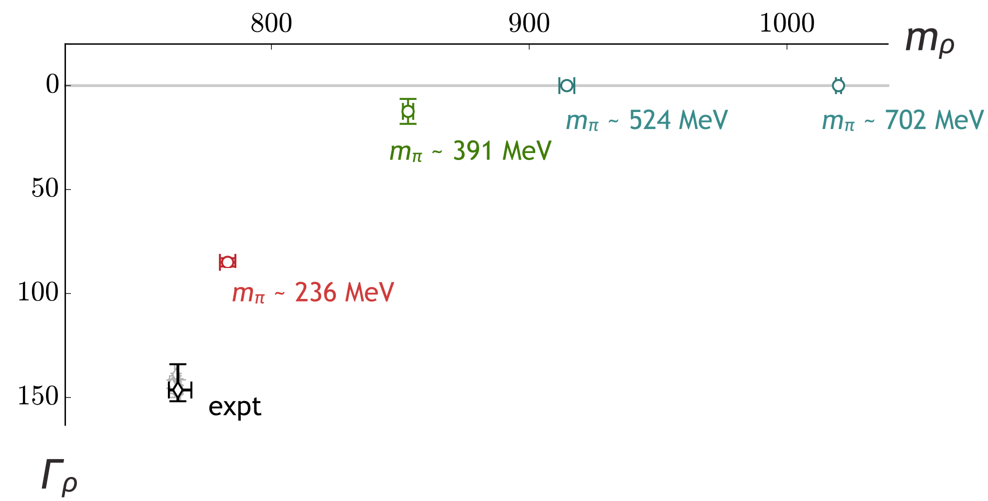
pole position independent of parameterization details

singularity structure from lattice calculations — elastic

$\pi\pi$ isospin=1



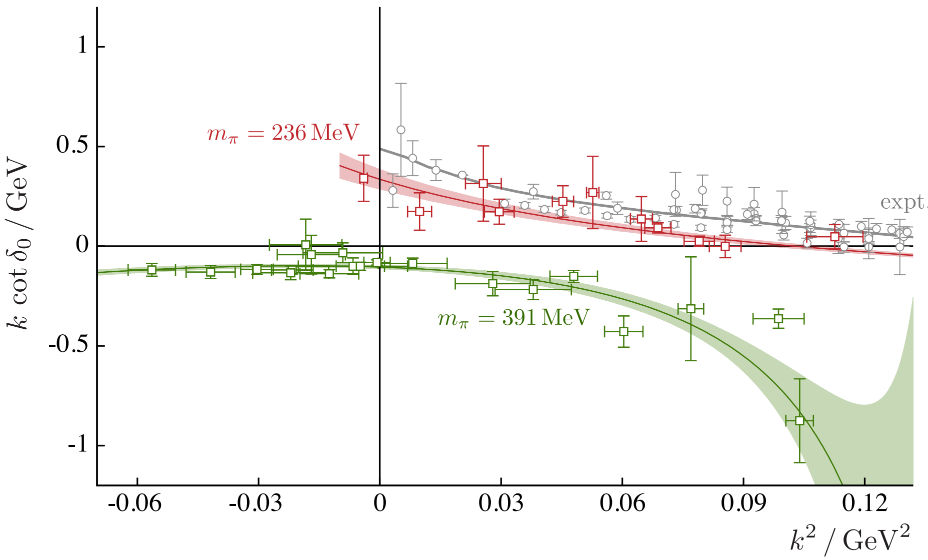
evolution with changing quark mass



poles don't 'appear' or 'disappear' with changing quark mass
— they smoothly move round the complex plane

singularity structure from lattice calculations — elastic

$\pi\pi$ isospin=0

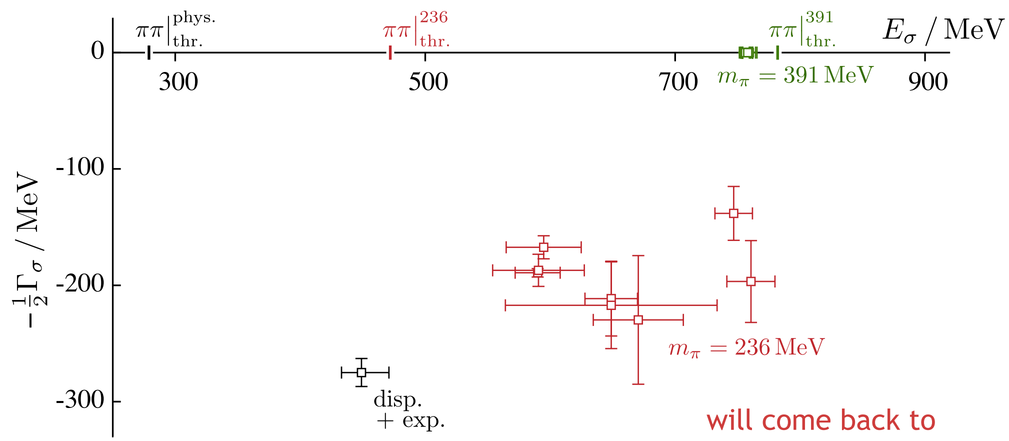


PRL118 022002 (2017)

$$t_{\ell=0} = \frac{\sqrt{s}}{2} \frac{1}{k \cot \delta_0 - ik}$$

m_π~391 MeV — a bound-state pole

m_π~236 MeV — a resonance pole

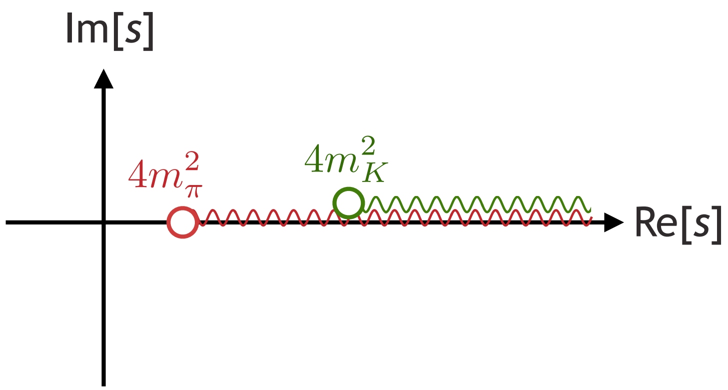


will come back to this scatter later if there's time

coupled-channels

for each new channel, each sheet splits in two ⇒ 2^N sheets for N channels

e.g. two channels ($\pi\pi$, $K\bar{K}$)



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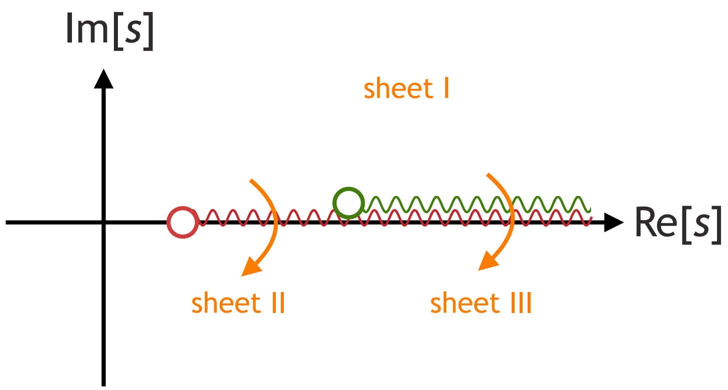
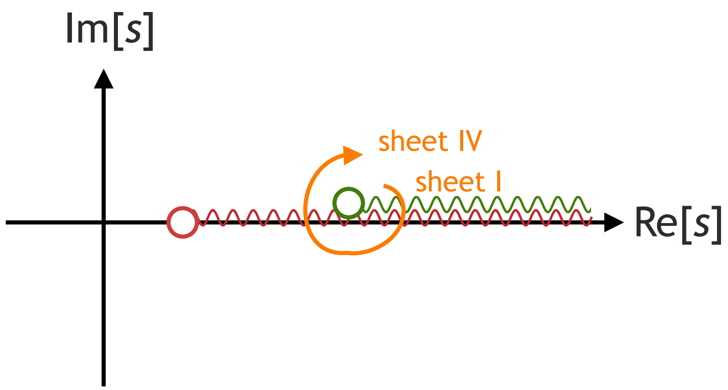


	Im[$k_{\pi\pi}$]	Im[k_{KK}]
sheet I	+	+
sheet II	−	+
sheet III	−	−
sheet IV	+	−

coupled-channels

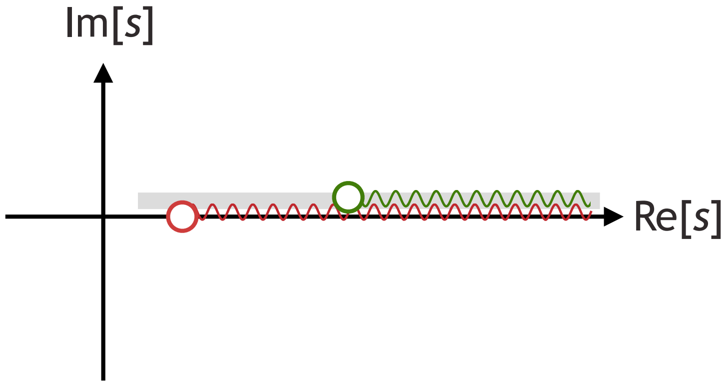
for each new channel, each sheet splits in two ⇒ 2^N sheets for N channels

e.g. two channels ($\pi\pi$, $K\bar{K}$)

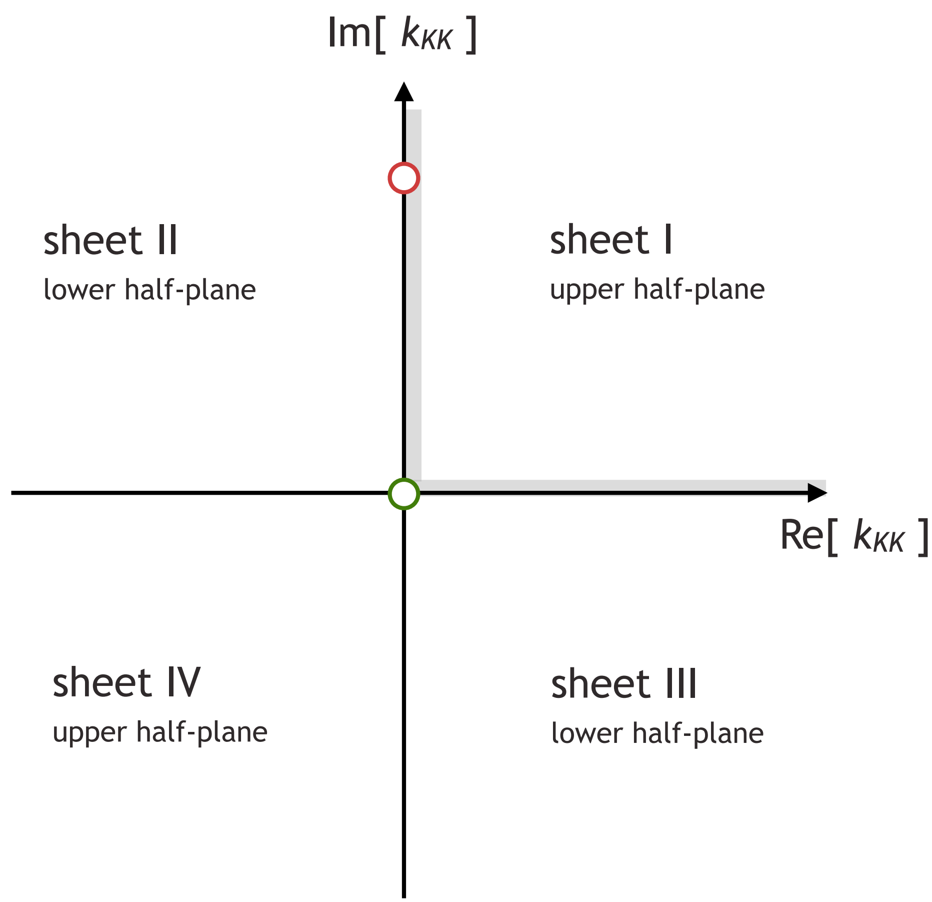


	Im[$k_{\pi\pi}$]	Im[k_{KK}]
sheet I	+	+
sheet II	−	+
sheet III	−	−
sheet IV	+	−

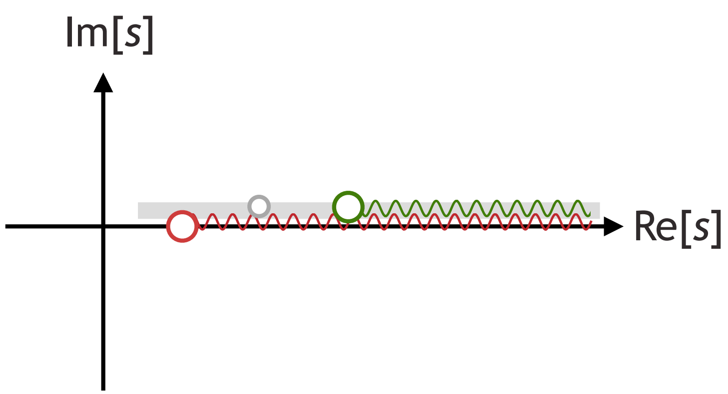
coupled-channels

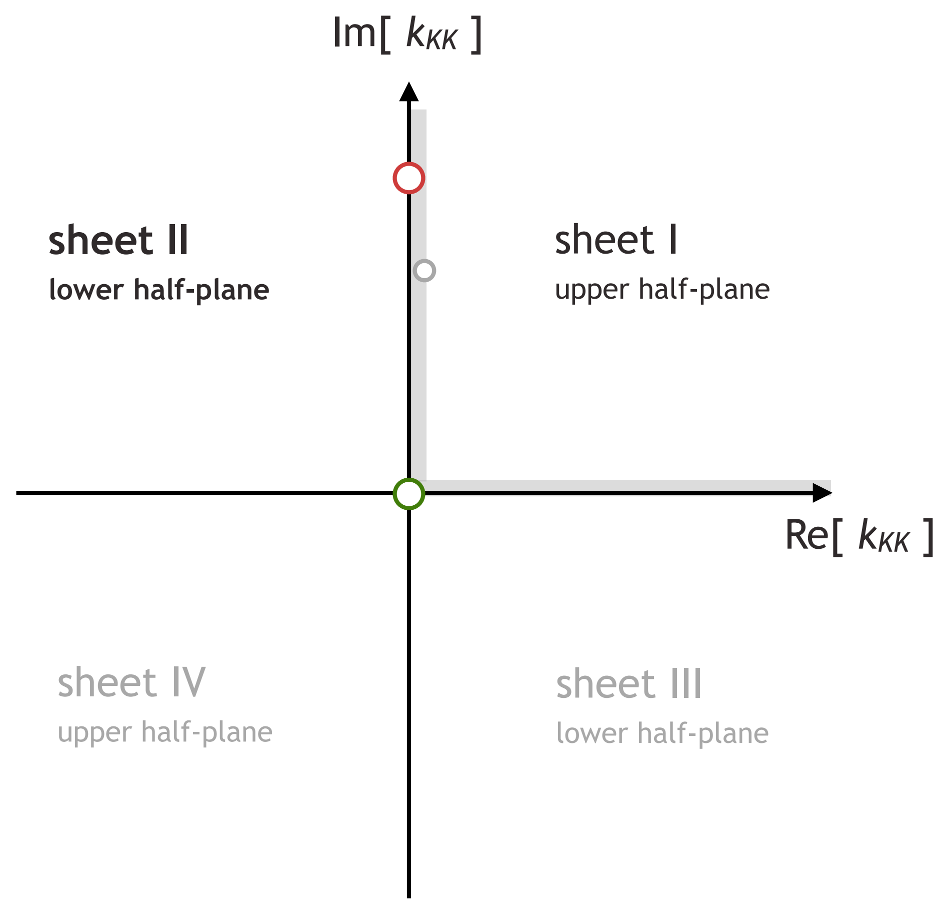


	Im[$k_{\pi\pi}$]	Im[k_{KK}]
sheet I	+	+
sheet II	−	+
sheet III	−	−
sheet IV	+	−

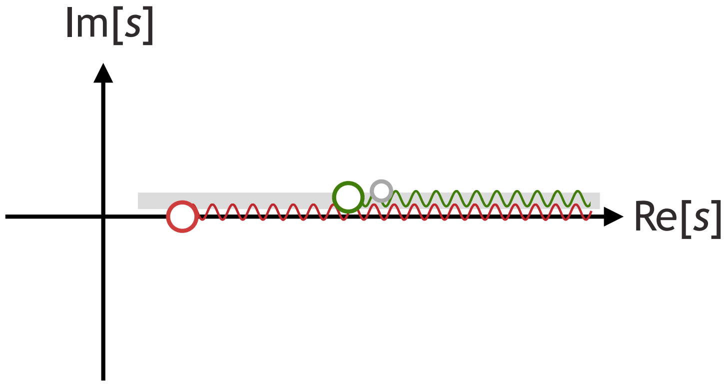


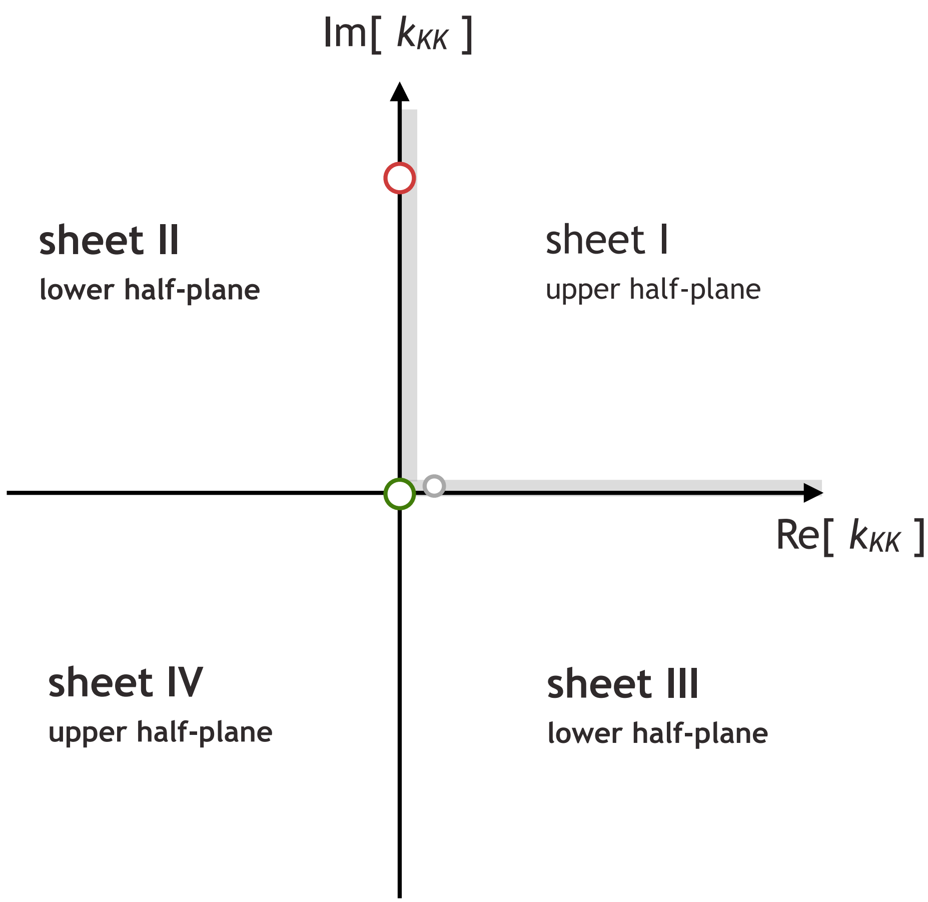
coupled-channels — between thresholds



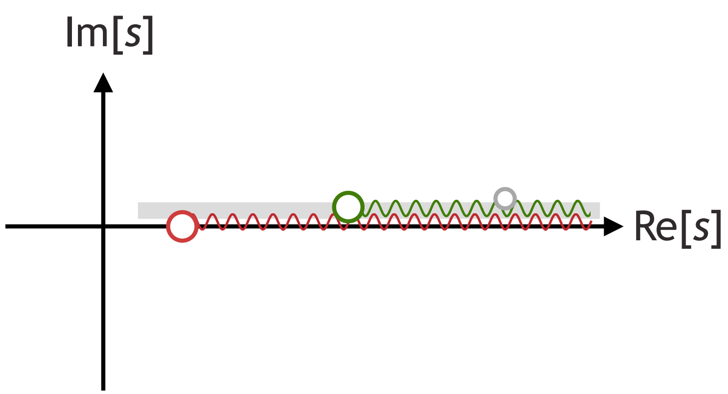


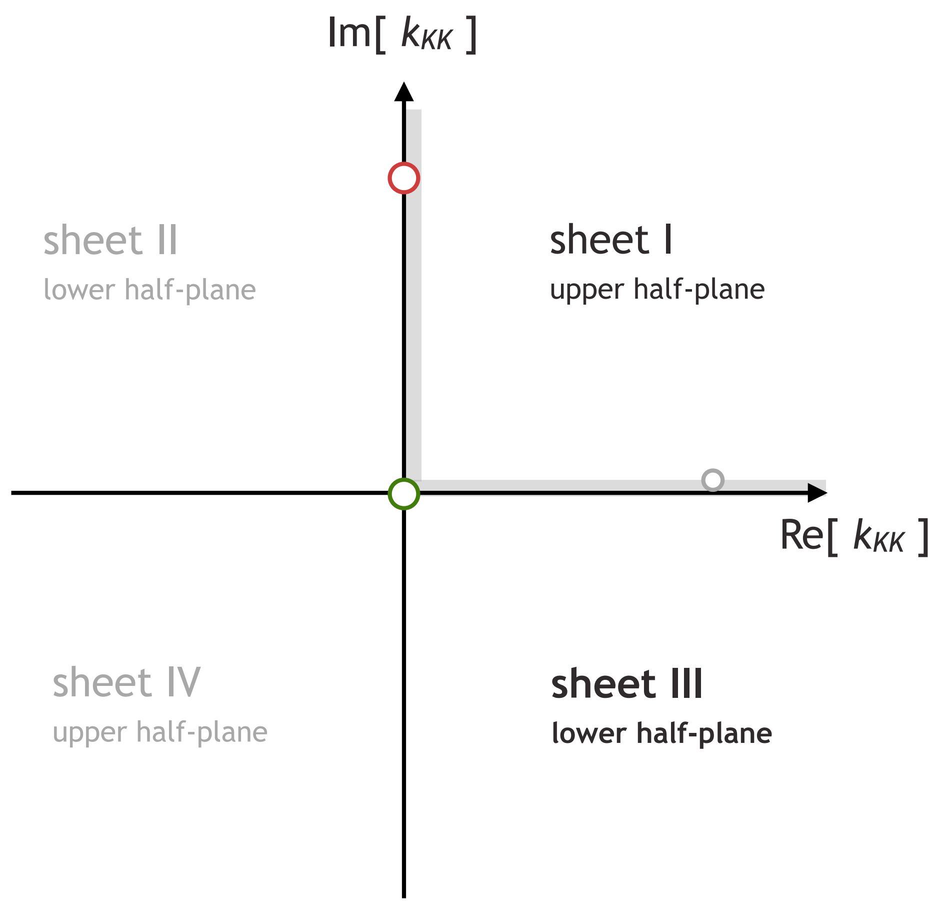
coupled-channels — near second threshold





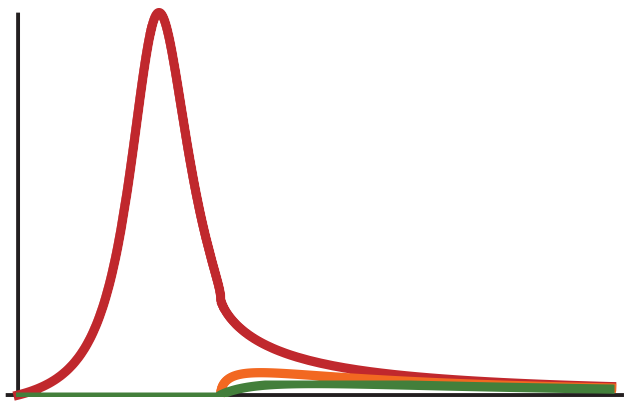
coupled-channels — well above both thresholds

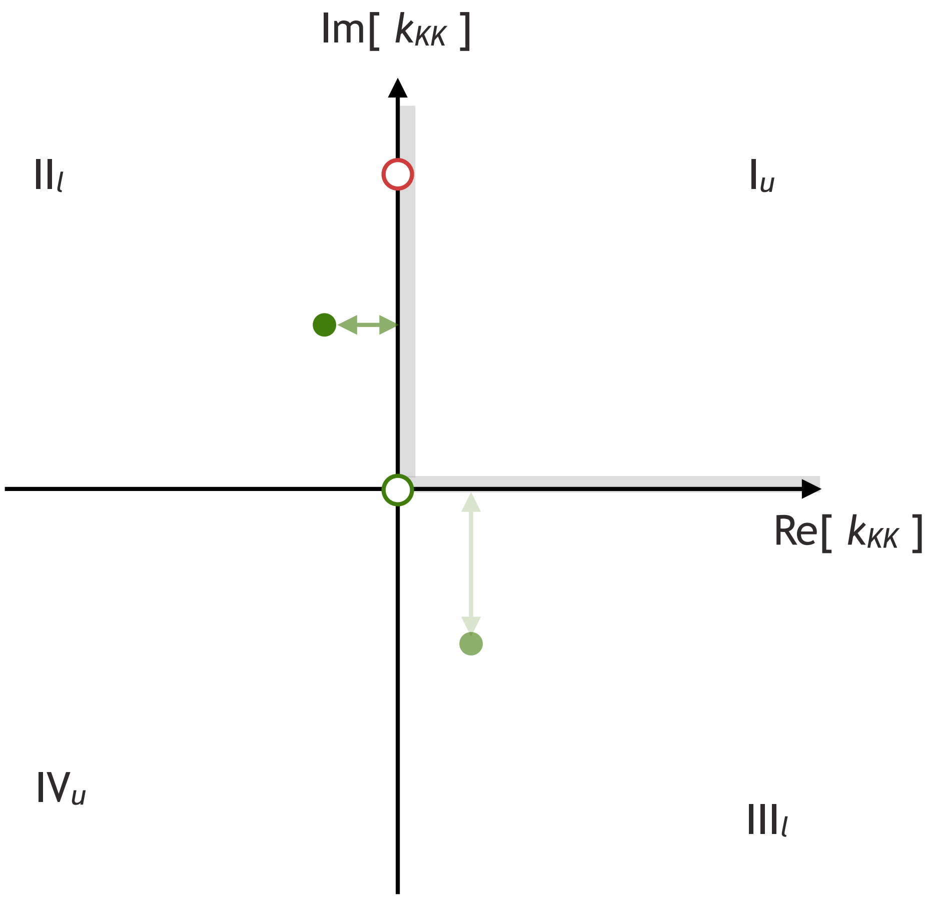




poles in two-channel scattering

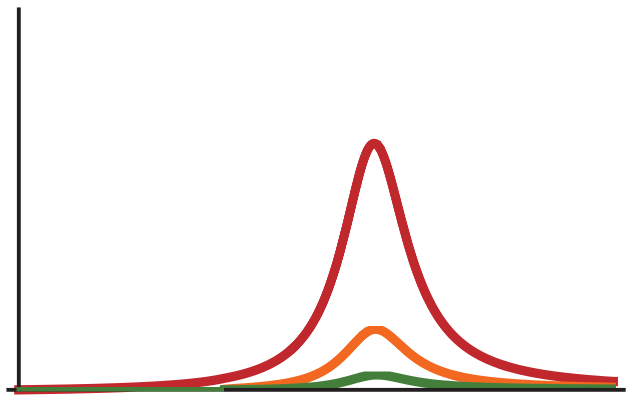
X

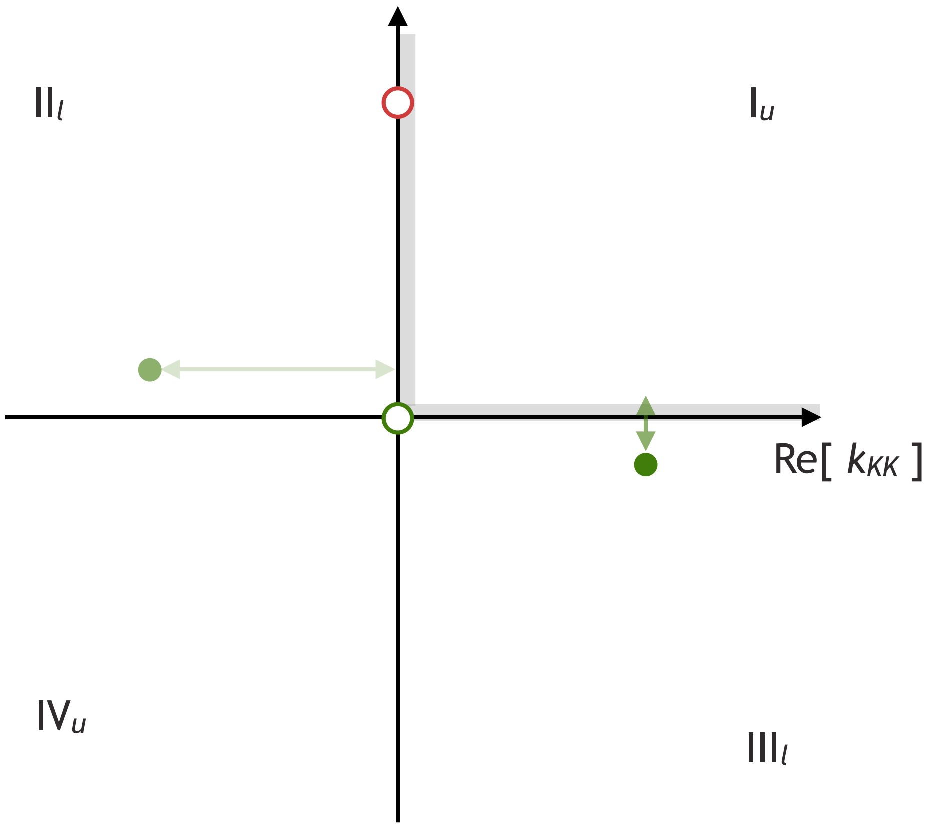




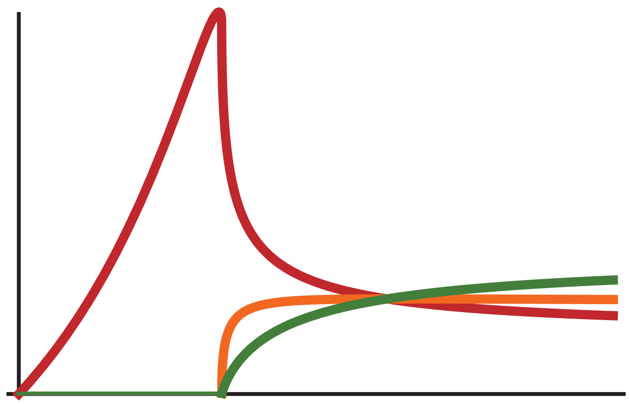
poles in two-channel scattering

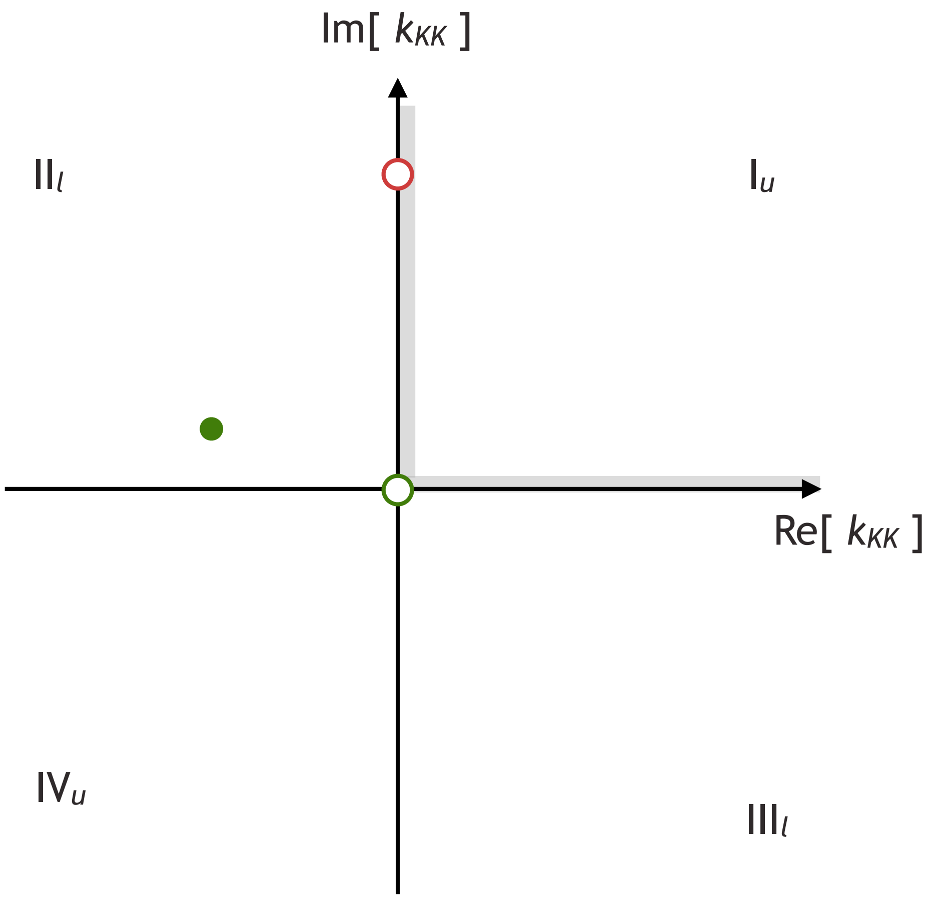
x



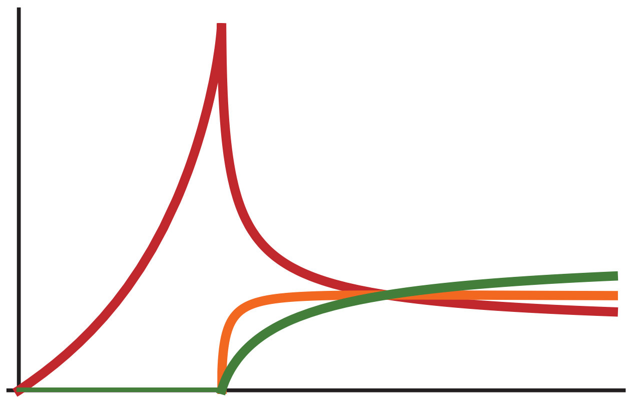


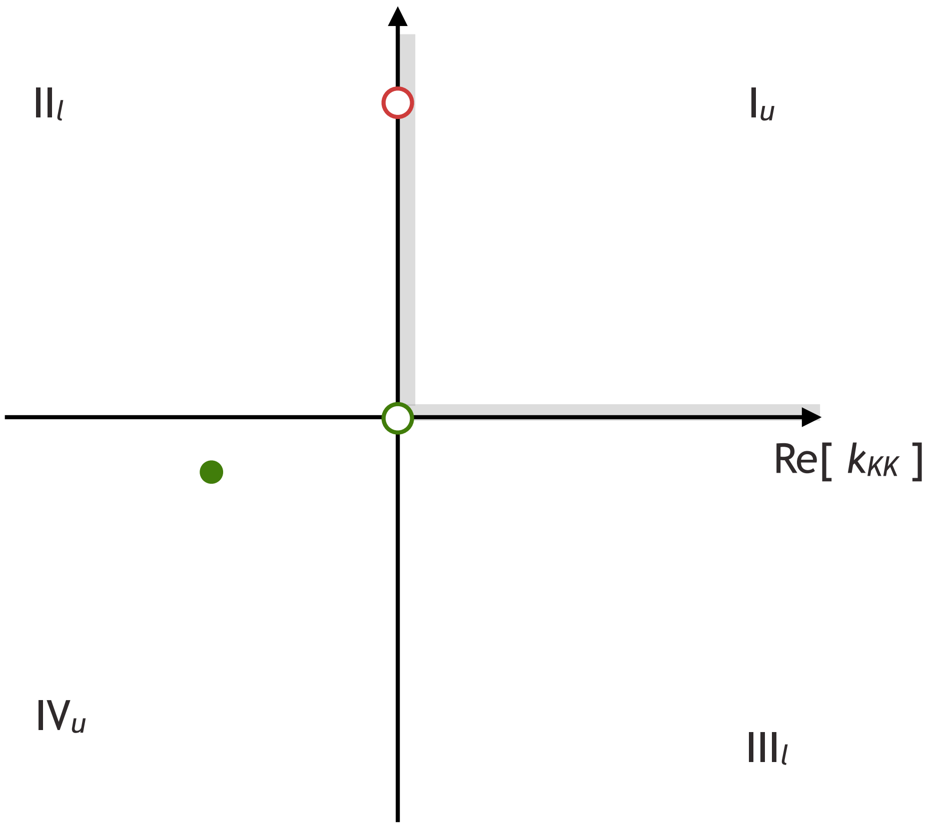
poles in two-channel scattering



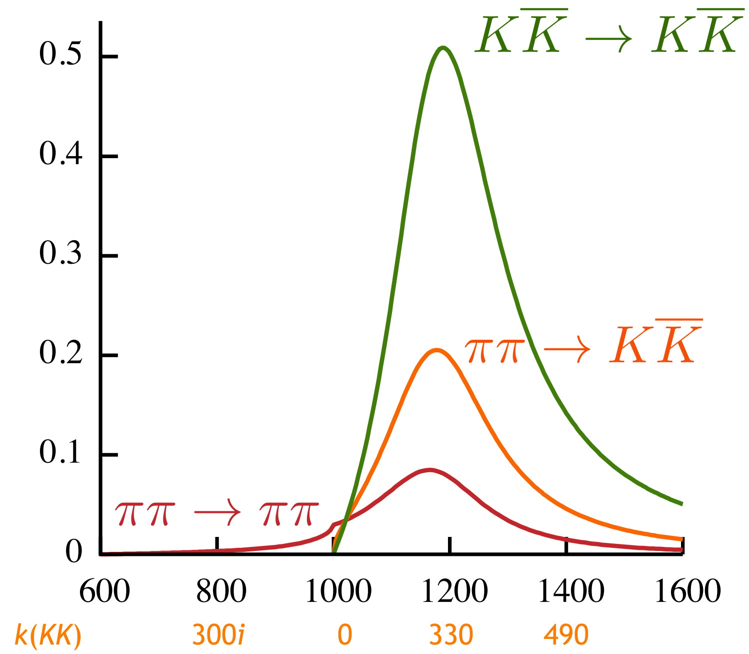


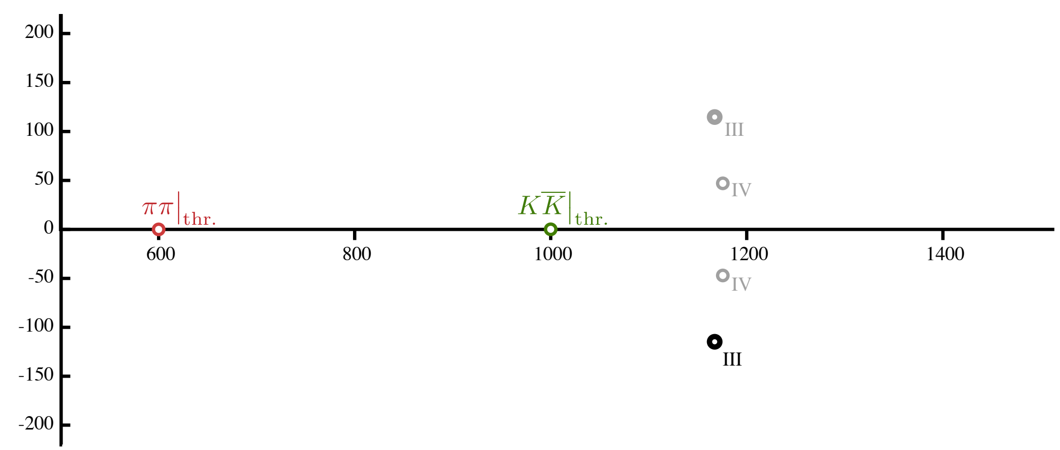
poles in two-channel scattering

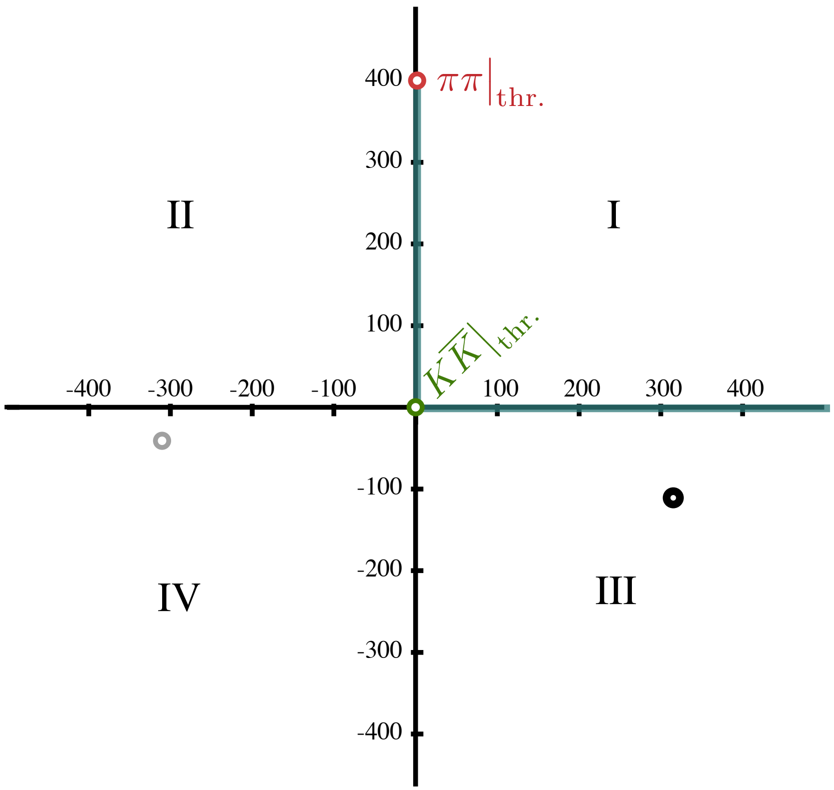




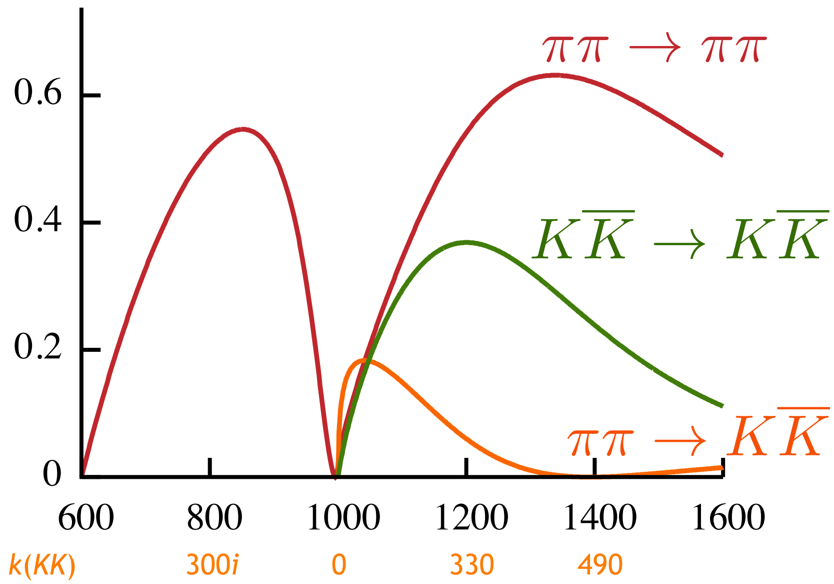
two-channel Flatté amplitude





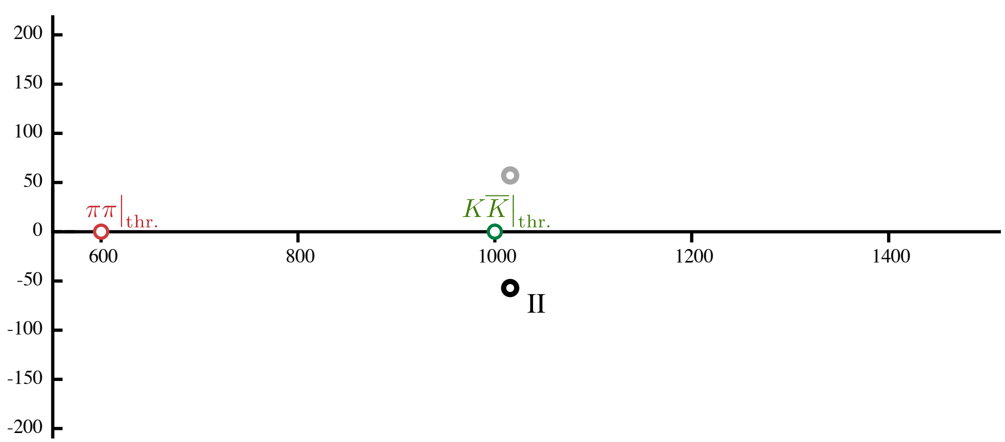


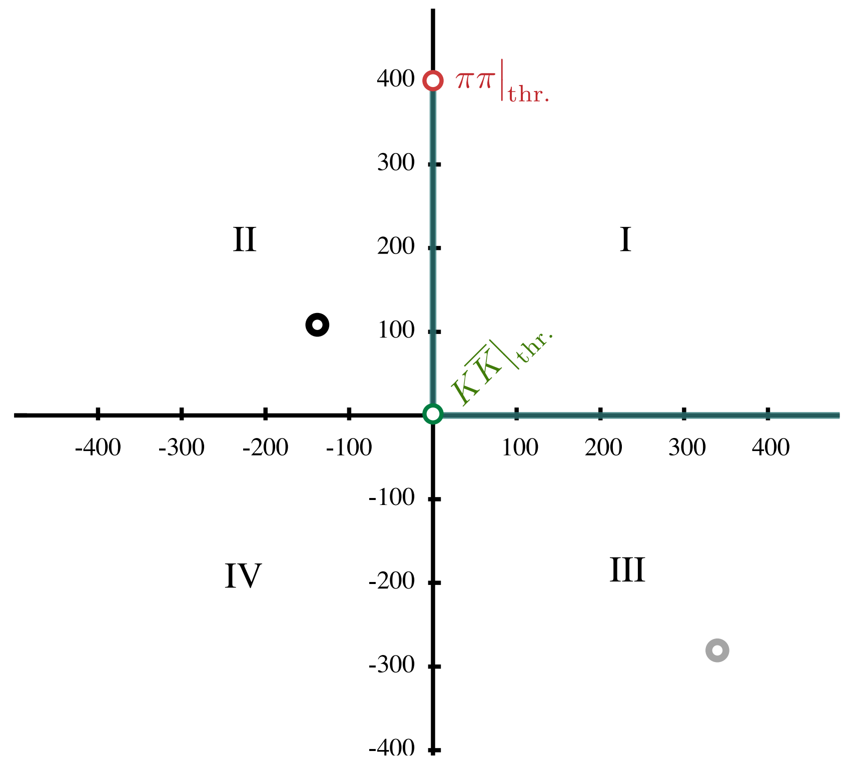
a less obviously resonant amplitude



$$\mathbf{K}^{-1}(s) = \begin{pmatrix} a & b + c\,s \\ b + c\,s & d + e\,s \end{pmatrix}$$

with Chew-Mandelstam phase-space





near the complex pole, s_0

$$t_{ij}(s \sim s_0) \sim \frac{c_i \, c_j}{s_0 - s}$$

pole position can be interpreted as **mass** and **width**

$$s_0 = \left(m_R \pm i\tfrac{1}{2}\Gamma_R\right)^2$$

information from the pole

near the complex pole, ***s*₀**

$$t_{ij}(s \sim s_0) \sim \frac{c_i\, c_j}{s_0 - s}$$

pole position can be interpreted as mass and width

$$s_0 = \left(m_R \pm i\tfrac{1}{2}\Gamma_R\right)^2$$

pole residue factorizes into a product of resonance **couplings** to the various decay channels

$$c_{\pi\pi},\; c_{K\bar{K}},\; \ldots$$

information from the pole

near the complex pole, s_0

$$t_{ij}(s \sim s_0) \sim \frac{c_i\, c_j}{s_0 - s}$$

pole position can be interpreted as mass and width

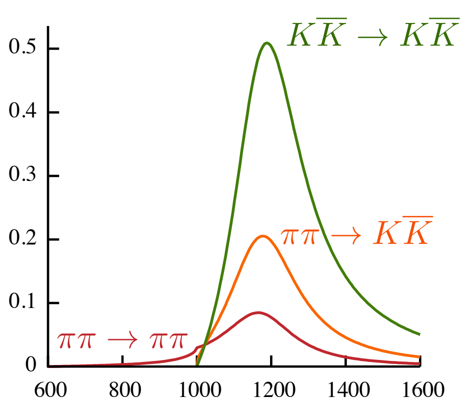
$$s_0 = \left(m_R \pm i\tfrac{1}{2}\Gamma_R\right)^2$$

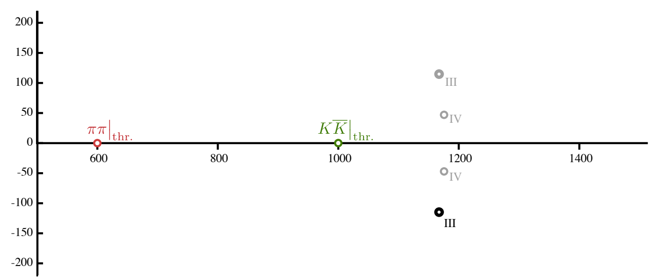
pole residue factorizes into a product of resonance couplings
to the various decay channels

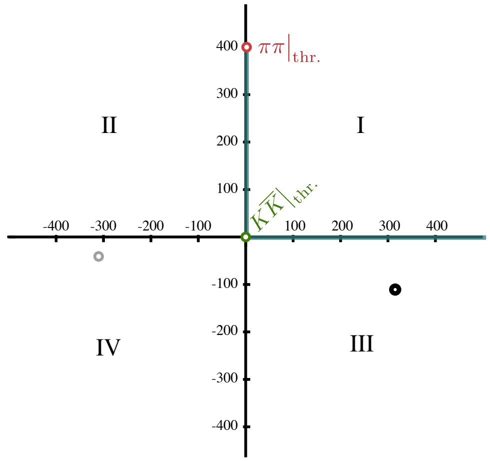
$$c_{\pi\pi},\; c_{K\bar{K}},\; \ldots$$

as we've seen a single resonance can be responsible
for poles on more than one sheet

– often only one is close enough to physical scattering to have a large effect

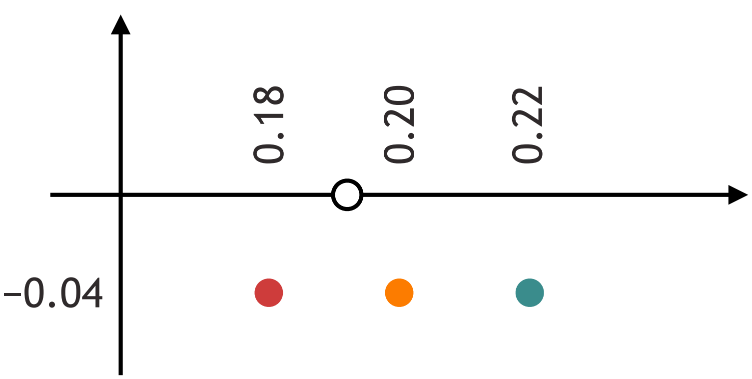


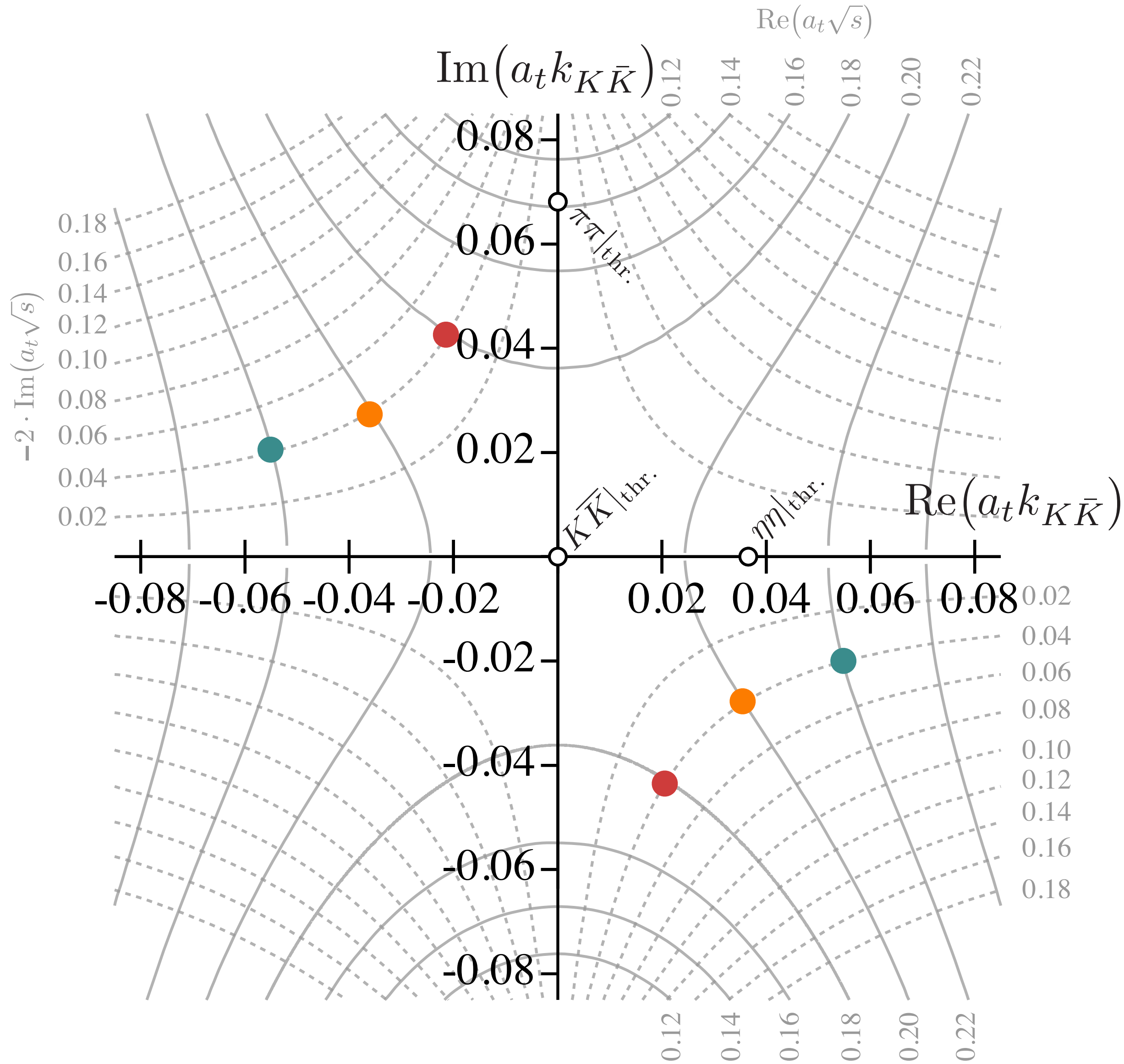




complex k-plane

slight complication — mapping of ‘distances’ is rather non-uniform



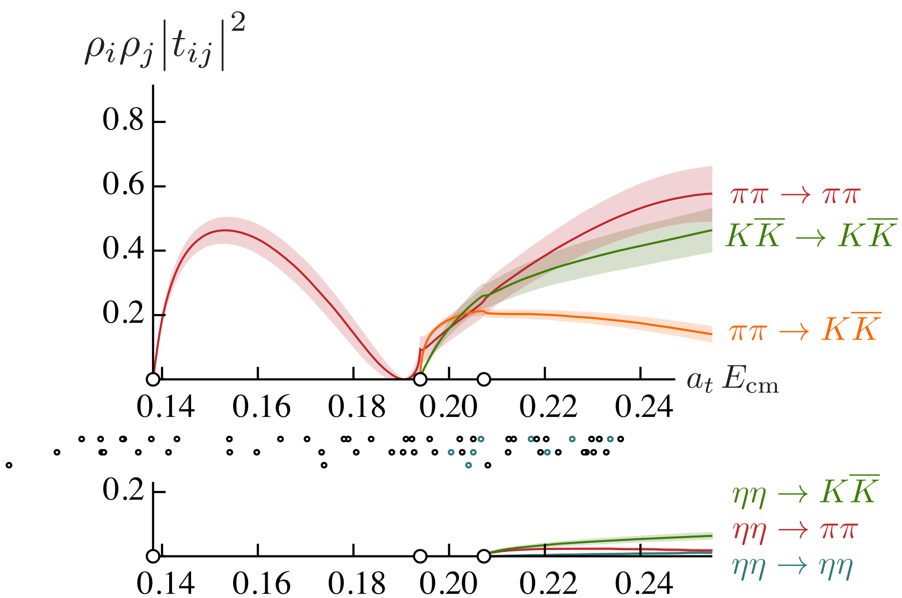


should really be using the s-plane rather than the $\surd s$ plane

$\pi\pi$, $K\bar{K}$, $\eta\eta$ scattering with m_π~391 MeV

PRD 97 054513 (2018)

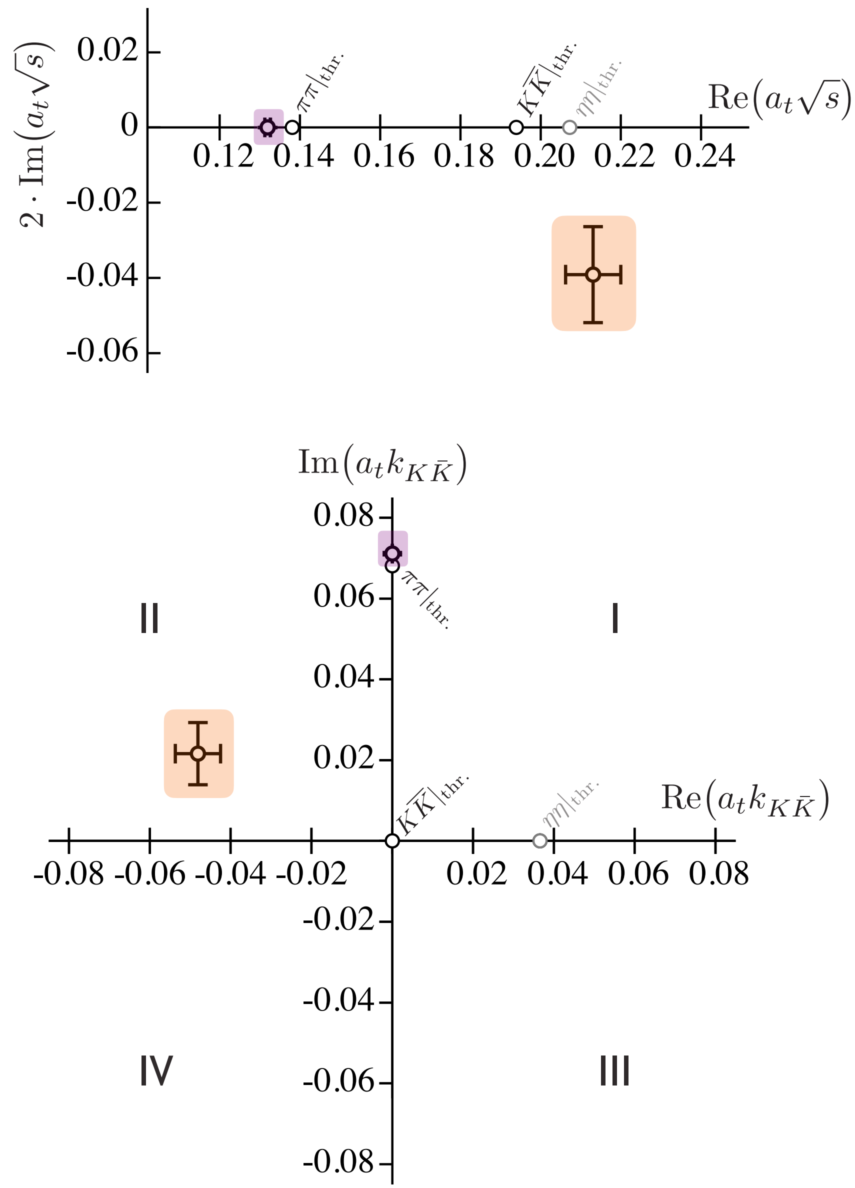
S-wave amplitudes



$$\mathbf{K}^{-1}(s) = \begin{pmatrix} a + b\,s & c + d\,s & e \\ c + d\,s & f & g \\ e & g & h \end{pmatrix}$$

with Chew-Mandelstam phase-space

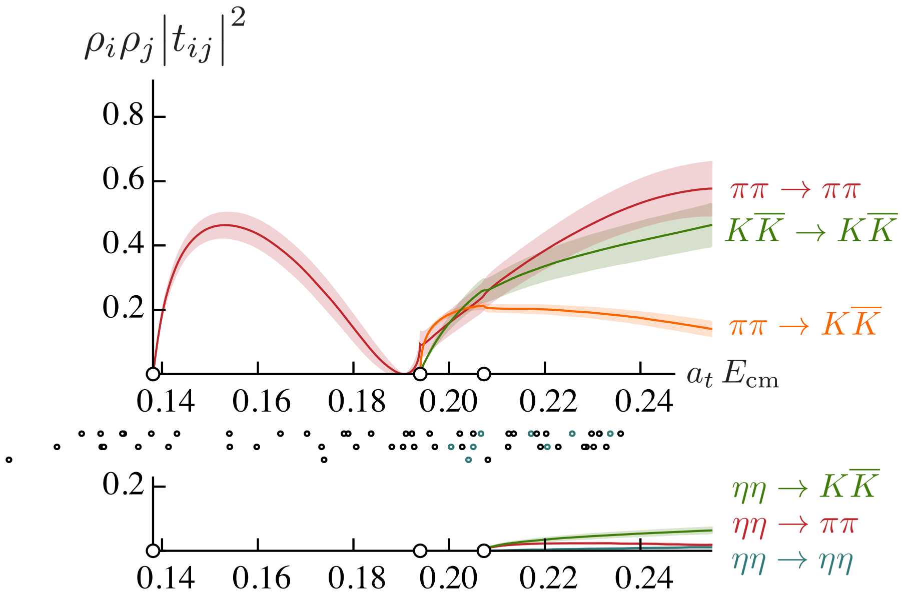
pole singularities



$\pi\pi$, $K\overline{K}$, $\eta\eta$ scattering with m_π~391 MeV

PRD 97 054513 (2018)

S-wave amplitudes



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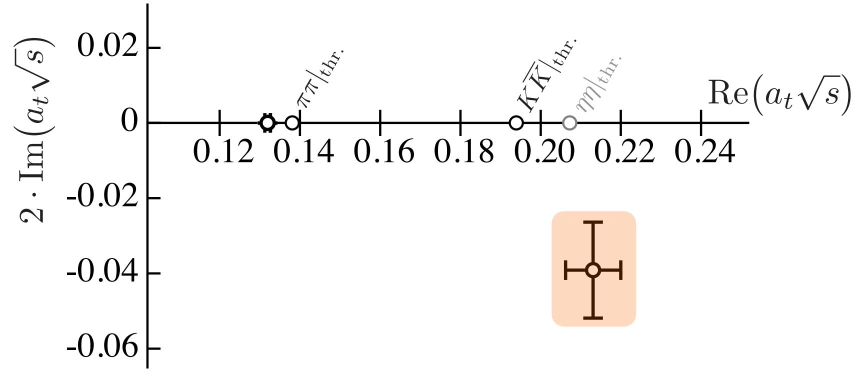
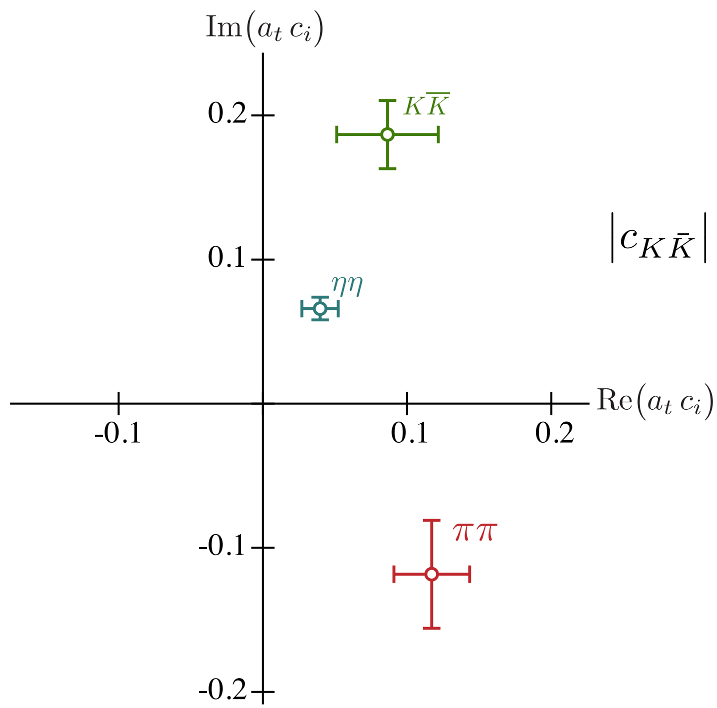


sheet II pole couplings

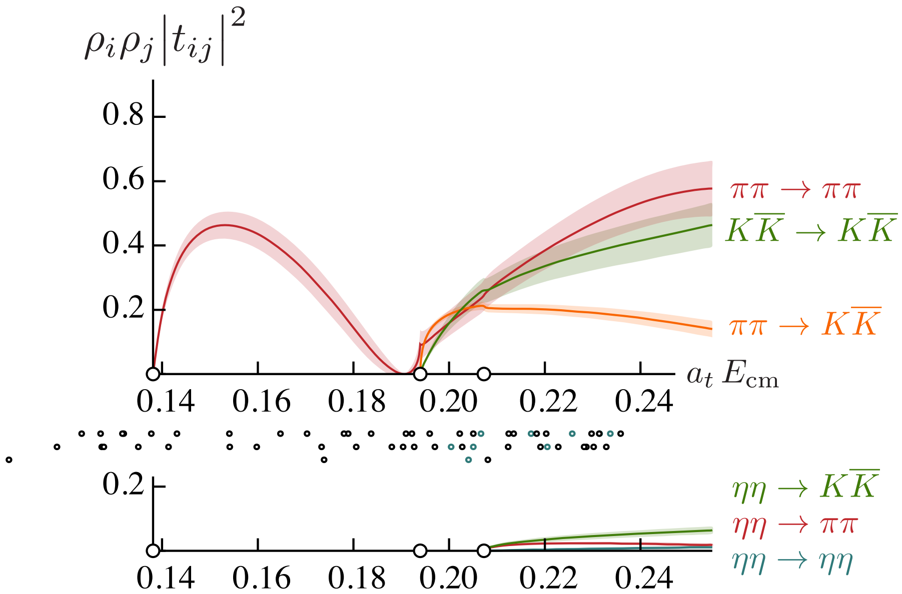


$$\left|c_{K\bar{K}}\right| \approx \left|c_{\pi\pi}\right|$$

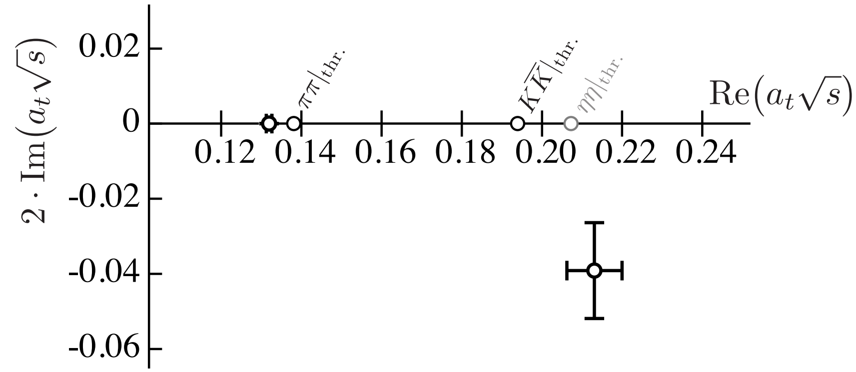
$\pi\pi$, $K\bar{K}$, $\eta\eta$ scattering with m_π~391 MeV

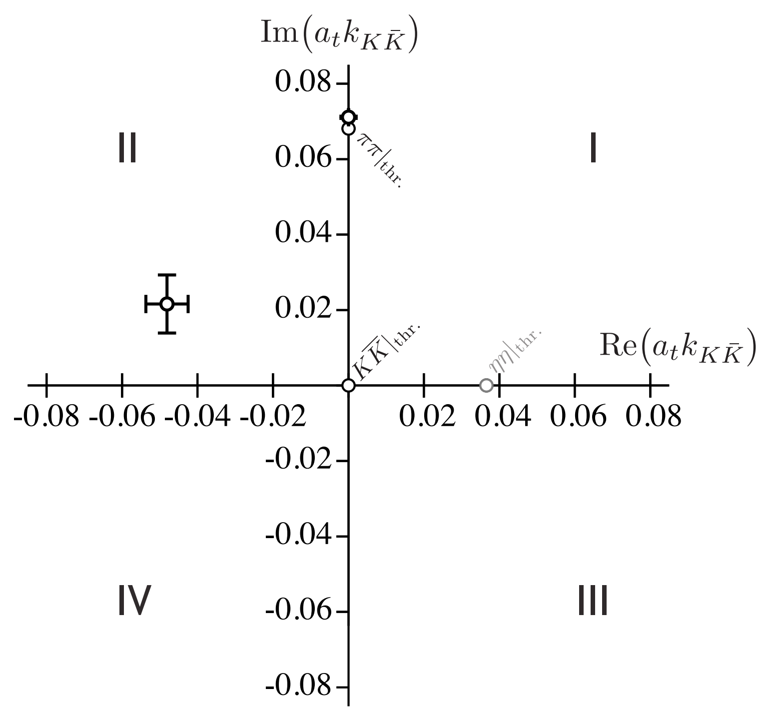
PRD 97 054513 (2018)

S-wave amplitudes



pole singularities





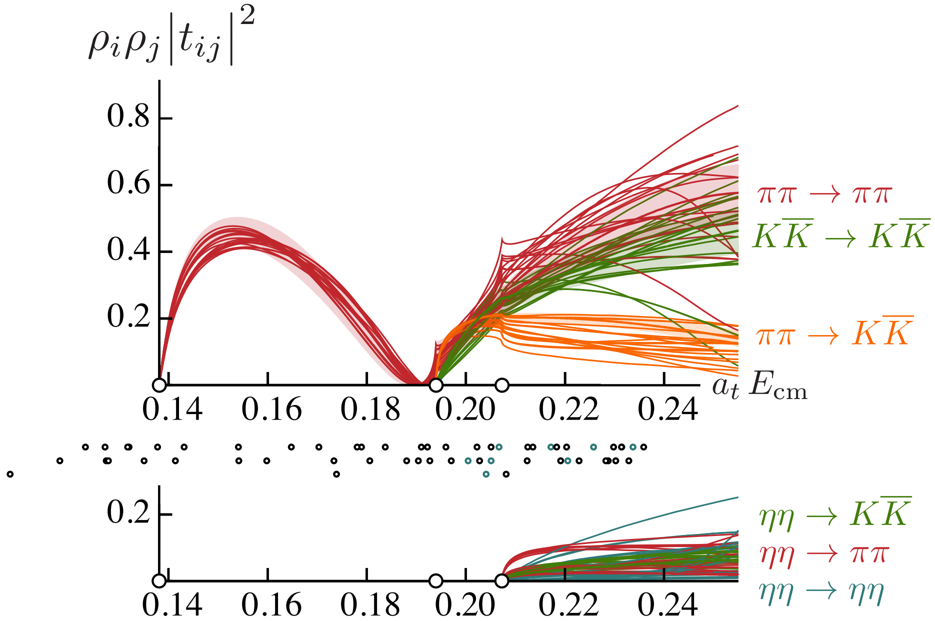
$$\mathbf{K}^{-1}(s) = \begin{pmatrix} a + b\,s & c + d\,s & e \\ c + d\,s & f & g \\ e & g & h \end{pmatrix}$$

with Chew-Mandelstam phase-space

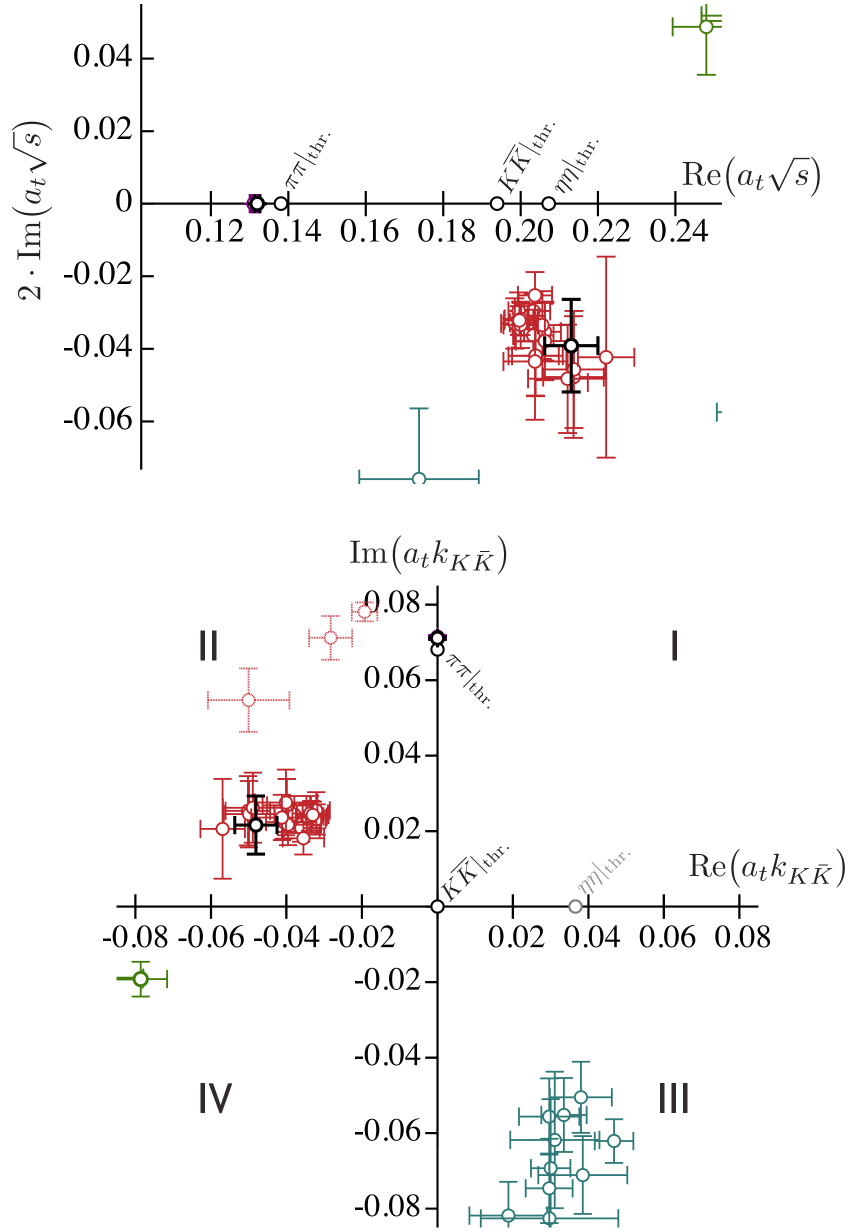
$\pi\pi$, $K\overline{K}$, $\eta\eta$ scattering with m_π~391 MeV

PRD 97 054513 (2018)

S-wave amplitudes



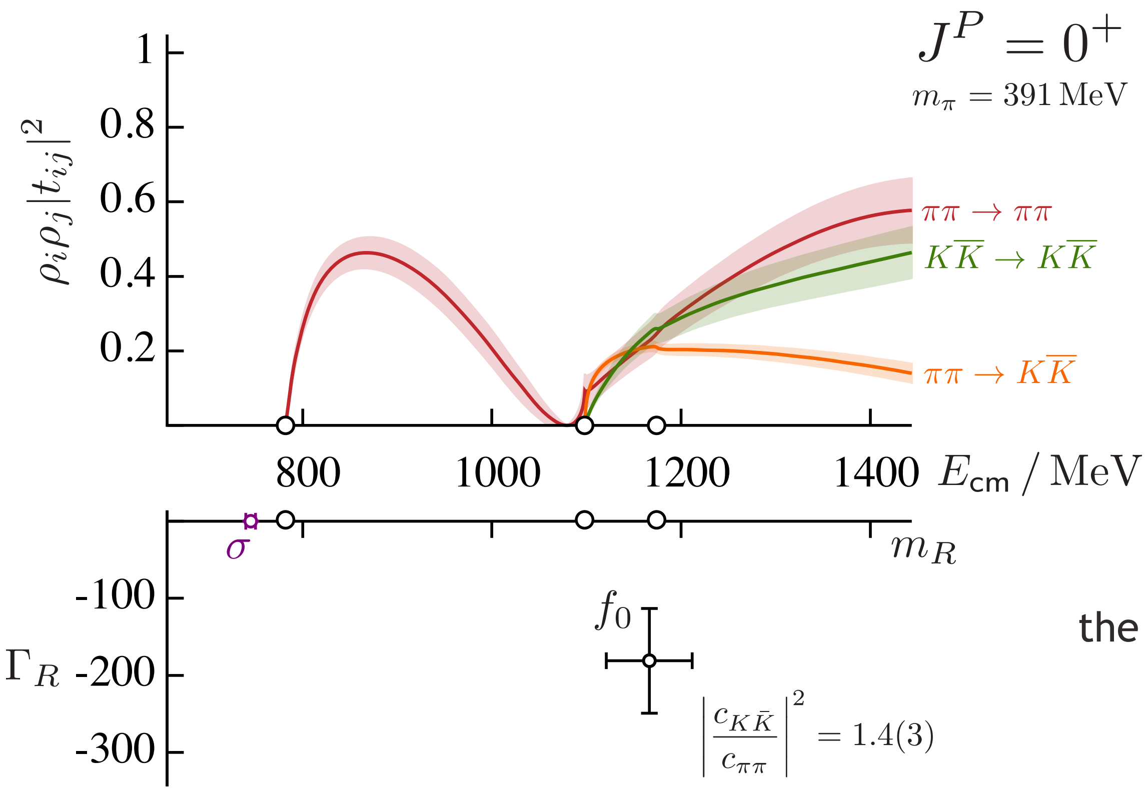
pole singularities



$\pi\pi$, $K\overline{K}$, $\eta\eta$ scattering with m_π~391 MeV

summary, including spread over parameterizations in pole uncertainty

S-wave amplitudes & poles



the f_0("980") ?

$\boxed{f_0(980)}$ $\qquad I^G(J^{PC}) = 0^+(0^{++})$

See the review on "Scalar Mesons below 2 GeV."

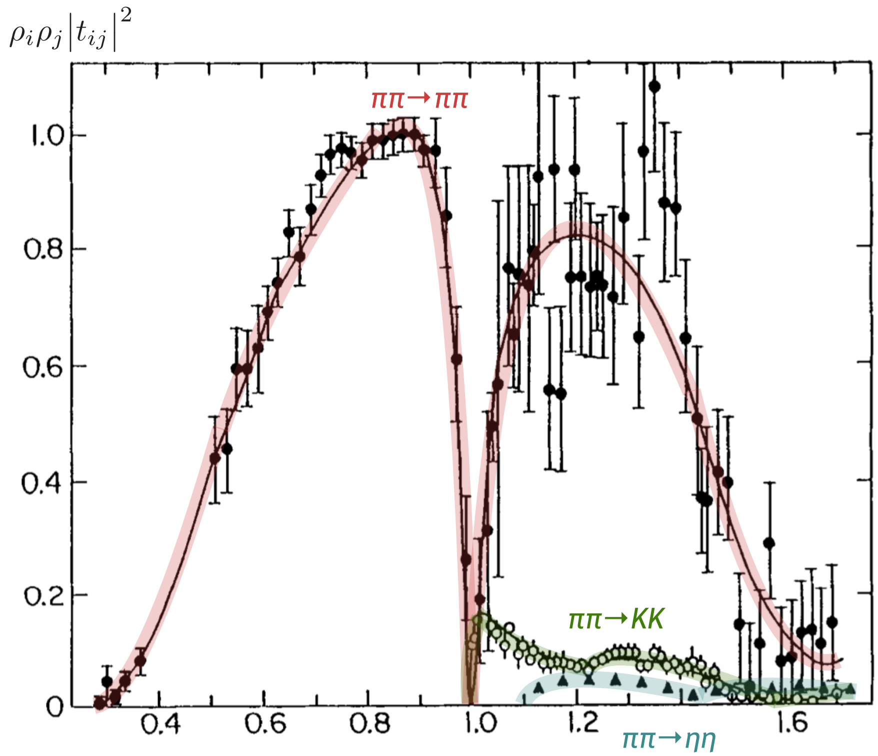
Mass $m = 990 \pm 20$ MeV

Full width $\Gamma = 10$ to 100 MeV

f_0(980) DECAY MODES	Fraction (Γ_i/Γ)	p (MeV/c)
$\pi\pi$	seen	476
$K\overline{K}$	seen	36
$\gamma\gamma$	seen	495

$f_0(980)$ as a dip in elastic scattering

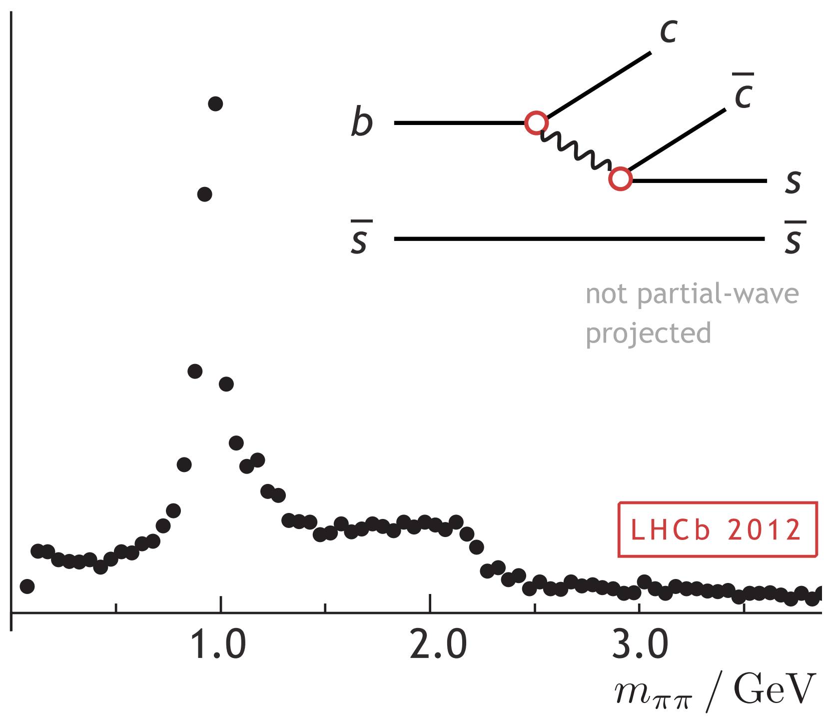
X



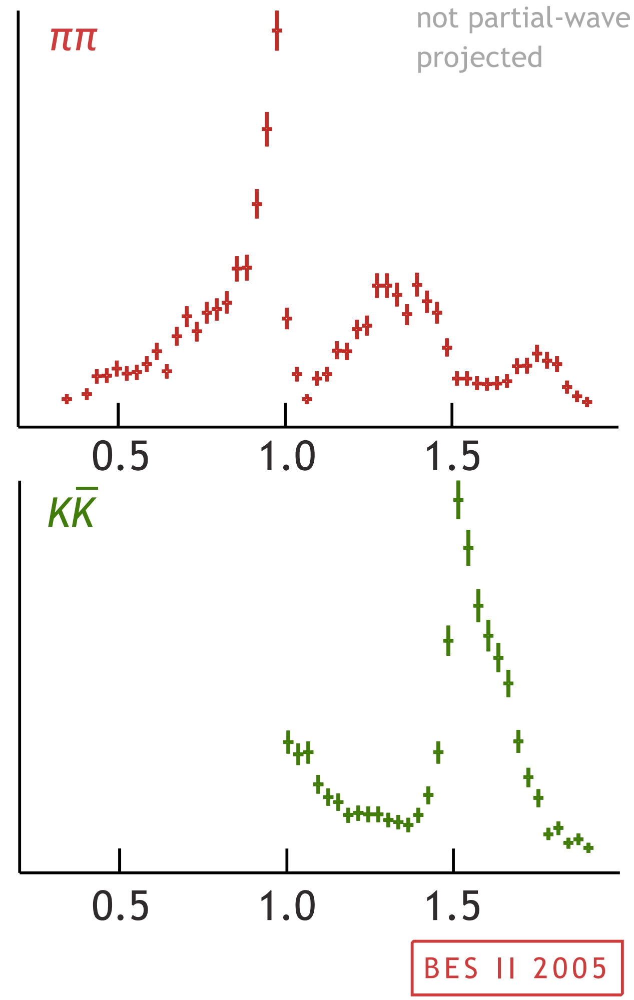
$f_0(980)$ as a peak in “$s\bar{s}$” production

x

$\bar{B}^0_s \to J/\psi\ \pi^+\pi^-$



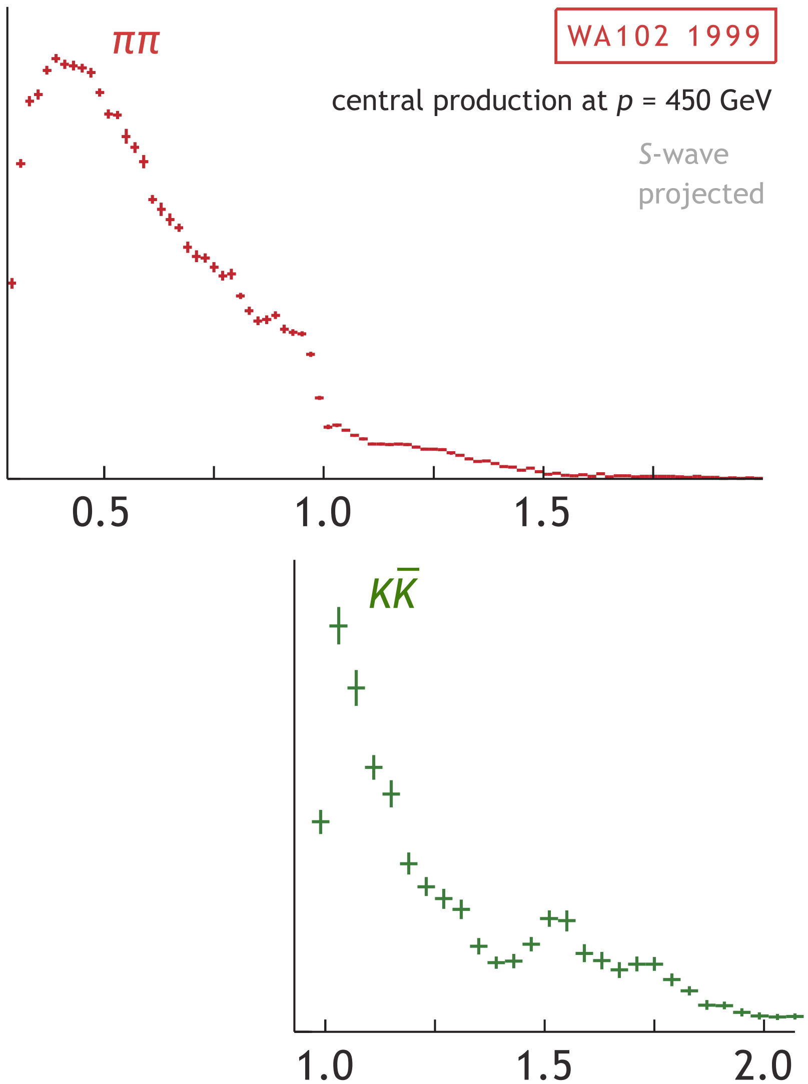
$J/\psi \to \varphi(\pi\pi, K\bar{K})$



note the rapid turn-on of $K\bar{K}$ at threshold

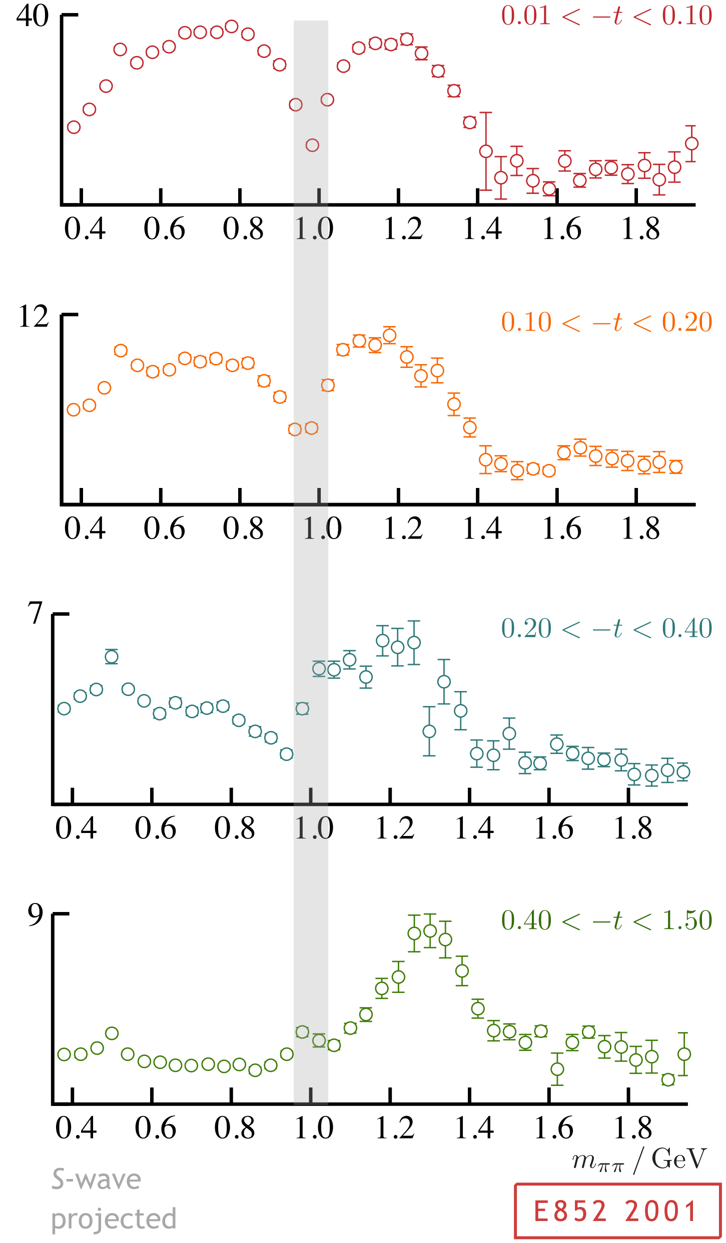
$f_0(980)$ as ?

$pp \to p(\pi\pi, K\bar{K})p$

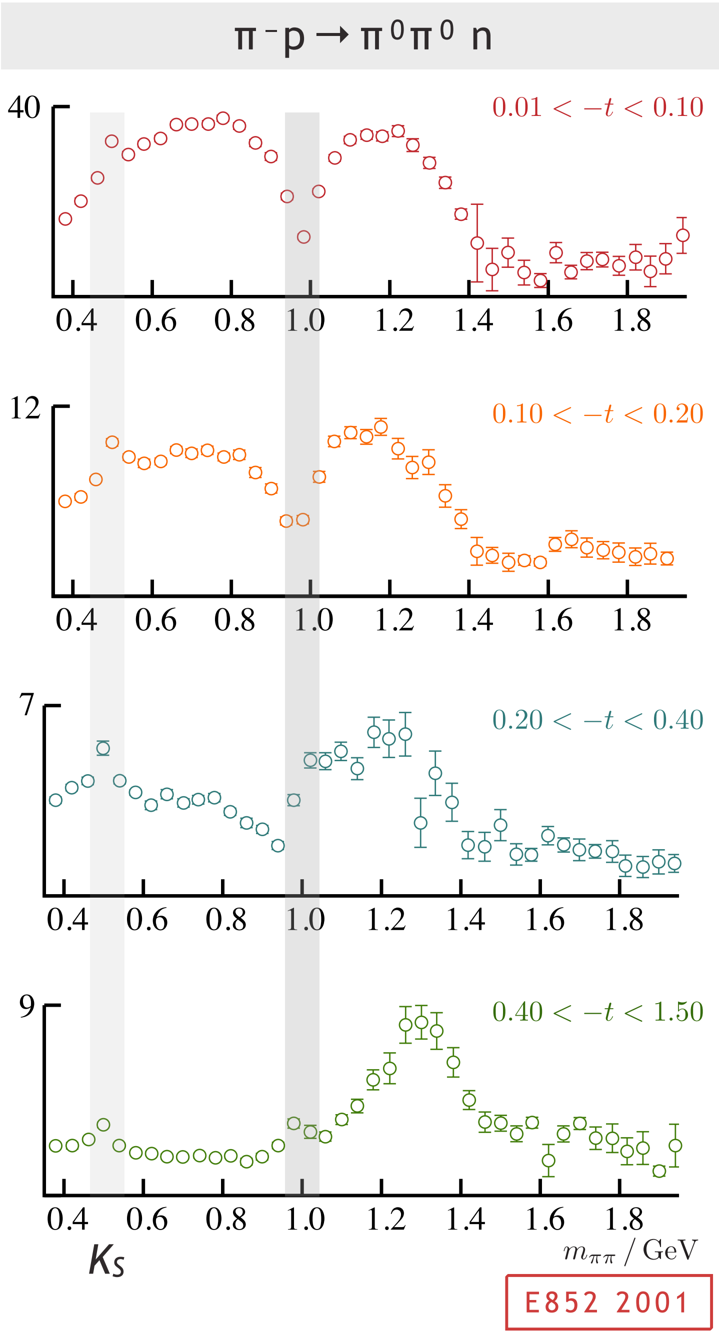


$f_0(980)$ as a shoulder on a large σ ‘background’

$\pi^- p \to \pi^0\pi^0\ n$



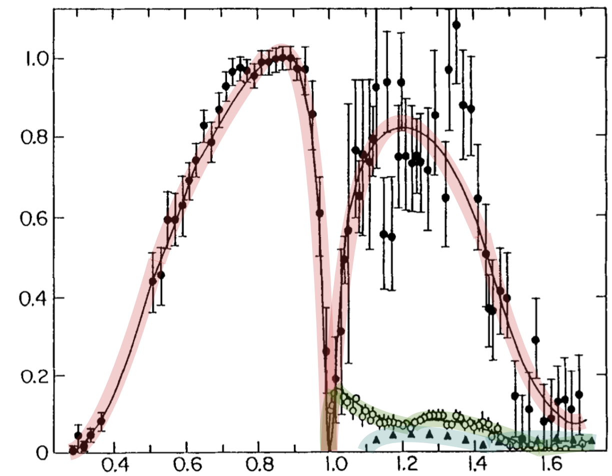
S—wave $\pi\pi$ production



dominated by π exchange
— looks like the the 1970s
elastic phase-shift data

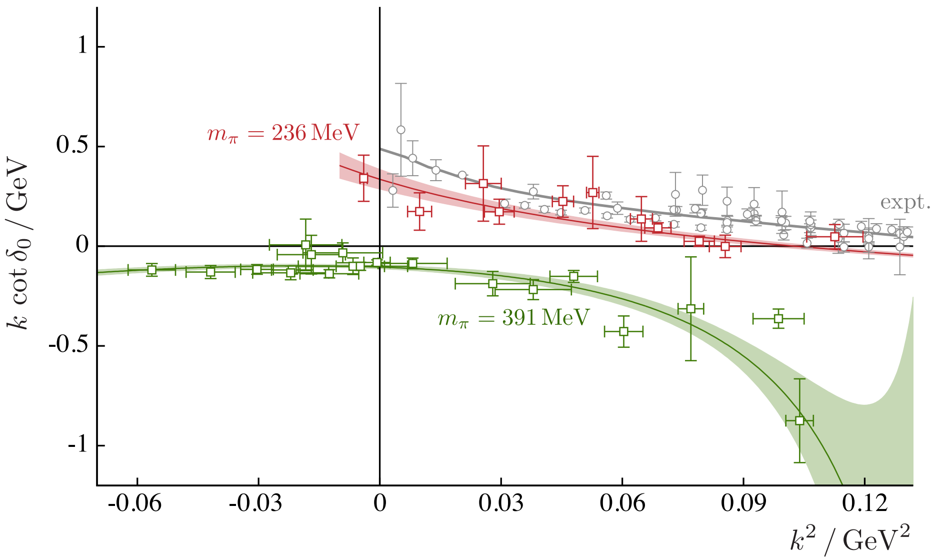
other (non-π) exchanges
becoming significant,
$f_0(980)$ dip less pronounced

σ no longer large,
$f_0(980)$ starting to be a peak ?



singularity structure from lattice calculations — elastic

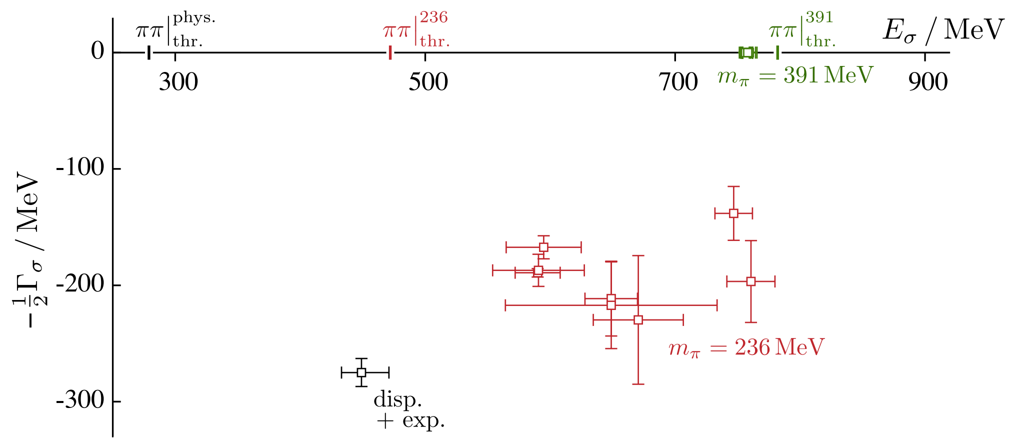
$\pi\pi$ isospin=0

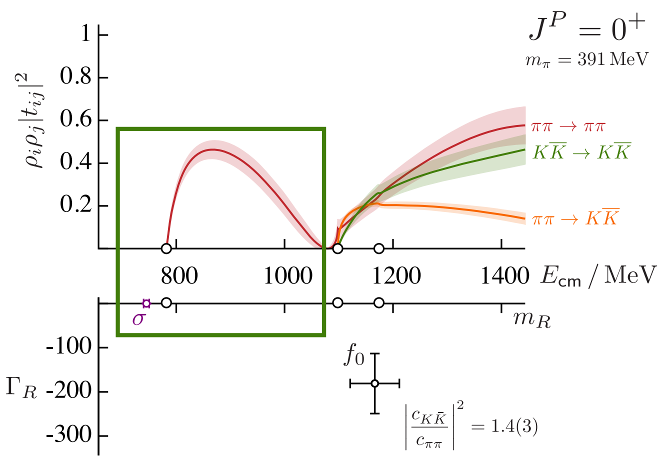


PRL118 022002 (2017)

m_π~391 MeV — a bound-state pole

m_π~236 MeV — a resonance pole

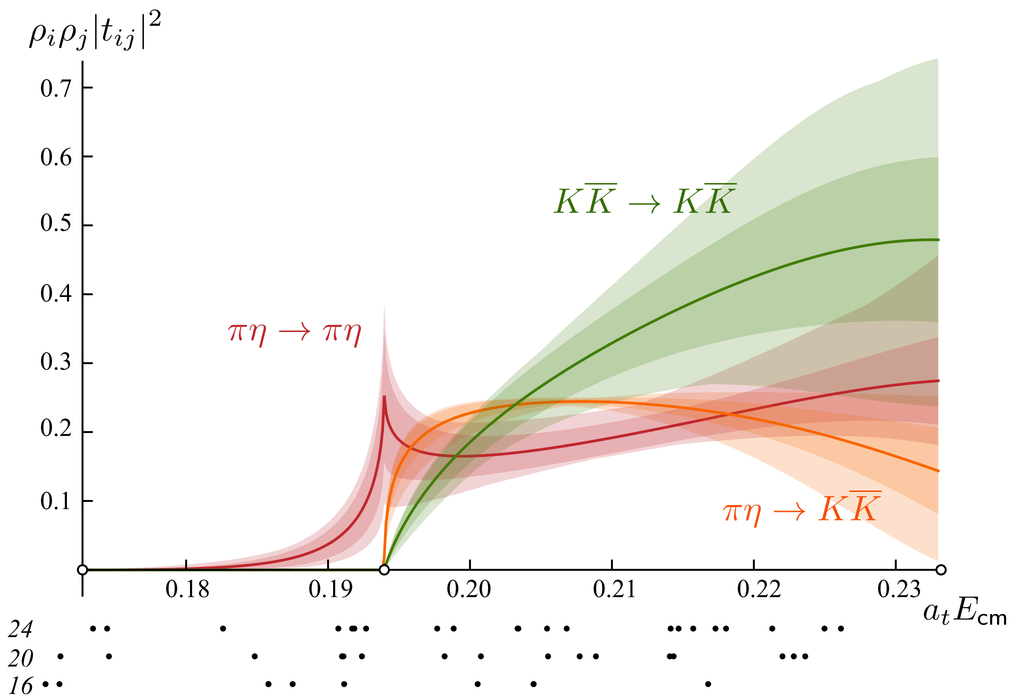




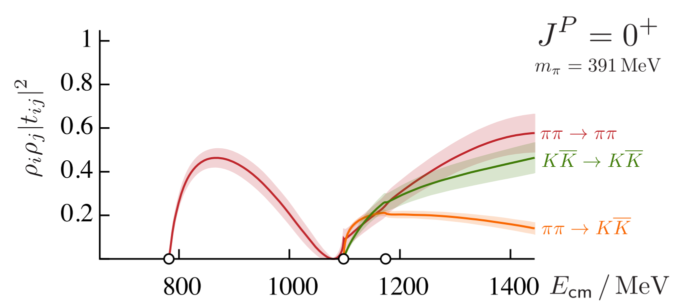
$\pi\eta$, $K\overline{K}$ scattering with m_π~391 MeV

similar calculation in **isospin=1**, ***G*-parity negative** channel

S-wave amplitudes



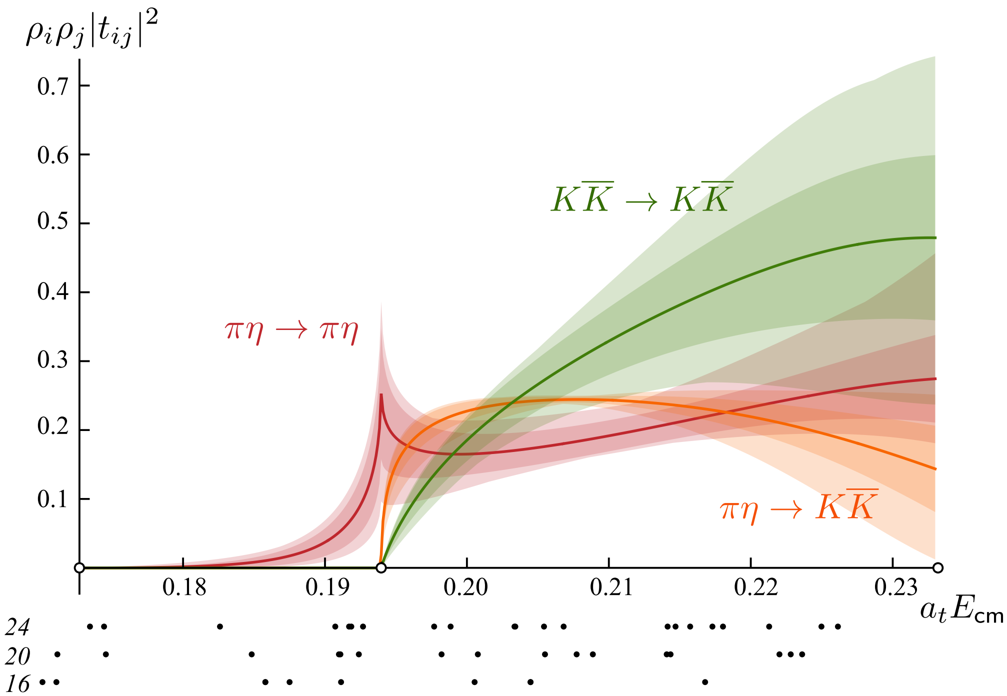
looks very different to isospin=0 case shown before



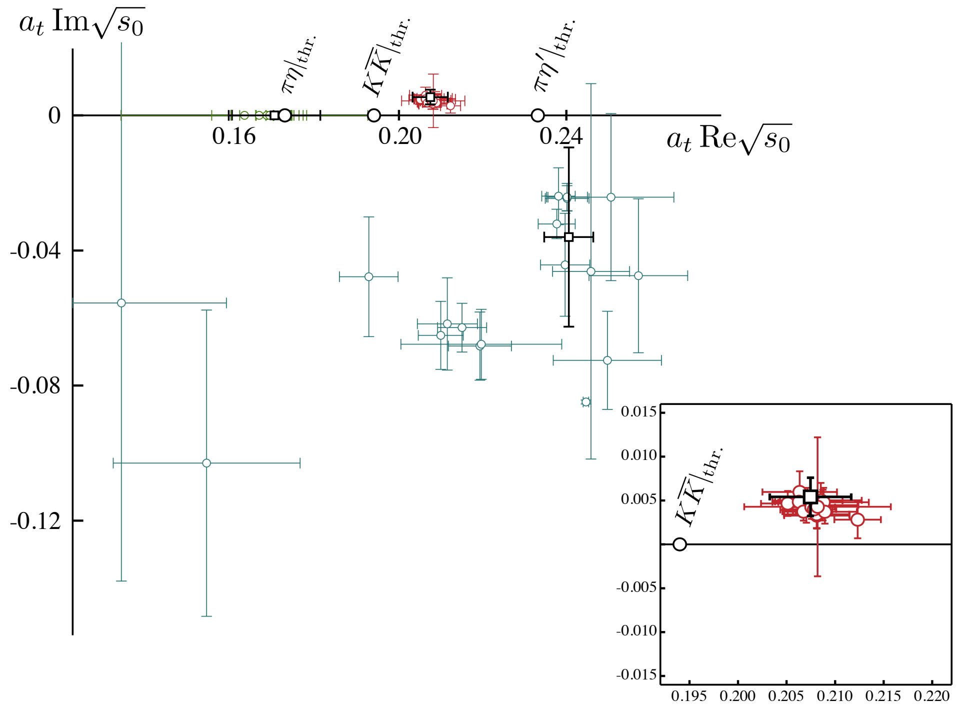
$\pi\eta$, $K\bar{K}$ scattering with m_π~391 MeV

PRD 93 094506 (2016)

S-wave amplitudes

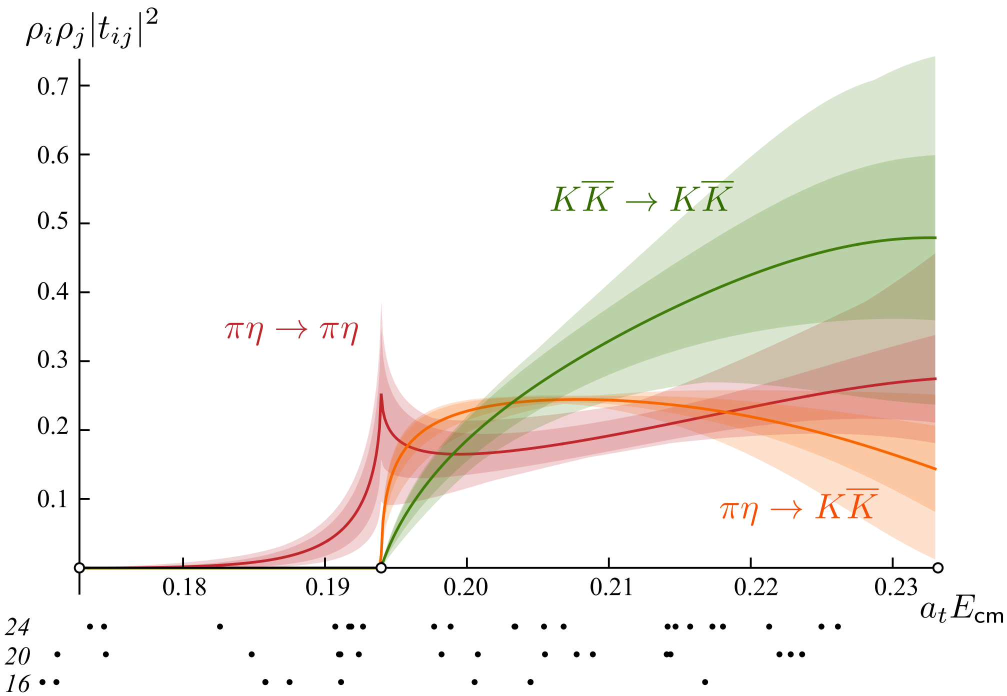


pole singularities

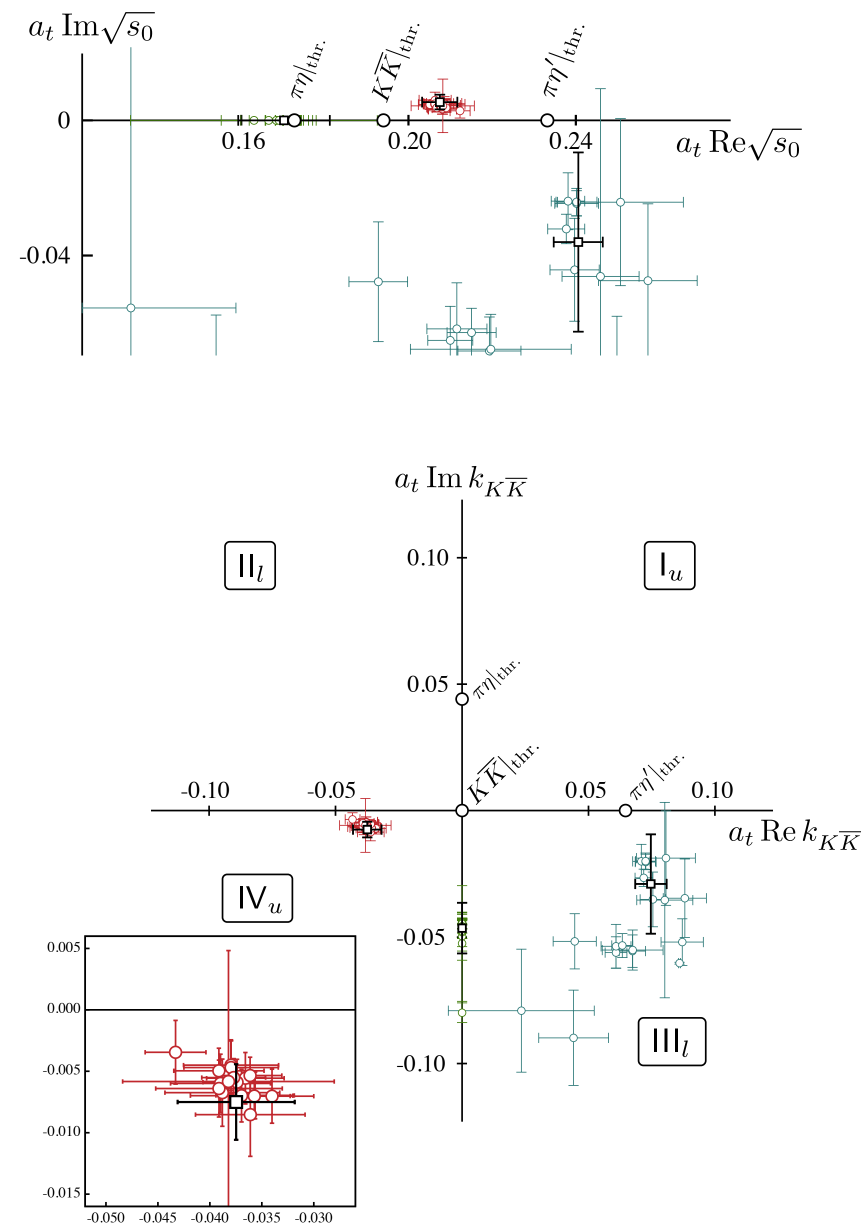


$\pi\eta$, $K\bar{K}$ scattering with m_π~391 MeV

S-wave amplitudes



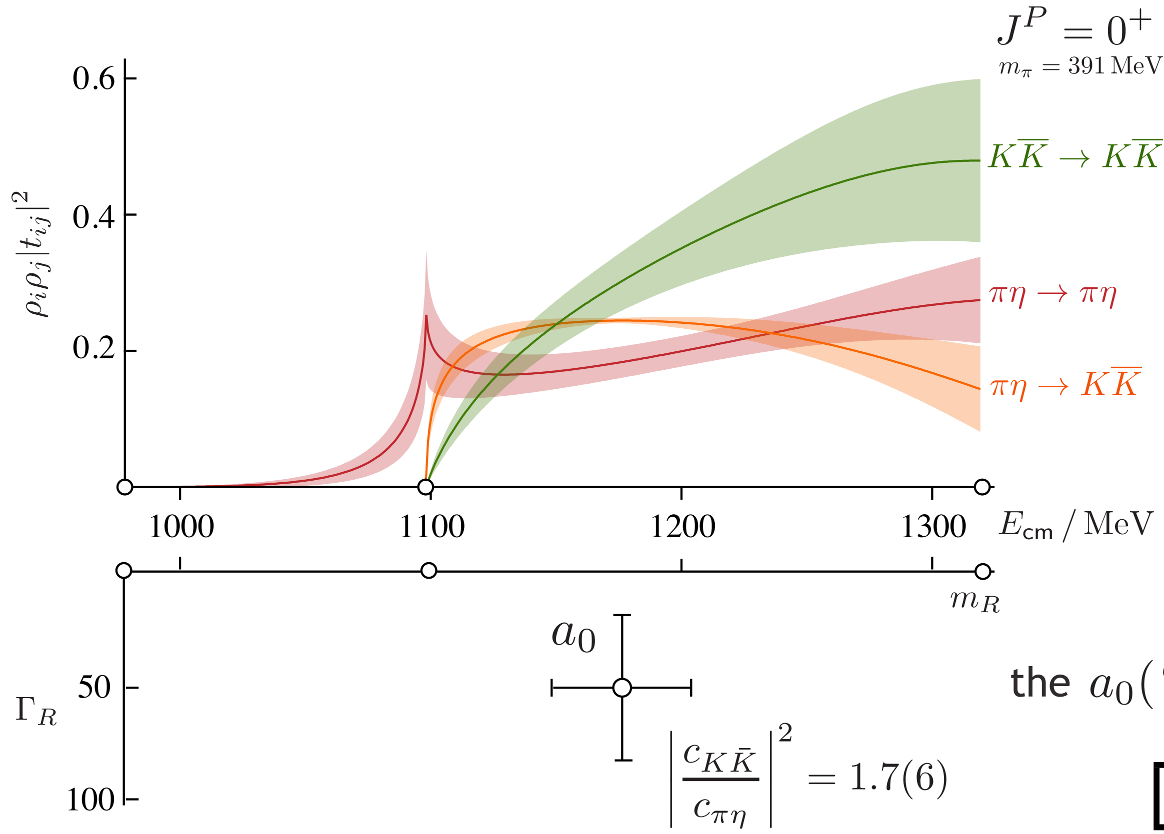
pole singularities



$\pi\eta$, $K\overline{K}$ scattering with m_π~391 MeV

PRD 93 094506 (2016)

S-wave amplitudes & poles



the a_0("980")?

a_0(980) $I^G(J^{PC}) = 1^-(0^{++})$

See the review on "Scalar Mesons below 2 GeV."

Mass $m = 980 \pm 20$ MeV

Full width $\Gamma = 50$ to 100 MeV

a_0(980) DECAY MODES	Fraction (Γ_i/Γ)	p (MeV/c)
$\eta\pi$	seen	319
$K\overline{K}$	seen	†
$\rho\pi$	not seen	137
$\gamma\gamma$	seen	490

lattice f_0, a_0 similarities ?

masses similar

$$m_R(f_0) = 1166(45)\,\mathrm{MeV},$$
$$m_R(a_0) = 1177(27)\,\mathrm{MeV},$$

widths a little different

$$\Gamma_R(f_0) = 181(68)\,\mathrm{MeV},$$
$$\Gamma_R(a_0) = 49(33)\,\mathrm{MeV}.$$

but channel couplings quite similar ?

$$\left|c(a_0 \to K\overline{K})\right| \approx \left|c(f_0 \to K\overline{K})\right| \quad \sim 850\,\mathrm{MeV}$$
$$\left|c(a_0 \to \pi\eta)\right| \approx \left|c(f_0 \to \pi\pi)\right| \quad \sim 700\,\mathrm{MeV}.$$

main difference is the **larger phase-space for $\pi\pi$ compared to $\pi\eta$**

can explore the effect using the simple Flatté amplitude

Flatté denominator $\quad D(s) = m_0^2 - s - i g_1^2\, \rho_1(s) - i g_2^2\, \rho_2(s)$

has zeros at

$$\sqrt{s_0} \approx m_0 \pm \frac{i}{2}\frac{g_2^2\,\rho_2}{m_0}\left[\left(\frac{g_1}{g_2}\right)^2 \frac{\rho_1}{\rho_2} - 1\right] \quad \text{on sheet II, if} \quad \left(\frac{g_1}{g_2}\right)^2 \frac{\rho_1}{\rho_2} > 1, \text{ or,}$$

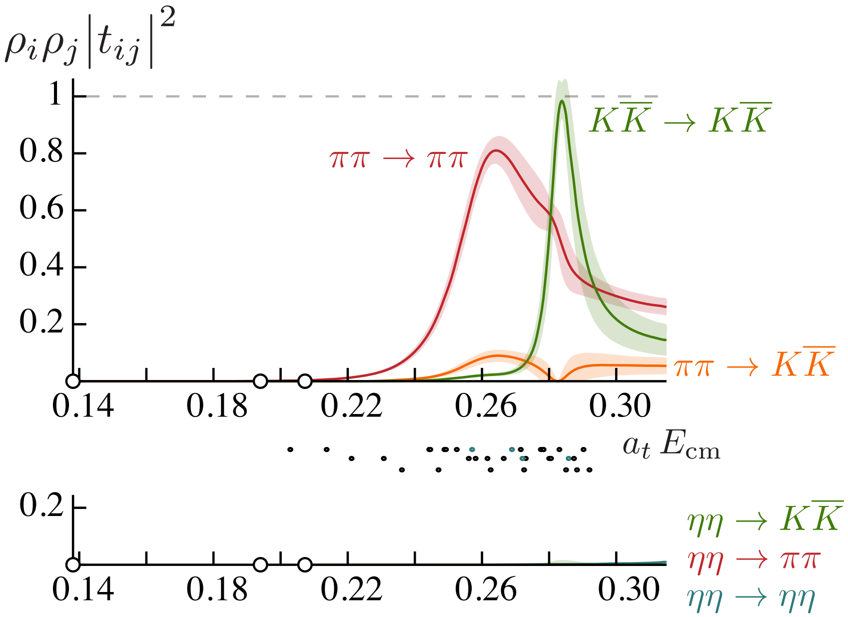
$$\sqrt{s_0} \approx m_0 \pm \frac{i}{2}\frac{g_2^2\,\rho_2}{m_0}\left[1 - \left(\frac{g_1}{g_2}\right)^2 \frac{\rho_1}{\rho_2}\right] \quad \text{on sheet IV, if} \quad \left(\frac{g_1}{g_2}\right)^2 \frac{\rho_1}{\rho_2} < 1, \text{ and,}$$

$$\sqrt{s_0} \approx m_0 \pm \frac{i}{2}\frac{g_2^2\,\rho_2}{m_0}\left[1 + \left(\frac{g_1}{g_2}\right)^2 \frac{\rho_1}{\rho_2}\right] \quad \text{on sheet III, in all cases,}$$

$\pi\pi$, $K\overline{K}$, $\eta\eta$ scattering with m_π~391 MeV

PRD 97 054513 (2018)

D-wave amplitudes

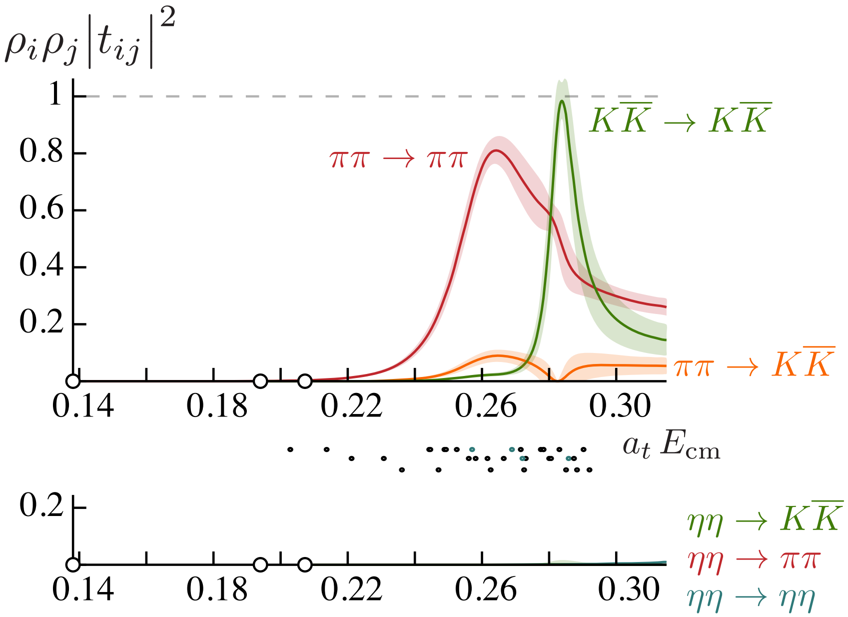


bumps are in the three-channel region ⇒ **8 sheets !**

won't burden you with the sheet details here ...

$\pi\pi$, $K\bar{K}$, $\eta\eta$ scattering with m_π~391 MeV

D-wave amplitudes



bumps are in the three channel region ⇒ **8 sheets !**

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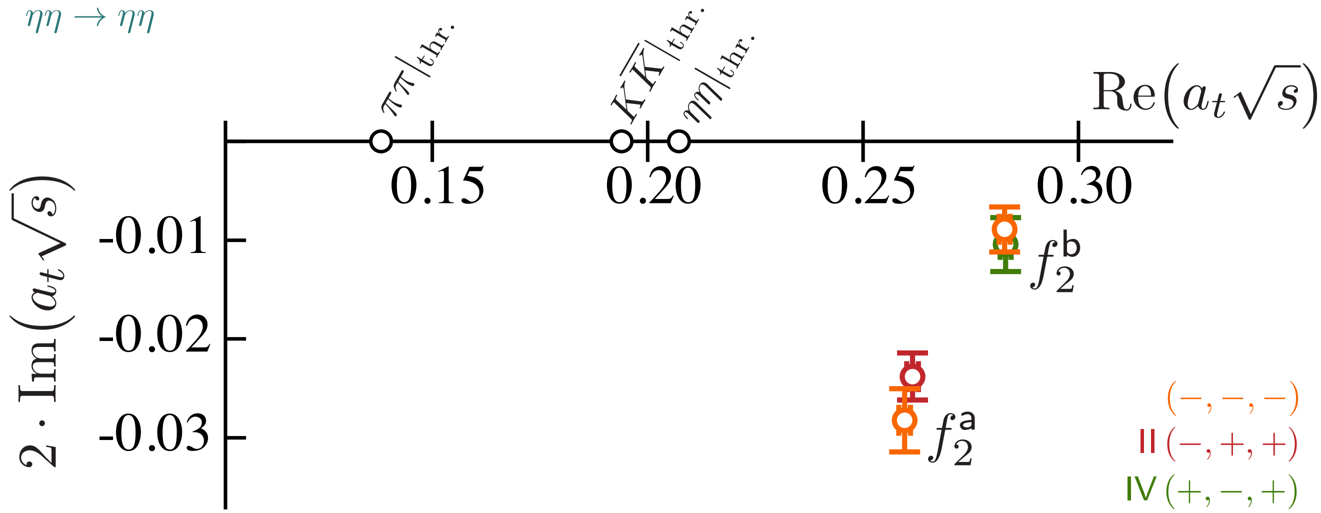
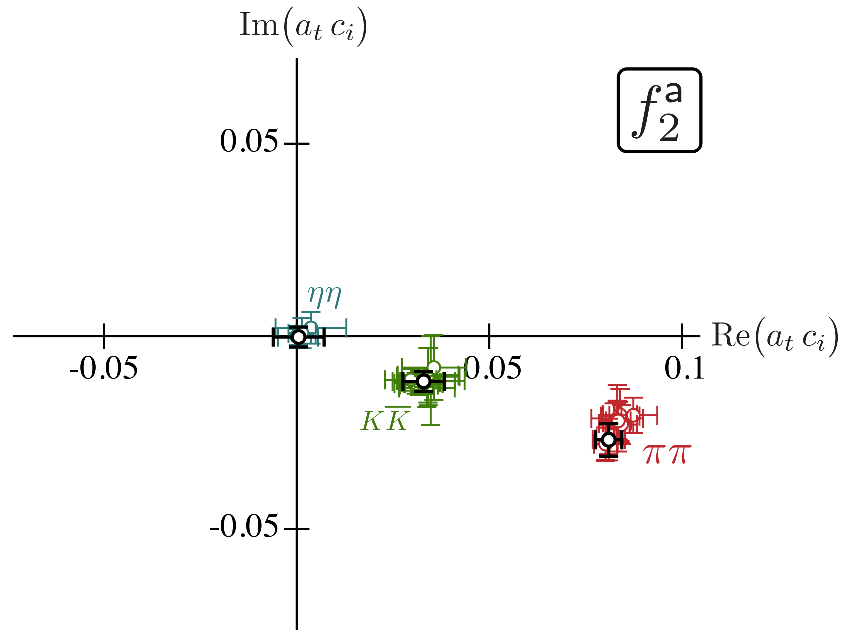
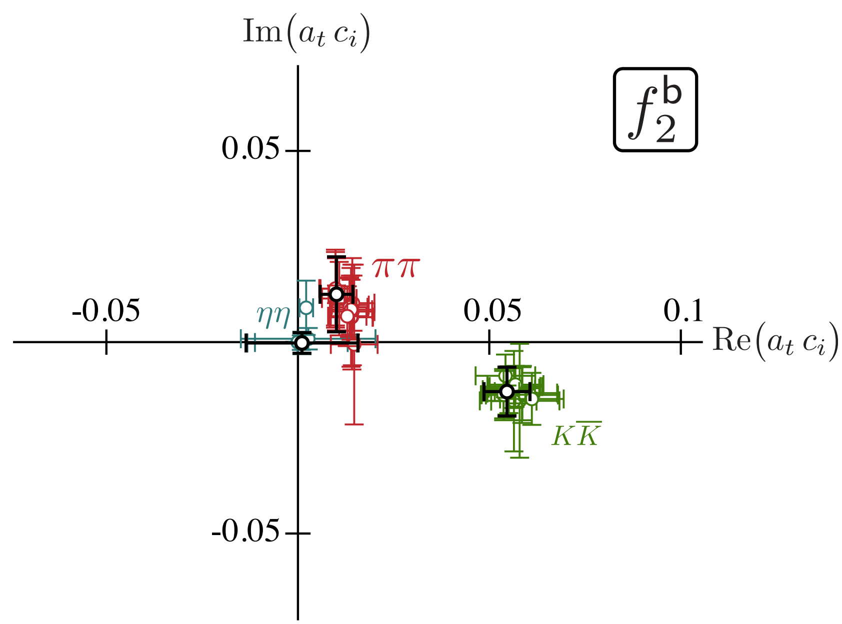


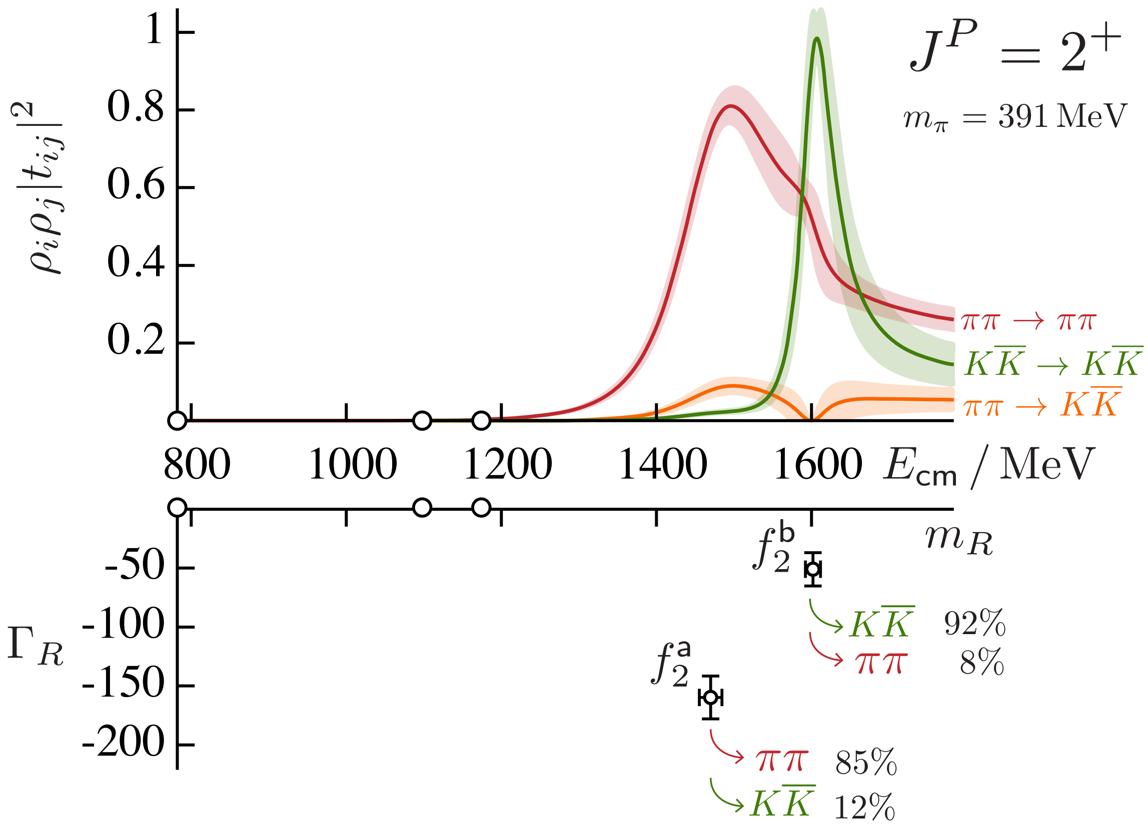
(−,−,−) is 'closest' sheet to physical scattering above all three thresholds


Im$(a_t\, c_i)$
Re$(a_t\, c_i)$
f_2^{a}
0.05
-0.05
-0.05
0.05
0.1
$\eta\eta$
$K\overline{K}$
$\pi\pi$


Im$(a_t\, c_i)$
Re$(a_t\, c_i)$
f_2^{b}
0.05
-0.05
-0.05
0.05
0.1
$\pi\pi$
$\eta\eta$
$K\overline{K}$

$\pi\pi$, $K\bar{K}$, $\eta\eta$ scattering with m_π~391 MeV

D-wave amplitudes & poles



pdg summary

$f_2(1270)$ $I^G(J^{PC}) = 0^+(2^{++})$

Mass $m = 1275.5 \pm 0.8$ MeV
Full width $\Gamma = 186.7^{+2.2}_{-2.5}$ MeV (S = 1.4)

$f_2(1270)$ DECAY MODES	Fraction (Γ_i/Γ)	Scale factor/ Confidence level	p (MeV/c)
$\pi\pi$	$(84.2\ ^{+2.9}_{-0.9})$ %	S=1.1	623
$\pi^+\pi^-2\pi^0$	$(7.7\ ^{+1.1}_{-3.2})$ %	S=1.2	563
$K\bar{K}$	$(4.6\ ^{+0.5}_{-0.4})$ %	S=2.7	404
$2\pi^+2\pi^-$	(2.8 ± 0.4) %	S=1.2	560
$\eta\eta$	$(4.0 \pm 0.8) \times 10^{-3}$	S=2.1	326
$4\pi^0$	$(3.0 \pm 1.0) \times 10^{-3}$		565

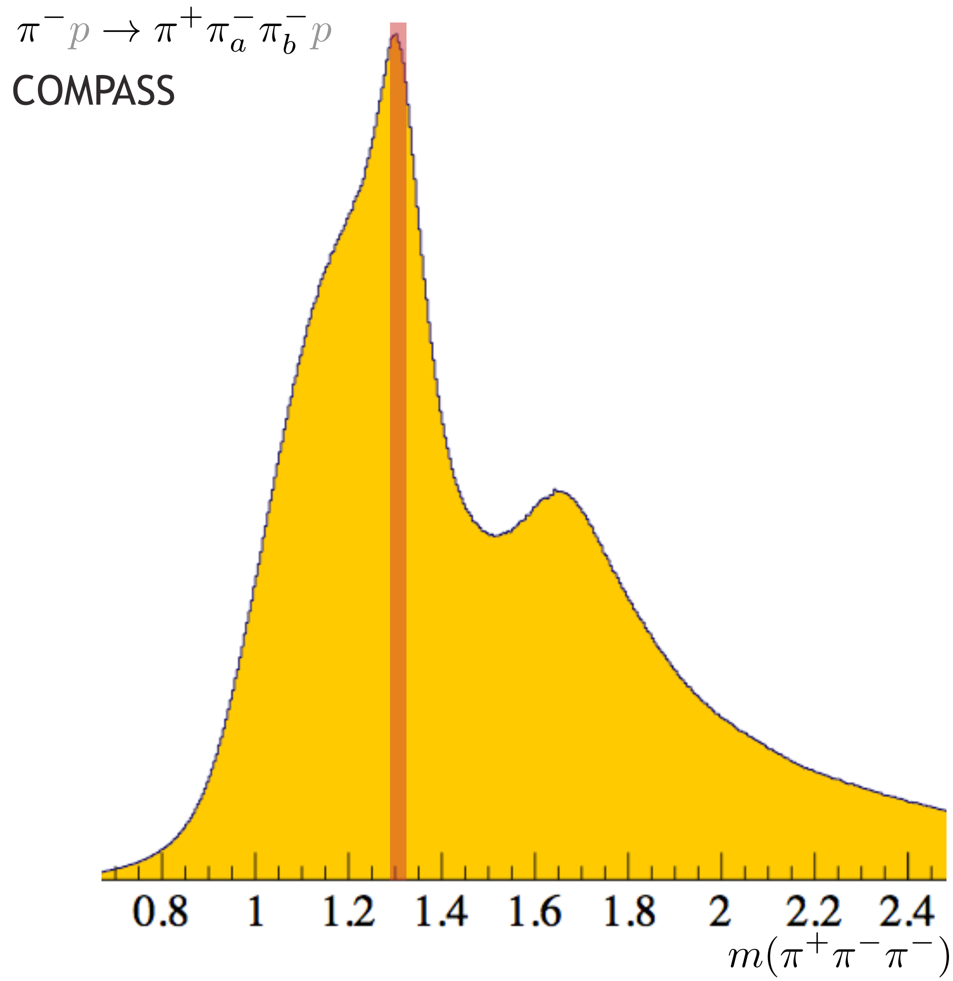
$f_2'(1525)$ $I^G(J^{PC}) = 0^+(2^{++})$

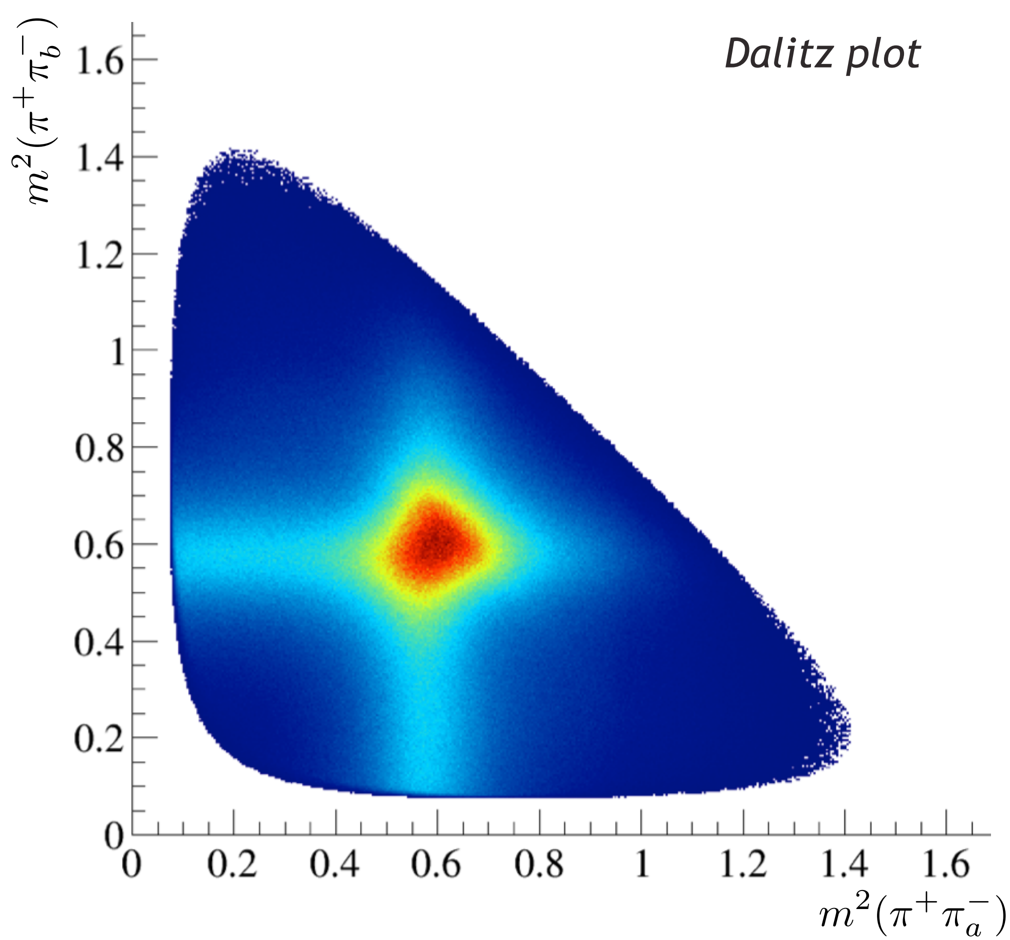
Mass $m = 1525 \pm 5$ MeV [l]
Full width $\Gamma = 73^{+6}_{-5}$ MeV [l]

$f_2'(1525)$ DECAY MODES	Fraction (Γ_i/Γ)	p (MeV/c)
$K\bar{K}$	(88.7 ± 2.2) %	581
$\eta\eta$	(10.4 ± 2.2) %	530
$\pi\pi$	$(8.2 \pm 1.5) \times 10^{-3}$	750

many-body decays tend to be dominated by **isobars**

e.g. $\pi\pi\pi$ dominated by $\pi\rho$





many-body decays tend to be dominated by isobars

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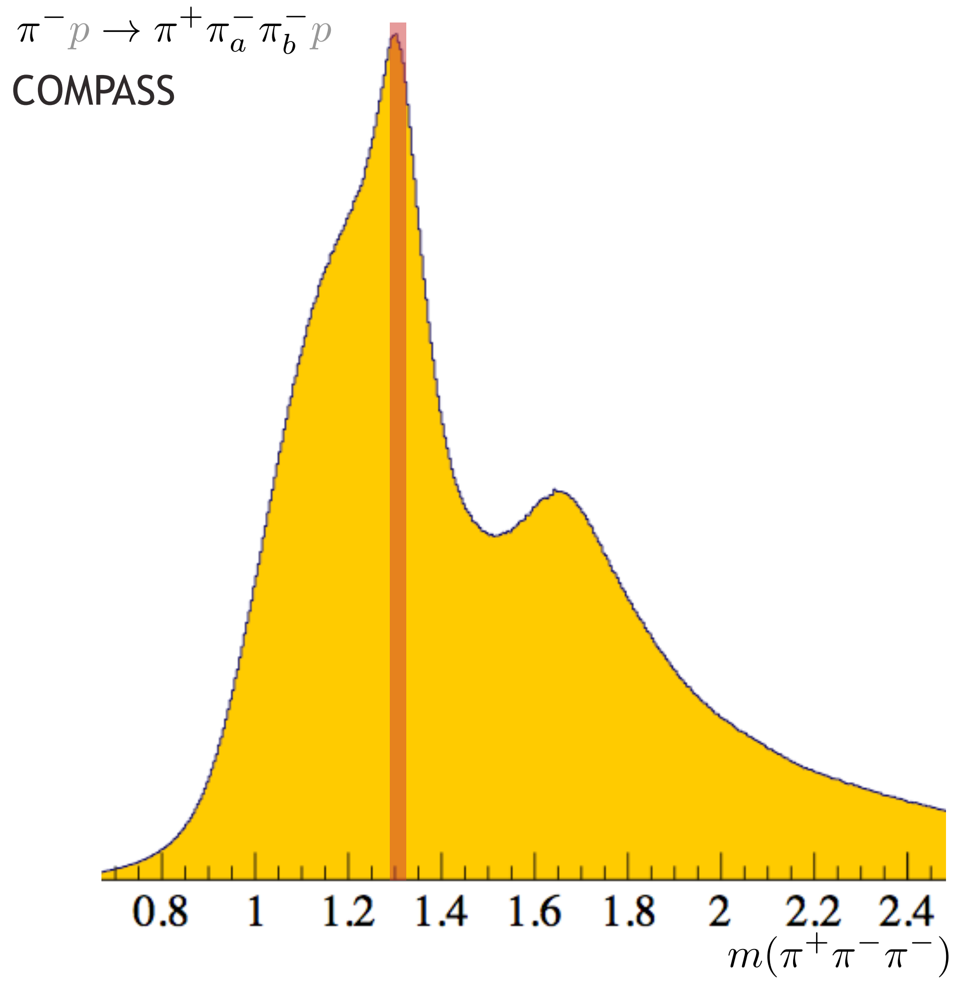
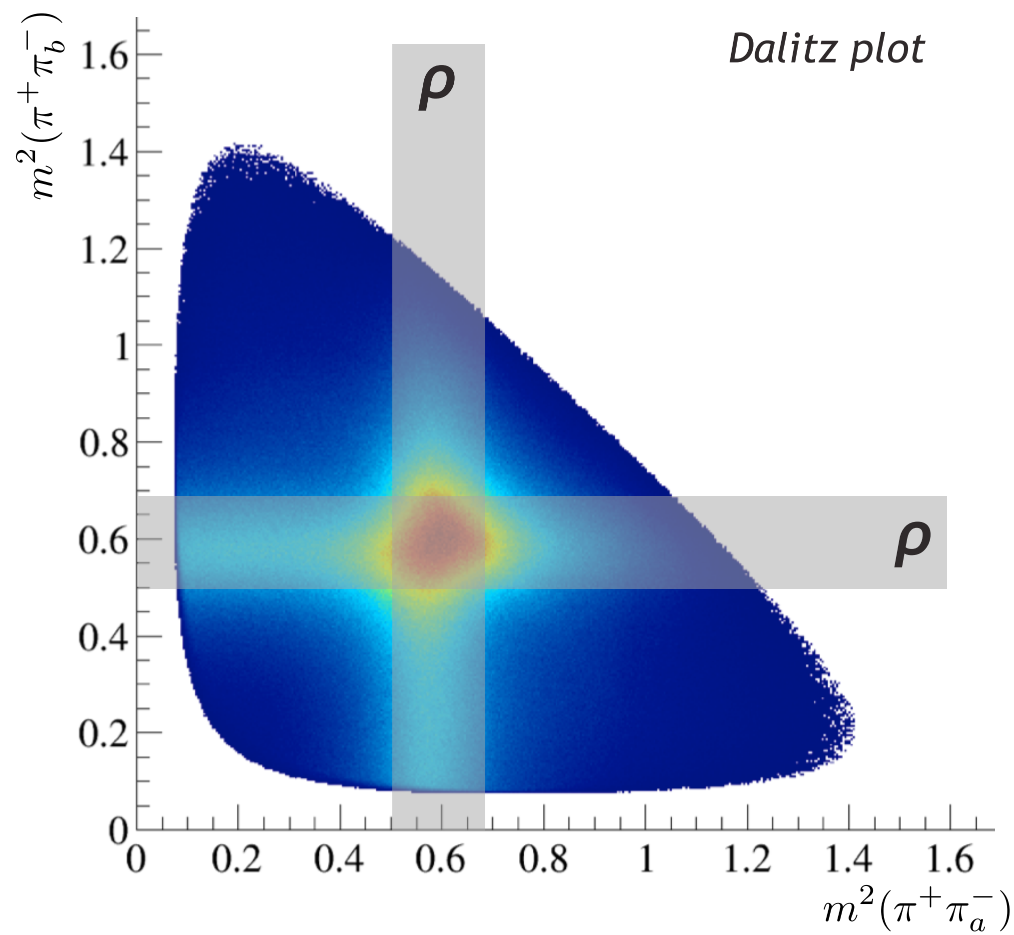
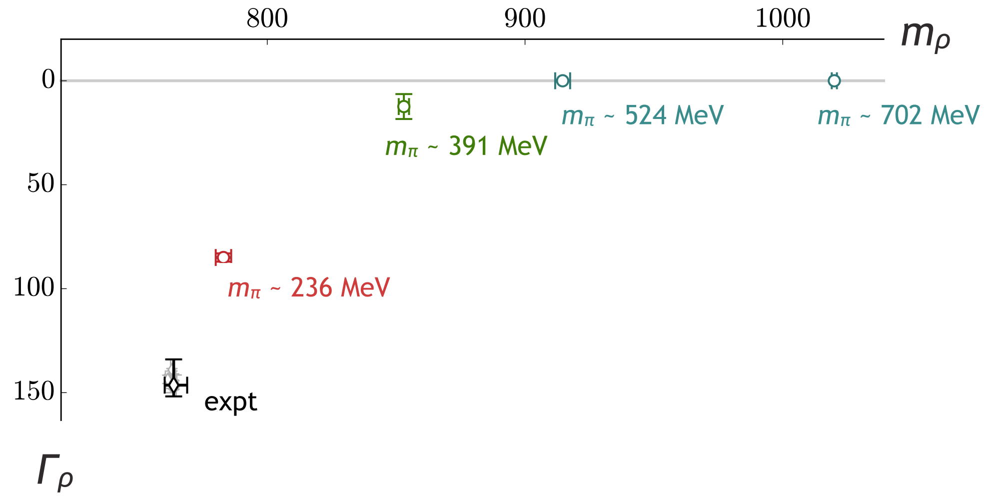




for heavier than physical light-quarks,
the ρ resonance becomes **stable**



can rigorously study vector-pseudoscalar scattering

complication: need to account for the vector **ρ spin**

helicity formalism is common experimental approach,
but **ℓS formalism** more convenient in finite-volume

$$\left|J, m\left[{}^{2S+1}\ell_J\right]\right\rangle = \sum_{m_\ell, m_S} \left\langle \ell m_\ell; S m_S \middle| J m\right\rangle \left|S, m_S\right\rangle \otimes \left|\ell, m_\ell\right\rangle$$

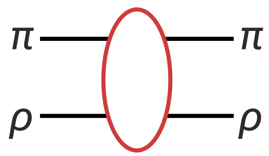
e.g. with S=1 can make J^P=1$^+$ in two ways: ${}^3S_1,\ {}^3D_1$

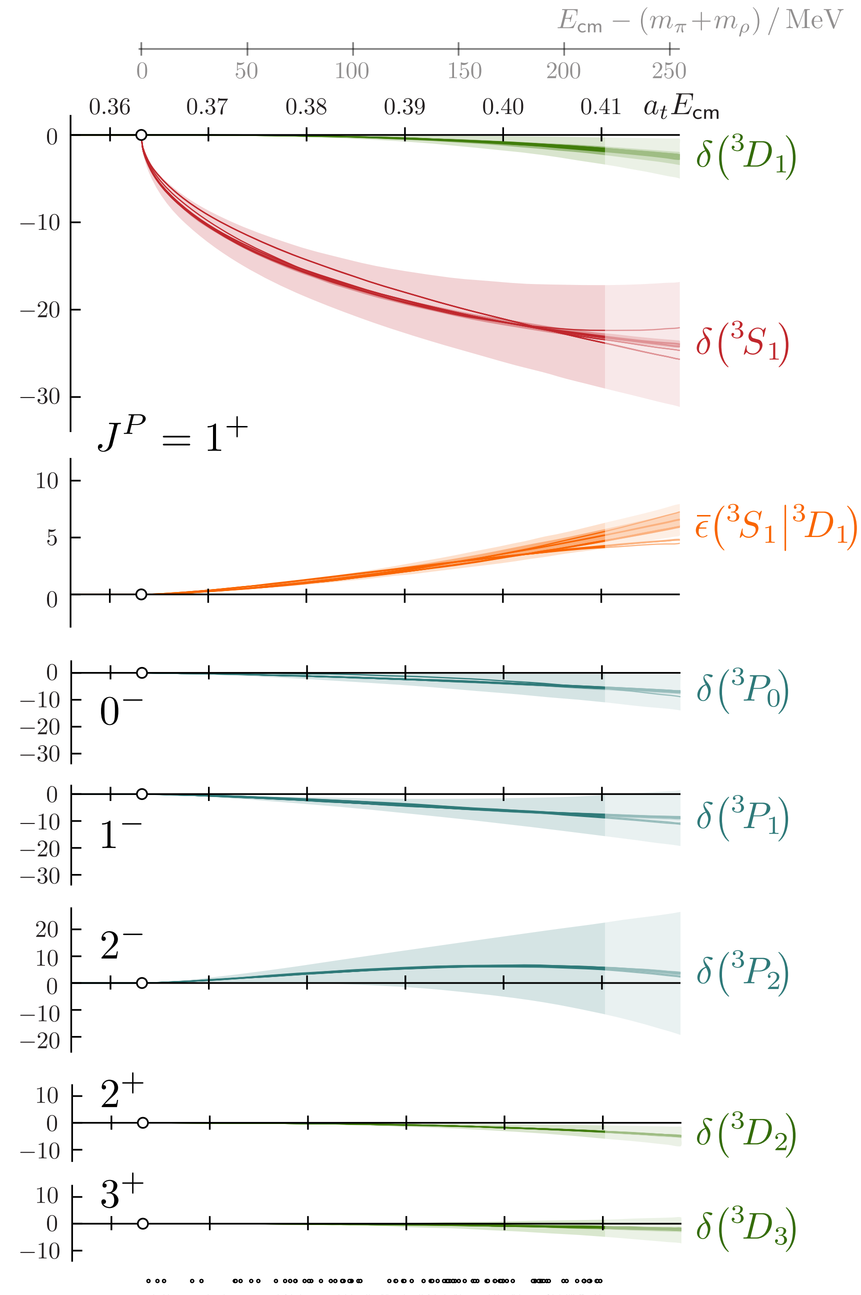
⇒ coupled partial-waves

$$\boldsymbol{t} = \begin{bmatrix} t({}^3S_1|{}^3S_1) & t({}^3S_1|{}^3D_1) \\ t({}^3S_1|{}^3D_1) & t({}^3D_1|{}^3D_1) \end{bmatrix}$$

finite-volume function basis changes too

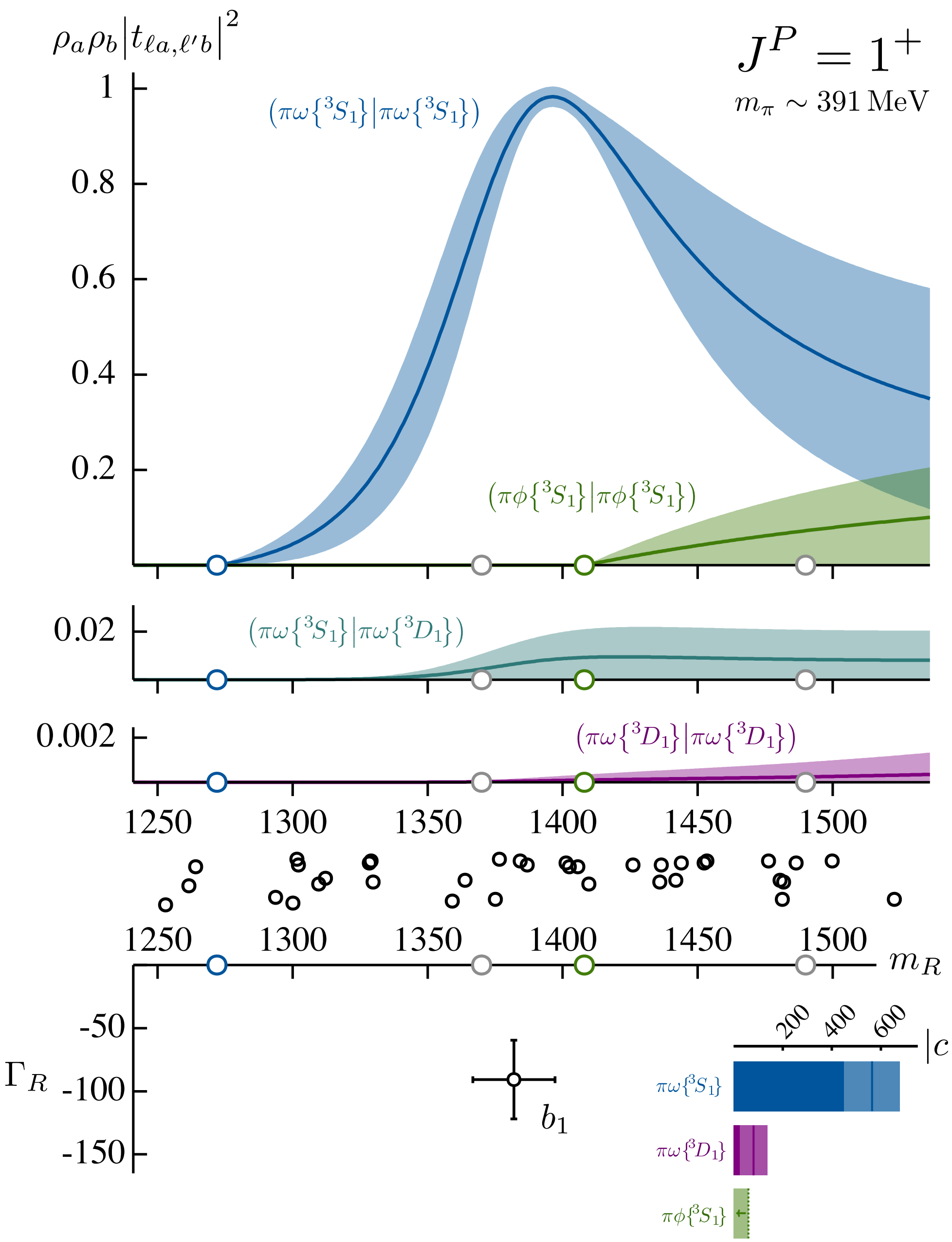
$$\overline{\mathcal{M}}_{\ell J m, \ell' J' m'} = \sum_{m_\ell, m'_\ell, m_S} \left\langle \ell m_\ell; 1 m_S \middle| J m\right\rangle \left\langle \ell' m'_\ell; 1 m_S \middle| J' m'\right\rangle \mathcal{M}_{\ell m_\ell, \ell' m'_\ell}$$


π
π
ρ
ρ


$E_{\mathsf{cm}} - (m_\pi + m_\rho)\,/\,\mathrm{MeV}$
0
50
100
150
200
250
0.36
0.37
0.38
0.39
0.40
0.41
$a_t E_{\mathsf{cm}}$
$\delta(^3D_1)$
$\delta(^3S_1)$
$J^P = 1^+$
$\bar\epsilon(^3S_1|^3D_1)$
0^-
$\delta(^3P_0)$
1^-
$\delta(^3P_1)$
2^-
$\delta(^3P_2)$
2^+
$\delta(^3D_2)$
3^+
$\delta(^3D_3)$

$\pi\omega$ isospin=1 — m_π~391 MeV

ω is stable down to quite low quark masses



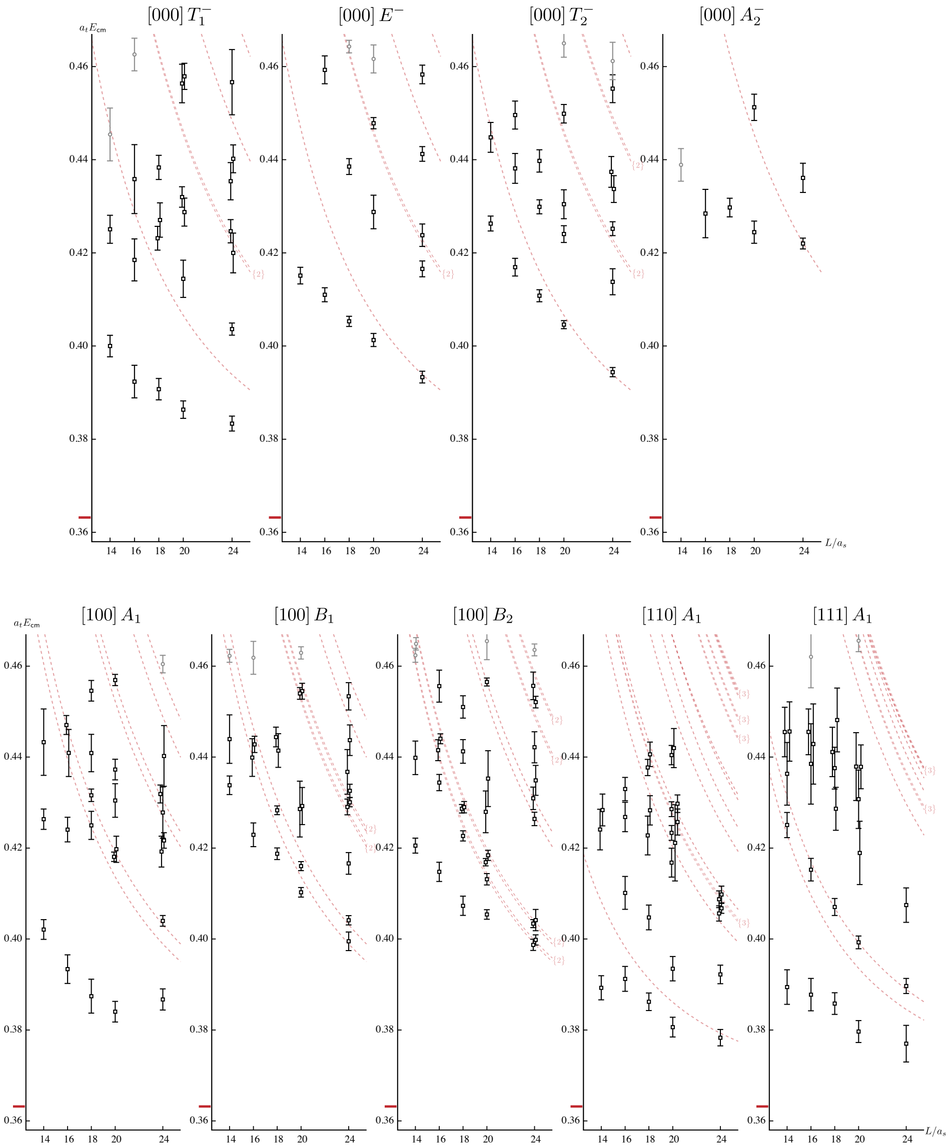
$b_1(1235)$ $I^G(J^{PC}) = 1^+(1^{+-})$

Mass $m = 1229.5 \pm 3.2$ MeV (S = 1.6)
Full width $\Gamma = 142 \pm 9$ MeV (S = 1.2)

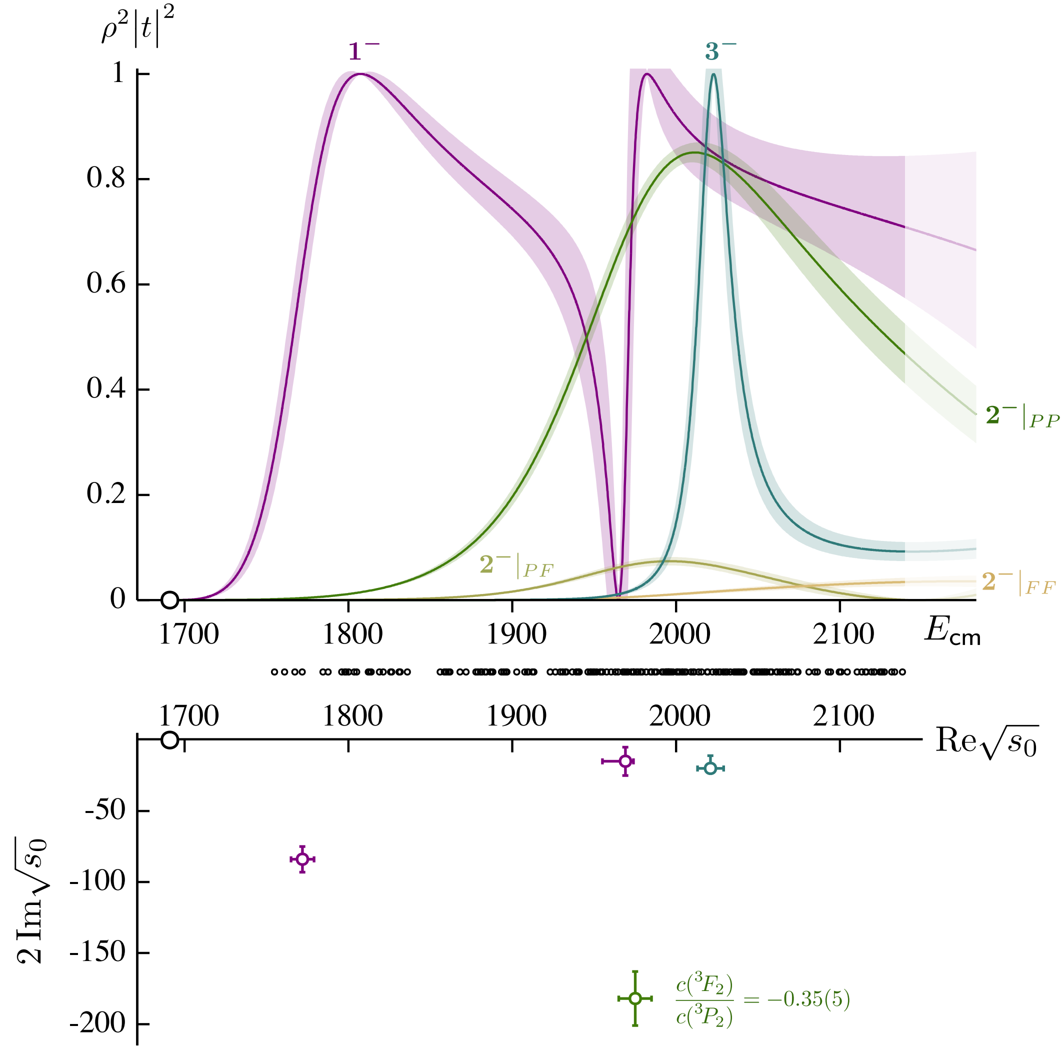
$b_1(1235)$ DECAY MODES	Fraction (Γ_i/Γ)	Confidence level	p (MeV/c)
$\omega\pi$	dominant		348
[D/S amplitude ratio = 0.277 ± 0.027]			

excited J^{--} meson resonances — m_π~700 MeV

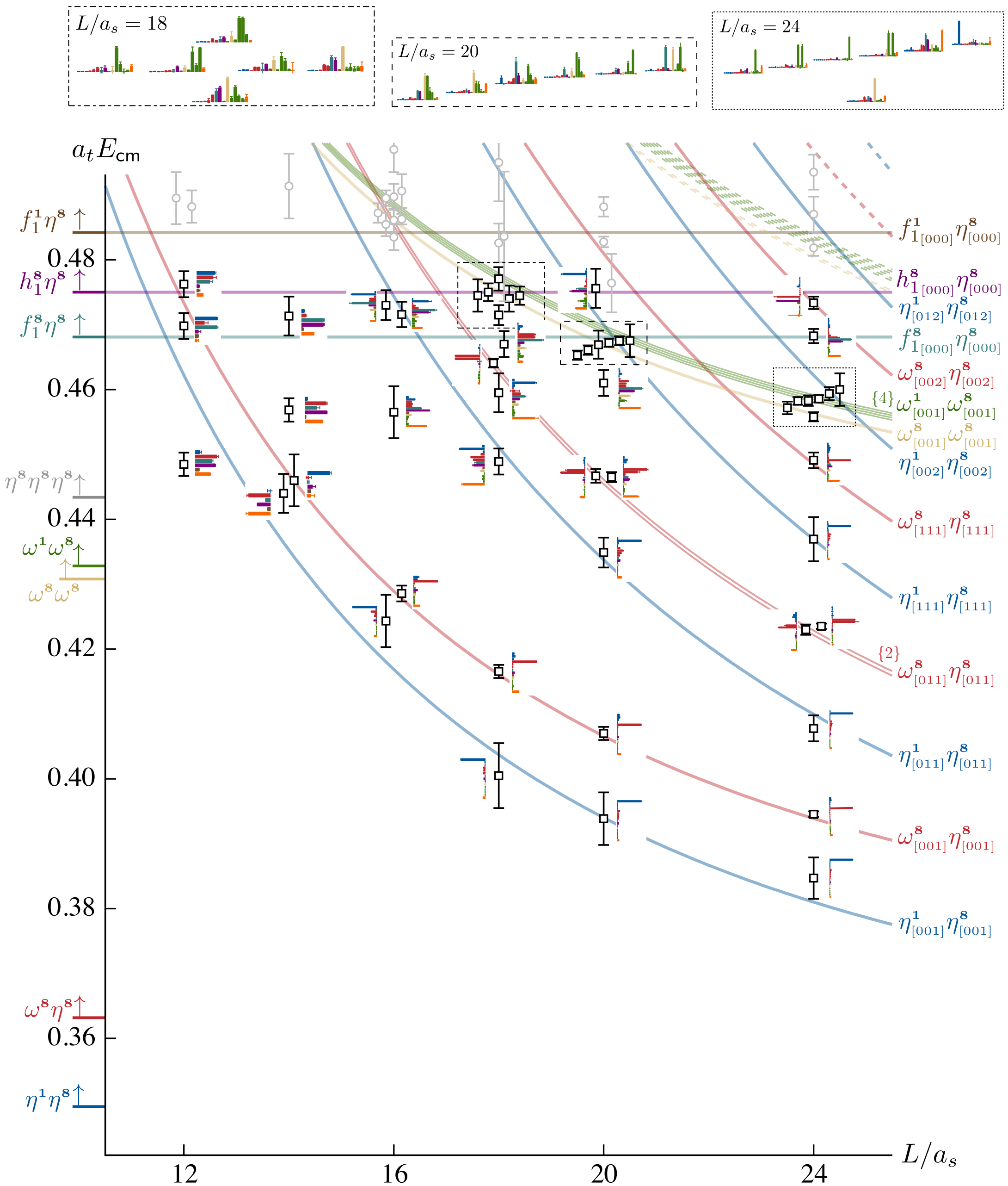
exact SU(3) flavor symmetry $\omega_J^{\mathbf{1}} \to \eta^{\mathbf{8}}\omega^{\mathbf{8}}$



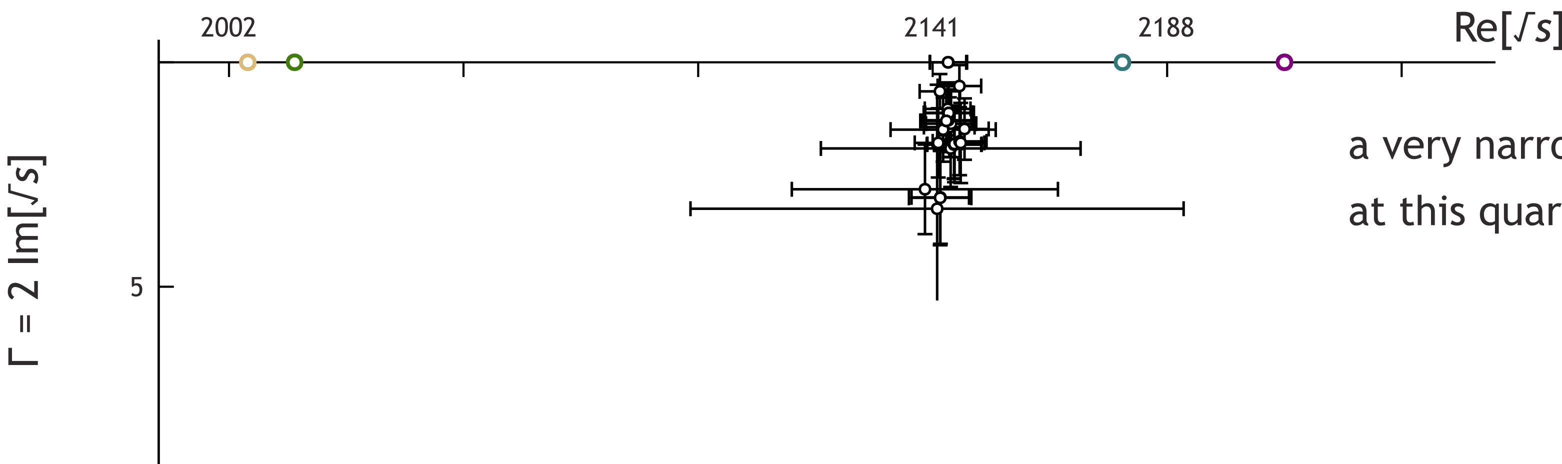
unprecedented number of energy levels



exotic 1^{-+} hybrid meson resonance — m_π~700 MeV

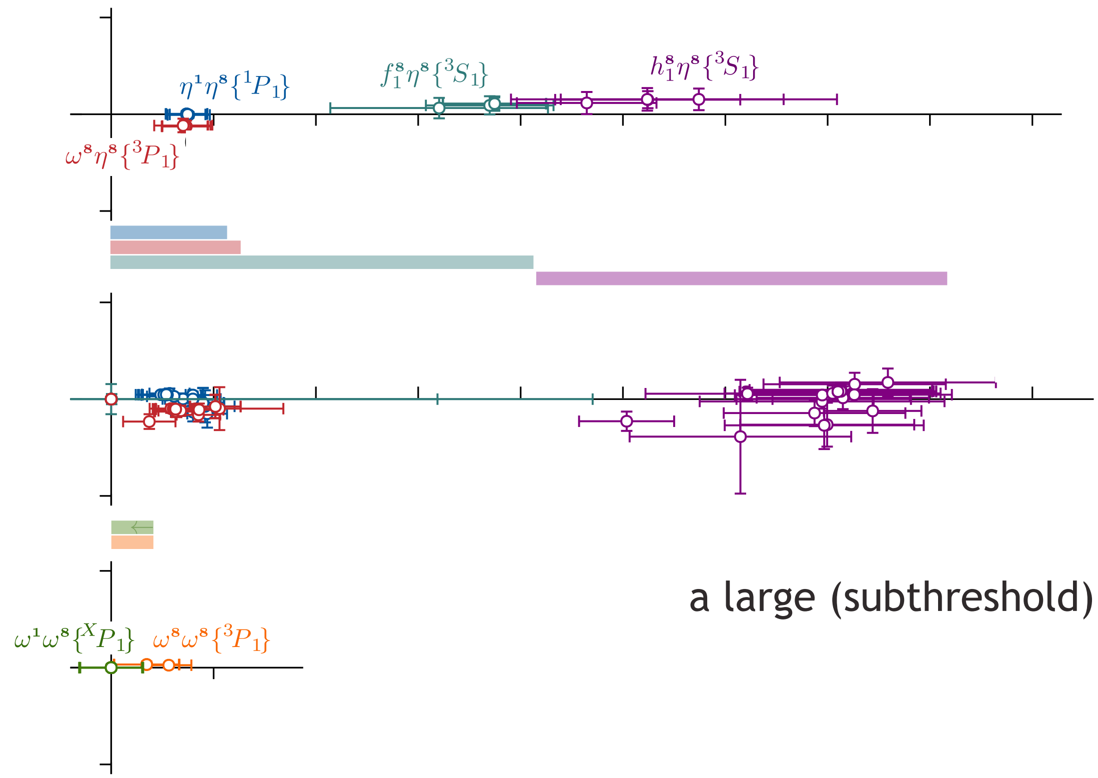


exotic 1^{-+} hybrid meson resonance — m_π~700 MeV



a very narrow resonance
at this quark mass

pole couplings



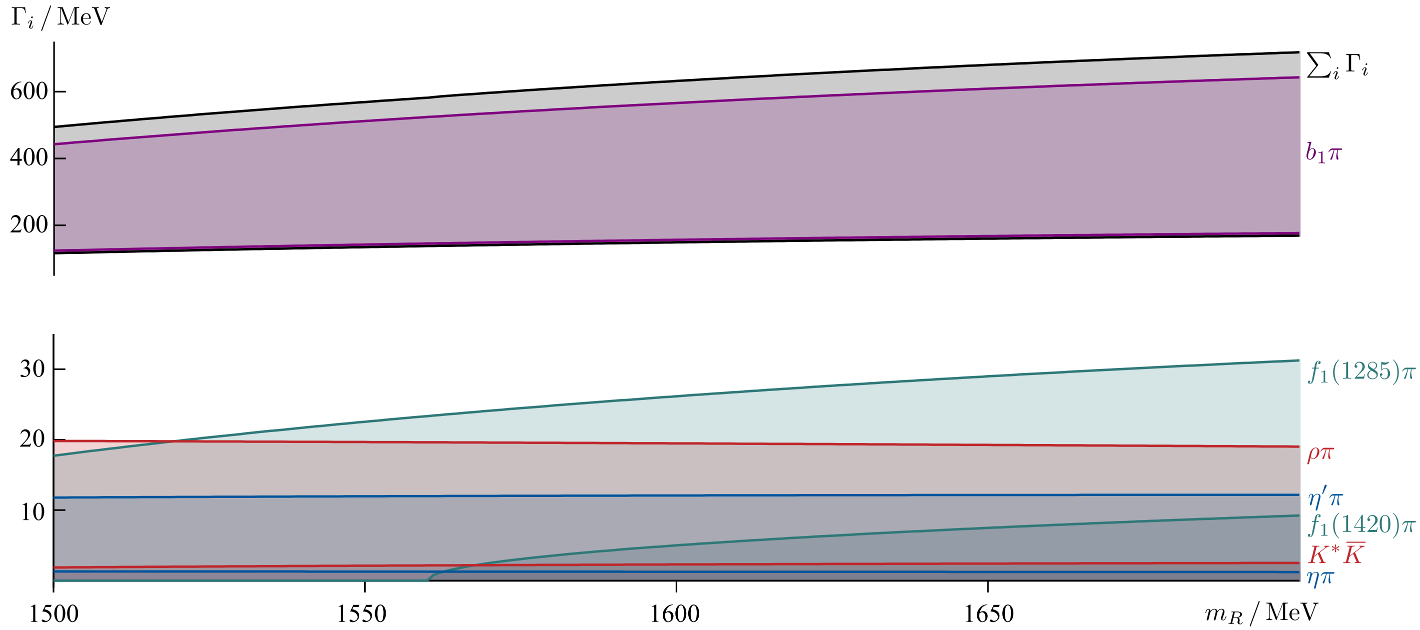
$$t_{ab}(s) \sim \frac{c_a c_b}{s_0 - s}$$

a large (subthreshold) coupling to

$h_1^8\eta^8 \sim \pi b_1$

exotic 1^{-+} hybrid meson resonance — m_π~700 MeV

a very crude extrapolation ...



Determination of the Pole Position of the Lightest Hybrid Meson Candidate

A. Rodas,[1,*] A. Pilloni,[2,3,†] M. Albaladejo,[2,4] C. Fernández-Ramírez,[5] A. Jackura,[6,7] V. Mathieu,[2] M. Mikhasenko,[8] J. Nys,[9] V. Pauk,[10] B. Ketzer,[8] and A. P. Szczepaniak[2,6,7]

(Joint Physics Analysis Center)

Poles	Mass (MeV)	Width (MeV)
$a_2(1320)$	$1306.0 \pm 0.8 \pm 1.3$	$114.4 \pm 1.6 \pm 0.0$
$a_2'(1700)$	$1722 \pm 15 \pm 67$	$247 \pm 17 \pm 63$
π_1	$1564 \pm 24 \pm 86$	$492 \pm 54 \pm 102$

Investigation of the Lightest Hybrid Meson Candidate with a Coupled-Channel Analysis of $\bar{p}p$-, π^-p- and $\pi\pi$-Data

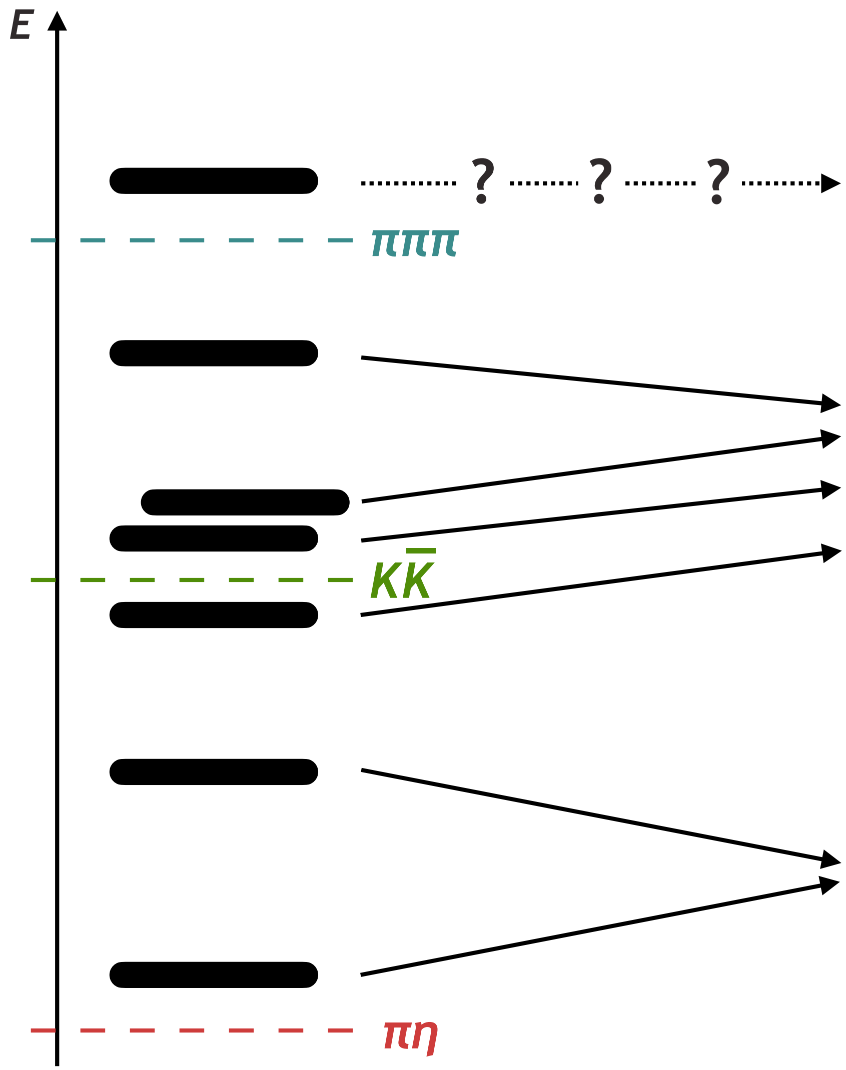
B. Kopf, M. Albrecht, H. Koch, J. Pychy, X. Qin,* and U. Wiedner
Ruhr-Universität Bochum, 44801 Bochum, Germany

name	pole mass [MeV/c^2]	pole width [MeV]
$a_2(1320)$	$1308.7 \pm 0.4^{+2.0}_{-4.2}$	$108.6 \pm 0.4^{+1.8}_{-12.9}$
$a_2(1700)$	$1669.2 \pm 1.0^{+20.2}_{-4.6}$	$429.0 \pm 1.7^{+44.4}_{-9.7}$
π_1	$1561.6 \pm 3.0^{+6.6}_{-2.6}$	$388.1 \pm 5.4^{+0.2}_{-14.1}$

why the heavier quark masses ?

we **can** compute spectra at lighter quark masses, but we wouldn't know what to do with them

the problem is three-body and higher channels…



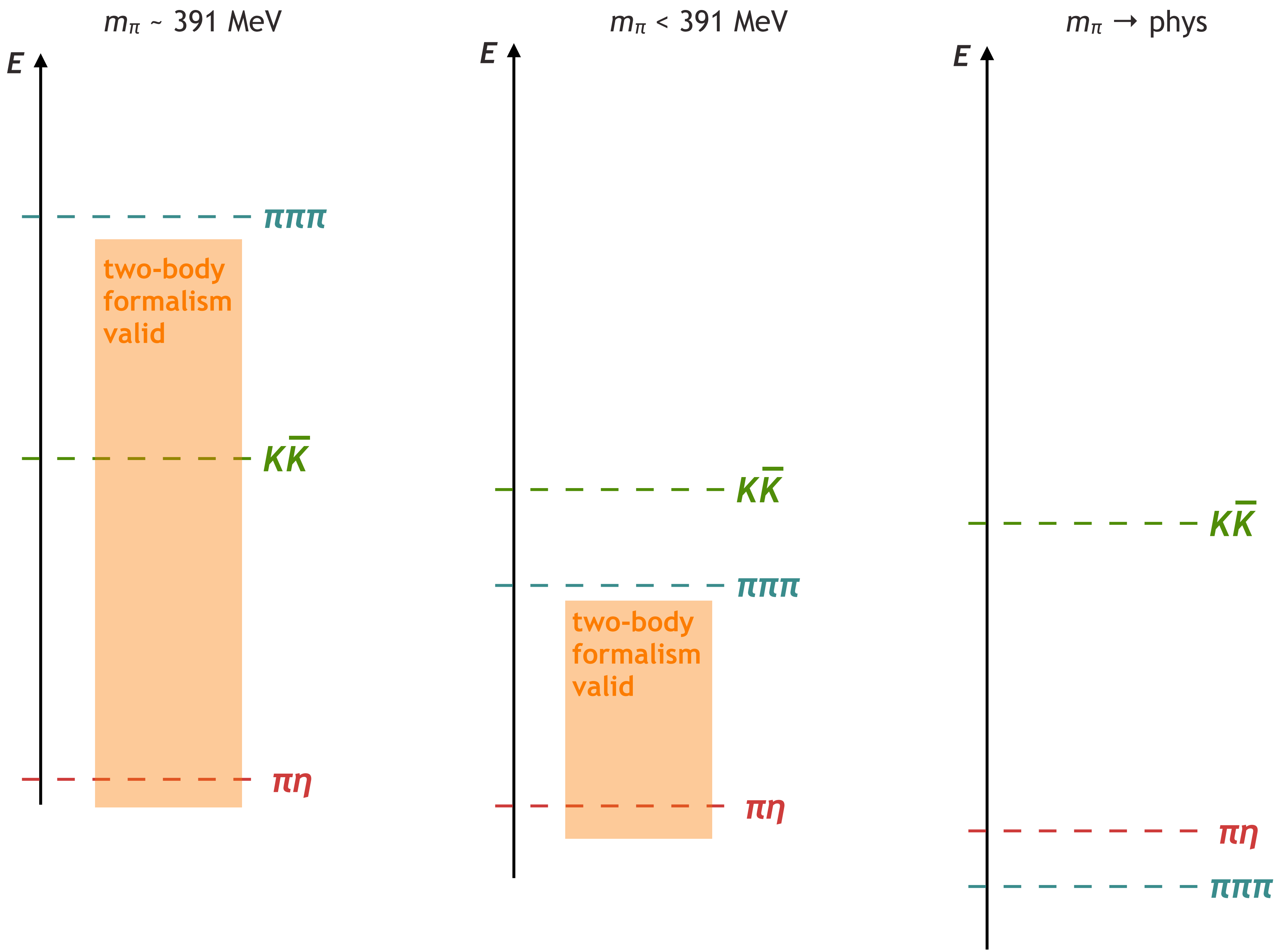
$$\begin{pmatrix} t_{\pi\eta,\pi\eta} & t_{\pi\eta,K\bar{K}} & t_{\pi\eta,\pi\pi\pi} \\ t_{\pi\eta,K\bar{K}} & t_{K\bar{K},K\bar{K}} & t_{K\bar{K},\pi\pi\pi} \\ t_{\pi\eta,\pi\pi\pi} & t_{K\bar{K},\pi\pi\pi} & t_{\pi\pi\pi,\pi\pi\pi} \end{pmatrix}$$

all the complications of infinite-volume three-body (+ more?)

$$\begin{pmatrix} t_{\pi\eta,\pi\eta} & t_{\pi\eta,K\bar{K}} \\ t_{\pi\eta,K\bar{K}} & t_{K\bar{K},K\bar{K}} \end{pmatrix}$$

$$t_{\pi\eta,\pi\eta}$$

physical pion masses = low-lying multipion channels

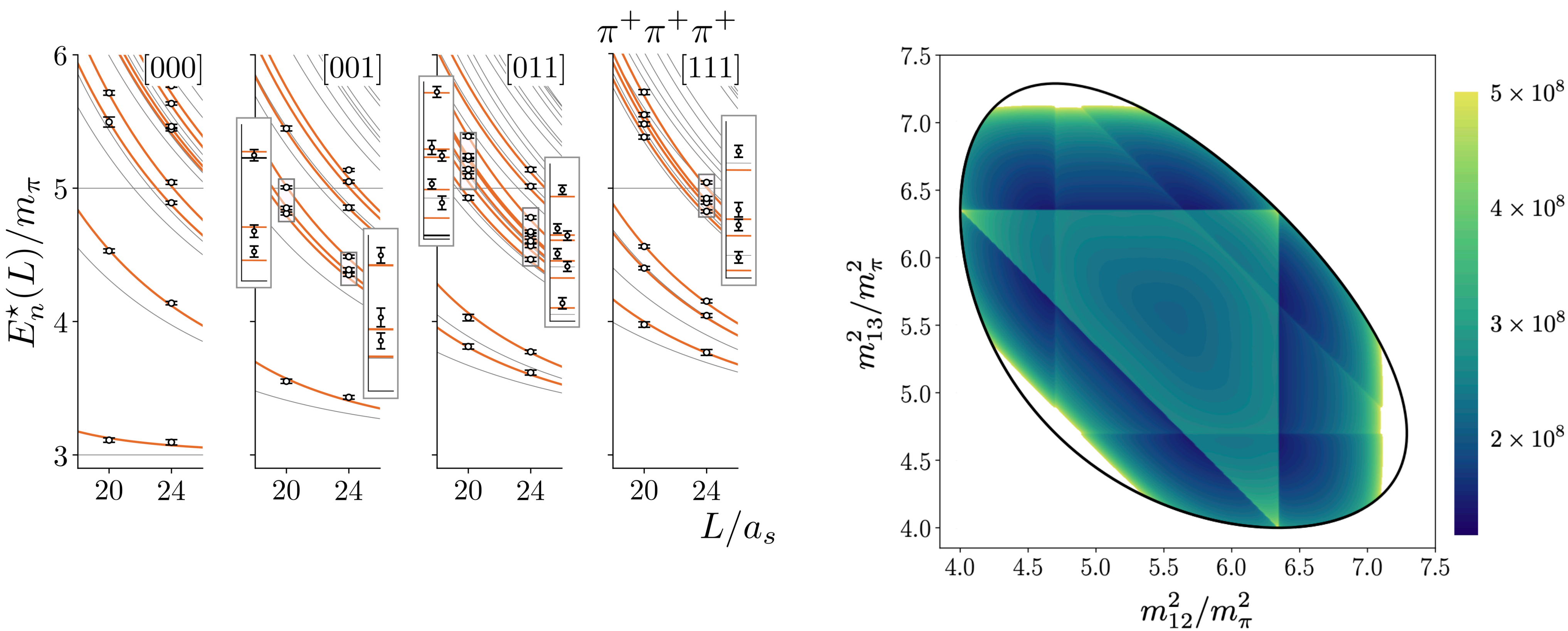


rigorous three body scattering

formalism is significantly more complicated — first applications have appeared

The energy-dependent $\pi^+\pi^+\pi^+$ scattering amplitude from QCD

Maxwell T. Hansen,[1,*] Raul A. Briceño,[2,3,†] Robert G. Edwards,[2,‡] Christopher E. Thomas,[4,§] and David J. Wilson[4,¶]
(for the Hadron Spectrum Collaboration)



coupling scattering systems to external currents

e.g. consider the process in which
a pion absorbs a photon* to become two pions $\gamma\pi \to \pi\pi$

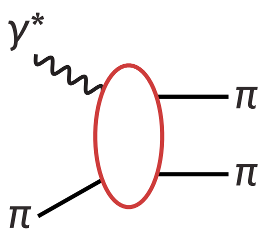
* could be virtual

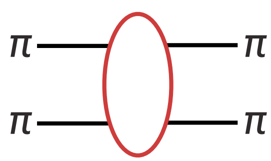
in infinite volume, described by a matrix element $\langle \pi\pi(E_{\mathsf{cm}}, \mathbf{P}) | j^\mu(0) | \pi(\mathbf{p}) \rangle$

$$\propto F(E_{\mathsf{cm}}, Q^2)$$

$\pi\pi$ state can be projected into a partial wave, e.g. ℓ=1

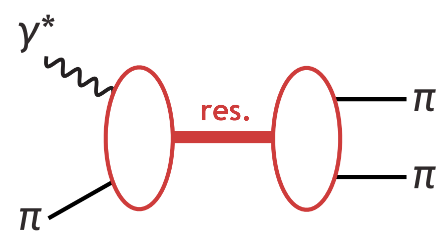
after the current produces $\pi\pi$ ... $\pi\pi$ will rescatter strongly





⇒ the matrix element is proportional to $t_\ell(E_{\mathsf{cm}})$

if there's a resonance $t_\ell(s \sim s_0) \sim \dfrac{c^2}{s_0 - s}$ and $F(s \sim s_0, Q^2) \sim \dfrac{c\, f(Q^2)}{s_0 - s}$



resonance transition form-factor $f(Q^2)$
rigorously defined at the complex pole position

e.g. $\rho \to \pi\gamma$

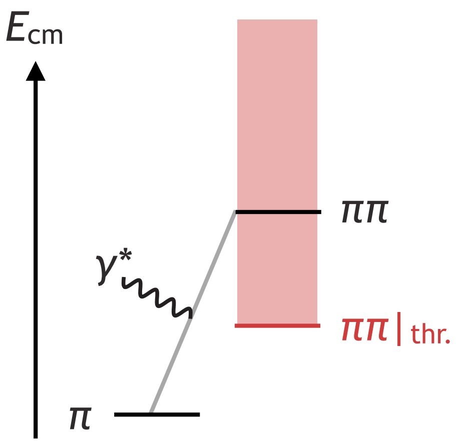
but what changes in a finite volume ... ?

coupling scattering systems to external currents — fin. vol.

e.g. consider the process in which
a pion absorbs a photon to become two pions

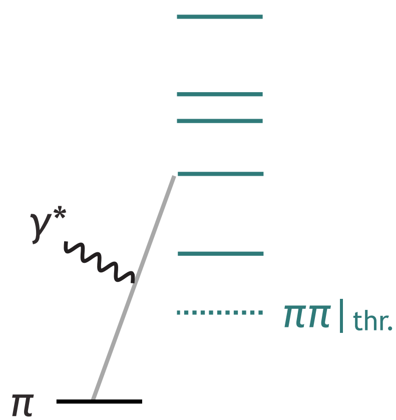
$$\gamma\pi \to \pi\pi$$

infinite volume



can transition
to any energy
in the $\pi\pi$ continuum

finite volume



can only transition
to one of the discrete
f.v. eigenstates

finite-volume matrix element

$$_L\langle \pi\pi(E_n(L),\mathbf{P}) | j^\mu(0) | \pi(\mathbf{p})\rangle_L$$

single hadron state

$$|\pi(\mathbf{p})\rangle_L = |\pi(\mathbf{p})\rangle_\infty + O(e^{-m_\pi L})$$

hadron-hadron state

$$|\pi\pi(E_n(L),\mathbf{P})\rangle_L \sim \boxed{\sqrt{\mathcal{R}_n}}\, |\pi\pi(E_{\text{cm}} = E_n(L),\mathbf{P})\rangle_\infty$$

effective f.v.
normalization

$$\mathcal{R}_n = 2E_n \lim_{E\to E_n} (E - E_n)\Big(F^{-1}(E,\mathbf{P};L) + M(E)\Big)^{-1}$$

$$F = \tfrac{1}{16\pi}\, i\rho\,(1 + i\mathcal{M})$$

$$M = 16\pi\, t$$

effective f.v. normalization depends
on the hadron-hadron scattering amplitude

a 3+1 field theory derivation … matrix elements

consider a two-point correlation function — operators with the quantum numbers of a two-hadron system

$$C_L(t,\mathbf{P}) = \int_L d^3\mathbf{x} \int_L d^3\mathbf{y}\; e^{-i\mathbf{P}\cdot(\mathbf{x}-\mathbf{y})} \langle 0 \big| A(\mathbf{x},t)\, B^\dagger(\mathbf{y},0) \big| 0 \rangle$$

$$C_L(t,\mathbf{P}) = L^3 \int \frac{dE}{2\pi}\, e^{iEt} \Big[C_\infty(E,\mathbf{P}) - \tilde{A}\, \big[F^{-1}(E,\mathbf{P},L) + M(E,\mathbf{P}) \big]^{-1} \tilde{B} \Big]$$

performing the energy integration

$$C_L(t,\mathbf{P}) = \sum_n e^{-E_n t}\, L^3 \tilde{A}_n \mathcal{R}_n \tilde{B}_n$$

a discrete spectral decomposition

$$\mathcal{R}_n = 2E_n \lim_{E\to E_n} (E - E_n) \Big(F^{-1}(E,\mathbf{P};L) + M(E) \Big)^{-1}$$

$$C_L(t,\mathbf{P}) = \sum_n e^{-E_n t}\, \big(L^{3/2} \tilde{A}_n \sqrt{\mathcal{R}_n} \big) \big(L^{3/2} \tilde{B}_n \sqrt{\mathcal{R}_n} \big)$$

operator dependent overlap

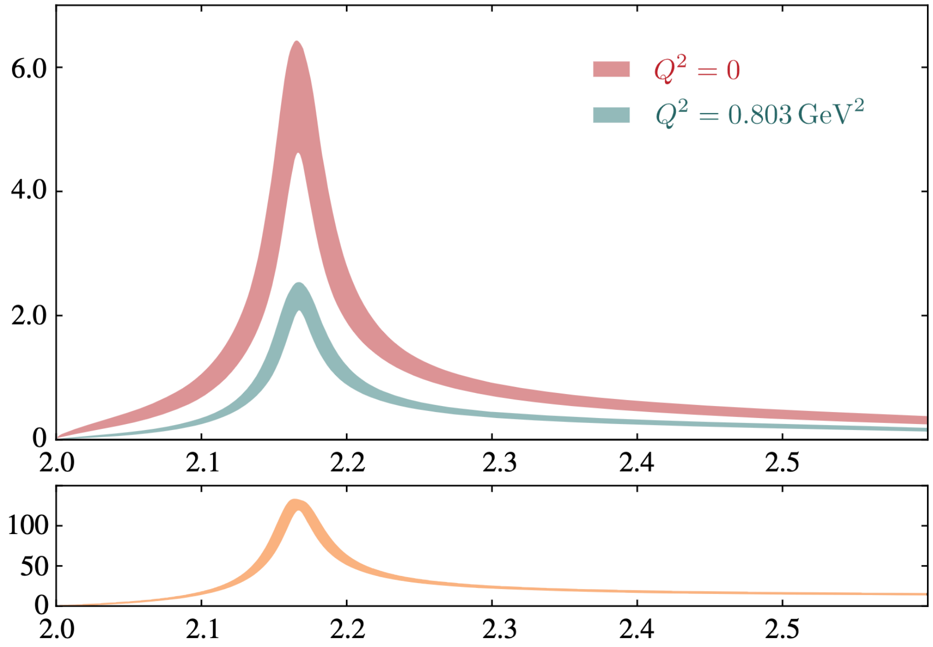
operator independent f.v. overlap

$\pi\pi \to \pi\gamma$ lattice QCD calculation

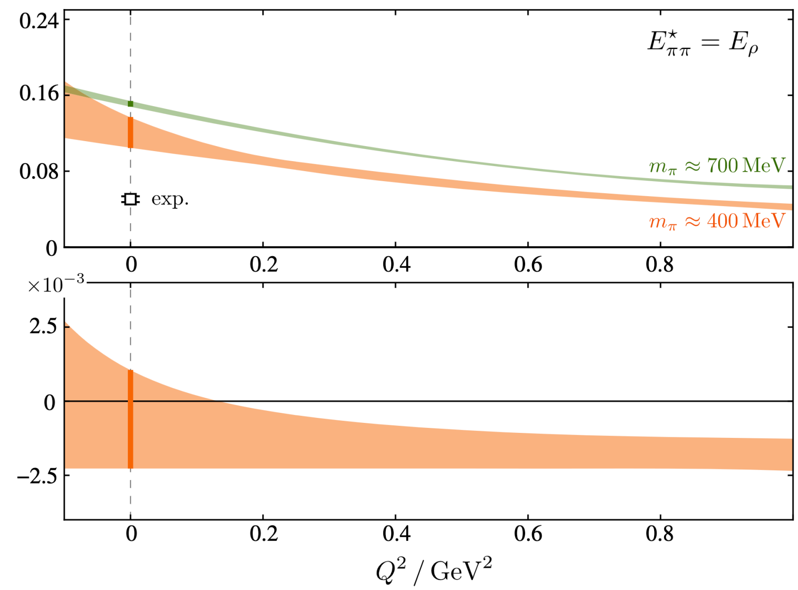
m_π~391 MeV

PRL115 242001 (2015)

PRD93 114508 (2016)



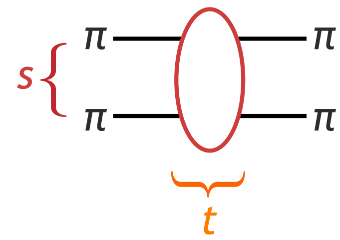
form-factor at the pole



the missing singularity – the ‘left-hand cut’

consider the amplitude **before** we partial-wave projected

a function of both ***s*** and ***t***



$T(s,t)$

the same amplitude should describe **crossed-channel** scattering

e.g. suppose a stable (scalar) hadron can be exchanged in the ***t*-channel**
what would that imply for the partial-wave amplitude ?

$$T(s,t) = \frac{g^2}{M^2 - t}$$

$$t_\ell(s) = \tfrac{1}{2}\int_{-1}^{1} dx\, P_\ell(x)\, T(s, t(x))$$

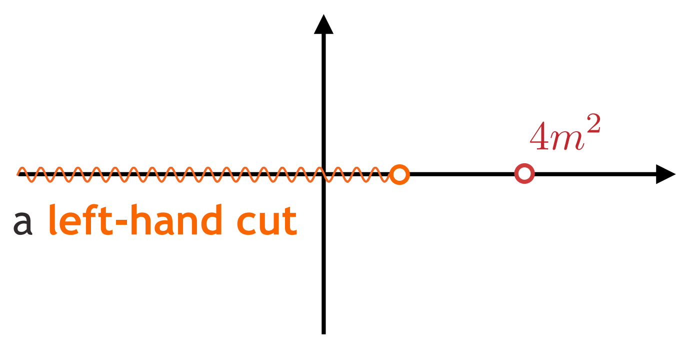
$$x = \cos\theta$$
$$t = -2k^2(1-x)$$

S-wave

$$t_0(s) = \tfrac{1}{2} g^2 \int_{-1}^{1} dx\, \frac{1}{M^2 + 2k^2(1-x)}$$

$$t_0(s) = \frac{g^2}{4k^2} \log\left[\frac{s - 4m^2 + M^2}{M^2}\right]$$

branch point at
$s = 4m^2 - M^2$



the missing singularity – the 'left-hand cut'

consider the amplitude **before** we partial-wave projected

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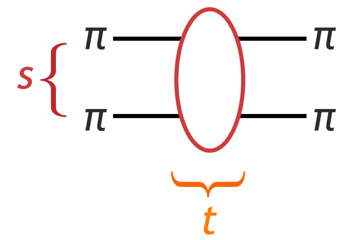


$T(s,t)$

more generally, unitarity in the crossed-channels demands a left-hand cut

dispersion at fixed *s*

$$T(s,t) = \frac{1}{2\pi i}\int_{4m^2}^{\infty} d\bar{t}\; \frac{\mathrm{disc}_t\, T(s,\bar{t})}{\bar{t}-t} + u\text{–channel}$$

$$t_\ell(s) = \tfrac{1}{2}\int_{-1}^{1} dx\, P_\ell(x)\, T(s,t(x))$$

$$x = \cos\theta \qquad t = -2k^2(1-x)$$

$$t_\ell(s) = \frac{1}{4\pi i k^2}\int_{4m^2}^{\infty} d\bar{t}\;\; \mathrm{disc}_t\, T(s,\bar{t})\; \frac{1}{2}\int_{-1}^{1} dx\, \frac{P_\ell(x)}{\left(1+\frac{\bar{t}}{2k^2}\right)-x}$$

$$= Q_\ell\left(1+\frac{\bar{t}}{2k^2}\right)$$

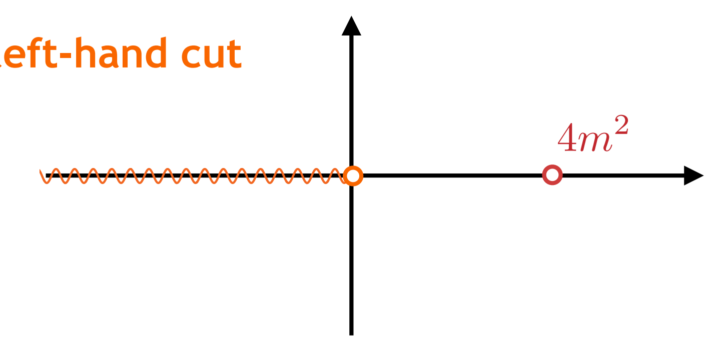
Legendre function of the second kind

$Q_\ell(z)$ has branch points at **z=±1**

singularity if $\bar{t} = -4k^2 = 4m^2 - s$

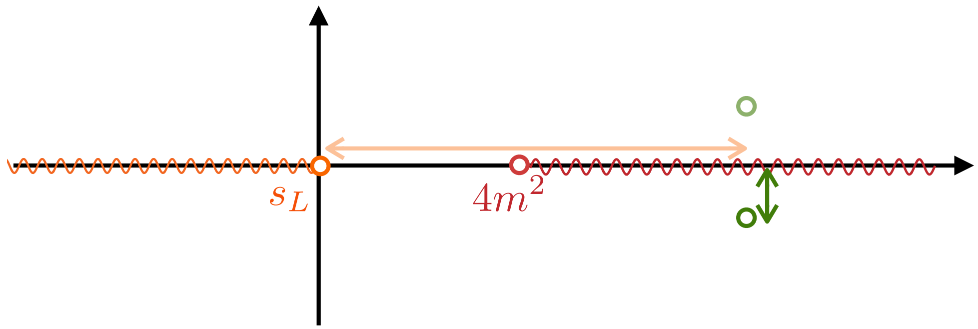
which is in the integration region if $s<0$

***t*-channel unitarity** generates a **left-hand cut**



importance of the left-hand cut ?

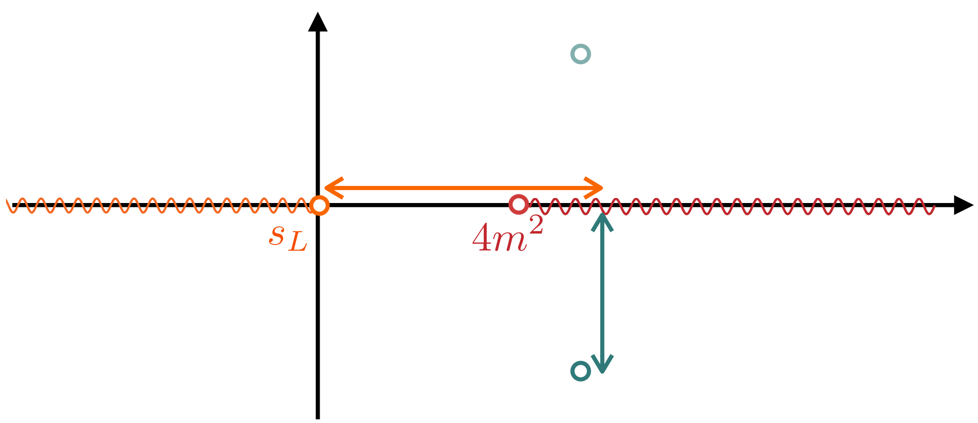
narrow resonance



resonance pole nearby
left-hand cut very distant

e.g. can describe scattering near the ρ resonance without describing the left-hand cut

broad resonance

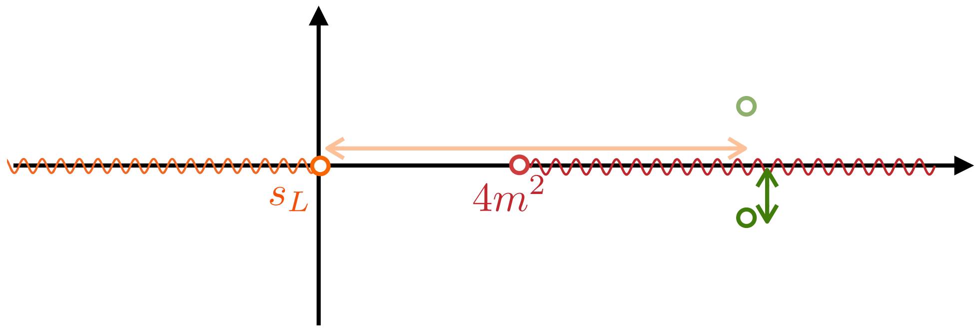


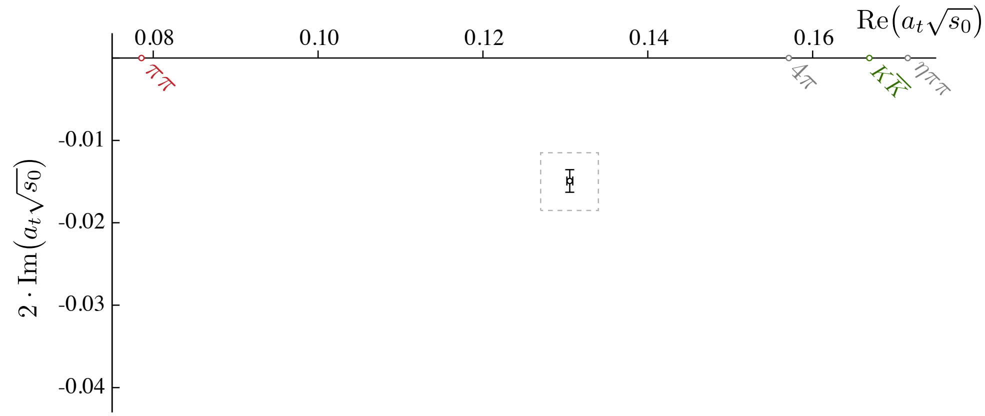
left-hand cut
may be as close as
resonance pole

e.g. the σ resonance in $\pi\pi$ I=0

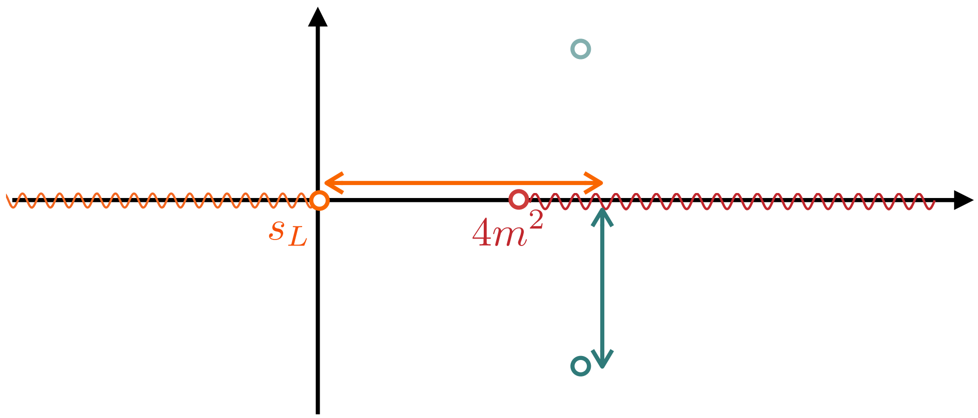
importance of the left-hand cut ?

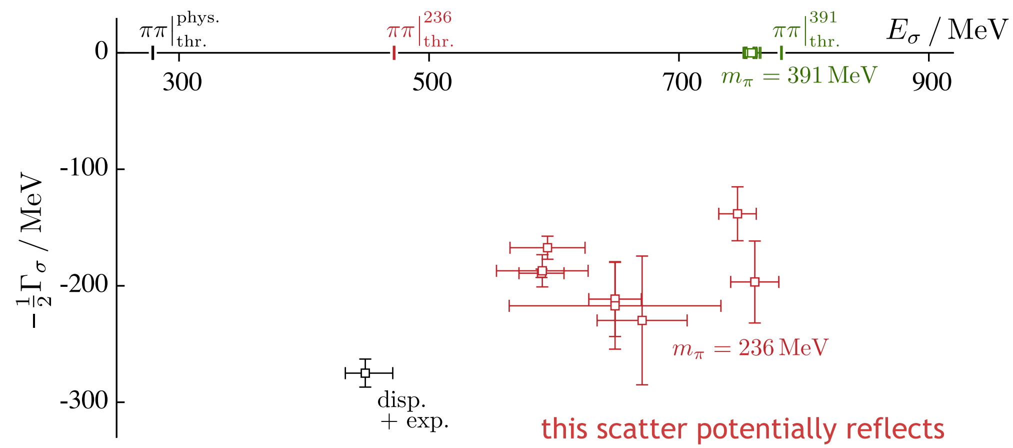
narrow resonance





broad resonance





this scatter potentially reflects the absence of accurate constraint on the left-hand cut …

channel decoupling below thresholds

coupled-channel quantization condition $\det\left[\mathbf{1} + i\boldsymbol{\rho t}(\mathbf{1} + i\boldsymbol{\mathcal{M}})\right] = 0$

e.g. two channels $\boldsymbol{\rho t} = \begin{pmatrix} \rho_1 t_{11} & \rho_1 t_{12} \\ \rho_2 t_{12} & \rho_2 t_{22} \end{pmatrix}$ $\quad \boldsymbol{\mathcal{M}} = \begin{pmatrix} \mathcal{M}(k_1) & 0 \\ 0 & \mathcal{M}(k_2) \end{pmatrix}$

$$\mathbf{1} + i\boldsymbol{\rho t}(\mathbf{1} + i\boldsymbol{\mathcal{M}}) = \begin{pmatrix} 1 + i\rho_1 t_{11}(1 + i\mathcal{M}_1) & i\rho_1 t_{12}(1 + i\mathcal{M}_2) \\ i\rho_2 t_{12}(1 + i\mathcal{M}_1) & 1 + i\rho_2 t_{22}(1 + i\mathcal{M}_2) \end{pmatrix}$$

---

e.g. consider the rest-frame A_1 irrep – below threshold: $\mathcal{M}(i\kappa) = i - \frac{i}{\kappa} \sum_{\mathbf{n} \neq \mathbf{0}} \frac{e^{-\kappa|\mathbf{n}|L}}{|\mathbf{n}|L}$

so **far below threshold** $\mathcal{M} \to i$

---

suppose we're above threshold 1, but well below threshold 2 $\mathbf{1} + i\boldsymbol{\rho t}(\mathbf{1} + i\boldsymbol{\mathcal{M}}) \to \begin{pmatrix} 1 + i\rho_1 t_{11}(1 + i\mathcal{M}_1) & 0 \\ i\rho_2 t_{12}(1 + i\mathcal{M}_1) & 1 \end{pmatrix}$

quantization condition → $1 + i\rho_1 t_{11}(1 + i\mathcal{M}_1) = 0$

which is the one-channel condition