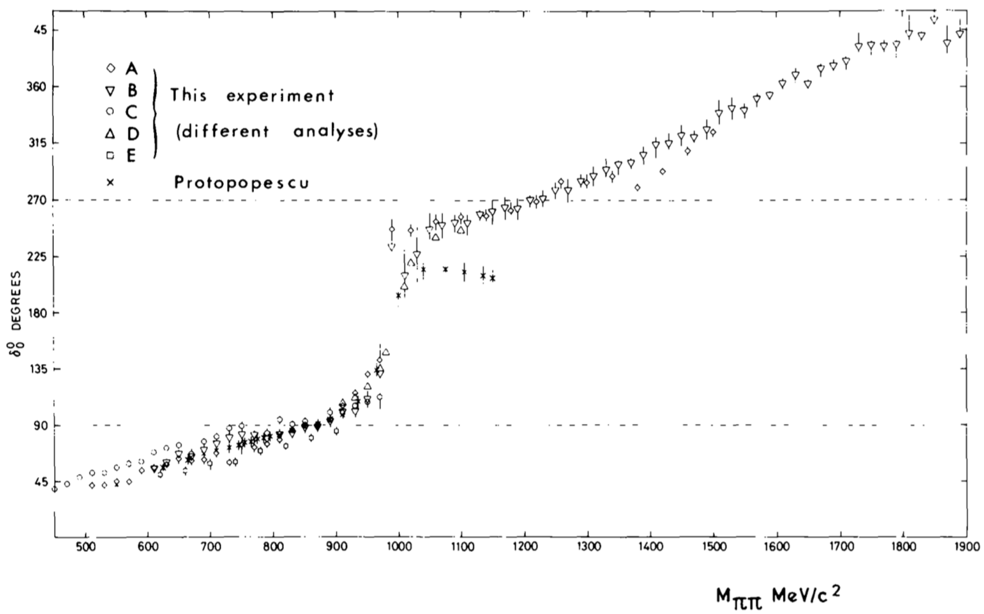
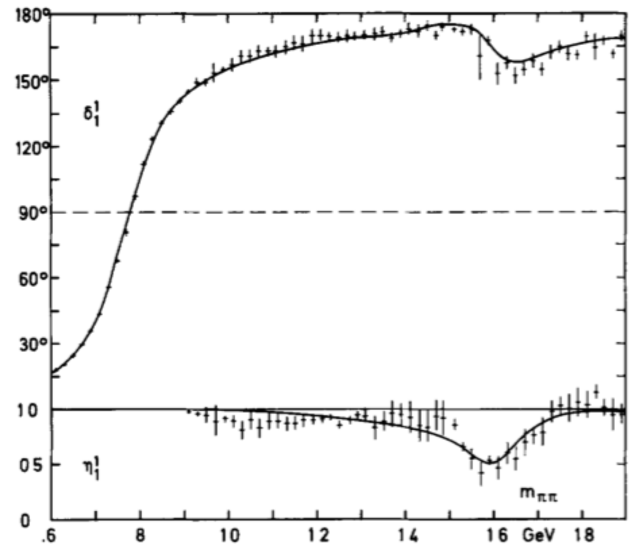
contents

the "simplest" case: $\pi\pi$ elastic scattering

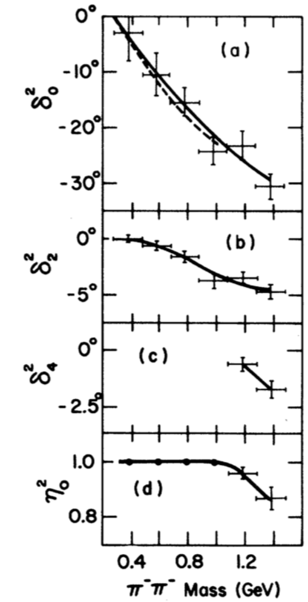
isospin=0



isospin=1



isospin=2



a first target: can a **first-principles QCD** calculation lead to these kinds of behaviour ?

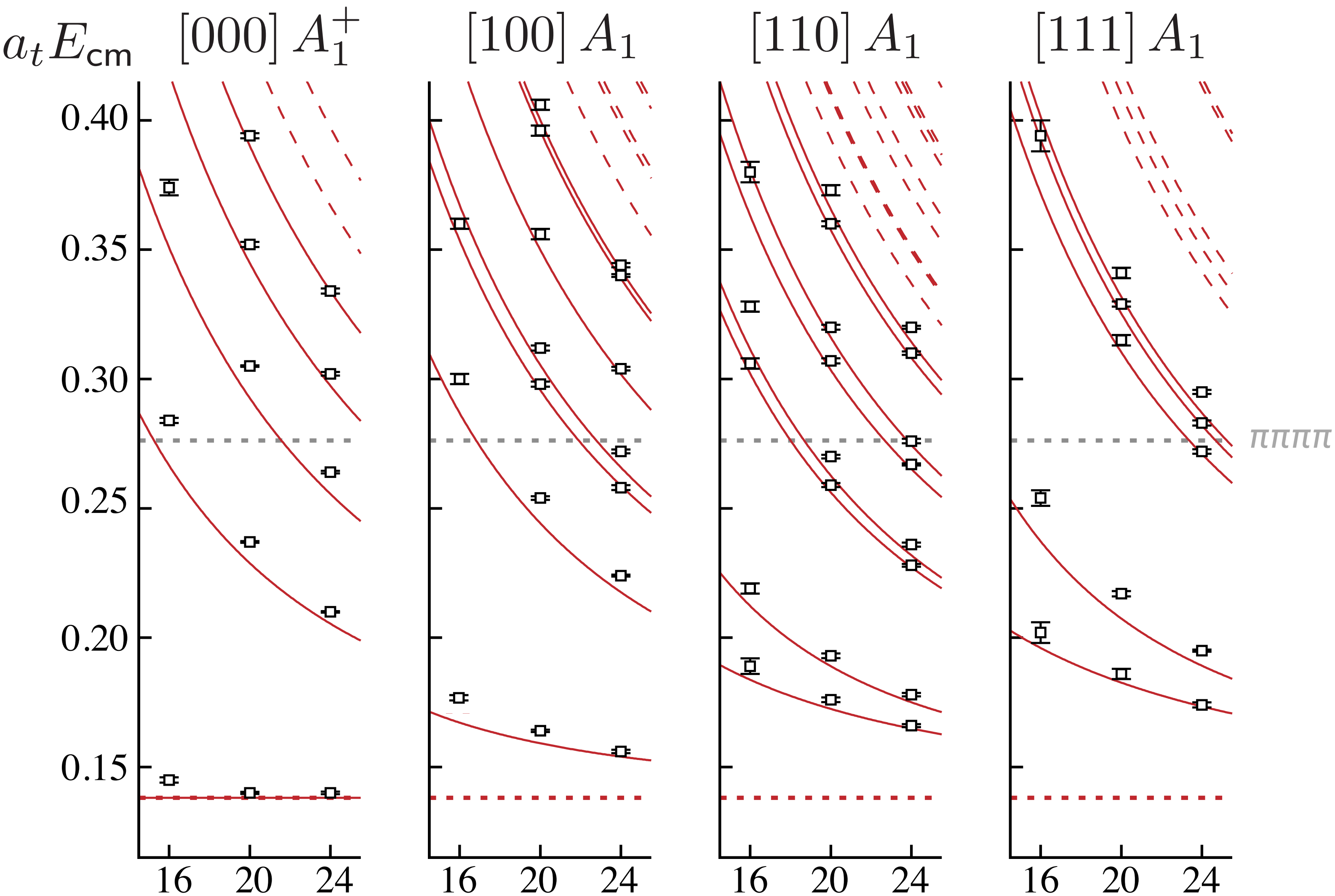
a next target: can we understand these behaviours in terms of **resonances** ?

an ultimate target: can we understand the **quark-gluon make-up** of these resonances ?

$\pi\pi$ isospin=2 — m_π~391 MeV

[basis of $\pi\pi$-like operators only]

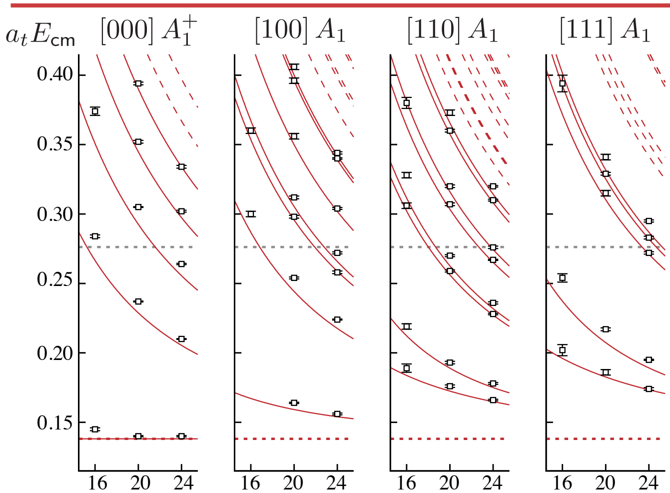
[computed in three volumes]

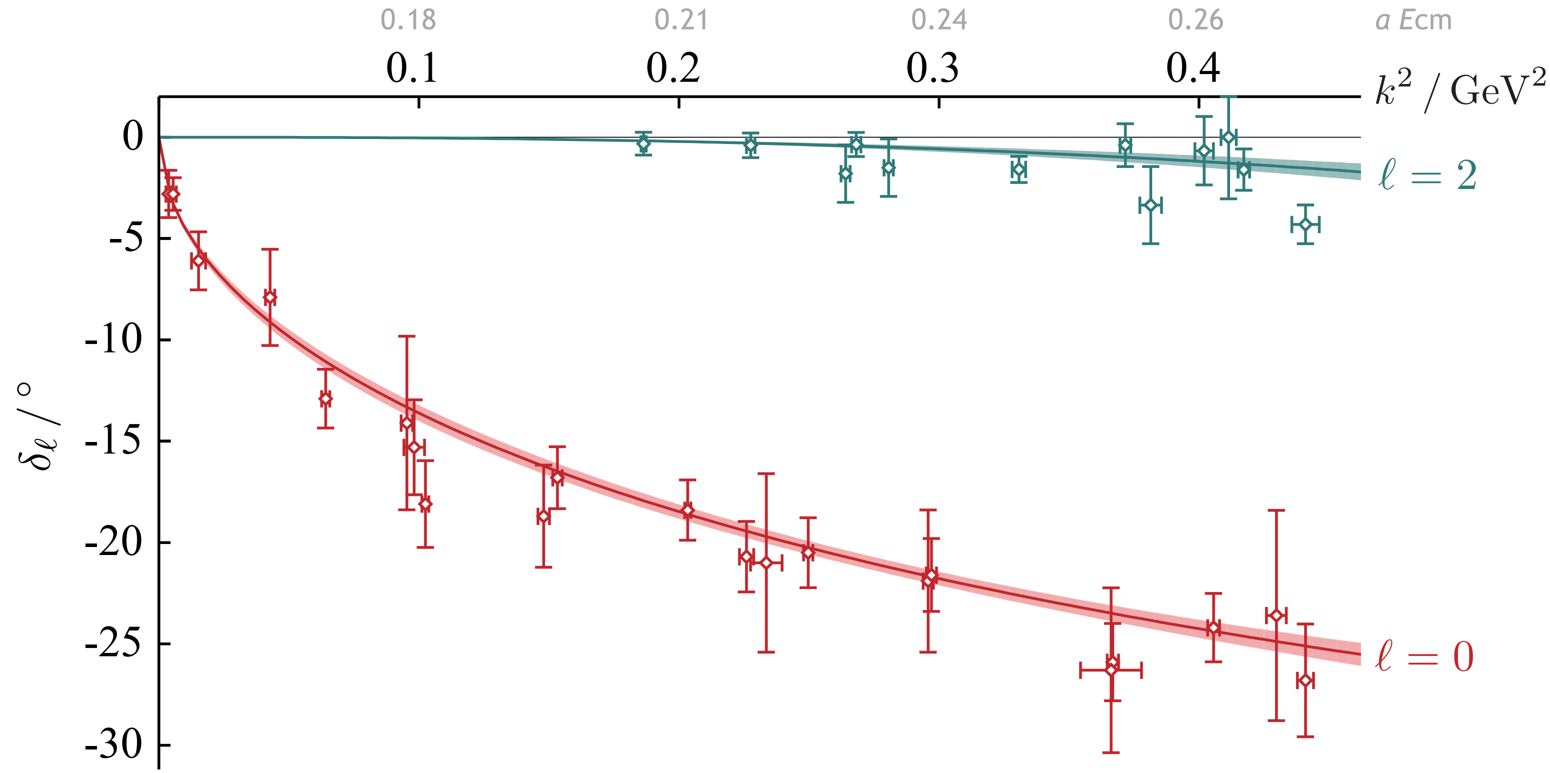


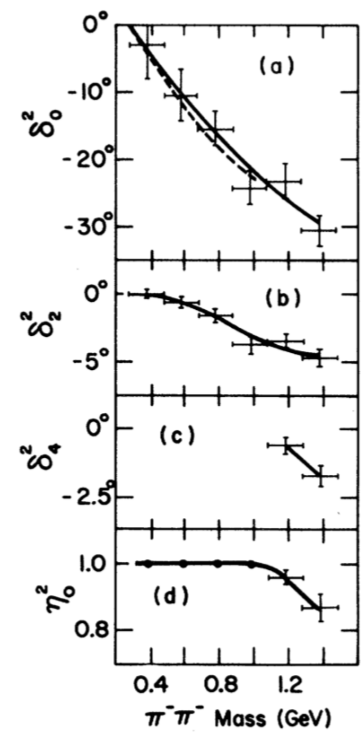
& spectra in irreps with lowest ℓ=2 (not shown here)

$\pi\pi$ isospin=2 — m_π~391 MeV

PRD86 034031 (2012)



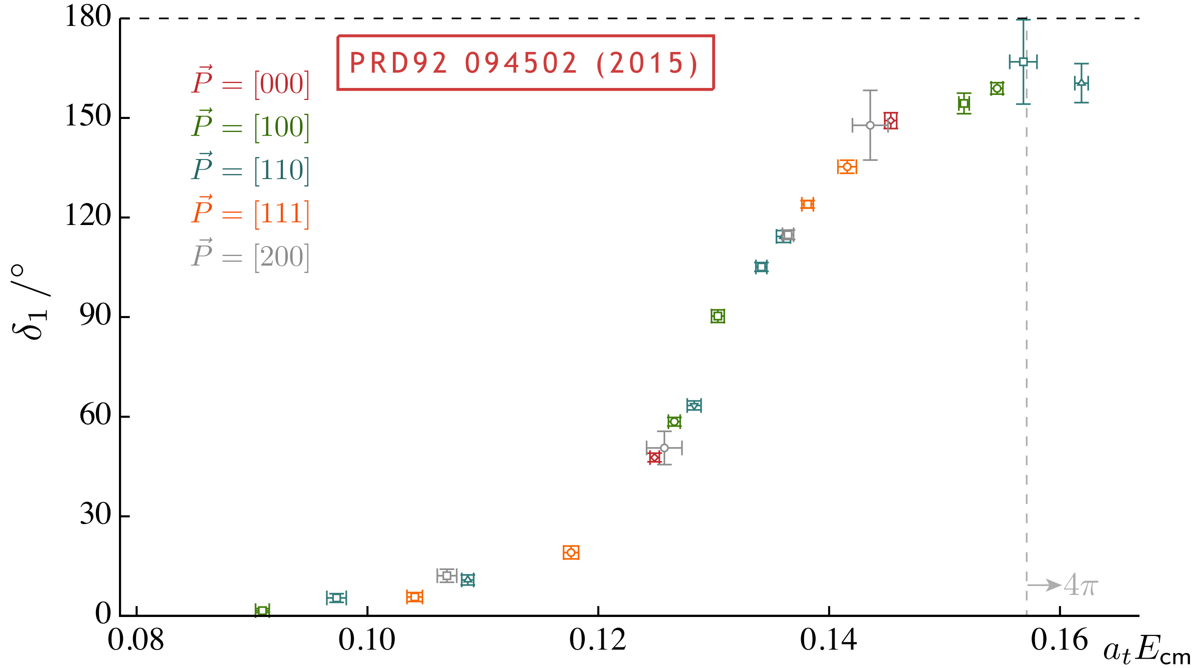




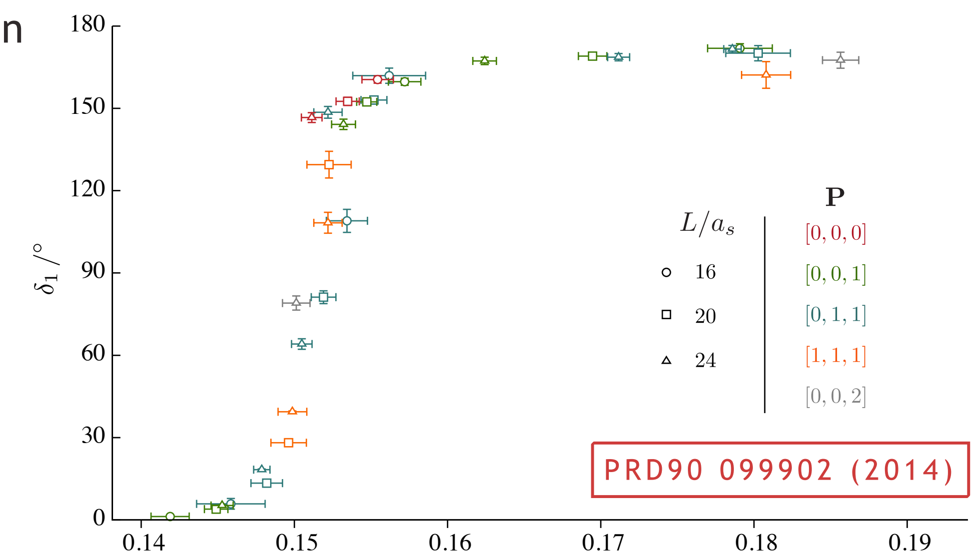
Cohen 1972

$\pi\pi$ isospin=1 — m_π~236 MeV, m_π~391 MeV

you saw this earlier …

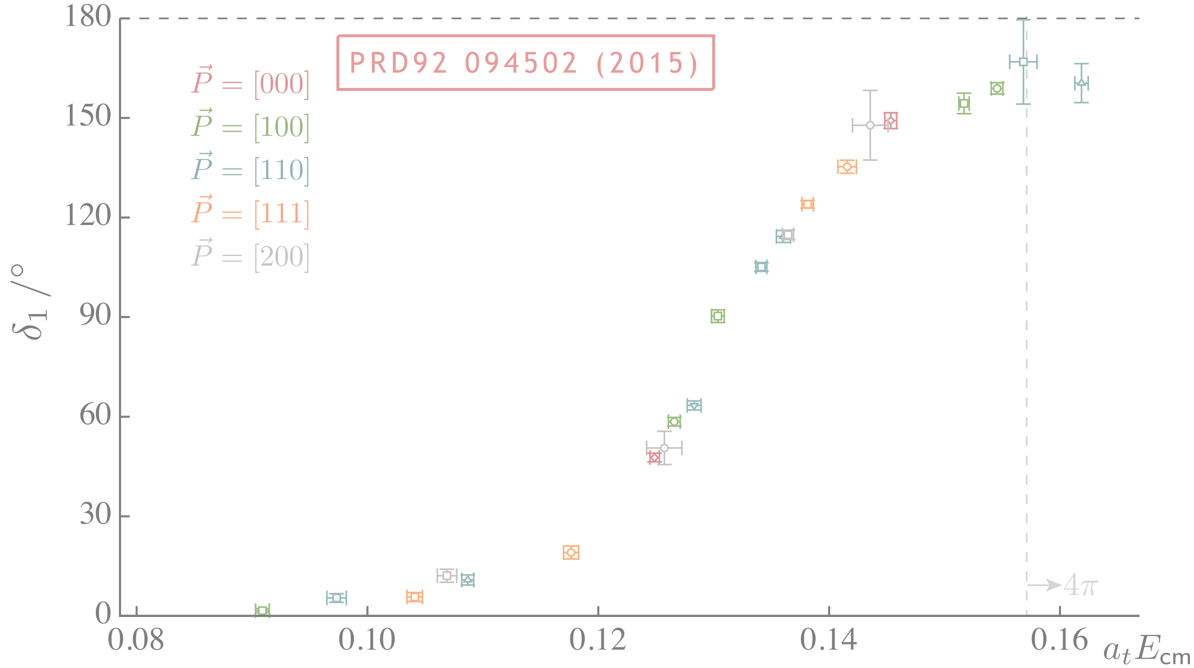


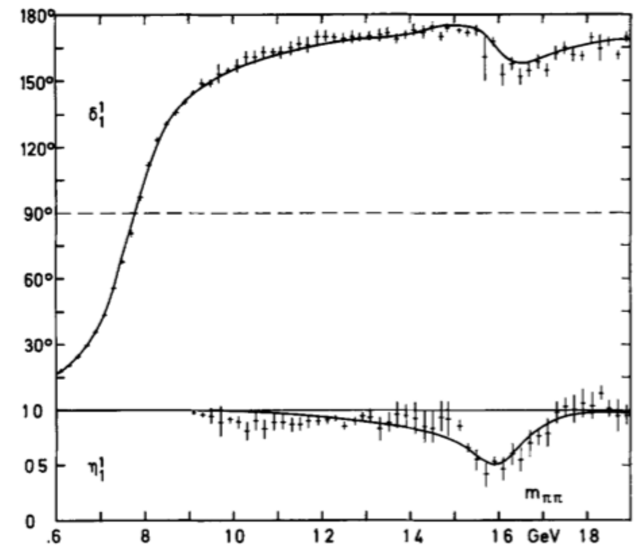
and a similar calculation at a heavier pion mass



$\pi\pi$ isospin=1 — m_π~236 MeV, m_π~391 MeV

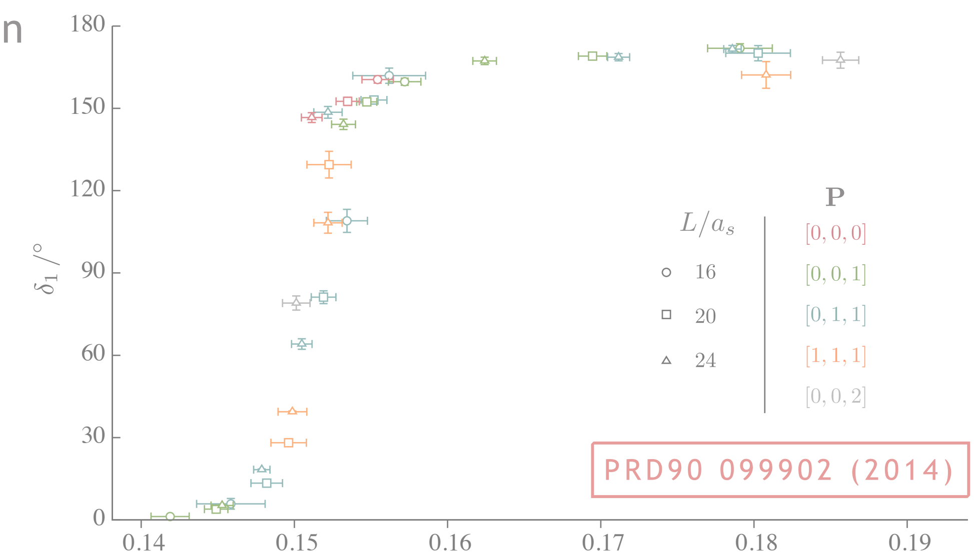
you saw this earlier …





Hyams 1973

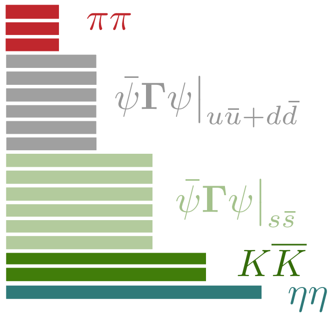
and a similar calculation
at a heavier pion mass



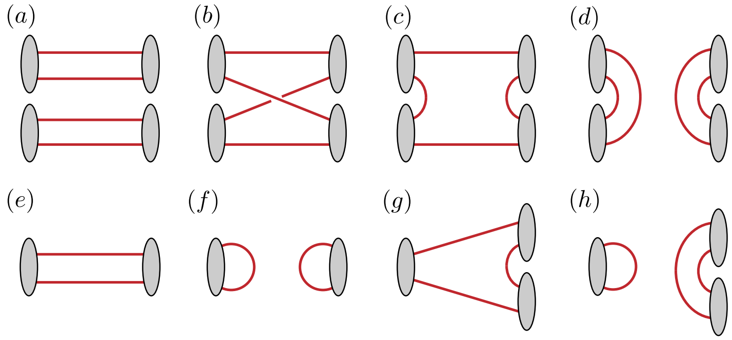
$\pi\pi$ isospin=0

this is the hardest one by far ...

operator basis



Wick diagrams

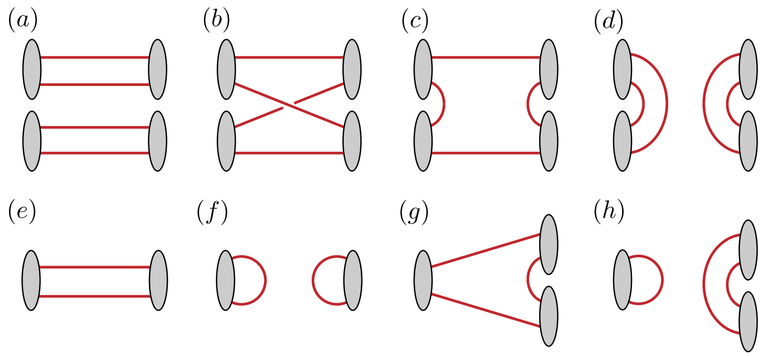


$\pi\pi$ isospin=0

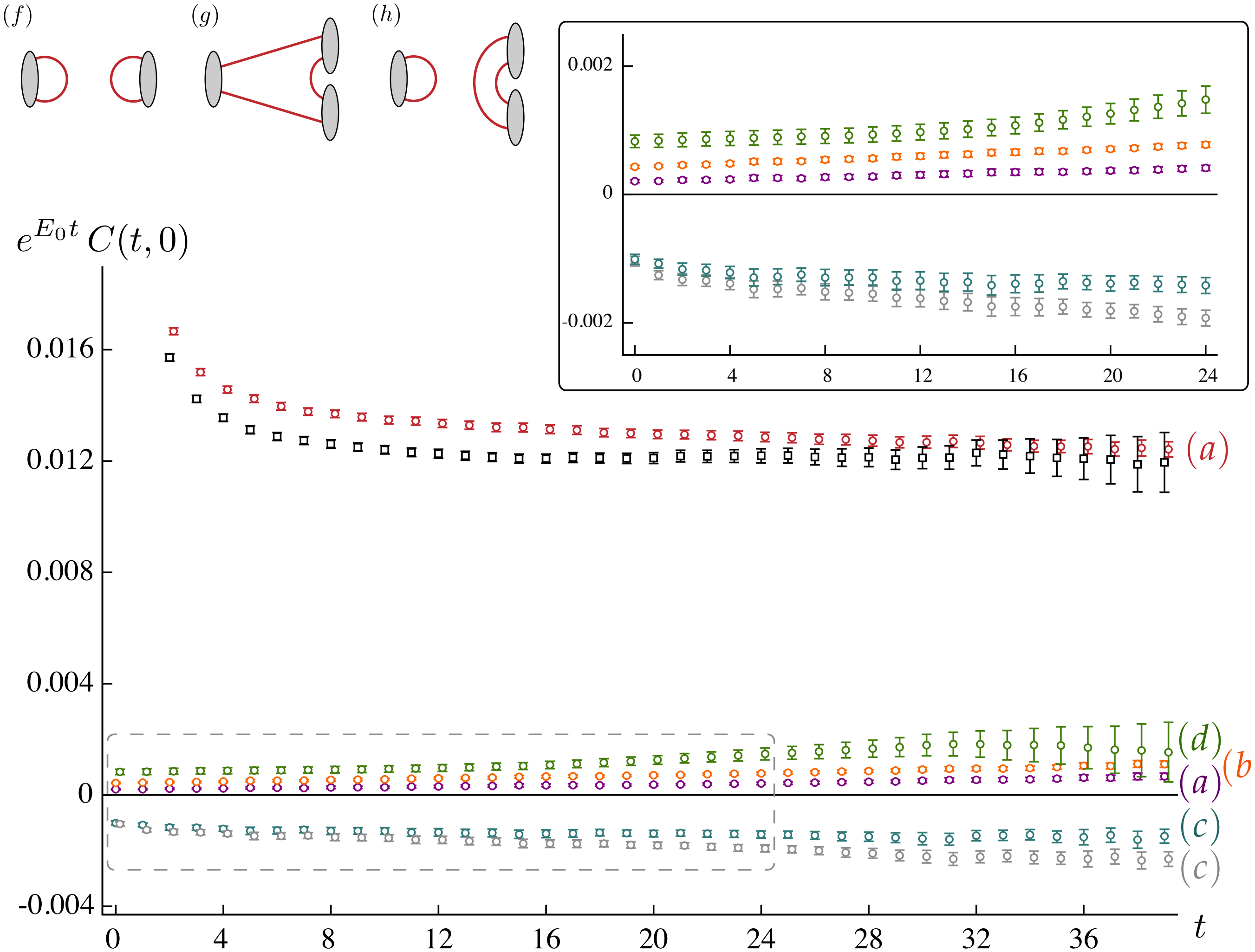
a single entry of the correlation matrix — $\pi\pi$-like operator :

m_π ~ 236 MeV

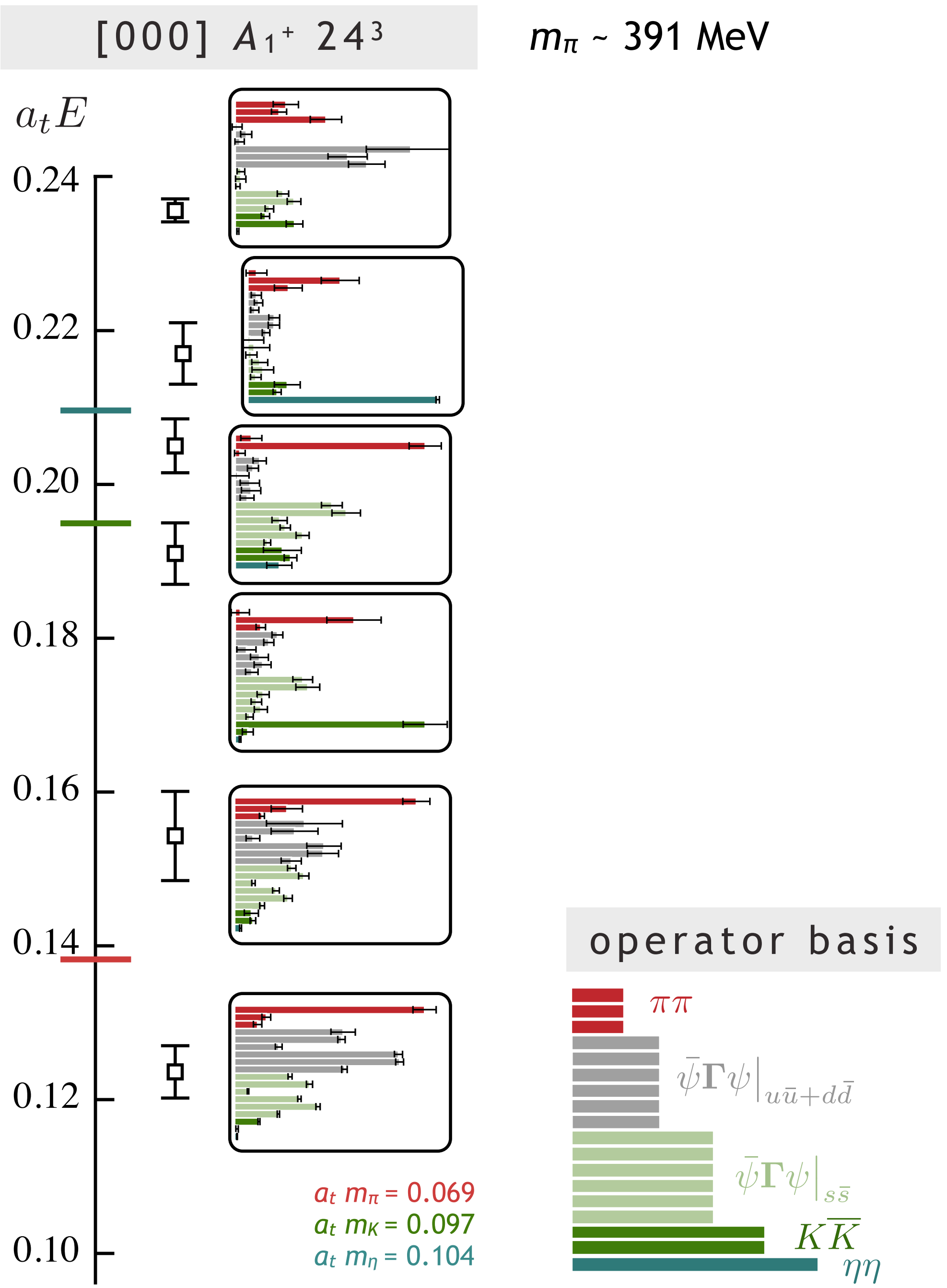
$32^3 \times 256$



P = [110]

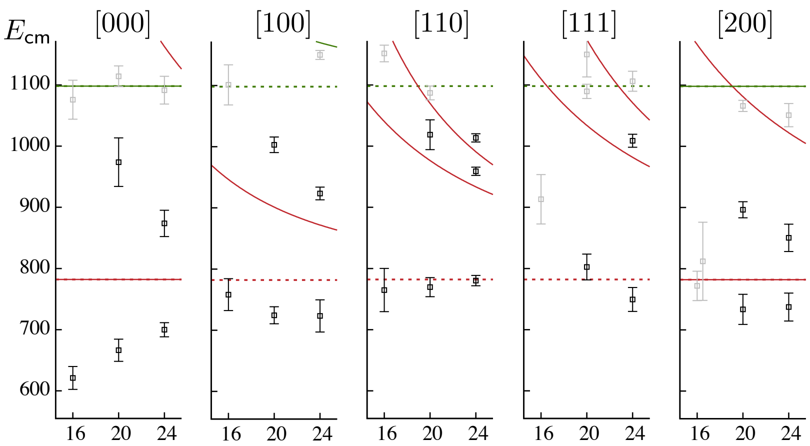


ππ isospin=0

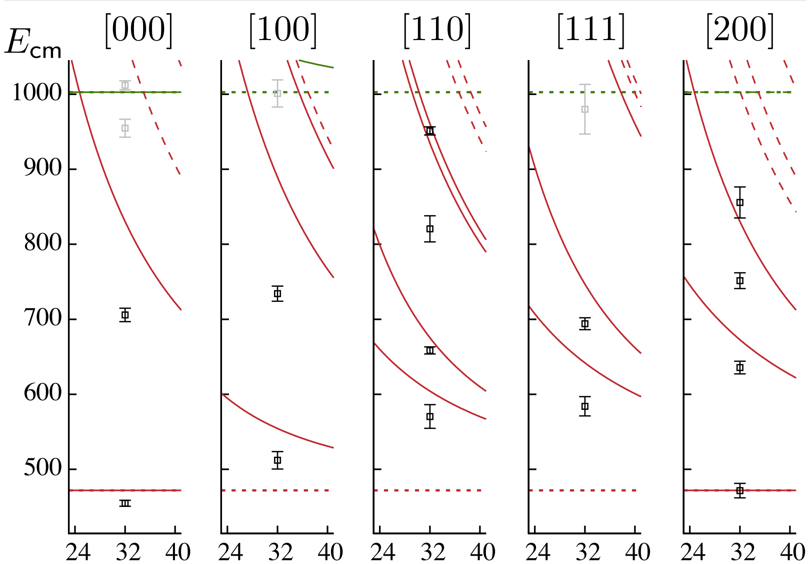

[000] A_1^+ 24^3
m_π ~ 391 MeV
$a_t E$
0.24
0.22
0.20
0.18
0.16
0.14
0.12
0.10
$a_t\, m_\pi$ = 0.069
$a_t\, m_K$ = 0.097
$a_t\, m_\eta$ = 0.104
operator basis
$\pi\pi$
$\bar{\psi}\mathbf{\Gamma}\psi\big|_{u\bar{u}+d\bar{d}}$
$\bar{\psi}\mathbf{\Gamma}\psi\big|_{s\bar{s}}$
$K\overline{K}$
$\eta\eta$

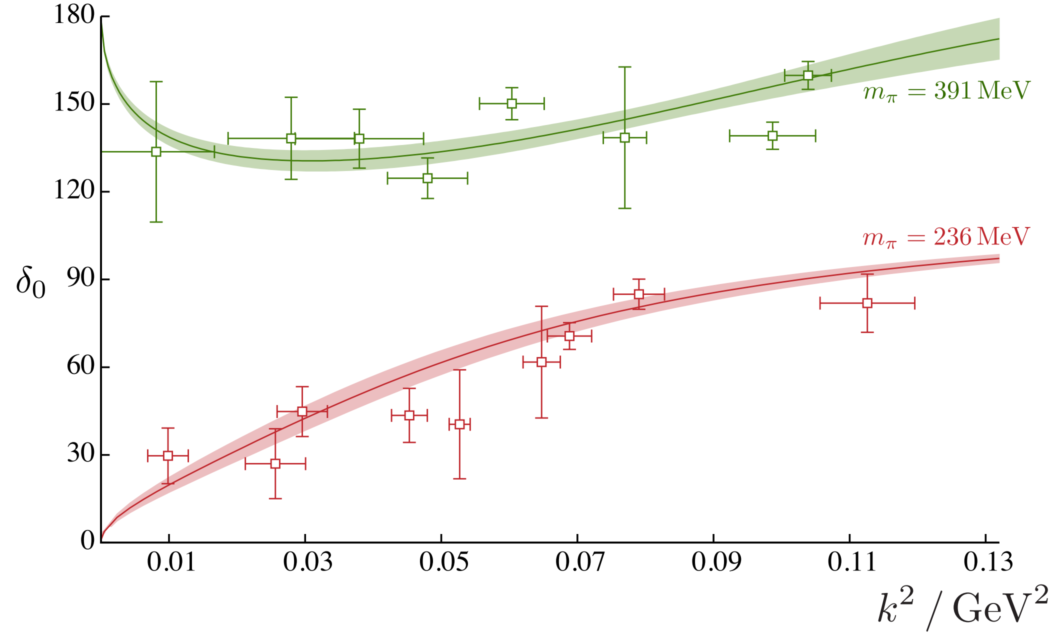
ππ isospin=0

m_π~391 MeV

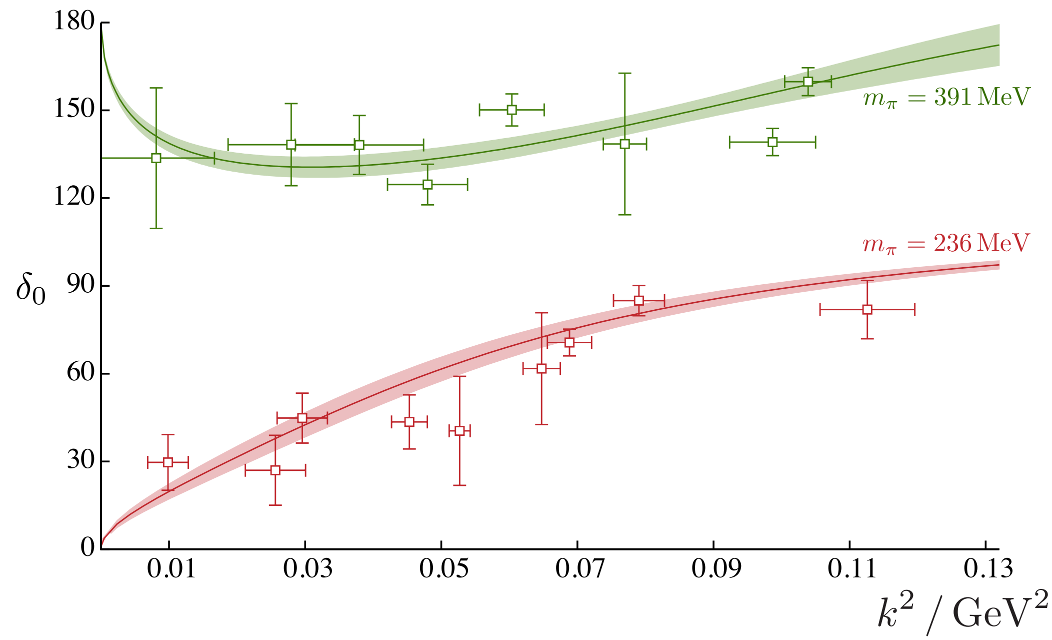


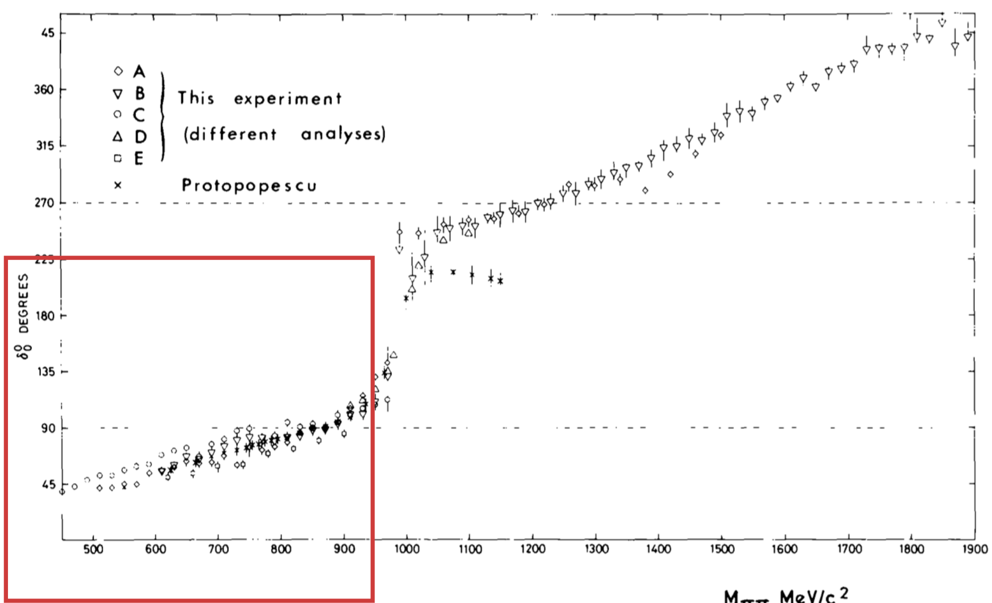
m_π~236 MeV



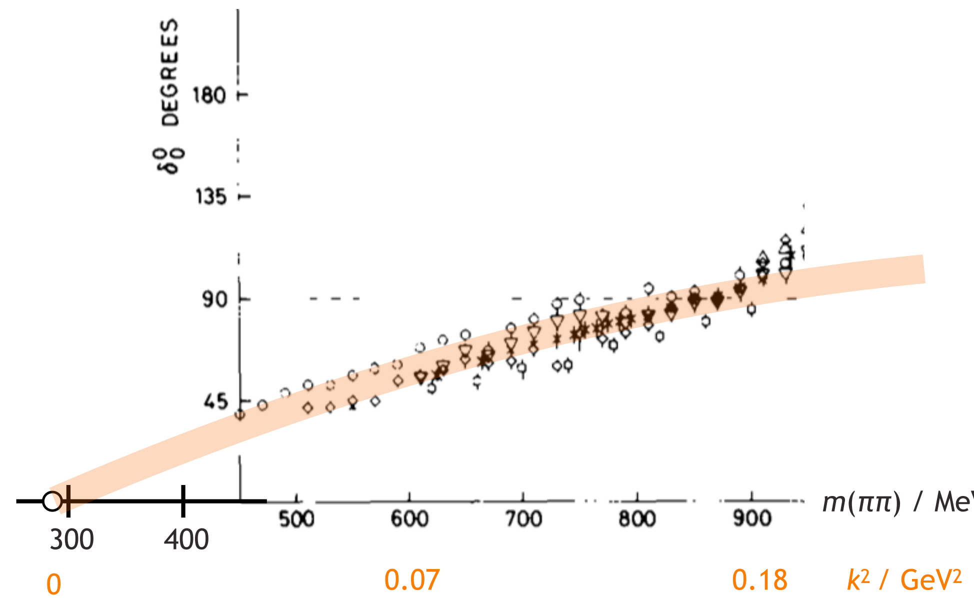


$\pi\pi$ isospin=0





Grayer 1974



contents

meson spectroscopy — *"illustrating the problem"*

resonances, scattering, elastic phase-shifts

lattice QCD — *"introducing the tool"*

discrete spectrum, finite volume, computing the spectrum

elastic scattering — *"solving the simplest problem"*

lattice QCD phase-shift results

coupled-channel scattering — *"a more realistic situation"*

mapping the discrete spectrum to the ***t***-matrix

lattice QCD calculation results

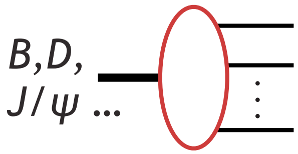
the complex energy plane — *"well-defined quantities"*

rigorously determining resonances

producing meson resonances

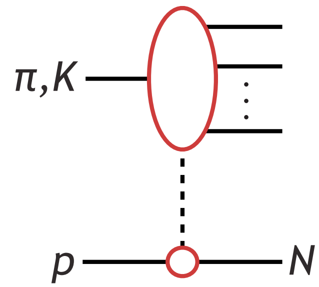
some example processes:

heavy flavour decays



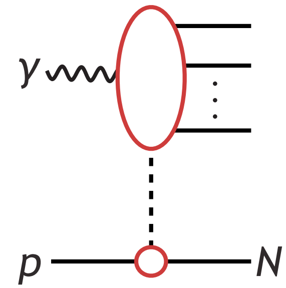
e.g. LHCb

peripheral meson hadroproduction



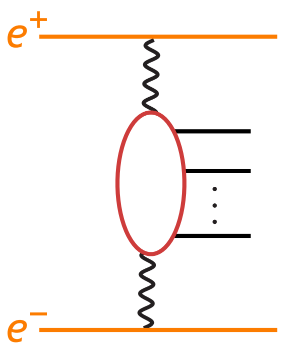
e.g. COMPASS

peripheral meson photoproduction



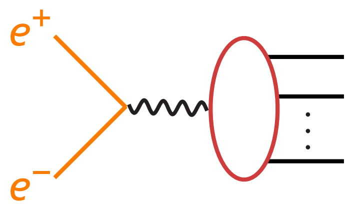
e.g. GlueX

two photon fusion



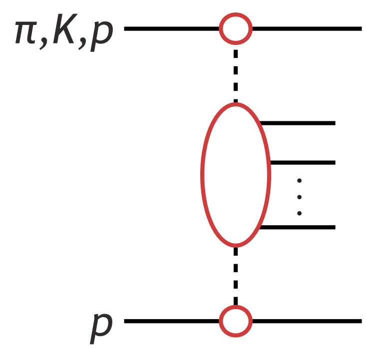
e.g. Belle

e^+e^- annihilation



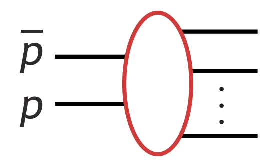
e.g. BES III

central production



e.g. WA102

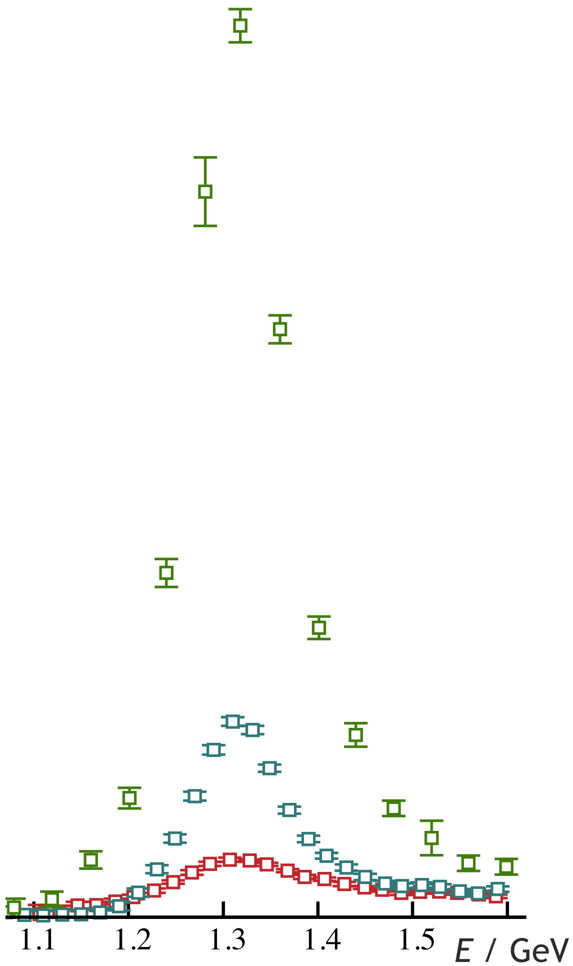
$p\bar{p}$ annihilation



e.g. Crystal Barrel

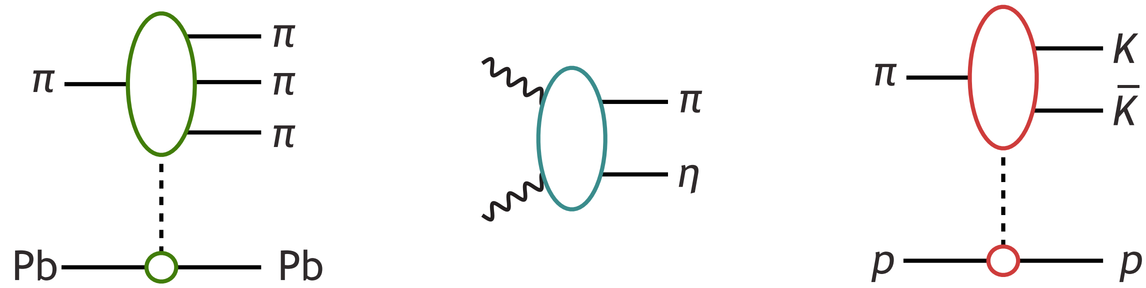
many decades of accumulated data ...

‘straightforward’ coupled-channel resonances



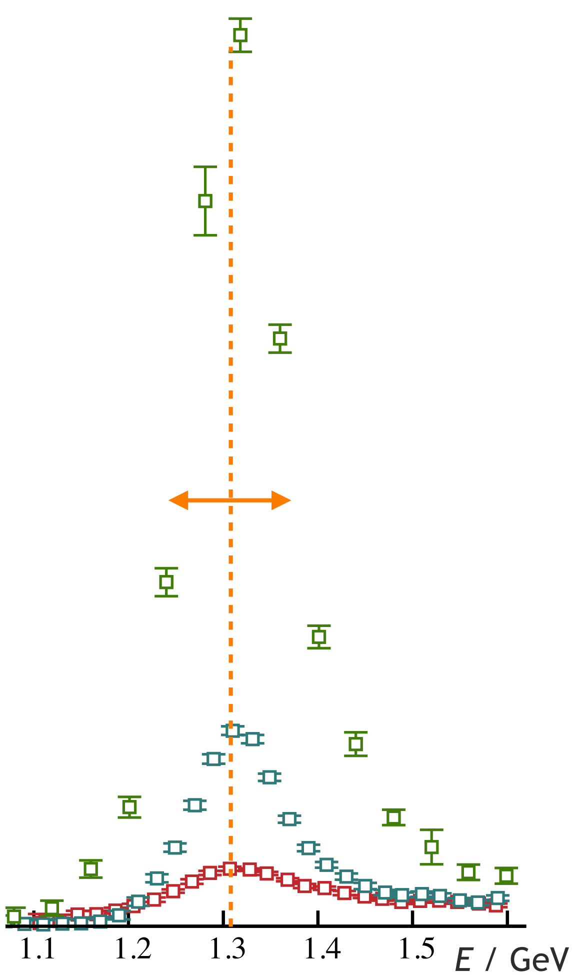
$\pi\,\mathrm{Pb} \to \pi\rho\,\mathrm{Pb}$ COMPASS
$\gamma\gamma \to \pi\eta$ Belle
$\pi p \to K\overline{K}\,p$ CERN SPS

same ‘bump’ appears in multiple different processes



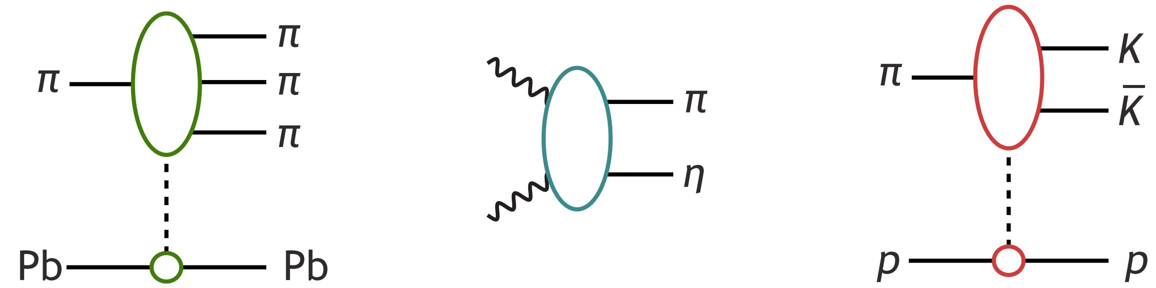
‘straightforward’ **coupled-channel** resonances

$a_2(1320)$

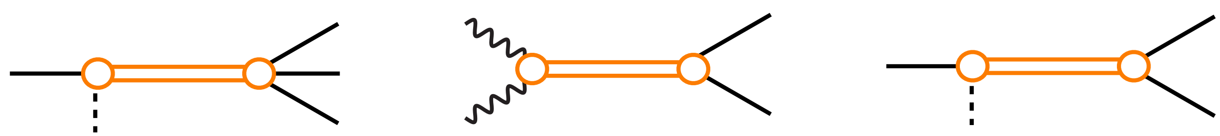


$\pi\,\mathrm{Pb} \to \pi\rho\,\mathrm{Pb}$	COMPASS
$\gamma\gamma \to \pi\eta$	Belle
$\pi p \to K\overline{K}\,p$	CERN SPS

same ‘bump’ appears in multiple different processes …



… due to same **a_2 resonance**



pdg summary entry

$a_2(1320)$ $\quad I^G(J^{PC}) = 1^-(2^{++})$

Mass $m = 1318.3^{+0.5}_{-0.6}$ MeV

Full width $\Gamma = 107 \pm 5$ MeV

$a_2(1320)$ DECAY MODES	Fraction (Γ_i/Γ)
3π	(70.1 ± 2.7) %
$\eta\pi$	(14.5 ± 1.2) %
$\omega\pi\pi$	(10.6 ± 3.2) %
$K\overline{K}$	(4.9 ± 0.8) %
$\eta'(958)\pi$	$(5.5 \pm 0.9) \times 10^{-3}$
$\pi^\pm\gamma$	$(2.91 \pm 0.27) \times 10^{-3}$
$\gamma\gamma$	$(9.4 \pm 0.7) \times 10^{-6}$

the experimental excited meson spectrum

pdg meson listings

LIGHT UNFLAVORED ($S = C = B = 0$)

	$I^G(J^{PC})$
• $\pi^\pm$	$1^-(0^-)$
• π^0	$1^-(0^{-+})$
• η	$0^+(0^{-+})$
• $f_0(500)$	$0^+(0^{++})$
• $\rho(770)$	$1^+(1^{--})$
• $\omega(782)$	$0^-(1^{--})$
• $\eta'(958)$	$0^+(0^{-+})$
• $f_0(980)$	$0^+(0^{++})$
• $a_0(980)$	$1^-(0^{++})$
• $\phi(1020)$	$0^-(1^{--})$
• $h_1(1170)$	$0^-(1^{+-})$
• $b_1(1235)$	$1^+(1^{+-})$
• $a_1(1260)$	$1^-(1^{++})$
• $f_2(1270)$	$0^+(2^{++})$
• $f_1(1285)$	$0^+(1^{++})$
• $\eta(1295)$	$0^+(0^{-+})$
• $\pi(1300)$	$1^-(0^{-+})$
• $a_2(1320)$	$1^-(2^{++})$
• $f_0(1370)$	$0^+(0^{++})$
$h_1(1380)$	$?^-(1^{+-})$
• $\pi_1(1400)$	$1^-(1^{-+})$
• $\eta(1405)$	$0^+(0^{-+})$
• $f_1(1420)$	$0^+(1^{++})$
• $\omega(1420)$	$0^-(1^{--})$
$f_2(1430)$	$0^+(2^{++})$

	$I^G(J^{PC})$
• $\rho_3(1690)$	$1^+(3^{--})$
• $\rho(1700)$	$1^+(1^{--})$
$a_2(1700)$	$1^-(2^{++})$
• $f_0(1710)$	$0^+(0^{++})$
$\eta(1760)$	$0^+(0^{-+})$
• $\pi(1800)$	$1^-(0^{-+})$
$f_2(1810)$	$0^+(2^{++})$
$X(1835)$	$?^?(0^{-+})$
$X(1840)$	$?^?(?^{??})$
$a_1(1420)$	$1^-(1^{++})$
• $\phi_3(1850)$	$0^-(3^{--})$
$\eta_2(1870)$	$0^+(2^{-+})$
• $\pi_2(1880)$	$1^-(2^{-+})$
$\rho(1900)$	$1^+(1^{--})$
$f_2(1910)$	$0^+(2^{++})$
$a_0(1950)$	$1^-(0^{++})$
• $f_2(1950)$	$0^+(2^{++})$
$\rho_3(1990)$	$1^+(3^{--})$
• $f_2(2010)$	$0^+(2^{++})$
$f_0(2020)$	$0^+(0^{++})$
• $a_4(2040)$	$1^-(4^{++})$
• $f_4(2050)$	$0^+(4^{++})$
$\pi_2(2100)$	$1^-(2^{-+})$
$f_0(2100)$	$0^+(0^{++})$
$f_2(2150)$	$0^+(2^{++})$

STRANGE ($S = \pm1,\ C = B = 0$)

	$I(J^P)$
• $K^\pm$	$1/2(0^-)$
• K^0	$1/2(0^-)$
• K^0_S	$1/2(0^-)$
• K^0_L	$1/2(0^-)$
$K^*_0(800)$	$1/2(0^+)$
• $K^*(892)$	$1/2(1^-)$
• $K_1(1270)$	$1/2(1^+)$
• $K_1(1400)$	$1/2(1^+)$
• $K^*(1410)$	$1/2(1^-)$
• $K^*_0(1430)$	$1/2(0^+)$
• $K^*_2(1430)$	$1/2(2^+)$
$K(1460)$	$1/2(0^-)$
$K_2(1580)$	$1/2(2^-)$
$K(1630)$	$1/2(?^?)$
$K_1(1650)$	$1/2(1^+)$
• $K^*(1680)$	$1/2(1^-)$
• $K_2(1770)$	$1/2(2^-)$
• $K^*_3(1780)$	$1/2(3^-)$
• $K_2(1820)$	$1/2(2^-)$
$K(1830)$	$1/2(0^-)$
$K^*_0(1950)$	$1/2(0^+)$
$K^*_2(1980)$	$1/2(2^+)$
• $K^*_4(2045)$	$1/2(4^+)$
$K_2(2250)$	$1/2(2^-)$
$K_3(2320)$	$1/2(3^+)$

CHARMED, STRANGE ($C = S = \pm1$)

	$I(J^P)$
• $D^\pm_s$	$0(0^-)$
• $D^{*\pm}_s$	$0(?^?)$
• $D^*_{s0}(2317)^\pm$	$0(0^+)$
• $D_{s1}(2460)^\pm$	$0(1^+)$
• $D_{s1}(2536)^\pm$	$0(1^+)$
• $D_{s2}(2573)$	$0(2^+)$
• $D^*_{s1}(2700)^\pm$	$0(1^-)$
$D^*_{s1}(2860)^\pm$	$0(1^-)$
$D^*_{s3}(2860)^\pm$	$0(3^-)$
$D_{sJ}(3040)^\pm$	$0(?^?)$

BOTTOM ($B = \pm1$)

• $B^\pm$	$1/2(0^-)$
• B^0	$1/2(0^-)$
• $B^\pm/B^0$ ADMIXTURE	
• $B^\pm/B^0/B^0_s/b$-baryon ADMIXTURE	
V_{cb} and V_{ub} CKM Matrix Elements	
• B^*	$1/2(1^-)$
• $B_1(5721)^+$	$1/2(1^+)$
• $B_1(5721)^0$	$1/2(1^+)$
$B^*_J(5732)$	$?(?^?)$
• $B^*_2(5747)^+$	$1/2(2^+)$
• $B^*_2(5747)^0$	$1/2(2^+)$

$c\bar{c}$

	$I^G(J^{PC})$
• $\eta_c(1S)$	$0^+(0^{-+})$
• $J/\psi(1S)$	$0^-(1^{--})$
• $\chi_{c0}(1P)$	$0^+(0^{++})$
• $\chi_{c1}(1P)$	$0^+(1^{++})$
• $h_c(1P)$	$?^?(1^{+-})$
• $\chi_{c2}(1P)$	$0^+(2^{++})$
• $\eta_c(2S)$	$0^+(0^{-+})$
• $\psi(2S)$	$0^-(1^{--})$
• $\psi(3770)$	$0^-(1^{--})$
• $\psi(3823)$	$?^?(2^{--})$
• $X(3872)$	$0^+(1^{++})$
• $X(3900)$	$1^+(1^{+-})$
• $X(3915)$	$0^+(0/2^{++})$
• $\chi_{c2}(2P)$	$0^+(2^{++})$
$X(3940)$	$?^?(?^{??})$
• $X(4020)$	$1(?^?)$
• $\psi(4040)$	$0^-(1^{--})$
$X(4050)^\pm$	$?(?^?)$
$X(4055)^\pm$	$?(?^?)$
• $X(4140)$	$0^+(1^{++})$
• $\psi(4160)$	$0^-(1^{--})$
$X(4160)$	$?^?(?^{??})$
$X(4200)^\pm$	$?(1^+)$
$X(4230)$	$?^?(1^{--})$
$X(4240)^\pm$	$?^?(0^-)$
$X(4250)^\pm$	$?(?^?)$

pdg.lbl.gov

evolution from scattering '*in*' state to scattering '*out*' state given by *S*-matrix elements $S_{ij} = \langle \mathrm{out}, i \,|\, \mathrm{in}, j \rangle$

e.g. in coupled *ππ*, *K*$\overline{K}$ scattering $\mathbf{S} = \begin{pmatrix} S_{\pi\pi,\pi\pi} & S_{\pi\pi,K\overline{K}} \\ S_{K\overline{K},\pi\pi} & S_{K\overline{K},K\overline{K}} \end{pmatrix}$

more convenient to work with *t*-matrix $\mathbf{S} = \mathbf{1} + 2i\sqrt{\boldsymbol{\rho}} \cdot \mathbf{t} \cdot \sqrt{\boldsymbol{\rho}}$ typically in partial-waves $t^{(\ell)}_{ij}(E)$

in time-reversal invariant theories, ***t*** is symmetric $\Rightarrow \tfrac{1}{2}N(N+1)$ *complex* numbers at each energy?

conservation of probability, a.k.a. **unitarity** is an important constraint

$$\left(S^\dagger S\right)_{ij} = \sum_k \langle \mathrm{in}, i \,|\, \mathrm{out}, k \rangle \langle \mathrm{out}, k \,|\, \mathrm{in}, j \rangle = \delta_{ij}$$

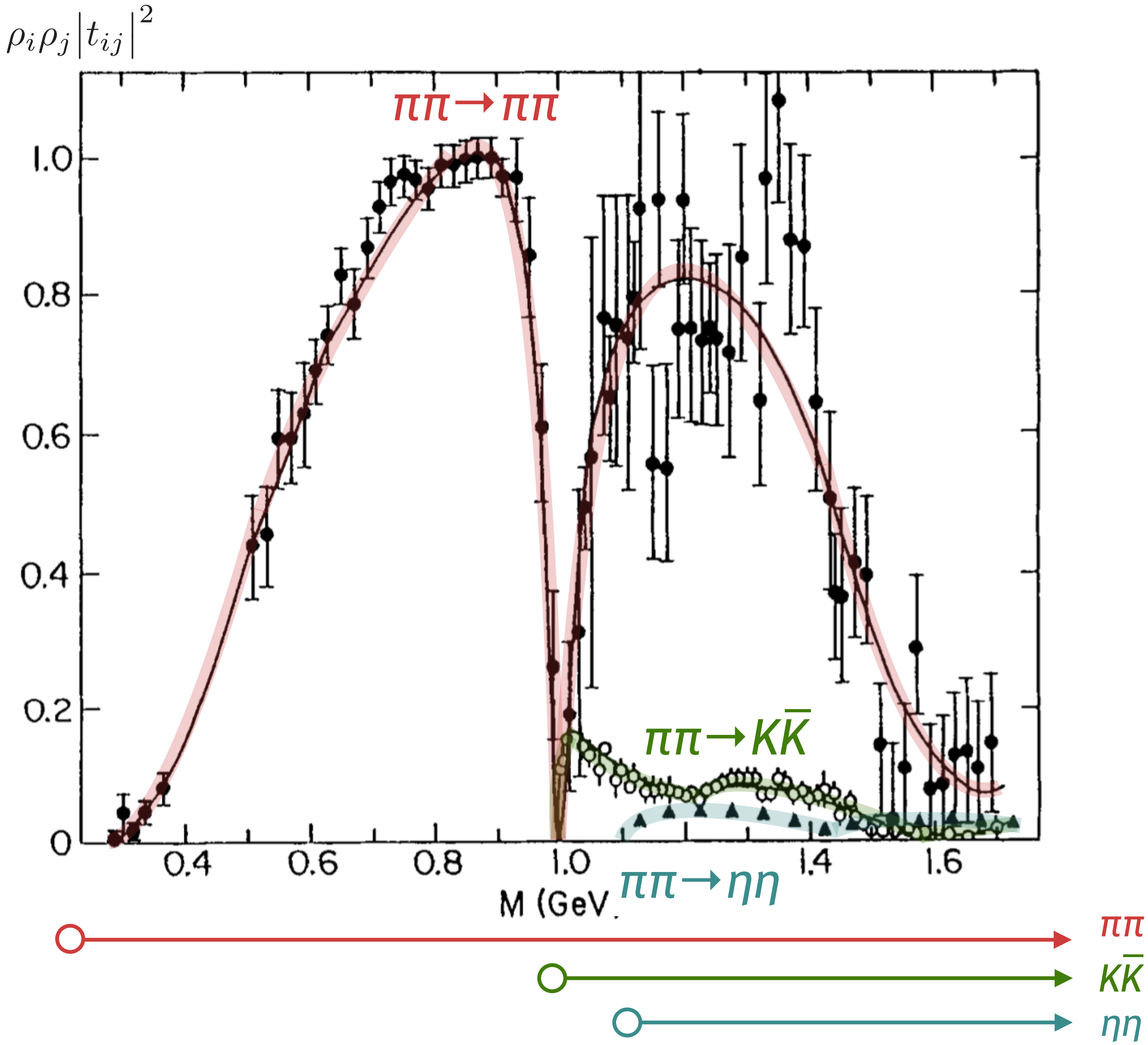
completeness of outgoing states $1 = \sum_k |\, \mathrm{out}, k \rangle \langle \mathrm{out}, k \,|$

$$\mathrm{Im}\, t_{ij} = \sum_k t^*_{ik}\, \rho_k\, t_{kj}$$ sum over channels kinematically open

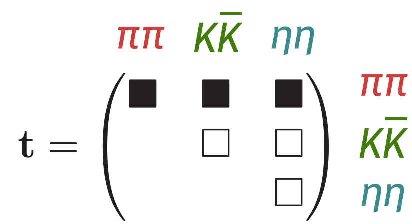
or $$\boxed{\mathrm{Im}\left(t^{-1}(E)\right)_{ij} = -\delta_{ij}\, \rho_i(E)\, \Theta(E - E_i^{\mathrm{thr.}})}$$

$\Rightarrow \tfrac{1}{2}N(N+1)$ ***real*** numbers at each energy

$\pi\pi$, $K\bar{K}$, $\eta\eta$ S-wave scattering



experimentally
quite difficult to fill out
the whole matrix



isolating kaon exchange hard
& η beams don't exist

normalization of $\pi\pi \to K\bar{K}$
also slightly uncertain ...

two-channel scattering

X

a common parameterization uses two phase-shifts, δ_1, δ_2, and an inelastiticity, η

$$S = \begin{pmatrix} \eta\, e^{2i\delta_1} & i\sqrt{1-\eta^2}\, e^{i(\delta_1+\delta_2)} \\ i\sqrt{1-\eta^2}\, e^{i(\delta_1+\delta_2)} & \eta\, e^{2i\delta_2} \end{pmatrix}$$

$$t_{11} = \frac{1}{\rho_1} e^{i\delta_1} \left[\tfrac{1}{2}(\eta+1)\sin\delta_1 - \tfrac{i}{2}(\eta-1)\cos\delta_1 \right]$$

elastic form regained if η→1

$$\rho_1 \rho_2 \left| t_{12} \right|^2 = 1 - \eta^2$$

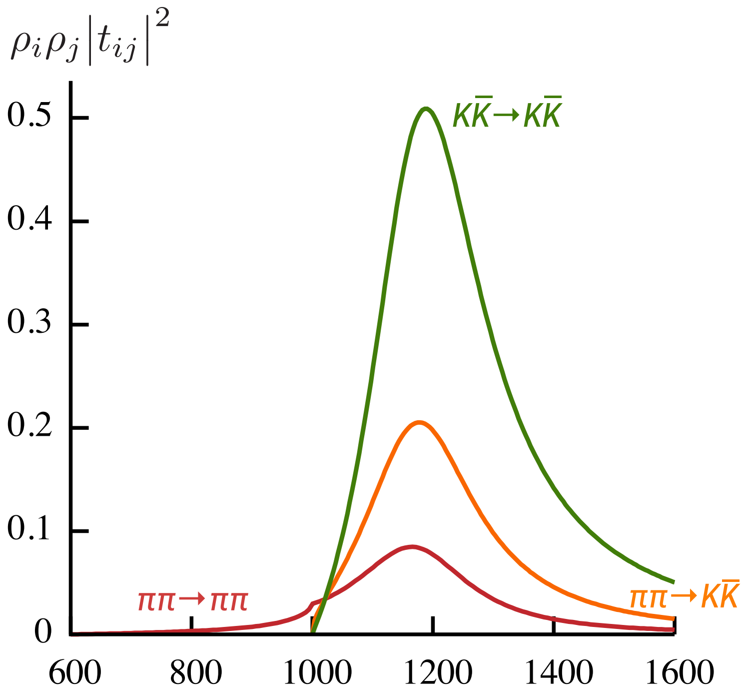
channel coupling given by η≠1

coupled-channel scattering — a simple resonance model

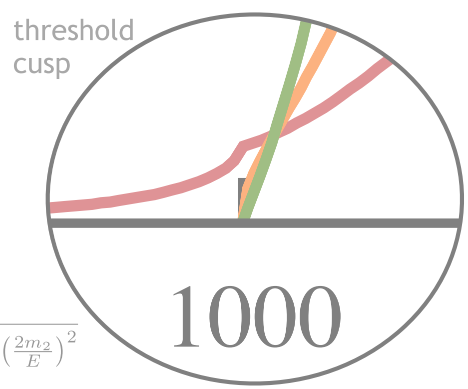
Flatté form — coupled-channel generalisation of Breit-Wigner

m_π = 300 MeV
m_K = 500 MeV

$$t_{ij}(E) = \frac{g_i\, g_j}{m^2 - E^2 - i g_1^2\, \rho_1 - i g_2^2\, \rho_2}$$



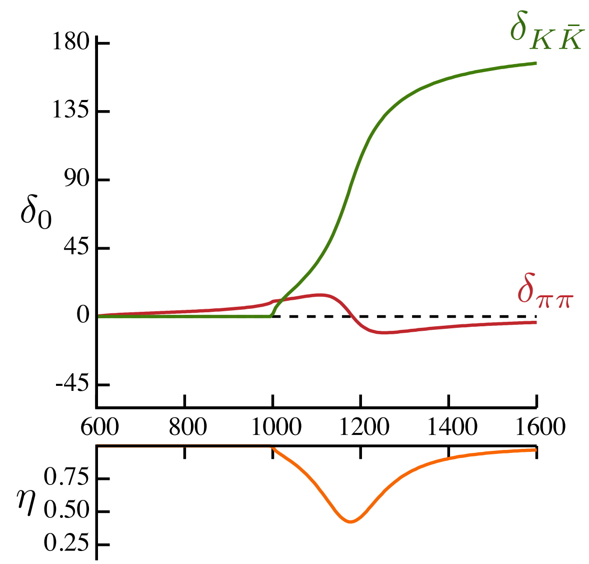
m = 1182 MeV
$g_{\pi\pi}$ = 296 MeV
g_{KK} = 592 MeV



$\rho_2(E) = \sqrt{1 - \left(\frac{2m_2}{E}\right)^2}$

‘phase-shifts’

$$S = \begin{pmatrix} \eta\, e^{2i\delta_1} & i\sqrt{1-\eta^2}\, e^{i(\delta_1+\delta_2)} \\ i\sqrt{1-\eta^2}\, e^{i(\delta_1+\delta_2)} & \eta\, e^{2i\delta_2} \end{pmatrix}$$



coupled-channel scattering in a finite-volume

the quantization condition generalizes to

$$0 = \det\left[\mathbf{1} + i\boldsymbol{\rho}\cdot\mathbf{t}\cdot\left(\mathbf{1} + i\boldsymbol{\mathcal{M}}\right)\right]$$

e.g. in A_1^+ irrep (ℓ = 0, 4 ...)

$$\mathbf{t} = \begin{pmatrix} \begin{pmatrix} t^{(0)}_{11} & t^{(0)}_{12} \\ t^{(0)}_{12} & t^{(0)}_{22} \end{pmatrix} & \mathbf{0} & \cdots \\ \mathbf{0} & \begin{pmatrix} t^{(4)}_{11} & t^{(4)}_{12} \\ t^{(4)}_{12} & t^{(4)}_{22} \end{pmatrix} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

dense in channel space
– infinite-volume dynamics mixes channels

diagonal in angular momentum space
– ℓ good q.n. in infinite-volume

$$\boldsymbol{\mathcal{M}} = \begin{pmatrix} \begin{pmatrix} \mathcal{M}^{A_1^+}_{00}(k_1) & 0 \\ 0 & \mathcal{M}^{A_1^+}_{00}(k_2) \end{pmatrix} & \begin{pmatrix} \mathcal{M}^{A_1^+}_{04}(k_1) & 0 \\ 0 & \mathcal{M}^{A_1^+}_{04}(k_2) \end{pmatrix} & \cdots \\ \begin{pmatrix} \mathcal{M}^{A_1^+}_{40}(k_1) & 0 \\ 0 & \mathcal{M}^{A_1^+}_{40}(k_2) \end{pmatrix} & \begin{pmatrix} \mathcal{M}^{A_1^+}_{44}(k_1) & 0 \\ 0 & \mathcal{M}^{A_1^+}_{44}(k_2) \end{pmatrix} & \cdots \\ \vdots & \vdots & \ddots \end{pmatrix}$$

diagonal in channel space
– no dynamics in $\boldsymbol{\mathcal{M}}$

dense in angular momentum
– cubic symmetry lives here

$$k_1 = \tfrac{1}{2}\sqrt{E^2 - 4m_1^2}$$

$$k_2 = \tfrac{1}{2}\sqrt{E^2 - 4m_2^2}$$

a 3+1 field theory derivation

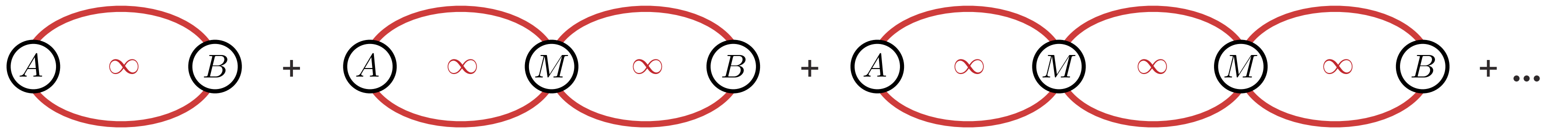
x

consider a two-point correlation function – operators with the quantum numbers of a two-hadron system

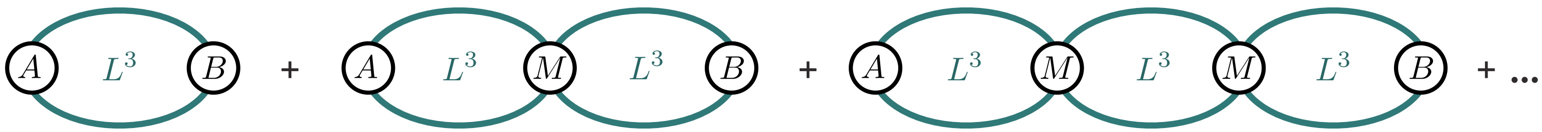
$$C_L(t, \mathbf{P}) = \int_L d^3\mathbf{x} \int_L d^3\mathbf{y}\; e^{-i\mathbf{P}\cdot(\mathbf{x}-\mathbf{y})} \langle 0 | A(\mathbf{x}, t)\, B^\dagger(\mathbf{y}, 0) | 0 \rangle$$

now consider the 'all-orders' skeleton perturbative expansion for this

in infinite volume



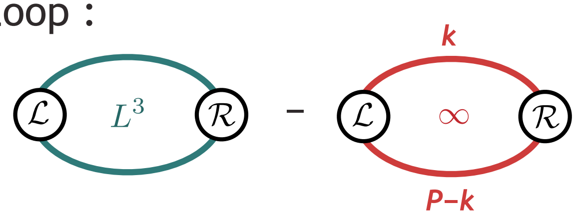
in finite volume



where the colored lines are fully-dressed propagators,
and where we are below three-hadron thresholds, so diagrams with three lines can't go on-shell

a 3+1 field theory derivation

basic loop :



$$= -\left[\frac{1}{L^3}\sum_{\mathbf{k}} - \int \frac{d^3\mathbf{k}}{(2\pi)^3}\right] \int \frac{dk_4}{2\pi}\, \mathcal{L}(P-k,k)\,\Delta(k)\Delta(P-k)\,\mathcal{R}^\dagger(P-k,k)$$

finite volume | infinite volume | dressed propagators [only the poles matter]

performing the k_4 integration

$$= -\left[\frac{1}{L^3}\sum_{\mathbf{k}} - \int \frac{d^3\mathbf{k}}{(2\pi)^3}\right] \frac{1}{2\omega_k}\frac{1}{2\omega_{P-k}}\,\mathcal{L}\,\frac{1}{E-\omega_k-\omega_{P-k}+i\epsilon}\,\mathcal{R}^\dagger\Big|_{k_4=i\omega_k}$$

for smooth functions of ***k***,
the difference between Σ and ∫
is exponentially suppressed

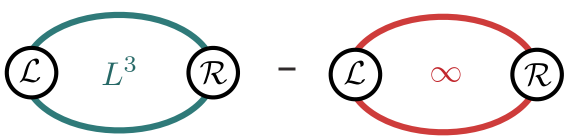
[Poisson summation formula]

but there is a **pole** at

$$E = \omega_k + \omega_{P-k}$$

this ensures **on-shell** dominance in $\mathcal{L}, \mathcal{R}^\dagger$

expanding in partial-waves



$$= -\mathcal{L}_{\ell m}(P)\, F_{\ell m,\ell' m'}(P,L)\, \mathcal{R}^\dagger_{\ell' m'}(P)$$

with $F_{\ell m,\ell' m'}(P,L) = -\left[\frac{1}{L^3}\sum_{\mathbf{k}} - \int \frac{d^3\mathbf{k}}{(2\pi)^3}\right] \frac{4\pi\, Y_{\ell m}(\hat{\mathbf{k}}^\star)\, Y^*_{\ell m}(\hat{\mathbf{k}}^\star)}{2\omega_k\, 2\omega_{P-k}\,(E-\omega_k-\omega_{P-k}+i\epsilon)} \left(\frac{k^\star}{q^\star}\right)^{\ell+\ell'}$

a 3+1 field theory derivation

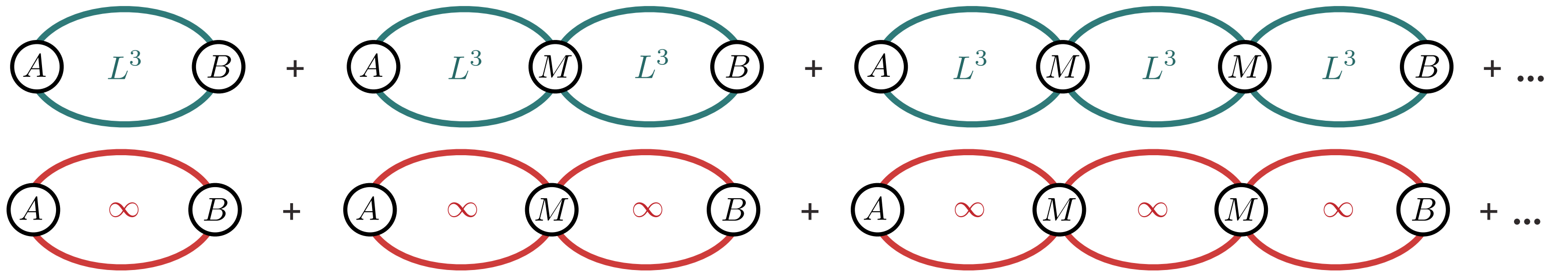
consider a two-point correlation function – operators with the quantum numbers of a two-hadron system

$$C_L(t,\mathbf{P}) = \int_L d^3\mathbf{x} \int_L d^3\mathbf{y}\; e^{-i\mathbf{P}\cdot(\mathbf{x}-\mathbf{y})} \langle 0 \big| A(\mathbf{x},t)\, B^\dagger(\mathbf{y},0) \big| 0 \rangle$$

now consider the 'all-orders' skeleton perturbative expansion for this

in finite volume

in infinite volume



$$C_L - C_\infty = \tilde{A}\,(-F)\,\tilde{B} + \tilde{A}\,(-F) M (-F)\,\tilde{B} + \tilde{A}\,(-F) M (-F) M (-F)\,\tilde{B} + \dots$$

a geometric series can be summed: $\tilde{A}\,\left[F^{-1} + M\right]^{-1}\tilde{B}$

giving $C_L(t,\mathbf{P}) = L^3 \int \frac{dE}{2\pi}\, e^{iEt} \Big[C_\infty(E,\mathbf{P}) - \tilde{A}\,\left[F^{-1}(E,\mathbf{P},L) + M(E,\mathbf{P})\right]^{-1} \tilde{B} \Big]$

discrete spectral decomposition for finite-volume requires poles in *E*

⇒ divergence of $\left[F^{-1}(E,\mathbf{P},L) + M(E,\mathbf{P})\right]^{-1}$

$$\Rightarrow \det\left[F^{-1}(E,\mathbf{P},L) + M(E,\mathbf{P})\right] = 0$$

a 3+1 field theory derivation

$$\det\left[F^{-1}(E,\mathbf{P},L)+M(E,\mathbf{P})\right]=0$$

$$0=\det\left[\mathbf{1}+i\boldsymbol{\rho}\cdot\mathbf{t}\cdot\left(\mathbf{1}+i\boldsymbol{\mathcal{M}}\right)\right]$$

formalism dictionary:

$$16\pi\, F_{\ell m,\ell' m'}=i\rho\,\delta_{\ell\ell'}\delta_{mm'}-\rho\mathcal{M}_{\ell m,\ell' m'}$$

$$M=16\pi\, t$$

coupled-channel scattering in a finite-volume

the quantization condition generalizes to $$0 = \det\left[\mathbf{1} + i\boldsymbol{\rho}\cdot\mathbf{t}\cdot\left(\mathbf{1} + i\boldsymbol{\mathcal{M}}\right)\right]$$

can also be expressed as $0 = \det\left[\mathbf{t}^{-1} + i\boldsymbol{\rho} - \boldsymbol{\mathcal{M}}\cdot\boldsymbol{\rho}\right]$

which exposes the role of unitarity $\mathrm{Im}\left(t^{-1}(E)\right)_{ij} = -\delta_{ij}\,\rho_i(E)\,\Theta(E - E_i^{\mathrm{thr.}})$

the quantization condition is a **single real condition**:

the zeroes E=E_n(L) of the function $\det\left[\mathbf{1} + i\boldsymbol{\rho}(E)\cdot\mathbf{t}(E)\cdot\left(\mathbf{1} + i\boldsymbol{\mathcal{M}}(E,L)\right)\right]$

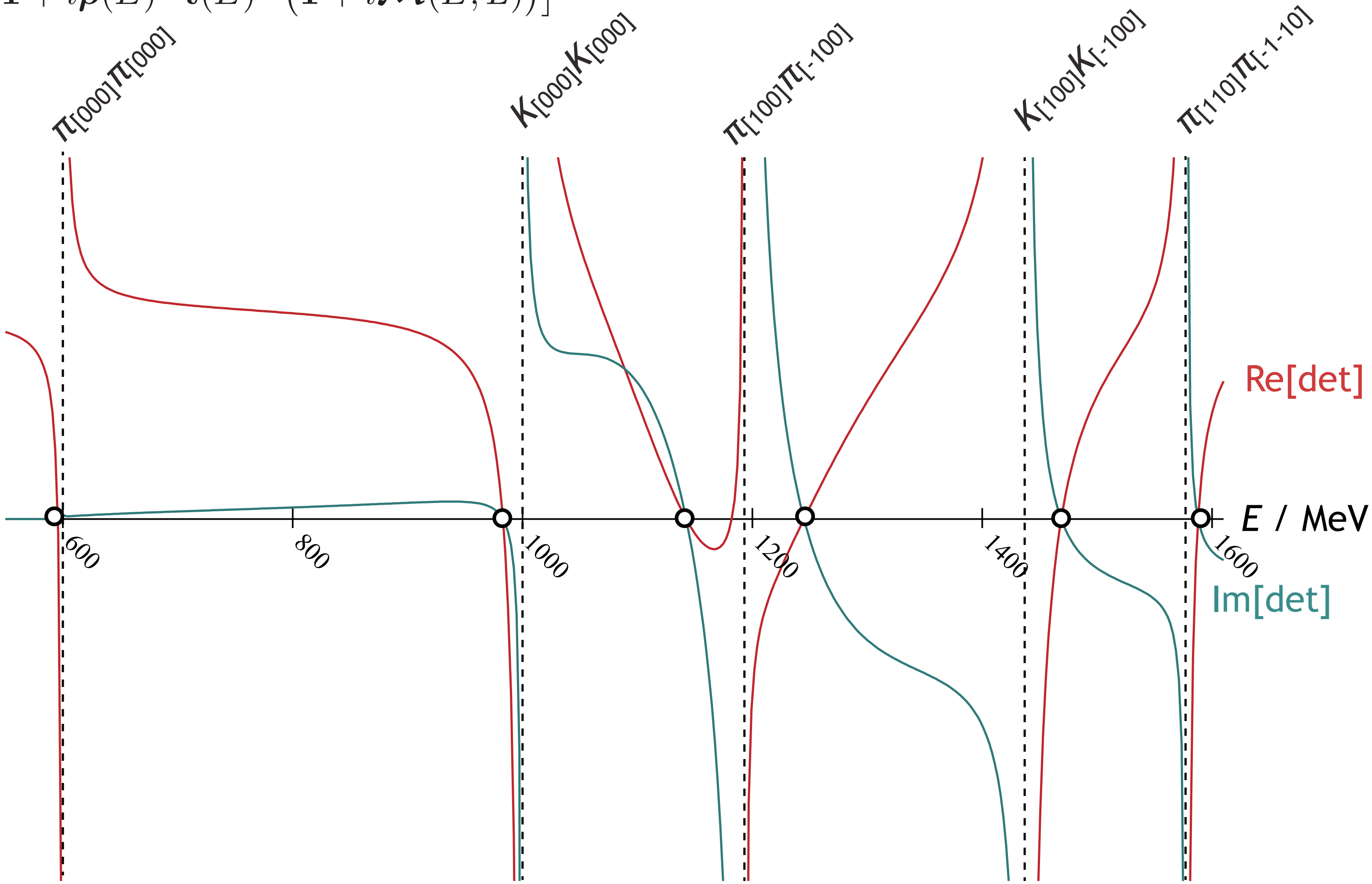
correspond to the spectrum in an $L\times L\times L$ volume

zeroes of the determinant

e.g. previously presented two-channel Flatté form — [000] A_1^+ irrep in L=2.4 fm box

m_π = 300 MeV
m_K = 500 MeV

$$\det\left[\mathbf{1} + i\boldsymbol{\rho}(E)\cdot\mathbf{t}(E)\cdot\big(\mathbf{1}+i\boldsymbol{\mathcal{M}}(E,L)\big)\right]$$



numerical root-finding exercise in practice

a more practical approach

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X

don't take the determinant — look at the matrix eigenvalues …

$$0 = \det \mathbf{D}(E_{\mathrm{cm}})$$

matrix $\mathbf{D} = \mathbf{1} + i\boldsymbol{\rho}\mathbf{t}\big(\mathbf{1} + i\boldsymbol{\mathcal{M}}\big)$

inconvenient — eigenvalues are unbounded, houses divergences

perform a transformation to a matrix with the same determinant

$$\mathbf{D} = \tfrac{1}{2}\boldsymbol{\rho}^{1/2}\big(\mathbf{1} + \mathbf{S}\mathbf{V}\big)\big(\mathbf{1} - i\boldsymbol{\mathcal{M}}\big)\boldsymbol{\rho}^{-1/2}$$

unitary matrices

$$\mathbf{S} = \mathbf{1} + 2i\boldsymbol{\rho}^{1/2}\,\mathbf{t}\,\boldsymbol{\rho}^{1/2}$$

$$\mathbf{V} = \big(\mathbf{1} + i\boldsymbol{\mathcal{M}}\big)\,\big(\mathbf{1} - i\boldsymbol{\mathcal{M}}\big)^{-1}$$

$$\mathbf{D}_V = \mathbf{1} + \mathbf{S}\,\mathbf{V}$$

eigenvalues are bounded

$$\lambda_p(E_{\mathsf{cm}}) = 2\,e^{i\frac{1}{2}\theta_p(E_{\mathsf{cm}})}\cos\tfrac{1}{2}\theta_p(E_{\mathsf{cm}})$$

$\mathbf{D}_V - \mathbf{1}$ *is unitary*

eigenvectors are orthogonal

find the zeroes of the eigenvalues …

a more practical approach

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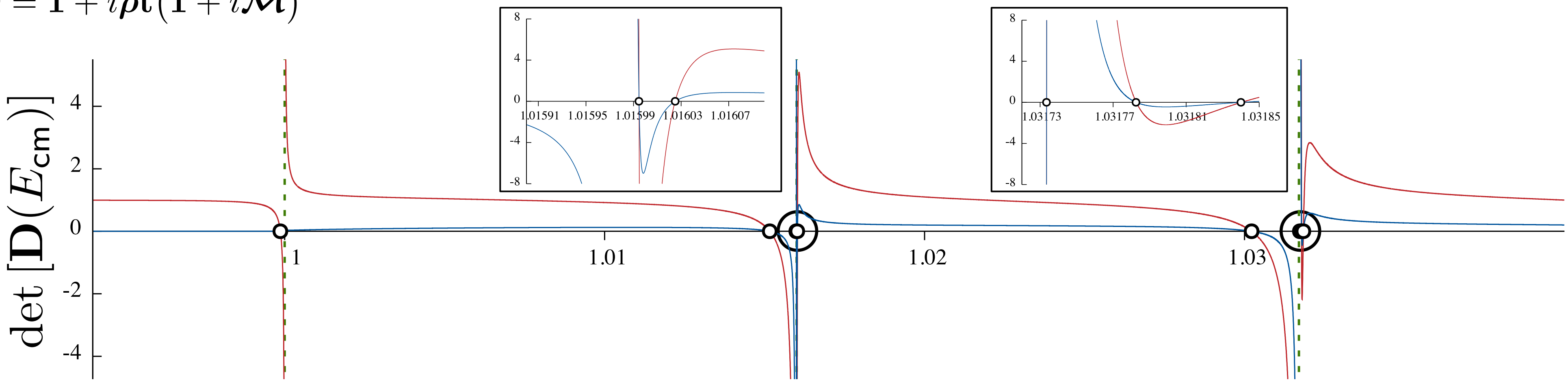
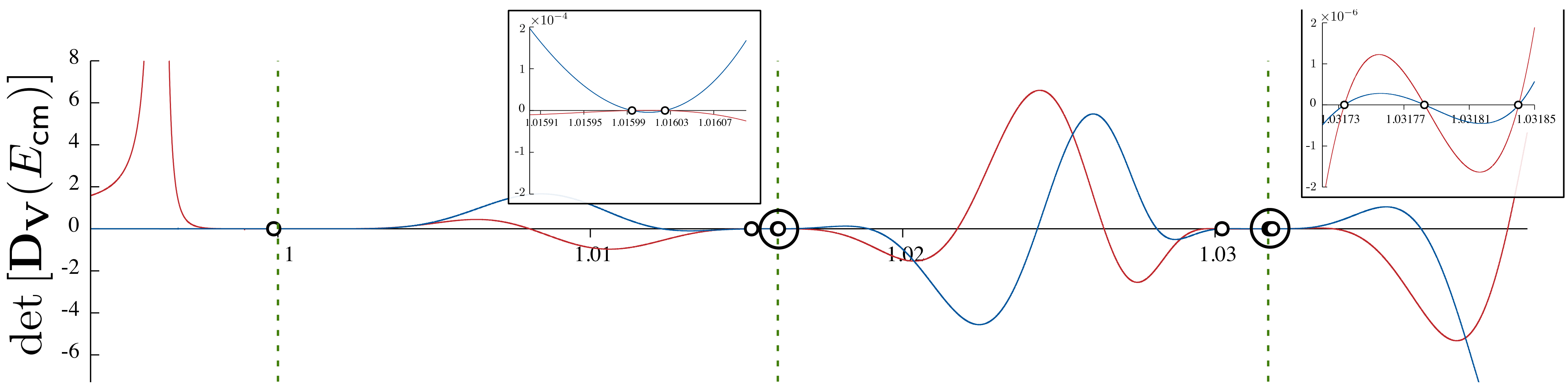
$$\mathbf{D} = \mathbf{1} + i\boldsymbol{\rho}\mathbf{t}(\mathbf{1} + i\boldsymbol{\mathcal{M}})$$



divergences & closely spaced solutions

$$\mathbf{D}_V = \mathbf{1} + \mathbf{S}\,\mathbf{V}$$



removed divergences, but still closely spaced solutions

a more practical approach

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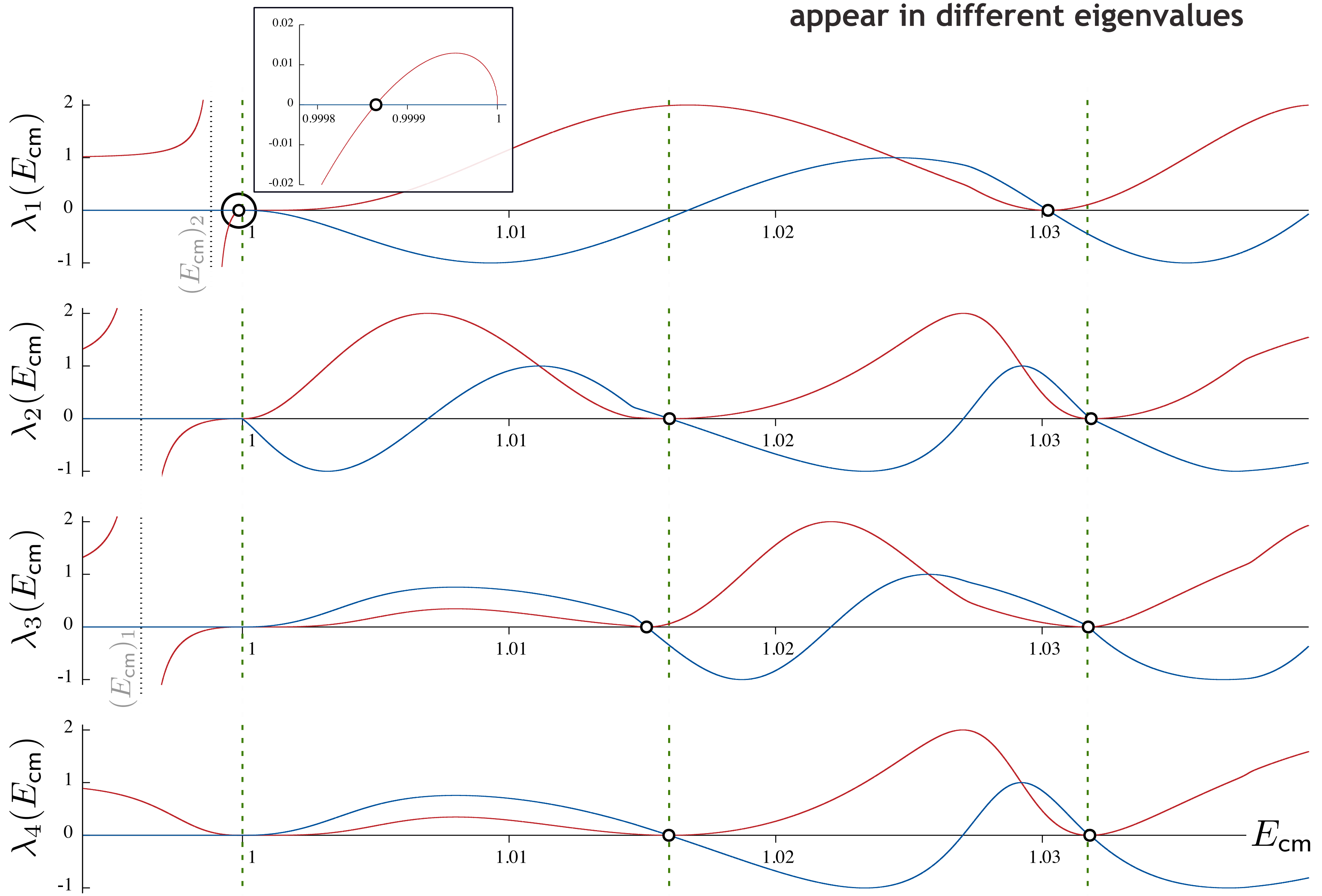
$$\mathbf{D}_V = \mathbf{1} + \mathbf{S}\,\mathbf{V}$$

closely spaced solutions
appear in different eigenvalues



a more practical approach

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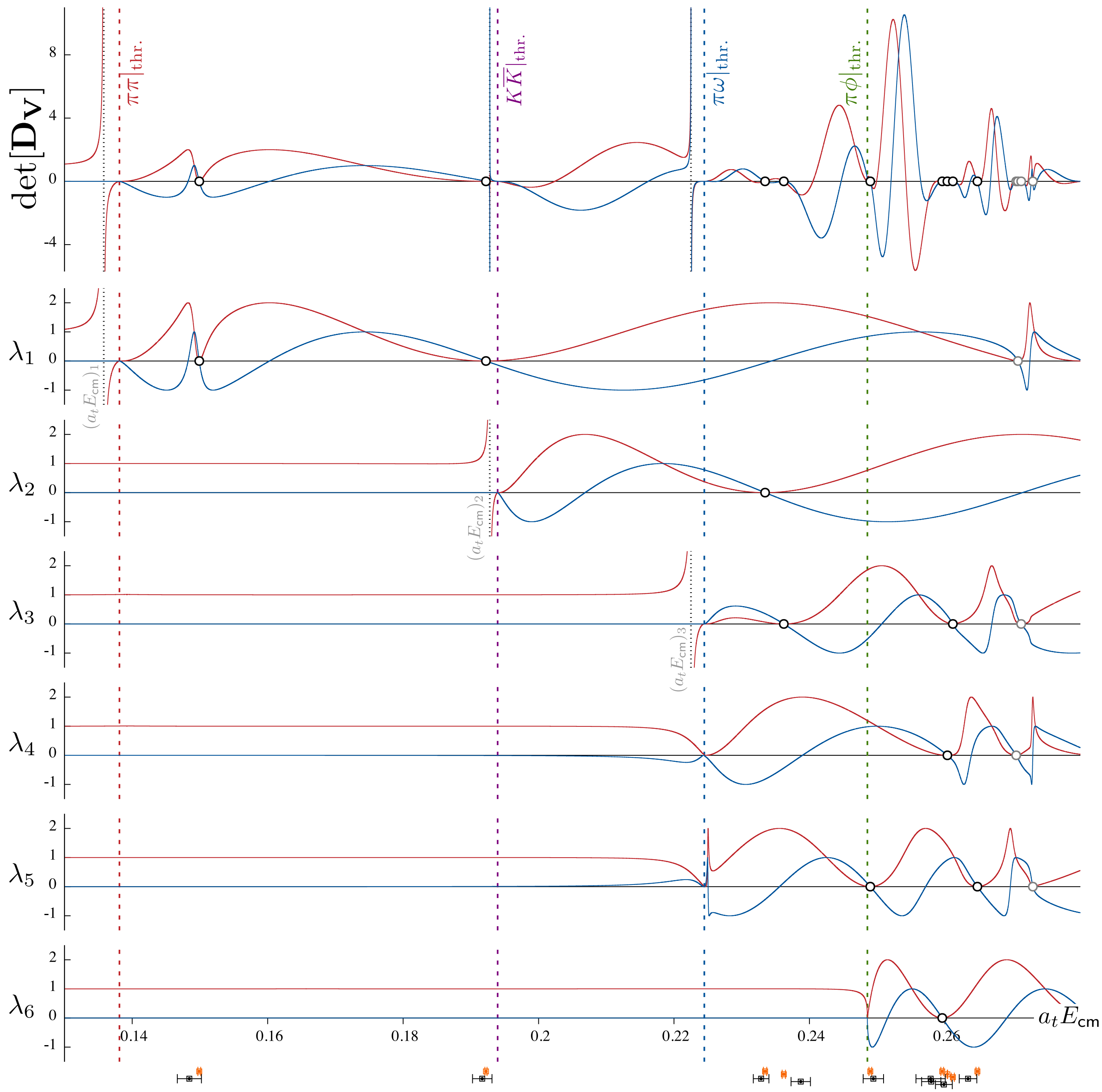
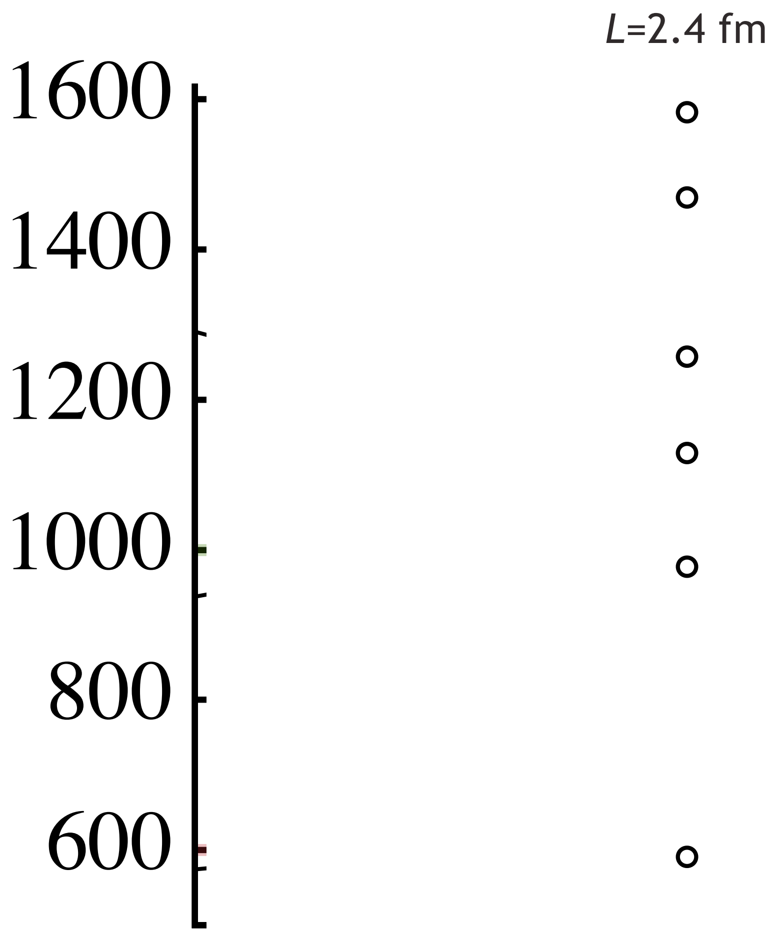
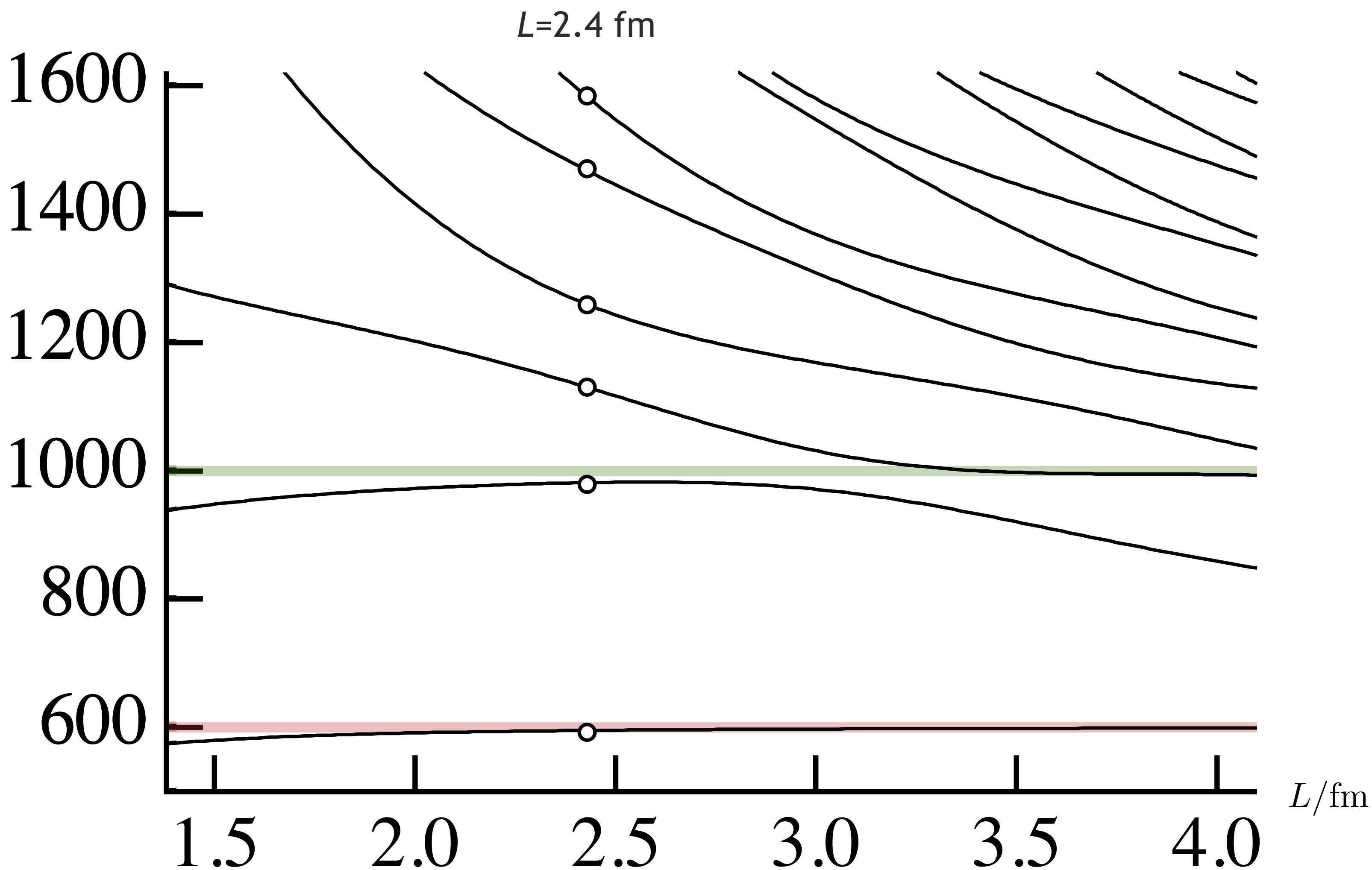
very powerful in
coupled-channel situation

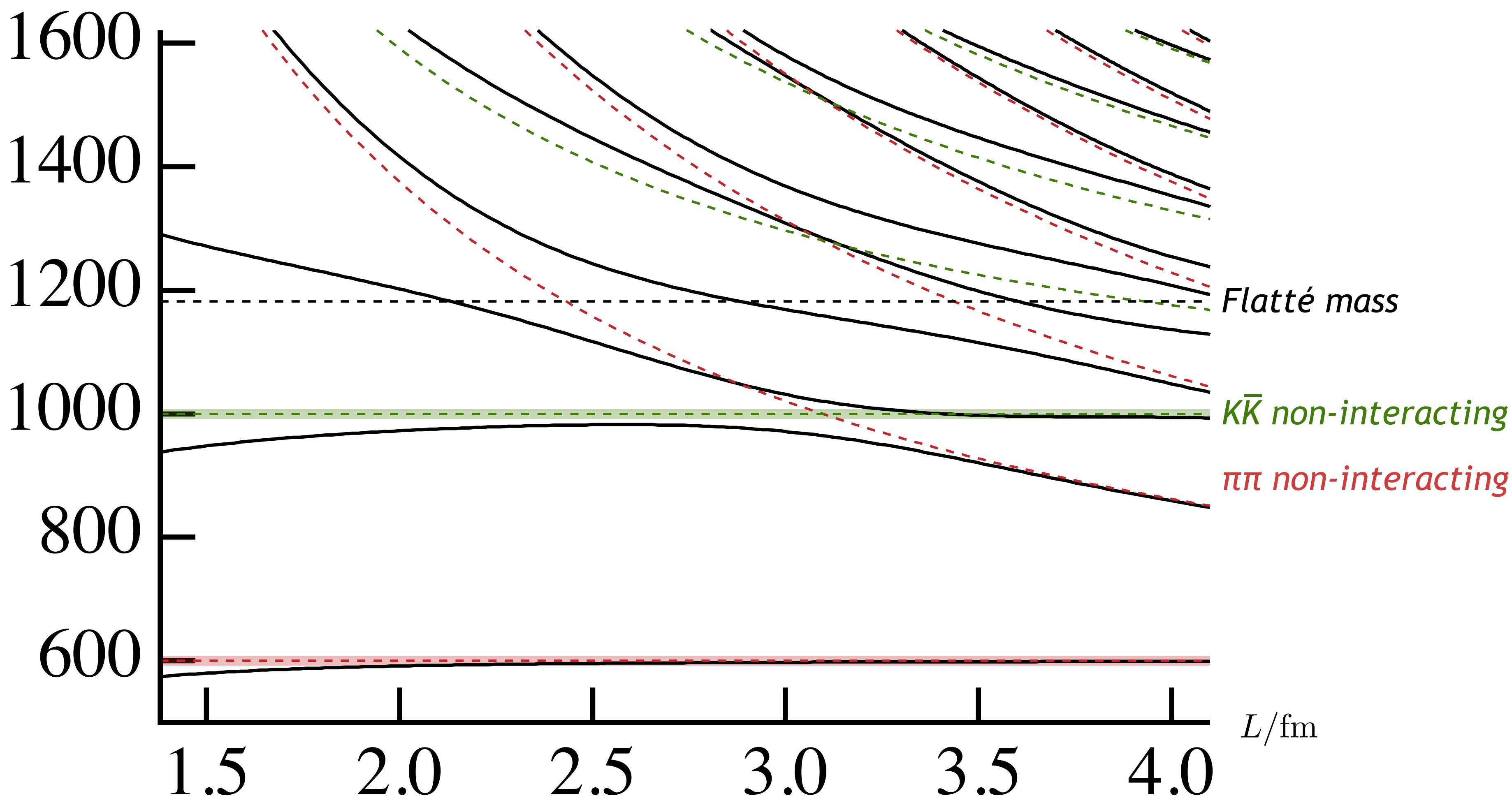


zeroes of the determinant

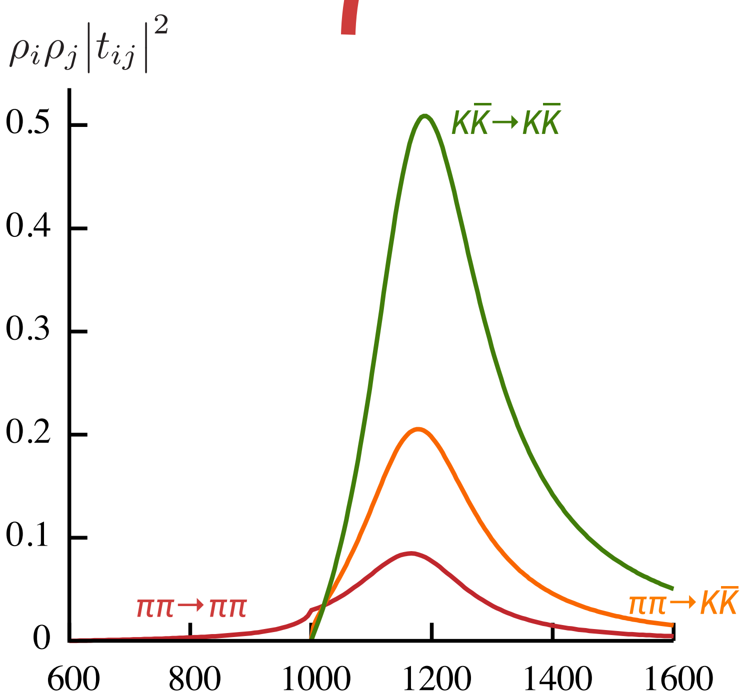


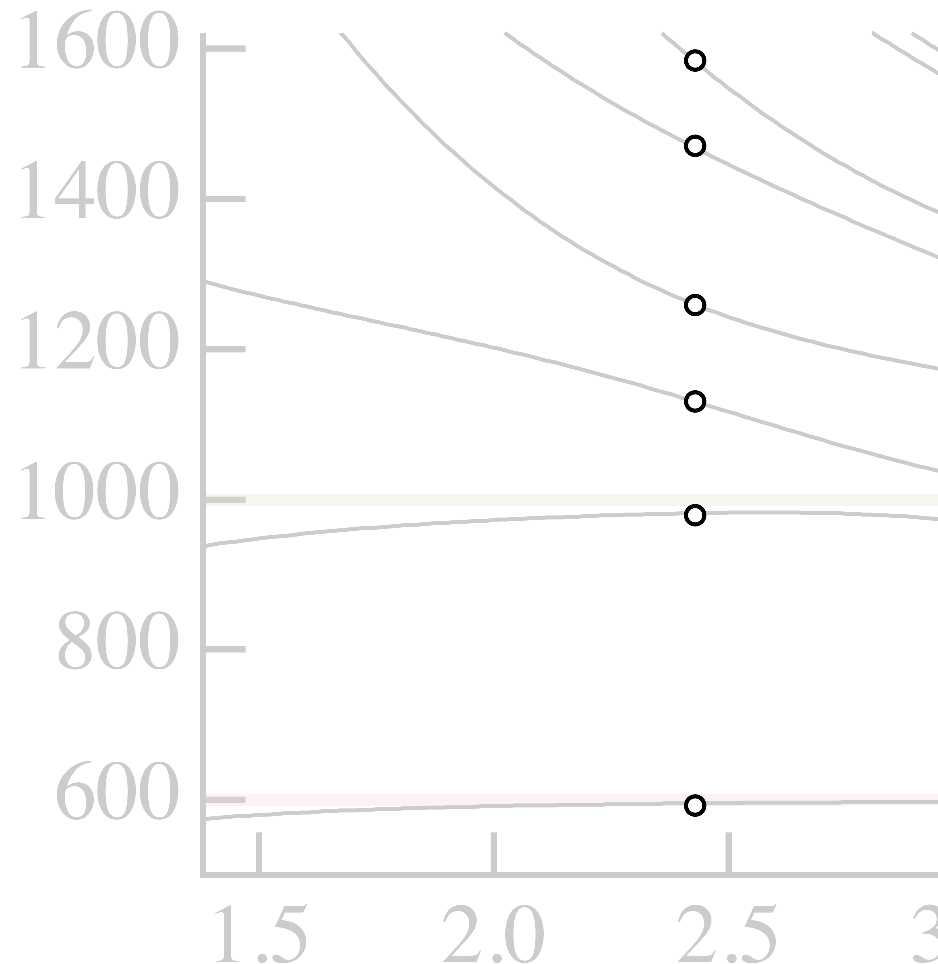

L=2.4 fm
1600
1400
1200
1000
800
600
1.5
2.0
2.5
3.0
3.5
4.0
L/fm

finite-volume spectrum



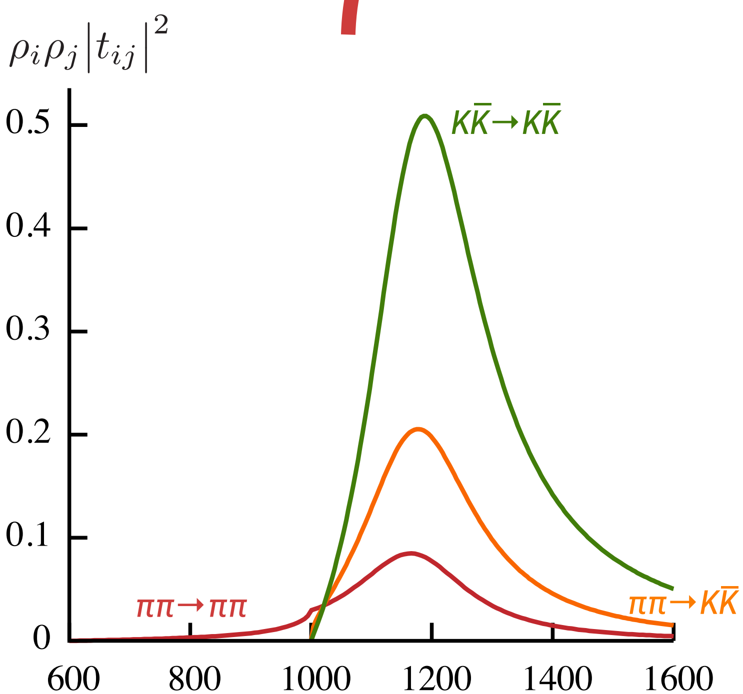
$$0 = \det\left[\mathbf{1} + i\boldsymbol{\rho}\cdot\mathbf{t}\cdot\left(\mathbf{1} + i\boldsymbol{\mathcal{M}}\right)\right]$$

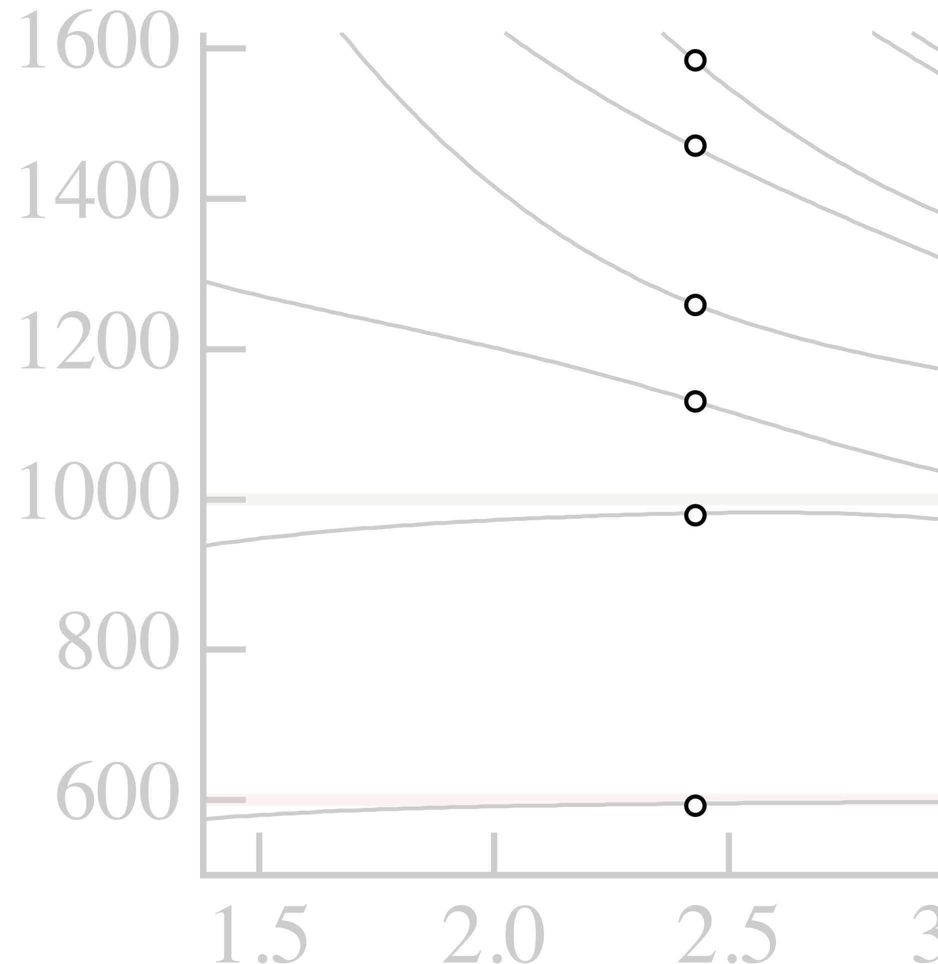




finite-volume approach

$$0 = \det\left[\mathbf{1} + i\boldsymbol{\rho}\cdot\mathbf{t}\cdot\left(\mathbf{1} + i\boldsymbol{\mathcal{M}}\right)\right]$$



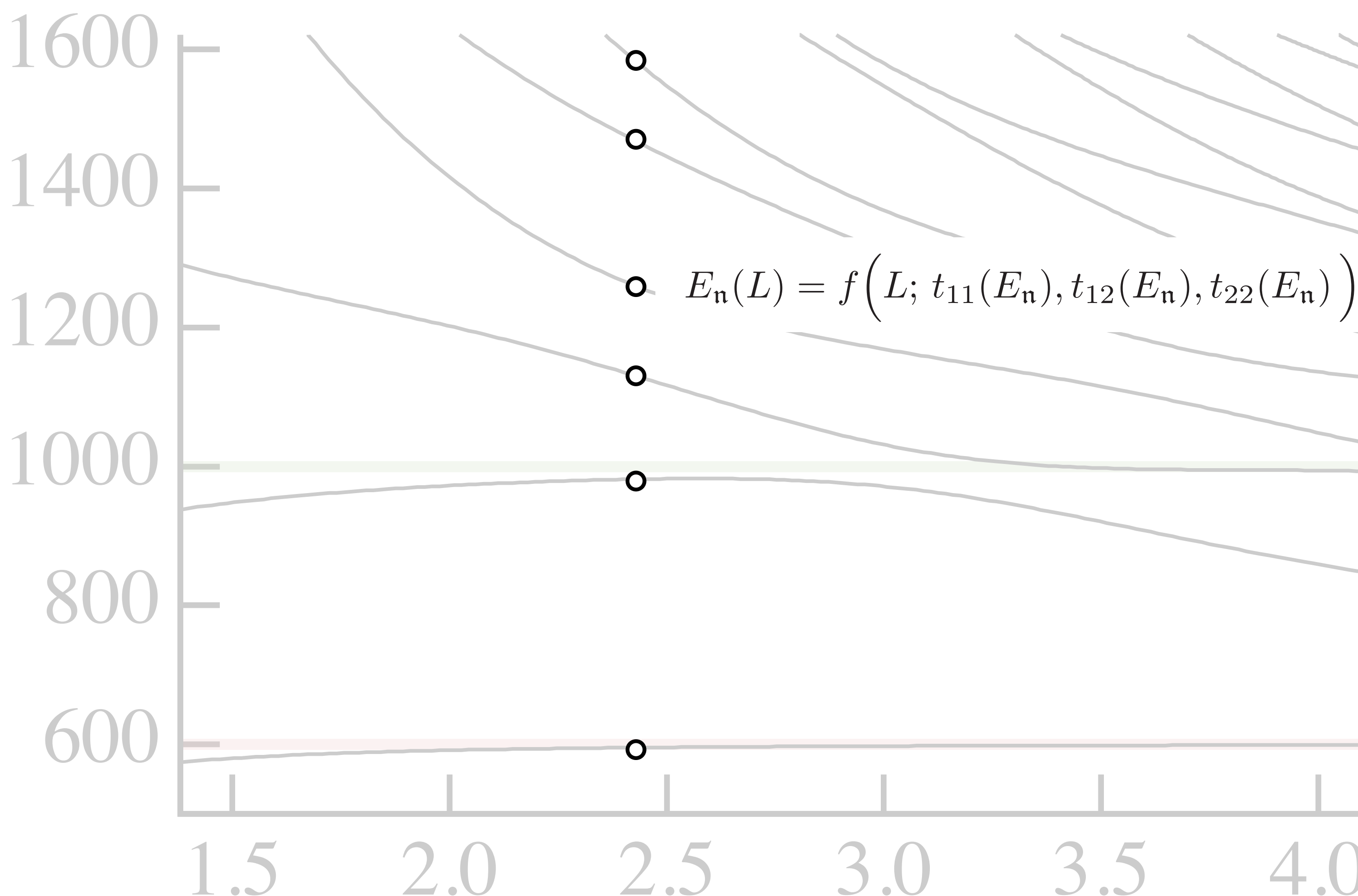


but in a lattice QCD calculation
we have the inverse problem ...

?

finite-volume approach

position of each energy level depends upon all elements of the ***t***-matrix



$$E_\mathfrak{n}(L) = f\Big(L;\, t_{11}(E_\mathfrak{n}), t_{12}(E_\mathfrak{n}), t_{22}(E_\mathfrak{n})\Big)$$

$$0 = \det\big[\mathbf{1} + i\boldsymbol{\rho}\cdot\mathbf{t}\cdot\big(\mathbf{1} + i\boldsymbol{\mathcal{M}}\big)\big]$$

at $E = E_n(L)$
is one equation in three unknowns ...

parameterizing the *t*-matrix

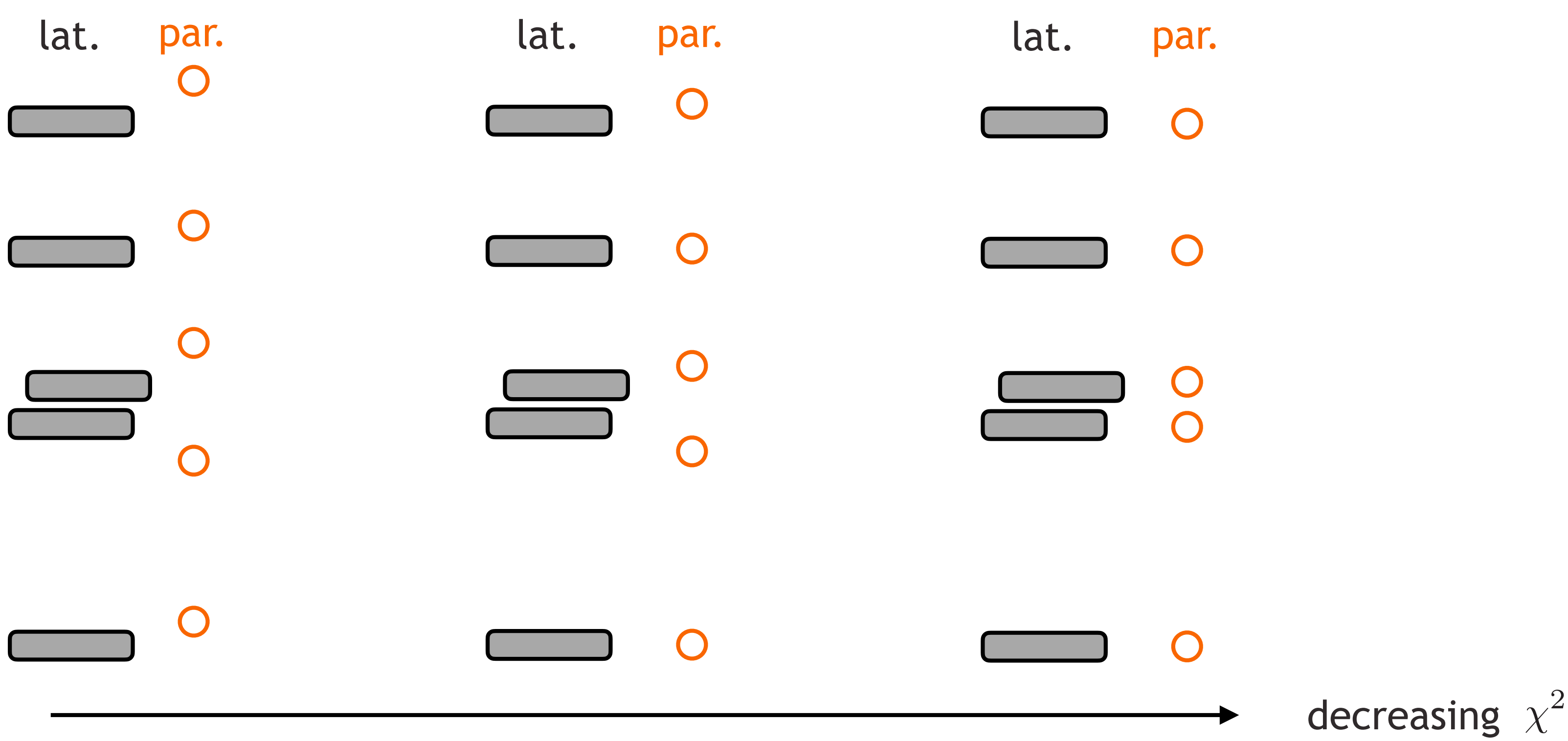
a solution is to propose that different energies are not unrelated – parameterize $\mathbf{t}\big(E;\{a_i\}\big)$

then can use many energy levels to constrain the parameters by minimising a χ^2

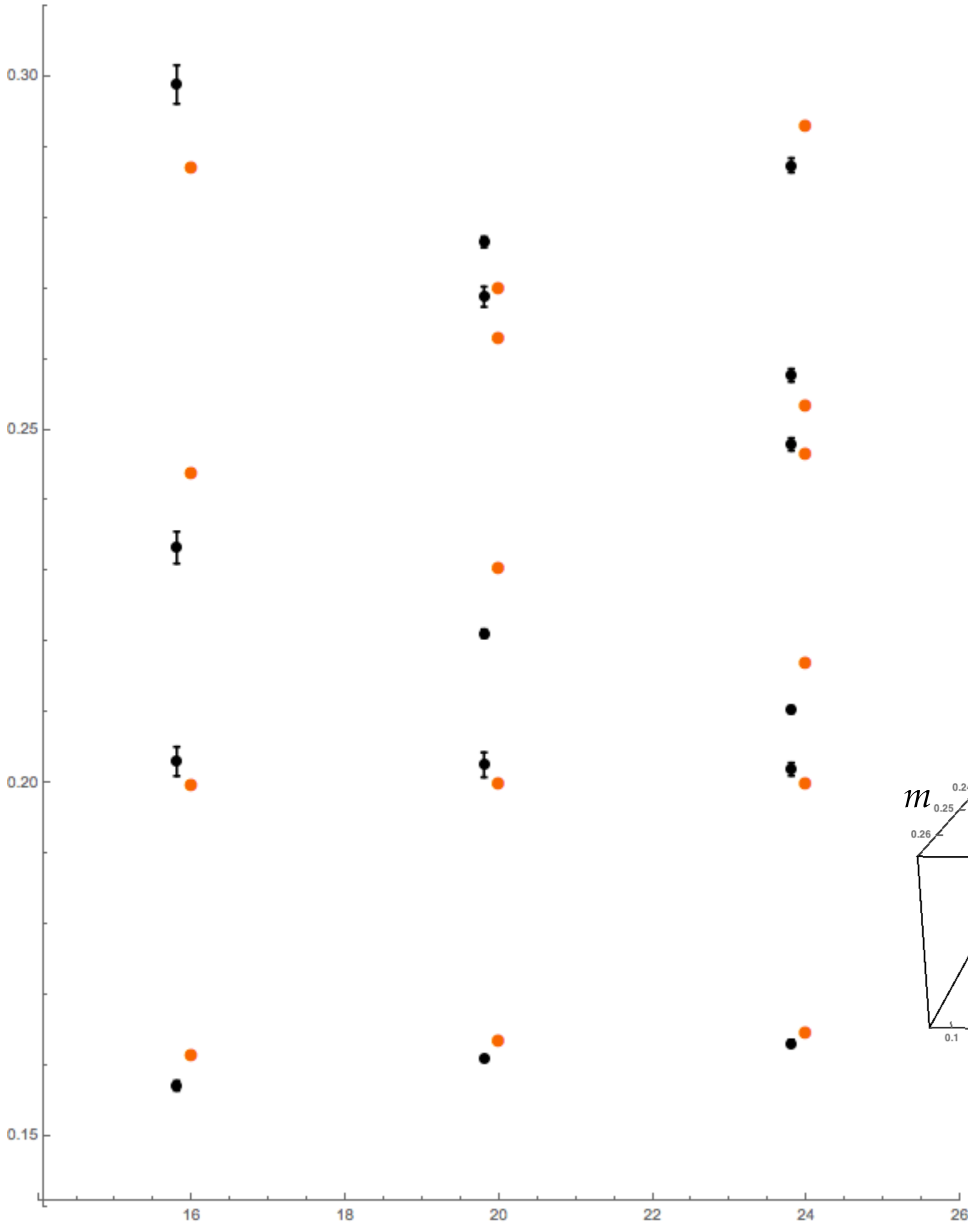
$$\chi^2\big(\{a_i\}\big) = \sum_{\mathfrak{n},\mathfrak{n}'} \Big(E^{\text{lat.}}_{\mathfrak{n}} - E^{\text{par.}}_{\mathfrak{n}}(L;\{a_i\})\Big)\, \mathbb{C}^{-1}_{\mathfrak{n},\mathfrak{n}'} \Big(E^{\text{lat.}}_{\mathfrak{n}'} - E^{\text{par.}}_{\mathfrak{n}'}(L;\{a_i\})\Big)$$

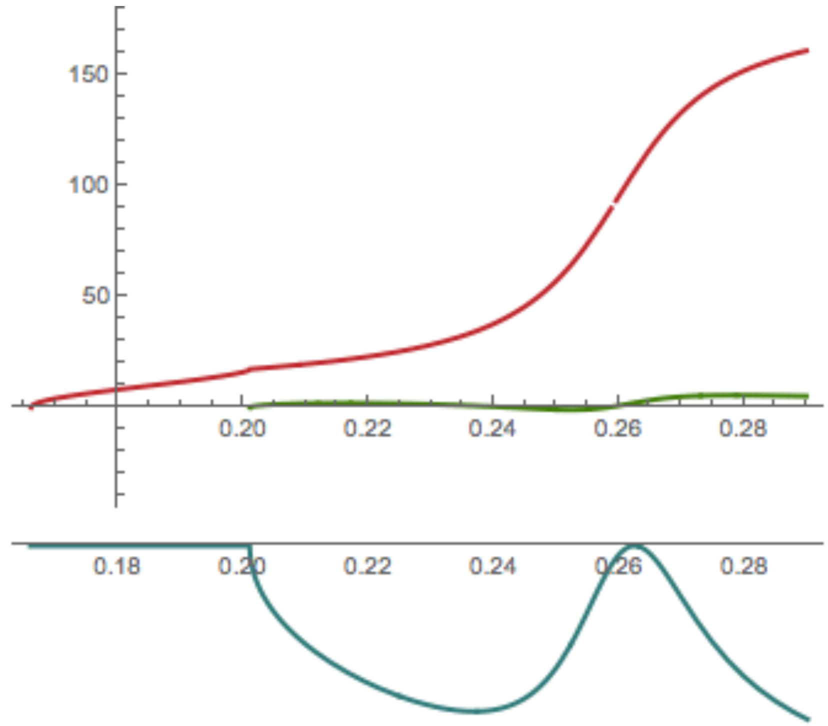
inverse data covariance

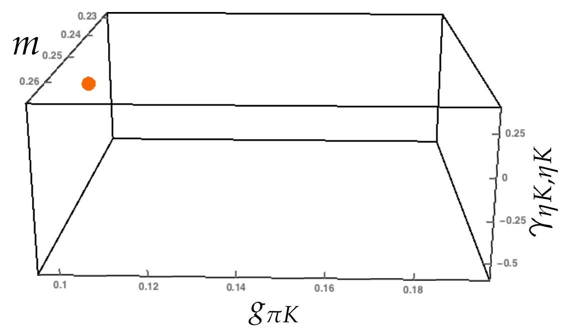
energy levels solving $0 = \det\big[\mathbf{1} + i\boldsymbol{\rho}\cdot\mathbf{t}\cdot(\mathbf{1}+i\boldsymbol{\mathcal{M}})\big]$ for $\mathbf{t}\big(E;\{a_i\}\big)$

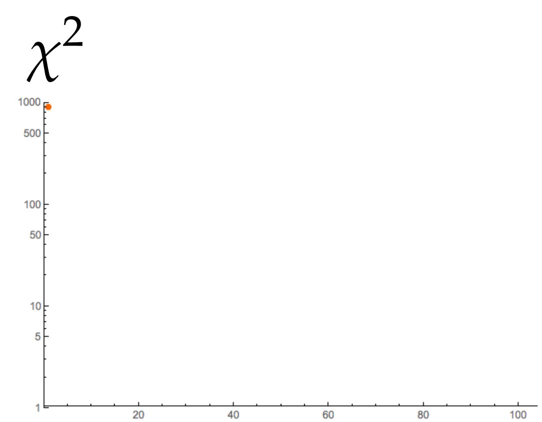


an example minimization









parameterizing the *t*-matrix

a solution is to propose that different energies are not unrelated – parameterize $\mathbf{t}\big(E;\{a_i\}\big)$

need to ensure multi-channel unitarity $\mathrm{Im}\,\big(t^{-1}(E)\big)_{ij} = -\delta_{ij}\,\rho_i(E)\,\Theta(E - E_i^{\mathrm{thr.}})$

– ***K*-matrix approach**

$$\mathbf{t}^{-1}(E) = \mathbf{K}^{-1}(E) + \mathbf{I}(E) \qquad \text{with} \qquad \mathrm{Im}\,\big(I(E)\big)_{ij} = -\delta_{ij}\,\rho_i(E)$$

simplest choice has $\mathrm{Re}\,\mathbf{I}(E) = 0$

a more sophisticated approach = "Chew-Mandelstam" phase-space

***K*(*E*)** should be a real symmetric matrix

for reasons you'll see later,
better to parameterize in terms of $s = E^2$

e.g. $K_{ij} = \dfrac{g_i\,g_j}{m^2 - s}$ gives the Flatté form

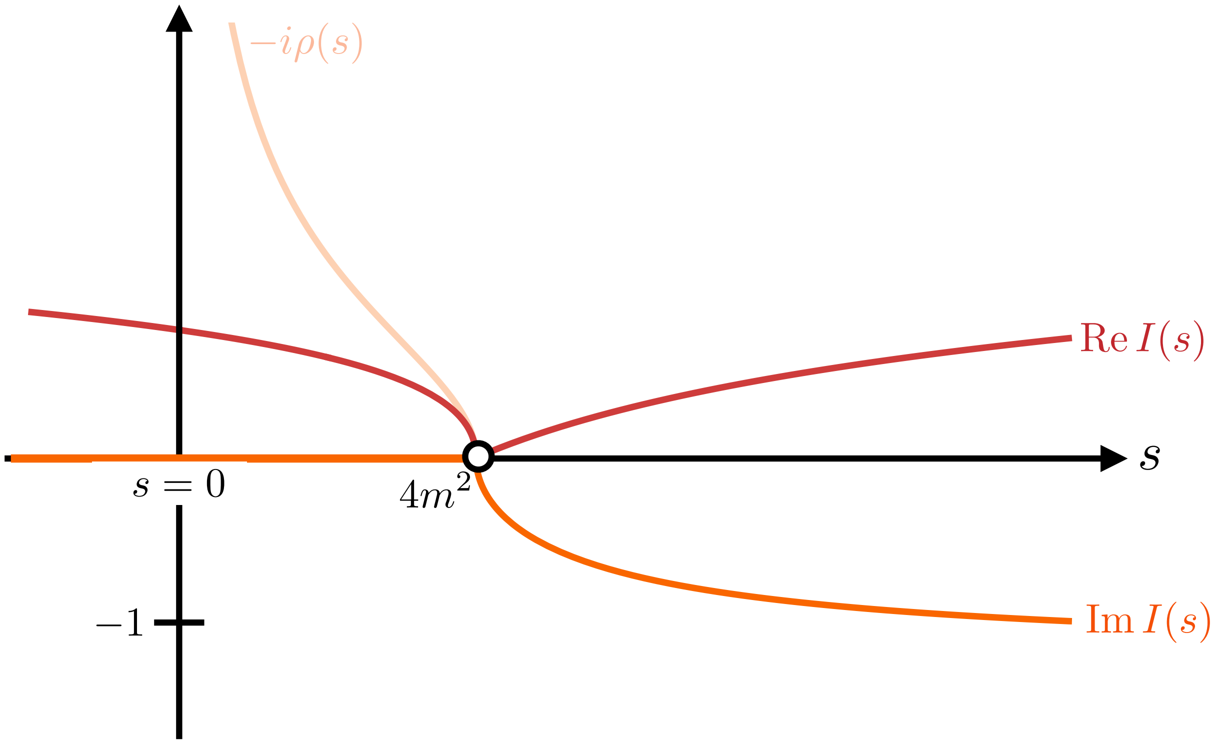
Chew-Mandelstam phase space

(subtracted) dispersion of the phase-space

$$I(s) = I(s_0) - \frac{s - s_0}{\pi} \int_{s_{\mathrm{thr}}}^{\infty} ds' \frac{\rho(s')}{(s' - s_0)(s' - s)}$$

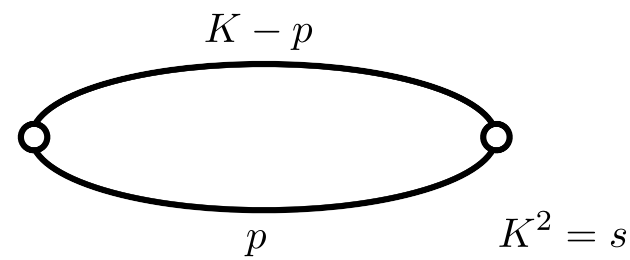
in the equal mass case evaluates to

$$I(s) = I(4m^2) - \frac{\rho(s)}{\pi} \log\left[\frac{1 - \rho(s)}{1 + \rho(s)}\right] - i\rho(s)$$



notice the smooth behavior below threshold
& absence of a singularity at *s*=0

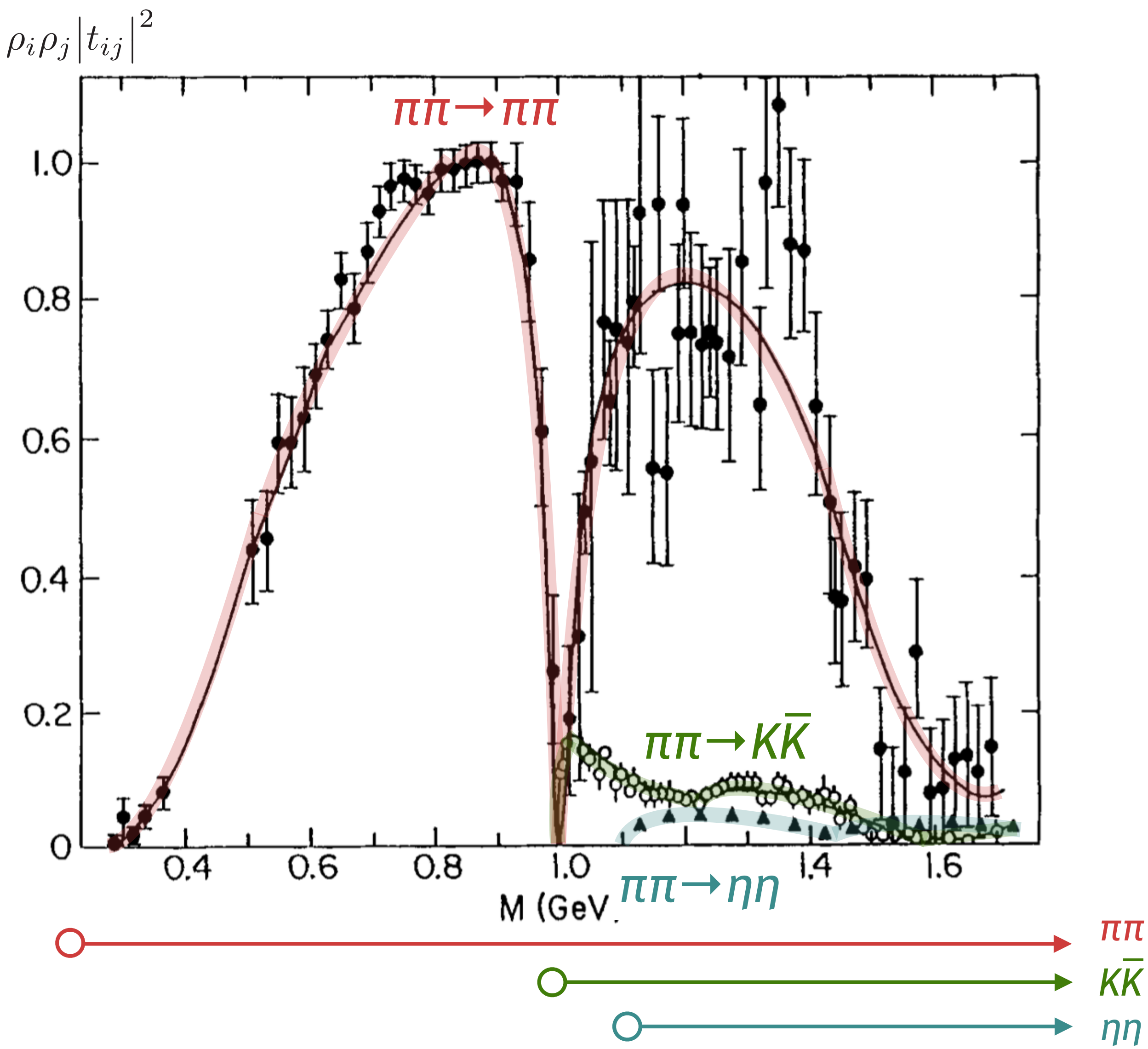
equivalent to the scalar loop integral



$$16\pi i \int \frac{d^4p}{(2\pi)^4} \frac{1}{p^2 - m^2 + i\epsilon} \frac{1}{(K-p)^2 - m^2 + i\epsilon}$$

[regularization → subtraction]

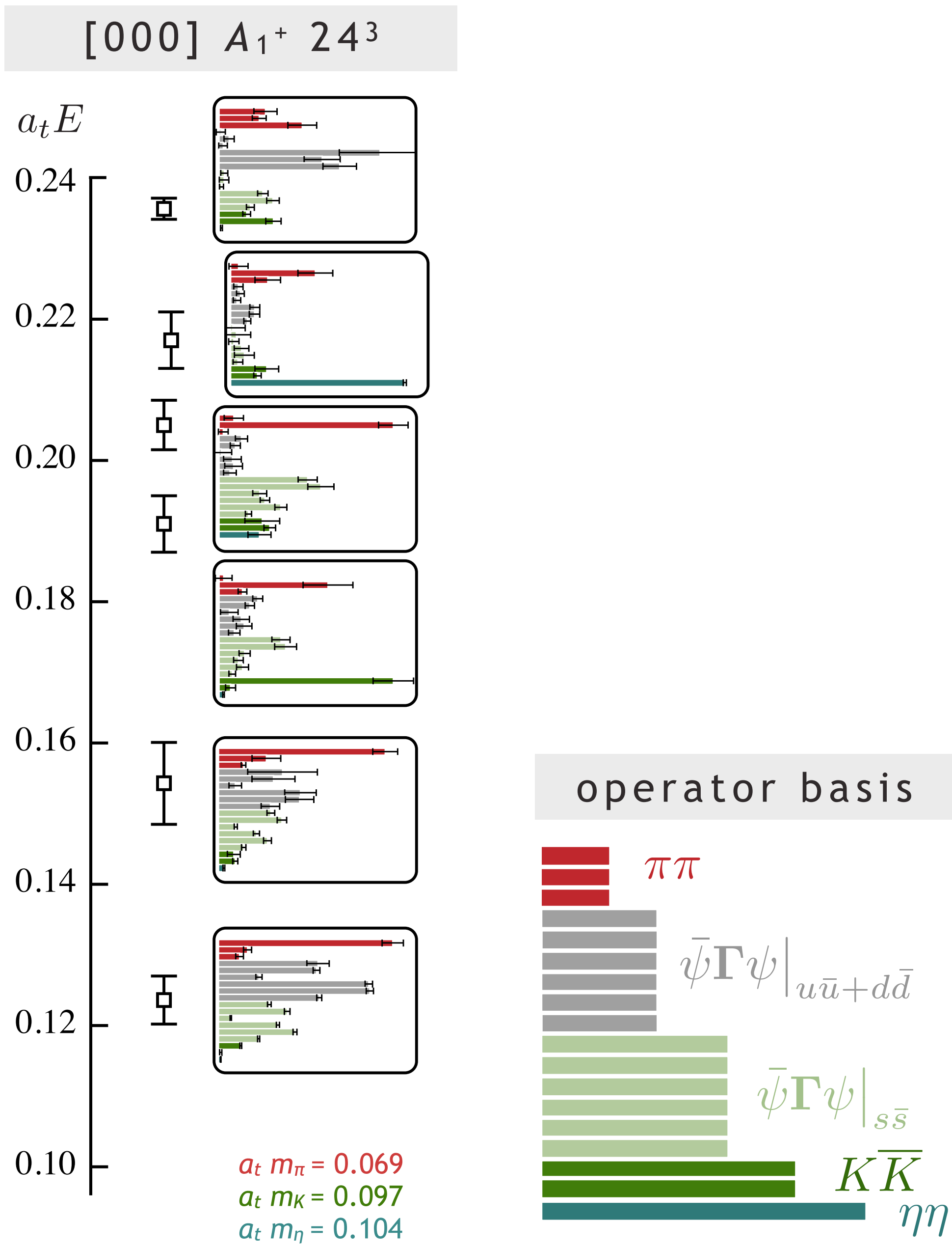
$\pi\pi$, $K\bar{K}$, $\eta\eta$ S-wave scattering



explore this non-trivial system …

… at a higher quark mass …

$\pi\pi$, $K\overline{K}$, $\eta\eta$ scattering with m_π~391 MeV



$\pi\pi$, $K\bar{K}$, $\eta\eta$ scattering with m_π~391 MeV

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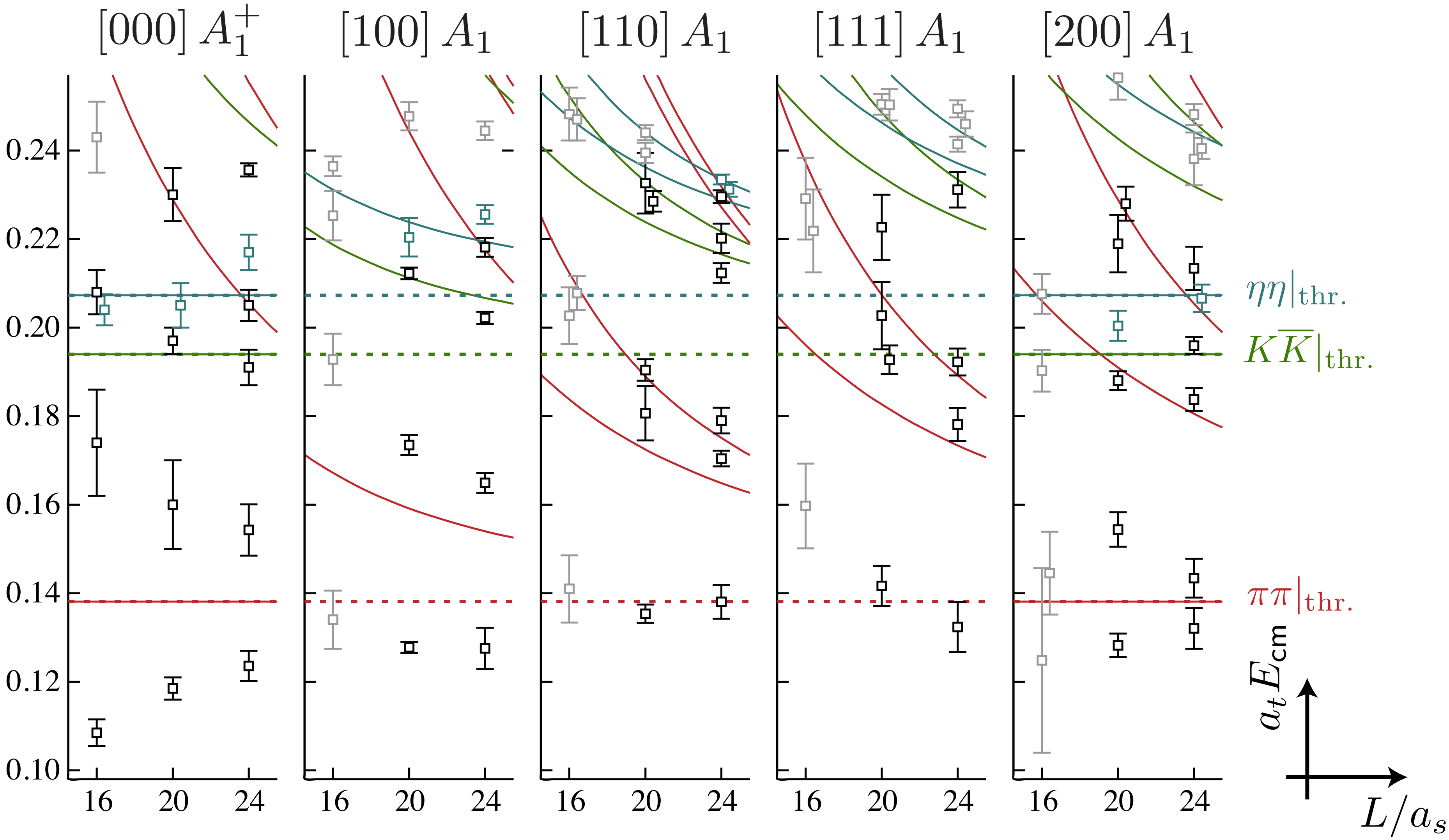

what ***t***-matrix gives these spectra ?

$\pi\pi$, $K\overline{K}$, $\eta\eta$ scattering with m_π~391 MeV

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not obvious what amplitude parameterization likely to describe the spectra well — try many …

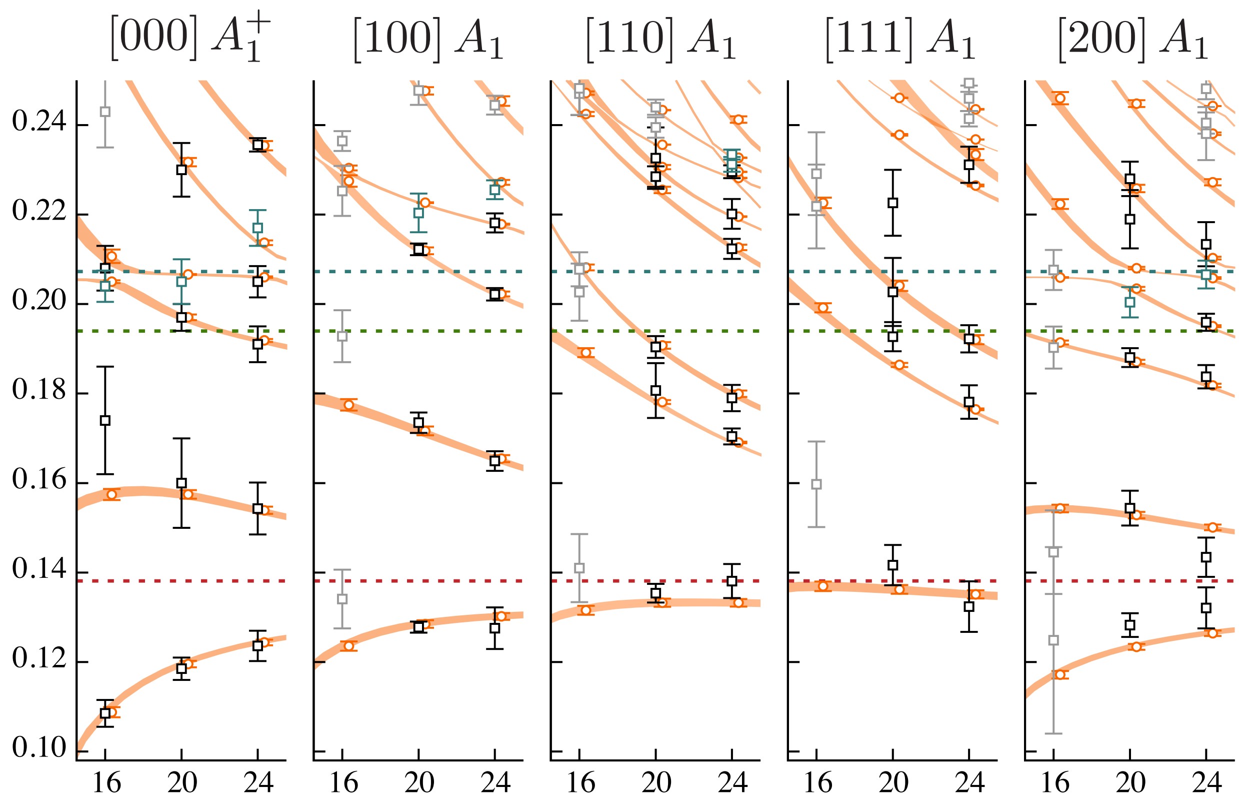
$$\text{e.g.}\quad \mathbf{K}^{-1}(s) = \begin{pmatrix} a+b\,s & c+d\,s & e \\ c+d\,s & f & g \\ e & g & h \end{pmatrix}$$

{ a … h } are free parameters

with Chew-Mandelstam phase-space

$$I(s) = -\frac{\rho(s)}{\pi}\log\left[\frac{\rho(s)-1}{\rho(s)+1}\right]$$

best fit to lattice spectra



$$\frac{\chi^2}{N_{\text{dof}}} = \frac{44.0}{57-8} = 0.90$$

$\pi\pi$, $K\overline{K}$, $\eta\eta$ scattering with m_π~391 MeV

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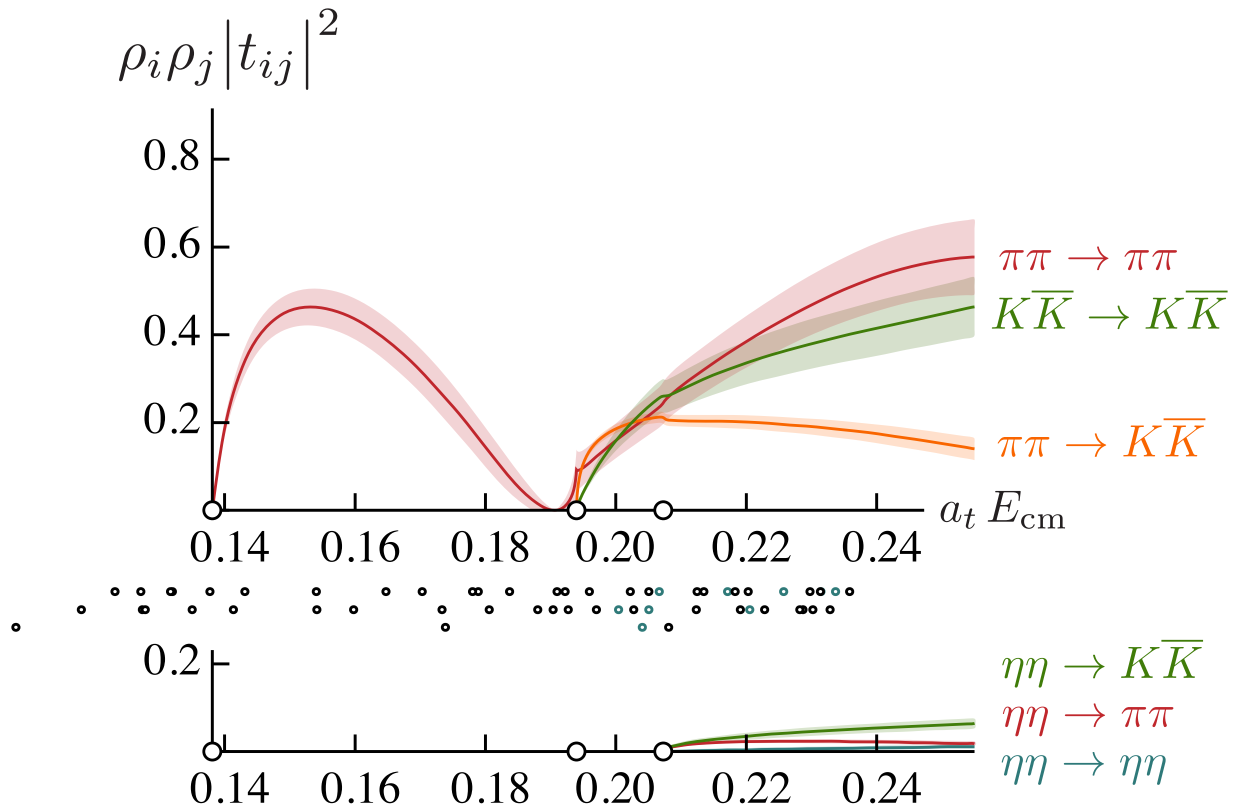
e.g. $$\mathbf{K}^{-1}(s) = \begin{pmatrix} a+b\,s & c+d\,s & e \\ c+d\,s & f & g \\ e & g & h \end{pmatrix}$$

{ *a* ... *h* } are free parameters

S-wave amplitudes



$\pi\pi$, $K\overline{K}$, $\eta\eta$ scattering with m_π~391 MeV

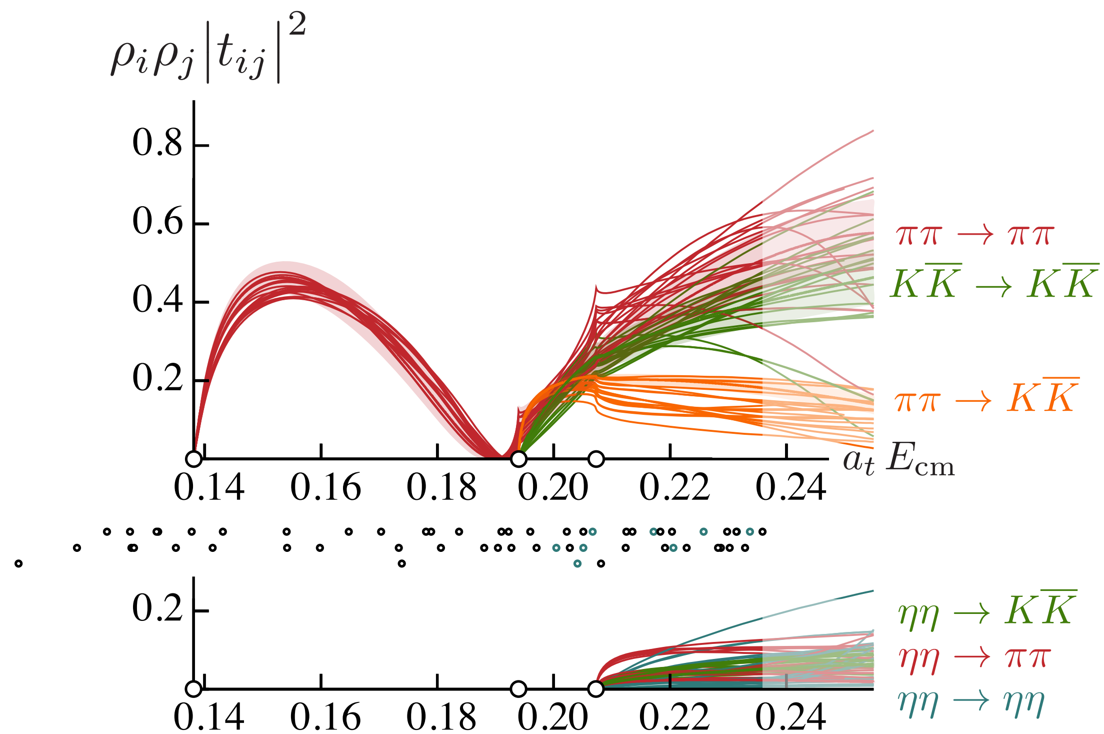
PRD 97 054513 (2018)

not obvious what amplitude parameterization likely to describe the spectra well — **try many …**

K^-1 as matrix of polynomials,
K as matrix of polynomials,
K as pole plus matrix of polynomials,
simple versus Chew-Mandelstam phase-space …

keep choices that can describe spectra with good χ^2

variation with parameterization



$\pi\pi$, $K\bar{K}$, $\eta\eta$ scattering with m_π~391 MeV

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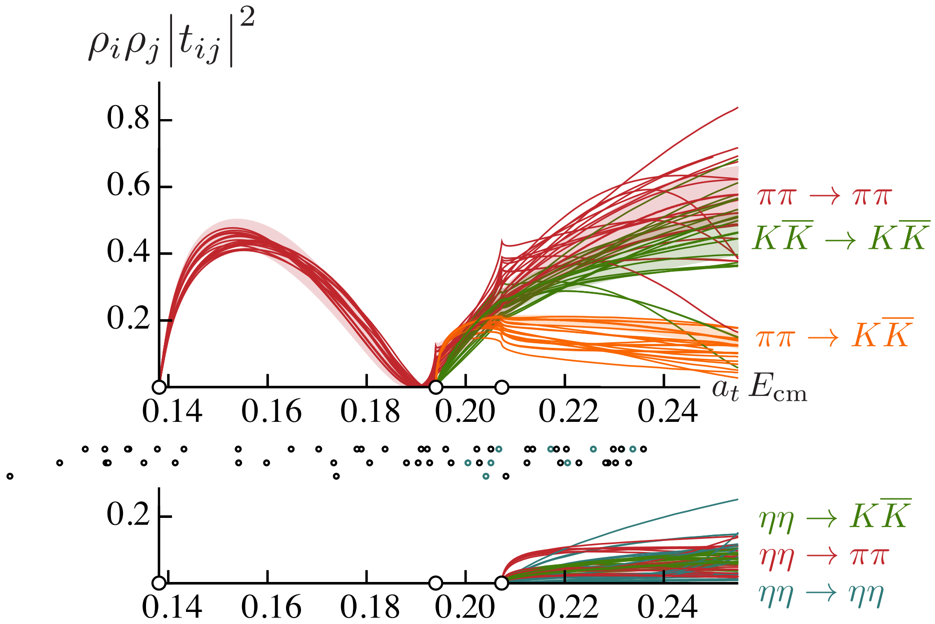
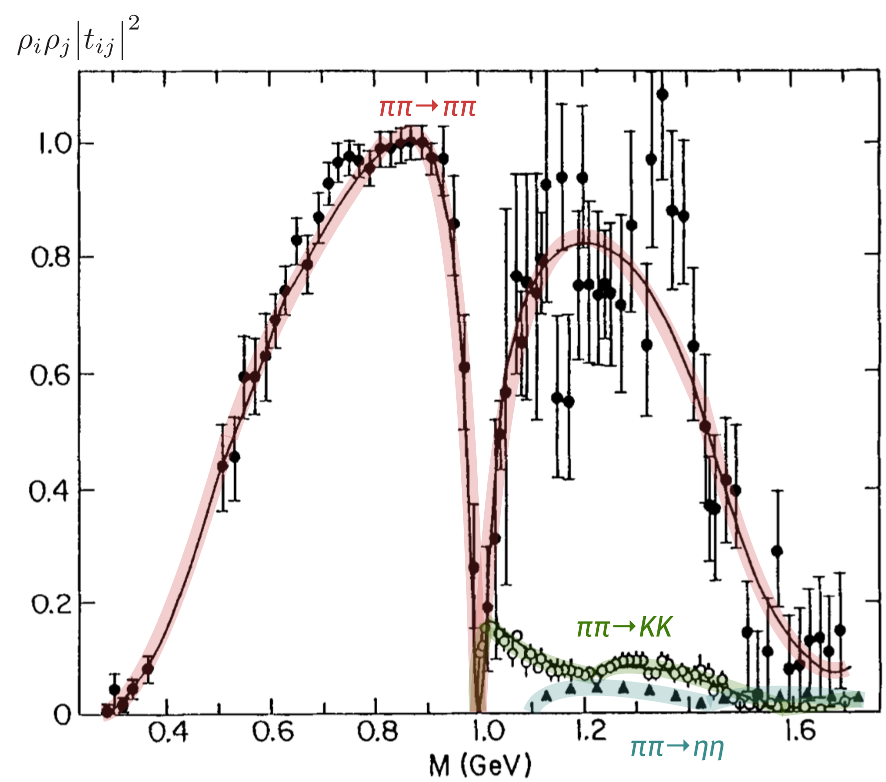
scattering amplitude 'prediction'



'analogous' experimental data



… but what do we do with this ?

… is this strange energy dependence due to resonances ?

$\pi\pi$, $K\bar{K}$, $\eta\eta$ scattering with m_π~391 MeV

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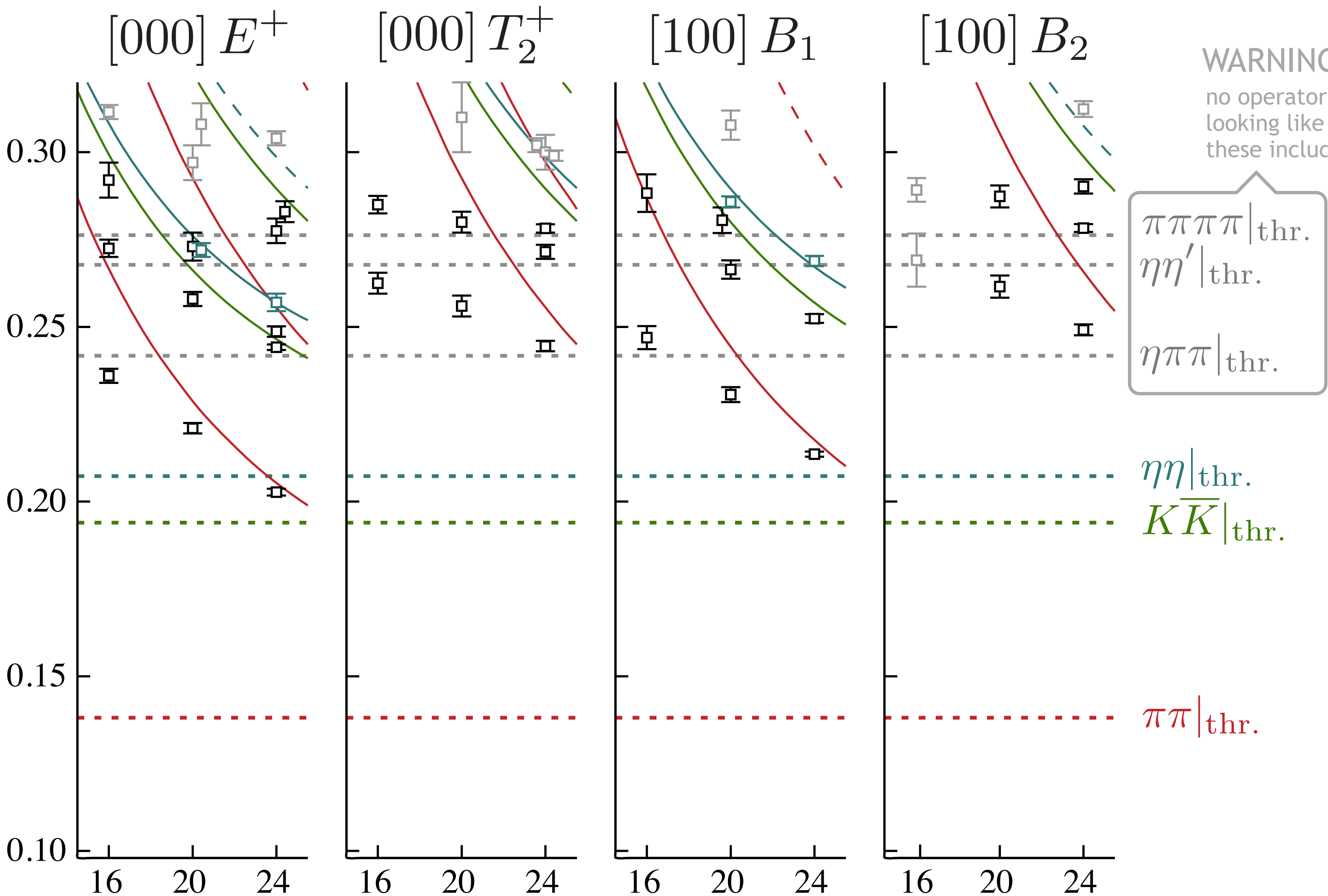
also computed spectra for irreps with lowest subduced spin ***J*=2**



$\pi\pi$, $K\overline{K}$, $\eta\eta$ scattering with m_π~391 MeV

PRD 97 054513 (2018)

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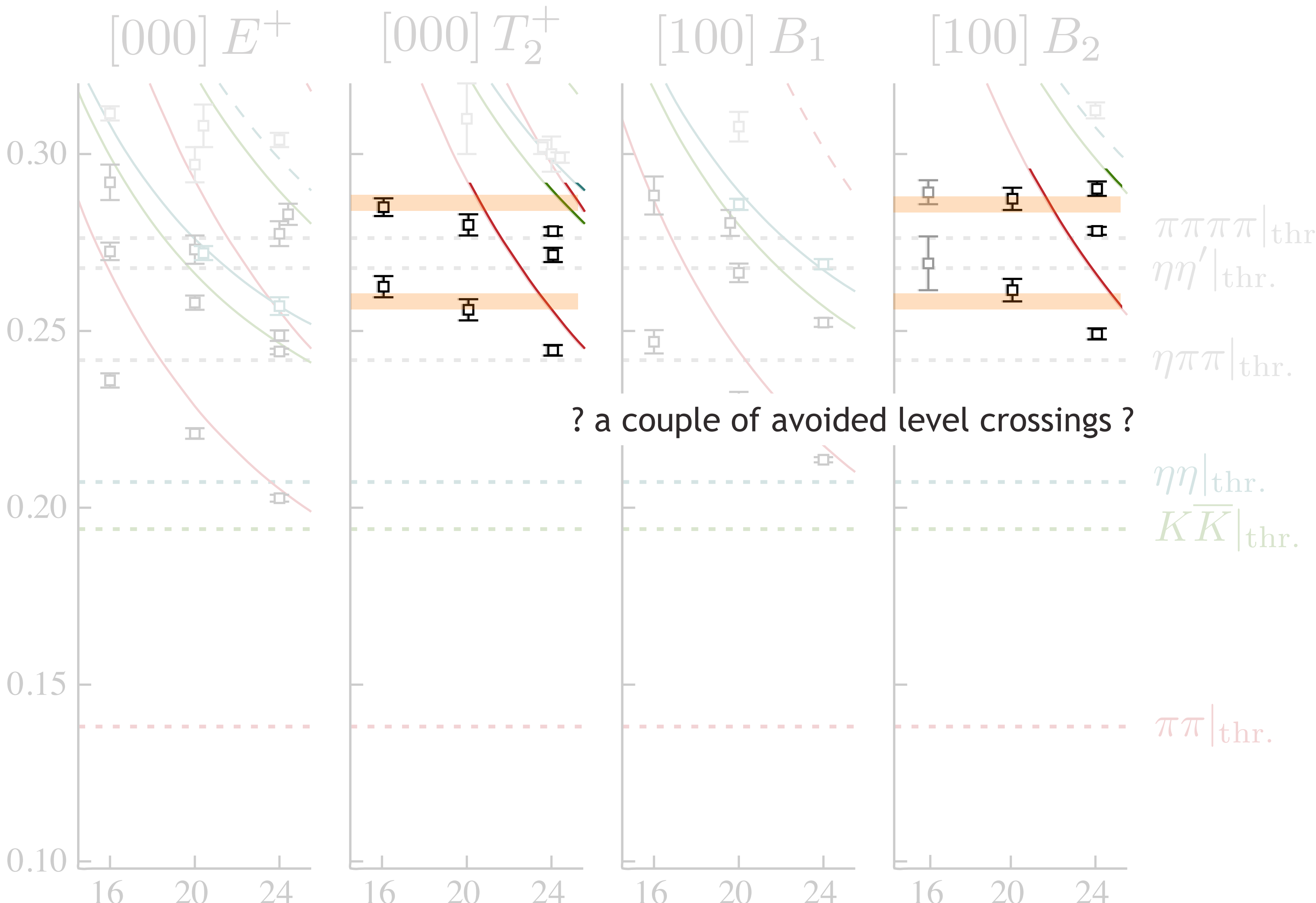


? a couple of avoided level crossings ?

$\pi\pi$, $K\overline{K}$, $\eta\eta$ scattering with m_π~391 MeV

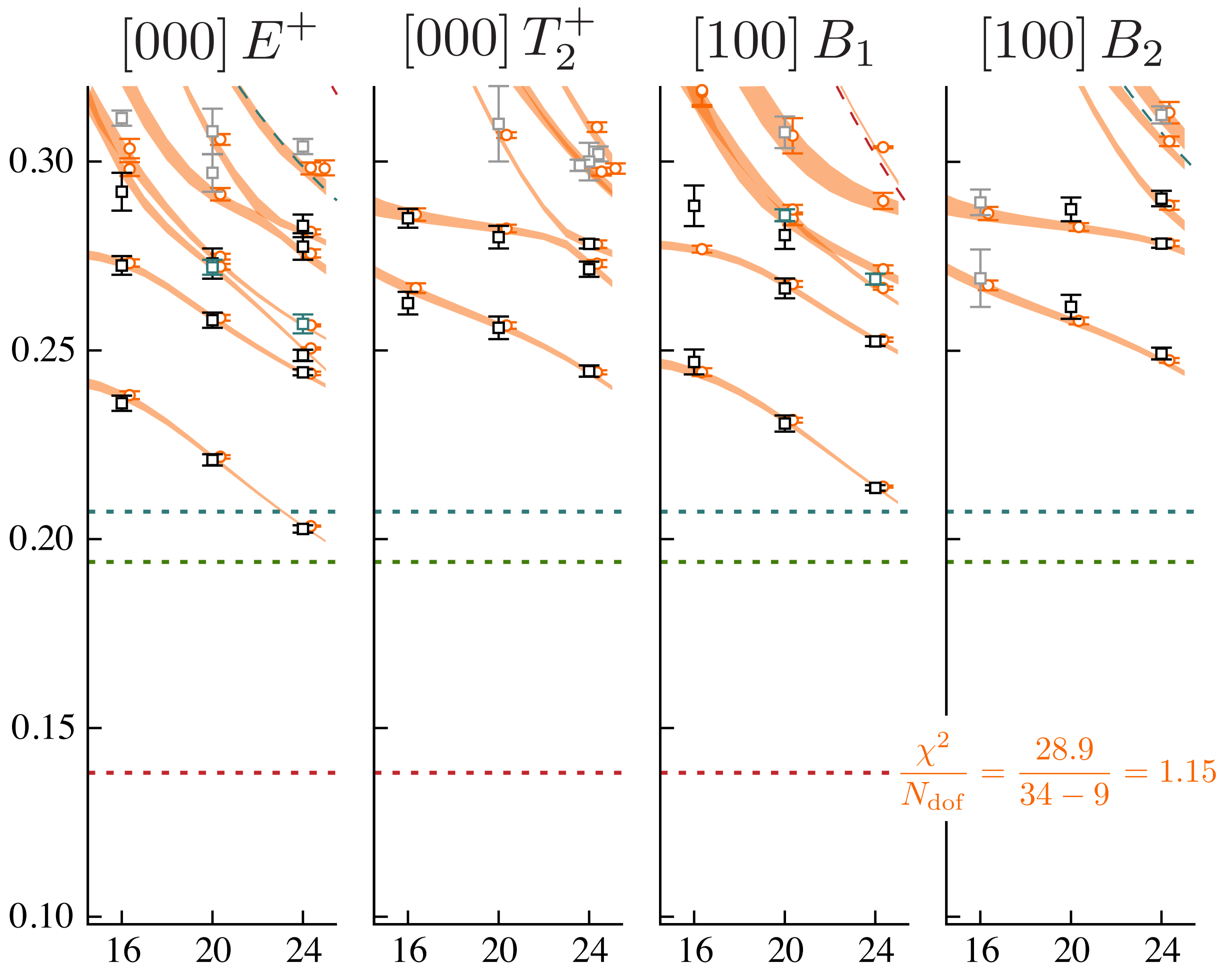
PRD 97 054513 (2018)

e.g. parameterize coupled *D*-wave ***t***-matrix with

$$K_{ij}(s) = \frac{g_i^{(1)} g_j^{(1)}}{m_1^2 - s} + \frac{g_i^{(2)} g_j^{(2)}}{m_2^2 - s} + \gamma_{ij} \qquad \gamma = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \gamma_{\eta\eta,\eta\eta} \end{pmatrix}$$

and the simple phase-space

best fit to lattice spectra



$\pi\pi$, $K\overline{K}$, $\eta\eta$ scattering with m_π~391 MeV

PRD 97 054513 (2018)

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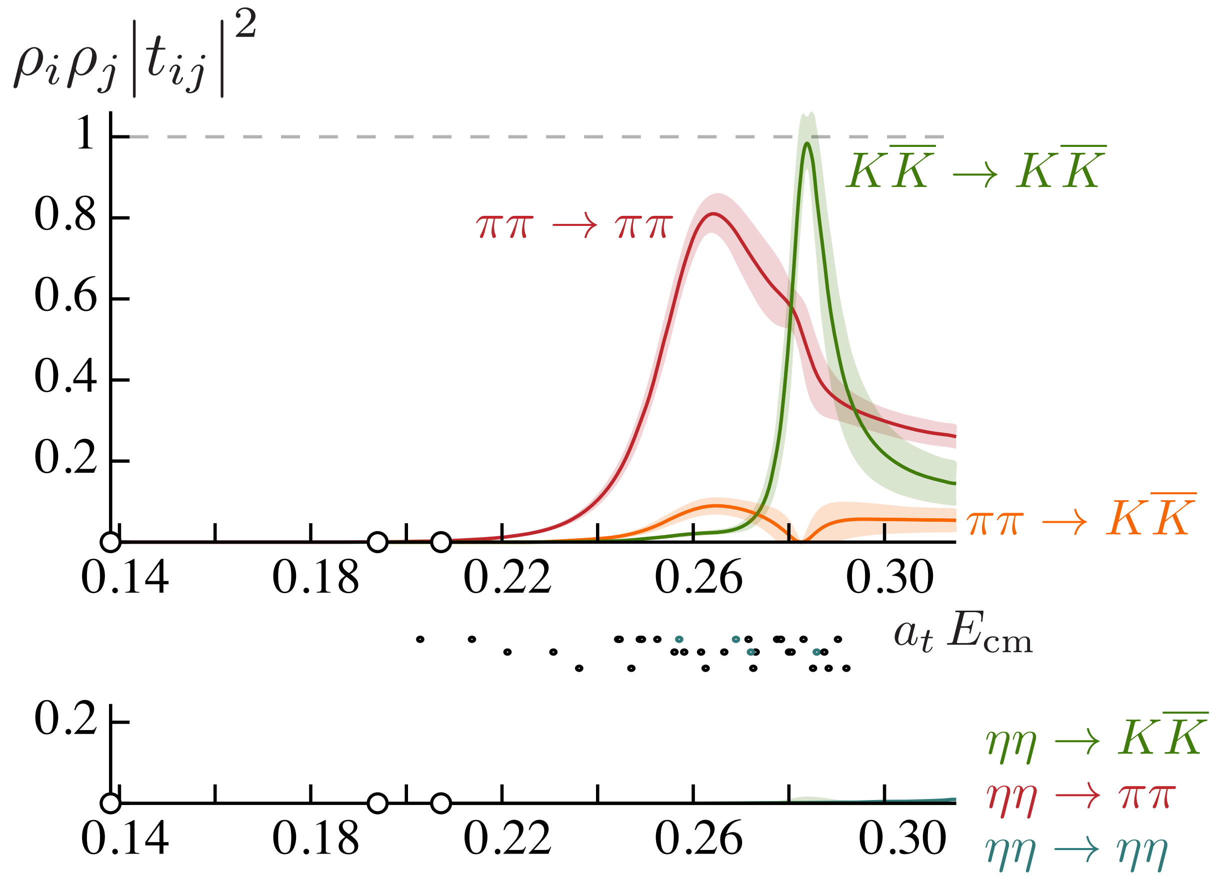
D-wave amplitudes



$\pi\pi$, $K\overline{K}$, $\eta\eta$ scattering with m_π~391 MeV

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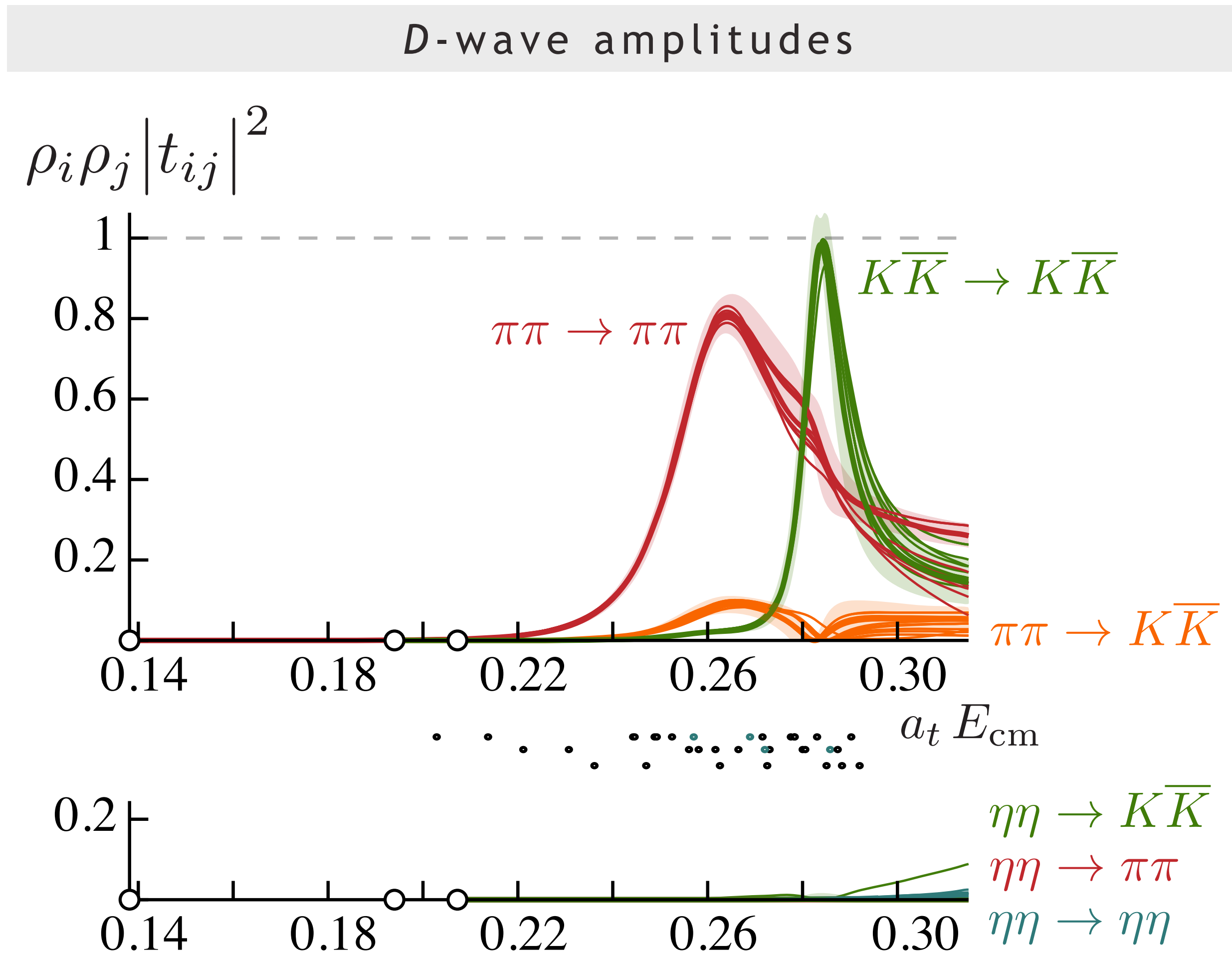
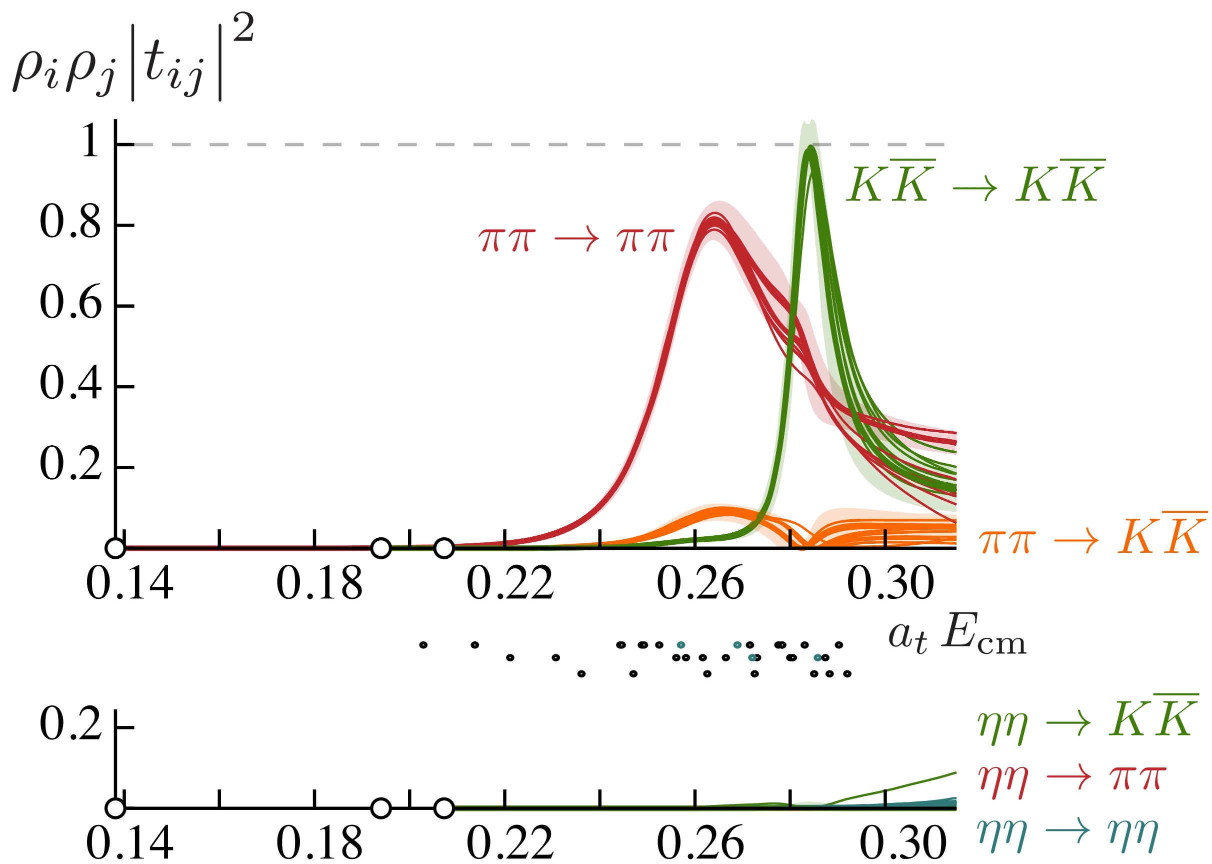
… and varying the particular choice of parameterization …



$\pi\pi$, $K\overline{K}$, $\eta\eta$ scattering with m_π~391 MeV

D-wave amplitudes



'looks like' two resonances

– lighter one has larger width, big coupling to $\pi\pi$

– heavier one has smaller width, big coupling to $K\overline{K}$

… there must be a more rigorous way to know the resonance content ?