Gluons in QCD

Lectures 3 - 4

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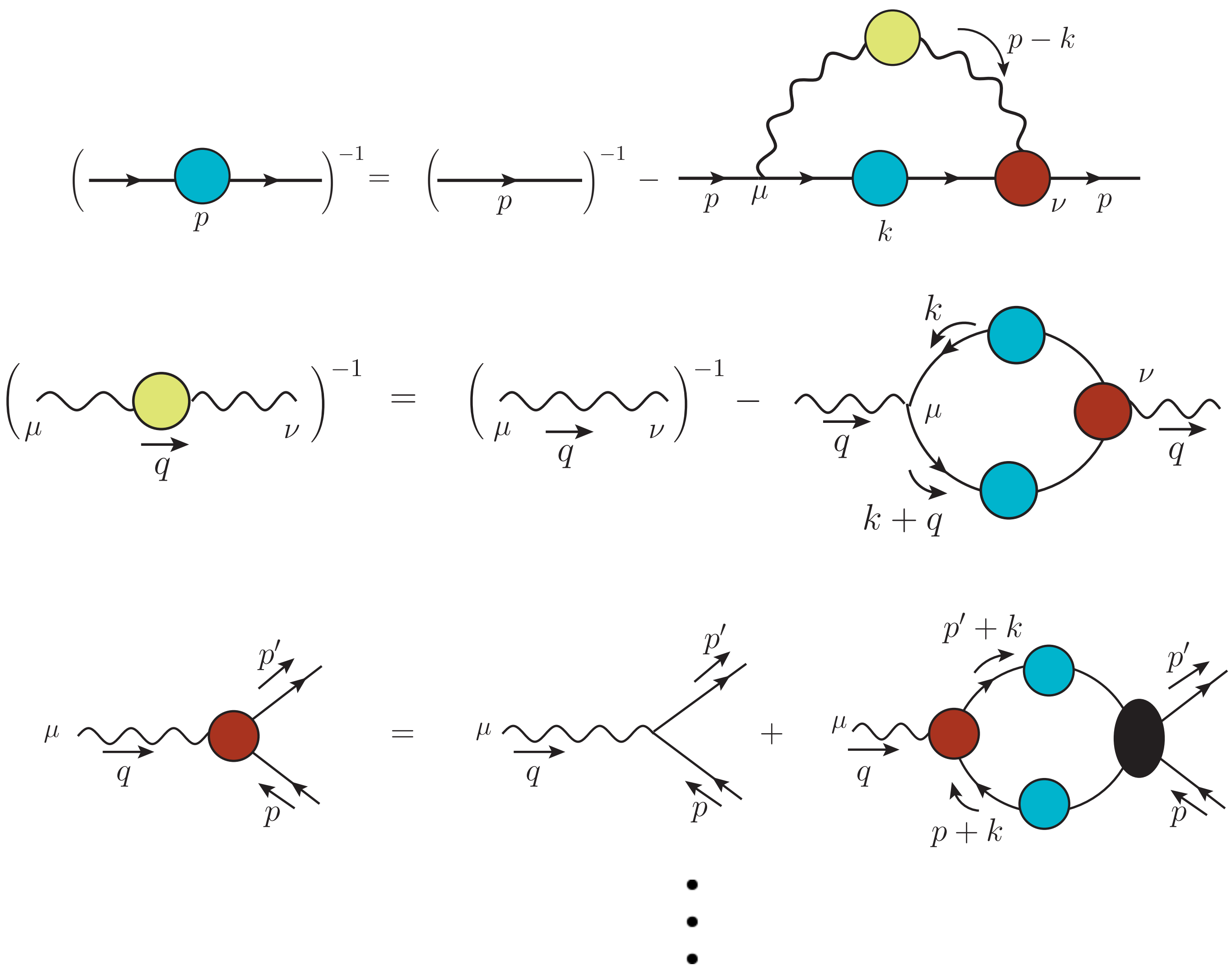
SDE tower for QED



Electron SDE

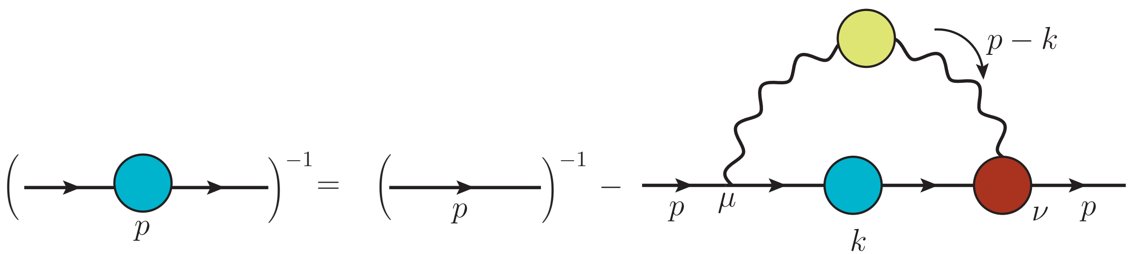
Photon SDE

Vertex SDE

- *They form an infinite set of coupled integral equation!*

Need for a truncation scheme

- First, let us examine the SDE for the fermion in isolation



$$S^{-1}(p) = (\not{p} - m) + ie_0^2 \int \frac{d^4k}{(2\pi)^4} \Delta_{\mu\nu}(q)\Gamma^{\nu}(p,k)S(k)\gamma^{\mu}$$

- This equation is more complicated than it seems.
- The full electron propagator (containing all order corrections) can be written as

$$S^{-1}(p) = A(p^2)\not{p} - B(p^2)\mathbb{I}$$

$$S_0^{-1}(p) = (\not{p} - m)$$

A and B are unknown functions

- Notice that at tree level $A(p^2) = 1$ and $B(p^2) = m$

$$\mathcal{M}(p) = \frac{B(p)}{A(p)}$$

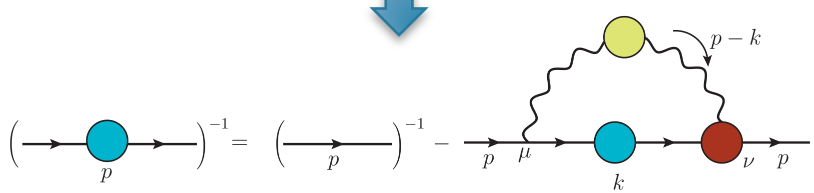
- The pole of the propagator defines the *mass of the particle.*

Dynamical mass

- To sum up, we have:

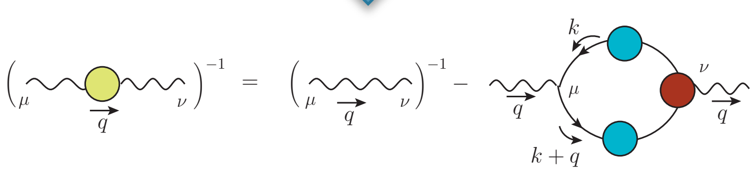
→ **2 unknowns functions**, A and B from full electron propagator, which

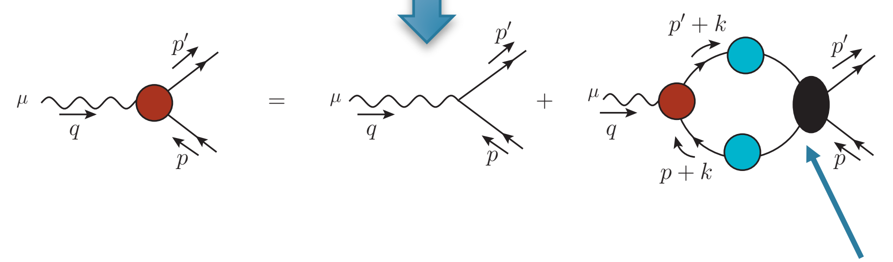
$$S^{-1}(p) = A(p^2)\not{p} - B(p^2)\mathbb{I}$$



are coupled **1 (photon) + 12 (form factors of the vertex)** unknowns functions**:**

$$\Delta_{\mu\nu}(q) = \left[g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right]\Delta(q^2) \quad + \quad \begin{aligned}\Gamma_\nu(p,k) = {} & \gamma_\nu\Gamma_1 + p_\nu\Gamma_2 + k_\nu\Gamma_3 + \gamma_\nu\not{p}\Gamma_4 + \gamma_\nu\not{k}\Gamma_5 + p_\nu\not{p}\Gamma_6 \\ & + p_\nu\not{k}\Gamma_7 + k_\nu\not{p}\Gamma_8 + k_\nu\not{k}\Gamma_9 + \gamma_\nu\not{p}\not{k}\Gamma_{10} + p_\nu\not{p}\not{k}\Gamma_{11} + k_\nu\not{p}\not{k}\Gamma_{12}\end{aligned}$$





+ ...

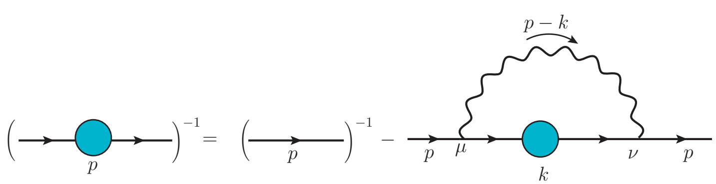
- To understand the basic principles of the dynamical mass generation, it is not necessary to solve this intricate coupled system.

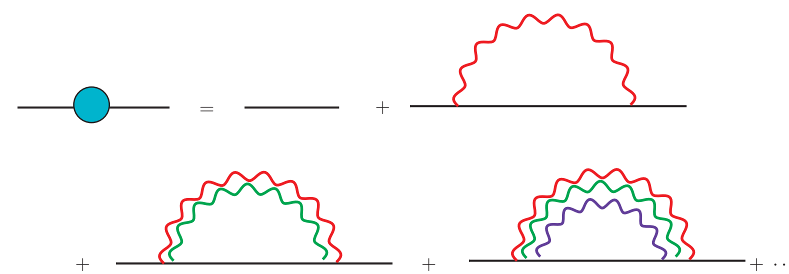
- Let's make some approximations to get the general idea of the problem.

- We will approximate the photon propagator and the vertex by their tree level values, i.e.

$$\Delta(q^2) \to \frac{1}{q^2} \qquad \Gamma^{\nu}(p,k) \to \gamma^{\nu}$$

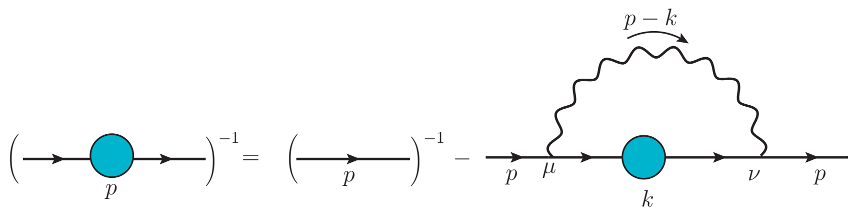
- Then, only the electron is treated nonperturbatively. Diagrammatically we have





Rainbow approximation

QED in the rainbow approximation



$$q = p - k$$

$$A(p^2)\not p - B(p^2) = \not p - m + ie_0^2 \int \frac{d^4k}{(2\pi)^4} \left[g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right] \gamma^\nu \frac{[A(k^2)\not k + B(k^2)]}{q^2[A^2(k^2)k^2 - B^2(k^2)]} \gamma^\mu$$

$$A(p^2)\not p - B(p^2) = \not p - m + ie_0^2 \int \frac{d^4k}{(2\pi)^4} \left[3B(k^2) - A(k^2) \left(2\not k + \frac{\not q \not k \not q}{q^2} \right) \right] \frac{1}{q^2[A^2(k^2)k^2 - B^2(k^2)]}$$

In Euclidean space, we have

$$[A(p^2) - 1]p^2 = e_0^2 \int \frac{d^4k}{(2\pi)^4} \frac{A(k^2)}{[A^2(k^2)k^2 + B^2(k^2)]} \left[(k \cdot p) + 2\frac{(q \cdot k)(q \cdot p)}{q^2} \right] \frac{1}{q^2}$$

The angular integral for the Dirac-vector component vanishes → $A(p^2)=1$

$$\underbrace{\mathcal{M}(p^2)}_{B(p^2)} = m + 3e_0^2 \int \frac{d^4k}{(2\pi)^4} \frac{\mathcal{M}(k^2)}{q^2[k^2 + \mathcal{M}^2(k^2)]}$$

$$\mathcal{M}(p) = \frac{B(p)}{A(p)}$$

$$\mathcal{M}(p^2) = \cancel{m}^{\,0} + 3e_0^2 \int \frac{d^4k}{(2\pi)^4} \frac{\mathcal{M}(k^2)}{q^2[k^2 + \mathcal{M}^2(k^2)]}$$

Measure in 4D spherical coordinates

$$\int d^4k := \int_0^\infty dk k^3 \int d\Omega$$
$$= \int_0^\infty dk k^3 \int_0^{2\pi} d\varphi \int_0^\pi d\beta \, \sin\beta \int_0^\pi d\theta \, \sin^2\theta$$
$$= \frac{1}{2} \int_0^\infty dk^2 k^2 \left(4\pi \int_0^\pi d\theta \, \sin^2\theta \right)$$

- The angular part for $B(p^2)$ is simple

Angular integration:

$$\int d\Omega \frac{1}{(p-k)^2} = 4\pi \int_0^\pi d\theta \, \frac{\sin^2\theta}{(p^2 + k^2 - 2|p||k|cos\theta)}$$
$$= 2\pi^2 \left[\frac{1}{p^2}\theta(p^2 - k^2) + \frac{1}{k^2}\theta(k^2 - p^2) \right]$$

$$\mathcal{M}(p^2) = \frac{3e_0^2}{16\pi^2} \left[\frac{1}{p^2} \int_0^{p^2} dk^2 \frac{k^2 \mathcal{M}(k^2)}{k^2 + \mathcal{M}^2(k^2)} + \int_{p^2}^{\Lambda^2} dk^2 \frac{\mathcal{M}(k^2)}{k^2 + \mathcal{M}^2(k^2)} \right]$$

- The integral equation can be converted in a differential equation + boundary conditions.

For more details see:

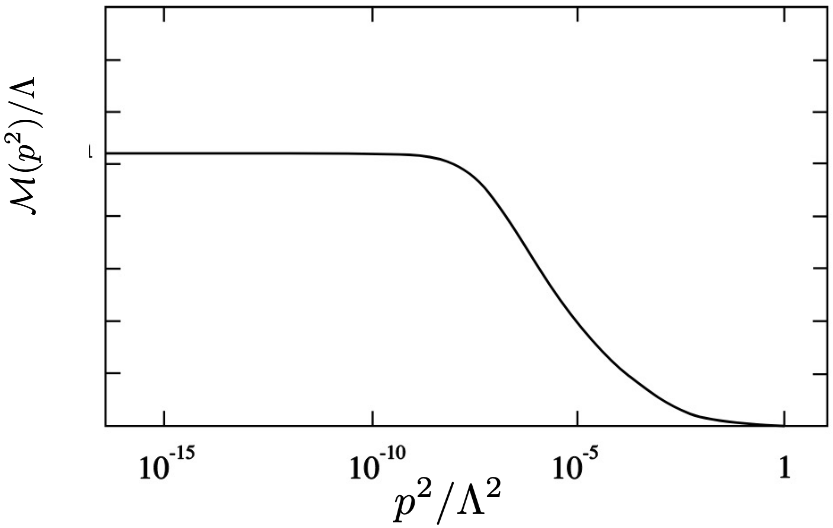
M.R.Pennington, J. Phys. Conf. Ser. 18, 1-73 (2005)

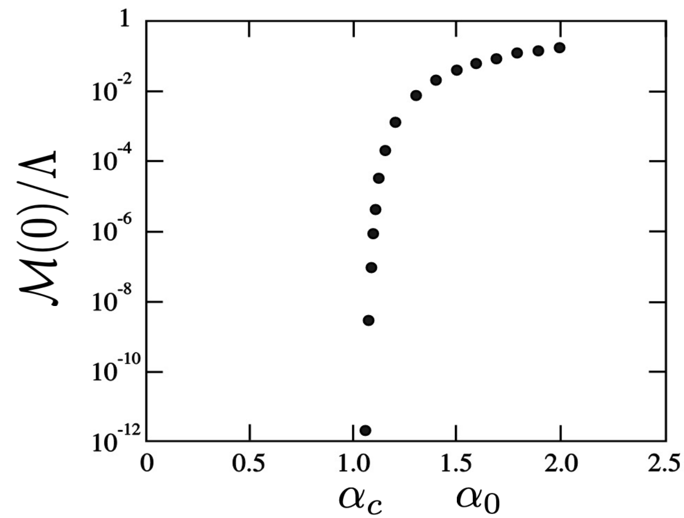
E. S. Swanson, AIP Conf. Proc. 1296, no.1, 75-121 (2010)

$$\mathcal{M}(p^2) = \frac{3e_0^2}{16\pi^2}\left[\frac{1}{p^2}\int_0^{p^2} dk^2 \frac{k^2\mathcal{M}(k^2)}{k^2+\mathcal{M}^2(k^2)} + \int_{p^2}^{\Lambda^2} dk^2 \frac{\mathcal{M}(k^2)}{k^2+\mathcal{M}^2(k^2)}\right]$$

- It is convenient to define the fine structure constant $\alpha_0 = \frac{e_0^2}{4\pi}$
- Then, we see that the nontrivial solution appears only when

$$\alpha_0 \geq \alpha_c \equiv \frac{\pi}{3}$$



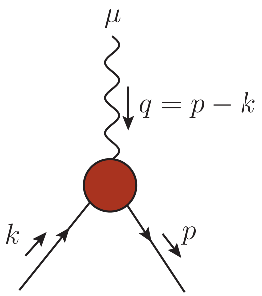


- The electron dynamical mass will be generated only when the constant will be greater than one.
- ...or we can interpret it as, we are consider that the electron is massless (no lagrangian mass), but it may develop dynamical mass if it will be immersed in the field of a heavy nucleus with charge Z > 144.

Ward-Takashashi Identity (WTI)

- The Ward-Takahashi identity (WTIs) is one of the most important consequences of gauge invariance in QED – valid to all orders. It states

q_μ



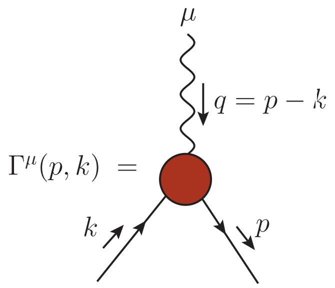
$$= S^{-1}(p) - S^{-1}(k)$$

Full vertex

Full inverse electron propagator

$$q_\mu \Gamma^\mu(p,k) = S^{-1}(p) - S^{-1}(k)$$

$$q_\mu \Gamma^\mu(p,k) = S^{-1}(p) - S^{-1}(k)$$

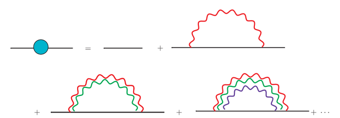


- Clearly, this relation is satisfied at tree level

$$\begin{aligned} q_\mu \Gamma^\mu(p,k) &= S^{-1}(p) - S^{-1}(k) \\ (p-k)_\mu \gamma^\mu &= (\not p - m) - (\not k - m) \\ \not p - \not k &= \not p - \not k \end{aligned}$$

$$\left\{ \begin{aligned} \Gamma_0^\mu(p,k) &= \gamma^\mu \\ S_0^{-1}(p) &= (\not p - m) \end{aligned} \right.$$

- But when we combine the vertex at tree level and the propagador dressed (as we have done in the rainbow approximation) the WTIs is not satisfied



$$\begin{aligned} q_\mu \Gamma^\mu(p,k) &= S^{-1}(p) - S^{-1}(k) \\ (p-k)_\mu \gamma^\mu &= [\not p - B(p^2)] - [\not k - B(k^2)] \\ \not p - \not k &\neq \not p - \not k - B(p^2) + B(k^2) \end{aligned}$$

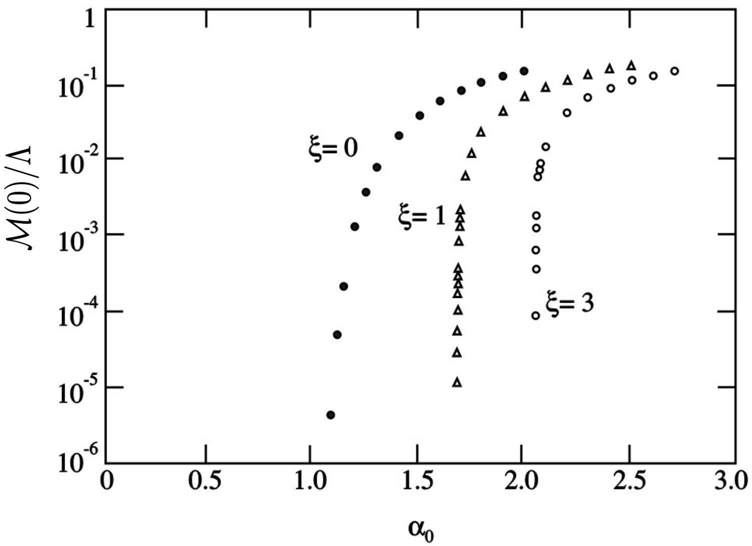
$$\left\{ \begin{aligned} S^{-1}(p) &= \not p - B(p^2) \\ \Gamma^\mu(p,k) &\to \gamma^\mu \end{aligned} \right.$$

The rainbow approximation violates gauge invariance.

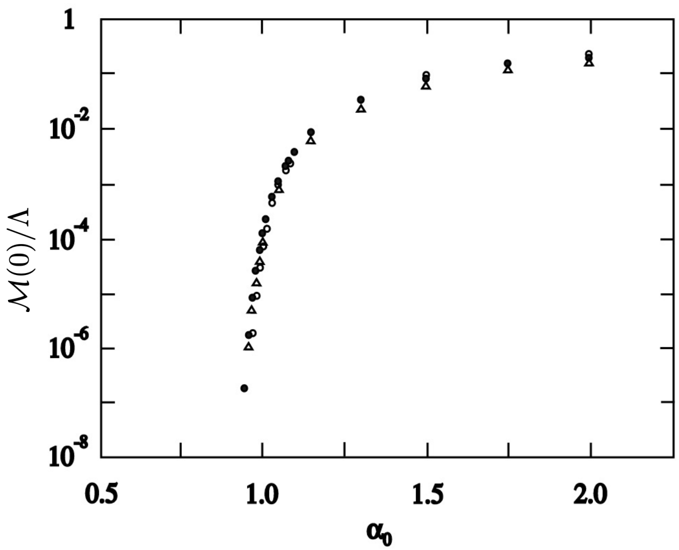
We have to use more sophisticated Ansatz for the electron-photon!

Consequences

- The result depends on the gauge (ξ-dependence)



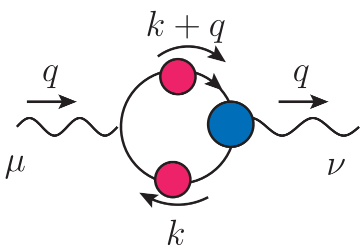
- With more sophisticated Ansatz which respect the WTI, it is possible to correct this behavior!



C.D. Curtis and M.R. Pennington, Phys. Rev. D46 2663 (1992).
A. Kizilersu and M. R. Pennington, Phys. Rev. D79, 125020 (2009)

Transversality of photon vaccum polarization

$\Pi_{\mu\nu}(q) =$



- Gauge invariance also imposes the transversality of the vaccum polarization, i.e.

$$q^{\mu}\Pi_{\mu\nu}(q) = q^{\nu}\Pi_{\mu\nu}(q) = 0$$

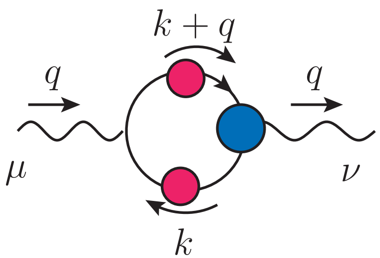
- We can show that full vaccum polarization is transverse using the WTI

$$q^{\mu}\Gamma_{\mu} = S^{-1}(k+q) - S^{-1}(k)$$

- The vaccum polarization can be written as

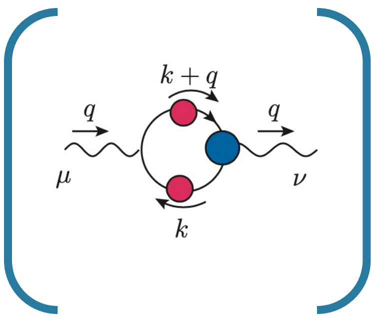
$$\Pi_{\mu\nu}(q) = C\int d^dk\,\mathrm{Tr}\left[\gamma_{\mu}S(k)\Gamma_{\nu}S(k+q)\right]$$

$$\int d^dk := \int \frac{d^dk}{(2\pi)^d}$$

$\Pi_{\mu\nu}(q) =$ 

$$\Pi_{\mu\nu}(q) = C \int d^d k \, \mathrm{Tr} \left[\gamma_\mu S(k) \Gamma_\nu S(k+q) \right]$$

- Contracting with the momenta of the photon q^ν

q^ν 

$$q^\nu \Pi_{\mu\nu}(q) = C \int d^d k \, \mathrm{Tr}[\gamma_\mu S(k) \underbrace{[q^\nu \Gamma_\nu]}_{\text{WTI}} S(k+q)]$$

$$q^\nu \Pi_{\mu\nu}(q) = C \int d^d k \, \mathrm{Tr}[\gamma_\mu S(k) \, [S^{-1}(k+q) - S^{-1}(k)] \, S(k+q)]$$

$$q^\nu \Pi_{\mu\nu}(q) = C \int d^d k \, \mathrm{Tr}[\gamma_\mu S(k)] - C \int d^d k \, \mathrm{Tr}[\underbrace{\gamma_\mu S(k+q)}_{\text{shift } k+q \to k}]$$

$$q^\nu \Pi_{\mu\nu}(q) = 0$$

Photon polarization is transverse to all orders!

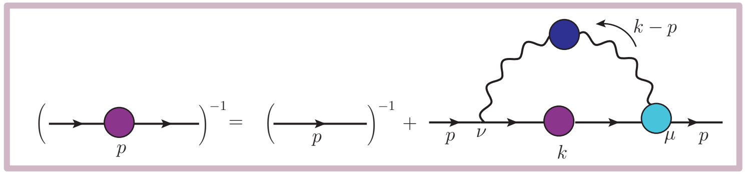
Similar property will hold for the gluon polarization!

What are the SDEs for QCD?

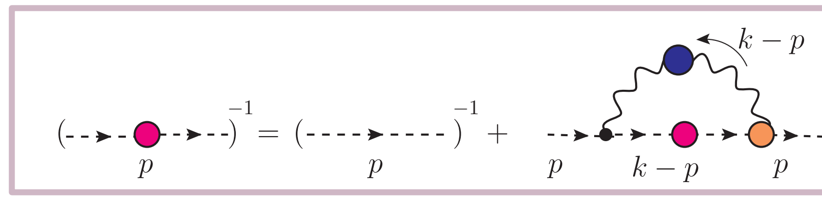
SDE for QCD

Two -point sector

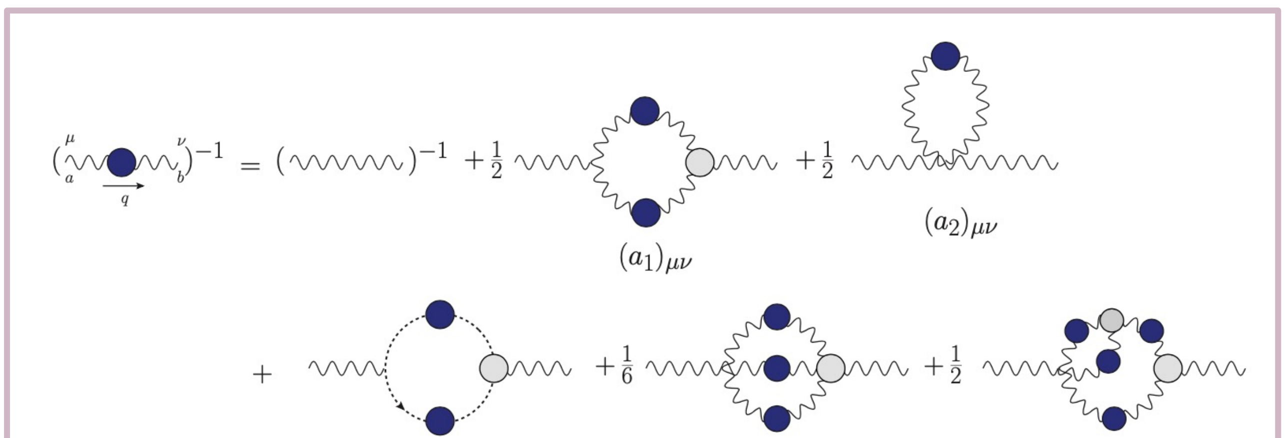
- Quark SDE or gap equation



- Ghost SDE

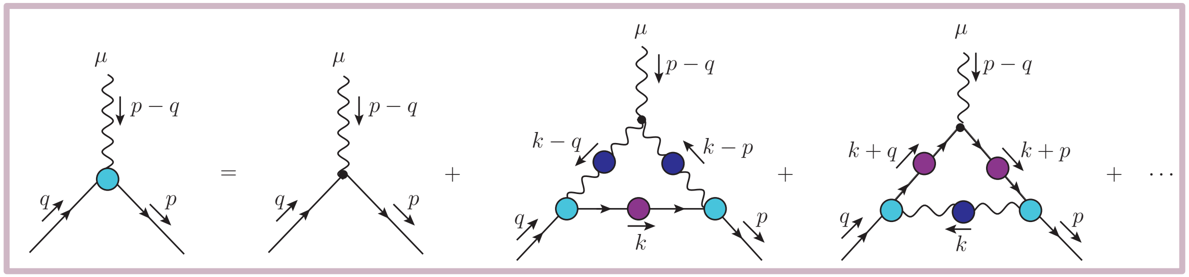


- Gluon SDE

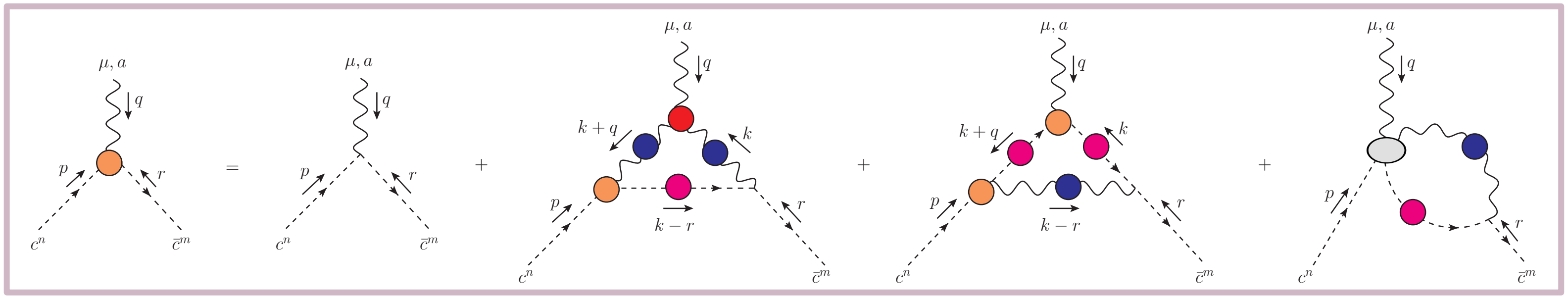


Three-point sector

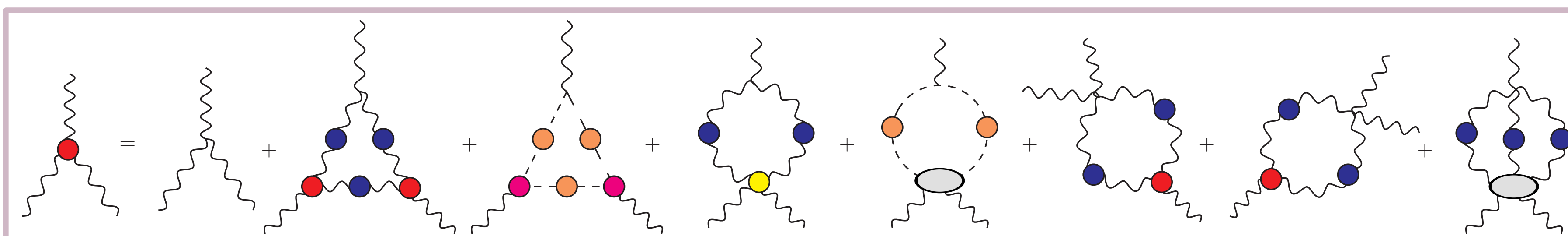
- Quark-gluon vertex SDE



- Ghost-gluon vertex SDE



- Three-gluon vertex SDE



Difficulties with SDEs

- The need for truncations is evident

  ✓ No obvious expansion parameter, so, no formal way of estimating the size of the omitted terms. However, it seems that the "projection" of higher Green's functions on the lower ones is "small".

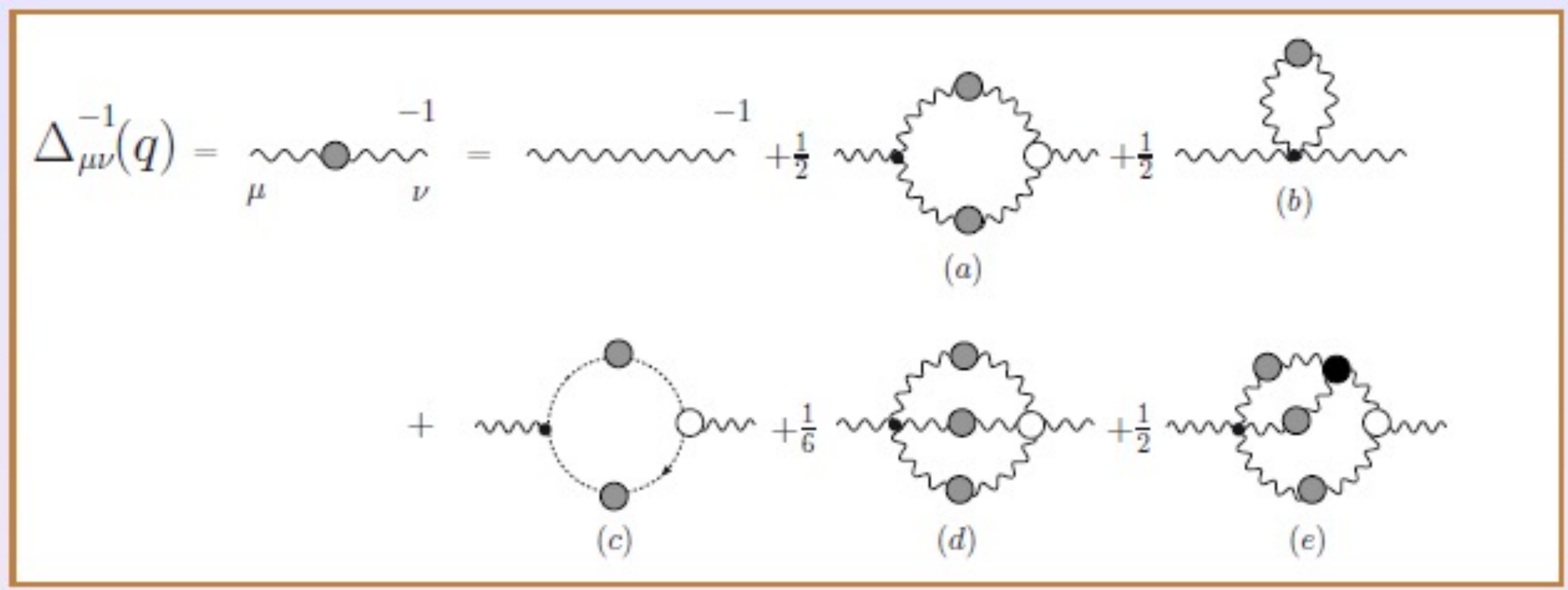
  ✓ Casual truncation interferes with the symmetries encoded in the form of the SDEs

$$q^\mu \Pi_{\mu\nu}(q) = 0 \quad \Longrightarrow \quad \Pi_{\mu\nu}(q) = \left[g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right]\Pi(q^2)$$

$\Pi_{\mu\nu}(q)$ *is the gluon self-energy*
It is transverse

- Self-consistent **truncation scheme** must be used.

The complete SDE for the gluon propagator



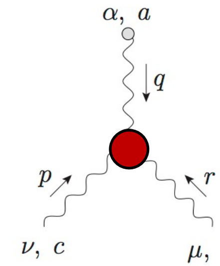
- ◎ Retaining only (a) and (b) is not correct even at one loop

$$q^\mu \Pi_{\mu\nu}(q)|_{(a)+(b)} \neq 0$$

- ◎ Adding (c) is not sufficient for a full analysis → beyond one loop

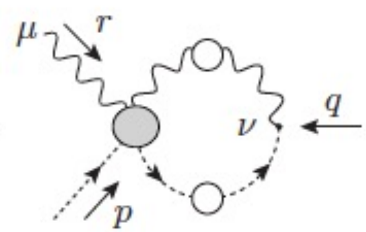
$$q^\mu \Pi_{\mu\nu}(q)|_{(a)+(b)+(c)} \neq 0$$

Slavnov Taylor identities



- The main problem is that fully dressed vertices satisfy STIs instead of WI.

$$q^\alpha \Gamma_{\alpha\mu\nu}(q,r,p) = F(q)\left[\Delta^{-1}(p)P_\nu^\alpha(p)H_{\mu\alpha}(r,p) - \Delta^{-1}(r)P_\mu^\alpha(r)H_{\nu\alpha}(p,r)\right]$$

$$H_{\nu\mu}(q,p,r) = g_{\mu\nu} +$$ 

$$D(q^2) = \frac{F(q^2)}{q^2}$$

$$\Delta^{-1}(q^2) = q^2 J(q^2)$$

$$P_{\mu\nu}(q) = g_{\mu\nu} - q_\mu q_\nu / q^2$$

- All diagrams must conspire to maintain intact crucial properties of the theory.
- If one truncates “naively”, i.e., just by dropping diagrams without a guiding principle → one will violate the fundamental transversality property.
- *To avoid that → use SDE in the Pinch Technique - Background field method (PT-BFM) formalism*

Background field method: Crash course

$$\mathcal{L}_{YM} = -\frac{1}{4} G_a^{\mu\nu} G^a_{\mu\nu} - \frac{1}{2\xi}(\partial^\mu A^a_\mu)^2 + \bar{c}^a(-\partial^\mu D^{ac}_\mu)c^c$$

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu$$

- The BFM is a special quantization scheme: Split the gauge field

$$A^a_\mu \to B^a_\mu + Q^a_\mu$$

B^a_μ → background field; Q^a_μ → quantum (fluctuating) field;

- In the generating functional integrate only over Q^a_μ
- The gauge-fixed $\mathcal{L}_{\mathrm{YM}}$ is invariant under the transformations

$$B^a_\mu \to B^a_\mu + D_\mu \theta^a$$

$$D_\mu = \partial_\mu - i B^a_\mu t^a$$

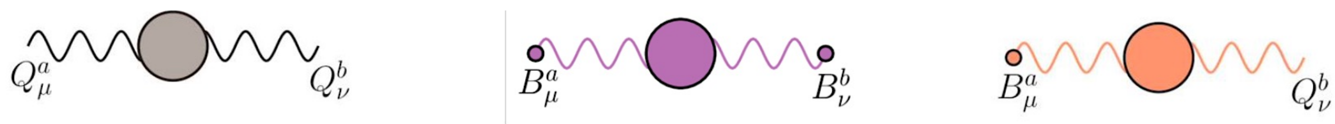
Simplifies the WTIs satisfied by these Green's functions

- Proliferation of vertices and propagators.

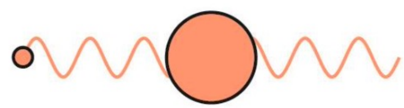
B. S. Witt, Phys. Rev. 162 (1967) 195—1239
G. 't Hooft, In *Karpacz 1975, Proceedings, Acta Universitatis Wratislaviensis No.368, 1976, 345-369
L. F. Abbott, Nucl. Phys. B185 (1981) 189

New Green's functions

- Three types of gluon propagators:



- Relation between gluon propagators

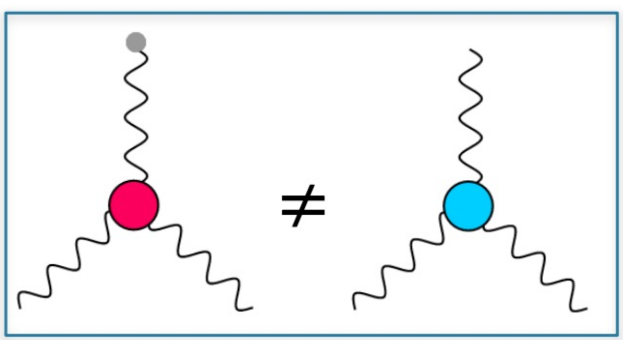
 $= (1 + G)\times$ 

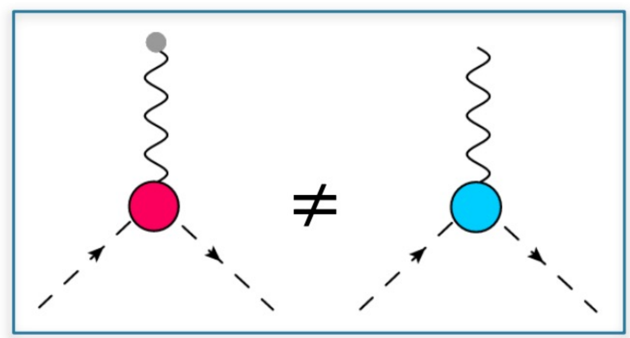
$$\widetilde{\Delta}(q) = [1 + G(q)]\,\Delta(q)$$

Auxiliary function

P. A. Grassi, T. Hurth and M. Steinhauser, Annals Phys. 288 , 197 (2001)
D. Binosi and J. Papavassiliou., Phys. Rev. D66, 025024 (2002)

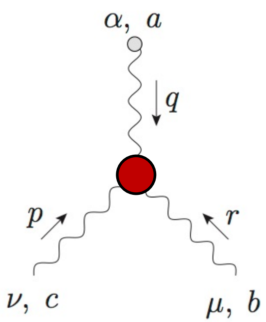
- New vertices:





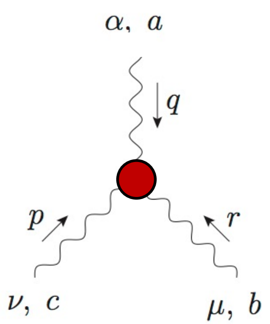
Special Properties

- QED-like (ghost free) Ward identities



$$q^{\alpha}\widetilde{\Gamma}_{\alpha\mu\nu}(q,r,p) = i\Delta^{-1}_{\mu\nu}(r) - i\Delta^{-1}_{\mu\nu}(p)$$

instead of the standard



$$q^{\alpha}\Gamma_{\alpha\mu\nu}(q,r,p) = iF(q)[\Delta^{-1}_{\sigma\nu}(r)H^{\sigma}_{\mu}(q,p,r) - \Delta^{-1}_{\sigma\mu}(p)H^{\sigma}_{\nu}(q,r,p)]$$

$H_{\mu\nu}(q,r,p)$: Ghost-gluon kernel

$D(q^2) = F(q^2)/q^2$: Ghost propagator

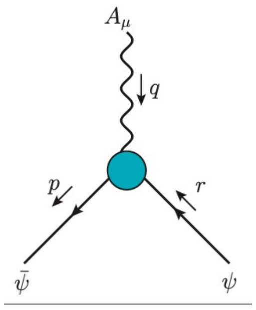
$F(q^2)$: Ghost dressing function

From Takahashi to Ward identities

- Takahashi Identities

QED :

$$q^{\mu}\Gamma_{\mu}(q,r,-p) = S^{-1}(p) - S^{-1}(r)$$



$p = q + r$

- Taylor expansion around $q = 0$, in the absence of poles $\sim \frac{1}{q^2}$ in the vertex:

$$f(q,r,p) = f(0,r,-r) + q^{\mu}\left\{\frac{\partial}{\partial q^{\mu}} f(q,r,p)\right\}_{q=0} + \mathcal{O}(q^2)$$

$$q^{\mu}\Gamma_{\mu}(0,r,-r) + \mathcal{O}(q^2) = q^{\mu}\left\{\frac{\partial}{\partial q^{\mu}} S^{-1}(q+r)\right\}_{q=0} + \mathcal{O}(q^2)$$

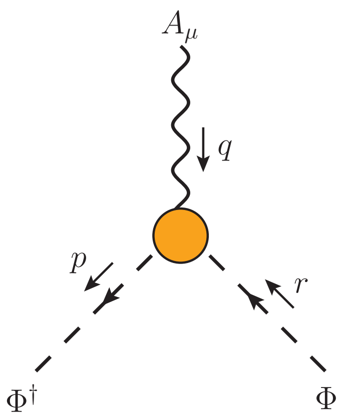
- We obtain

$$\boxed{\Gamma_{\mu}(0,r,-r) = \frac{\partial S^{-1}(r)}{\partial r^{\mu}}}$$

Ward identity

From Takahashi to the Ward identities

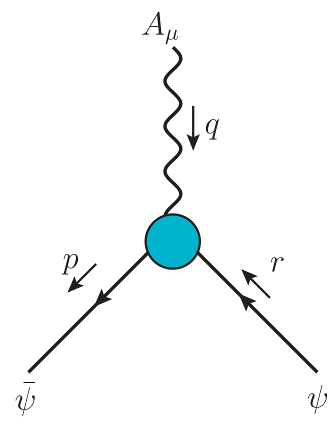
Scalar QED



$$\boxed{q^\mu \Gamma_\mu(q,r,-p) = \mathcal{D}^{-1}(p) - \mathcal{D}^{-1}(r)}$$

$$\Gamma_\mu(0,r,-r) = \frac{\partial}{\partial r^\mu}\mathcal{D}^{-1}(r)$$

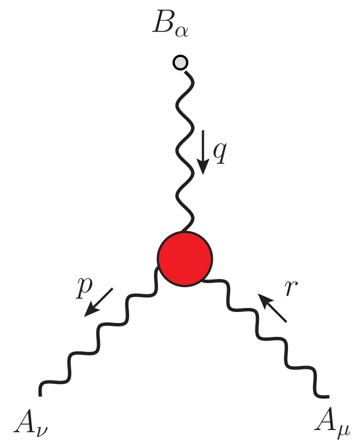
QED



$$\boxed{q^\mu \Gamma_\mu(q,r,-p) = S^{-1}(p) - S^{-1}(r)}$$

$$\Gamma_\mu(0,r,-r) = \frac{\partial}{\partial r^\mu} S^{-1}(r)$$

QCD (in the BFM)

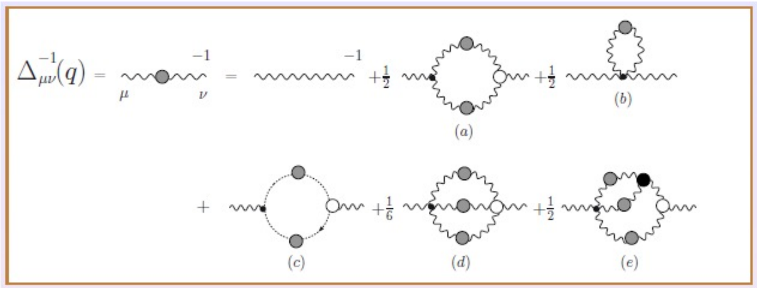


$$\boxed{q^\alpha \widetilde{\Gamma}_{\alpha\mu\nu}(q,r,p) = i\Delta^{-1}_{\mu\nu}(r) - i\Delta^{-1}_{\mu\nu}(p)}$$

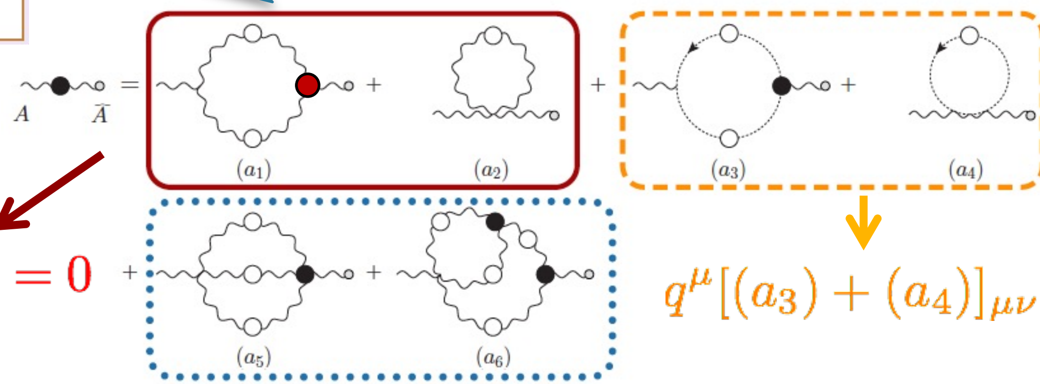
$$\widetilde{\Gamma}_{\alpha\mu\nu}(0,r,-r) = -i\frac{\partial}{\partial r^\alpha}\Delta^{-1}_{\mu\nu}(r)$$

Warning: Only valid in the absence of massless poles

Pinch Technique - Background Field Method



$$\Delta(q) = \widetilde{\Delta}(q)[1 + G(q)]$$



$$q^{\mu}[(a_1) + (a_2)]_{\mu\nu} = 0$$

$$q^{\mu}[(a_3) + (a_4)]_{\mu\nu} = 0$$

$$q^{\mu}[(a_5) + (a_6)]_{\mu\nu} = 0$$

- **Transversality** is enforced separately for gluon and ghost loops, and order by order in the "dressed-loop" expansion!

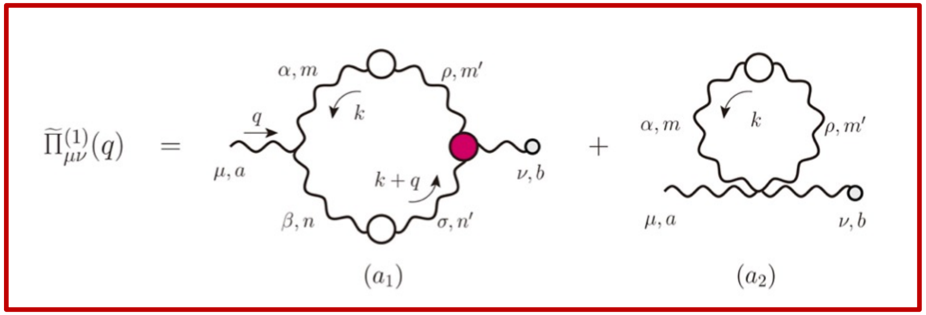
$$q^{\mu}\widetilde{\Gamma}_{\mu\alpha\beta}(q,r,p) = i\Delta^{-1}_{\alpha\beta}(r) - i\Delta^{-1}_{\alpha\beta}(p) \quad \Rightarrow \quad q^{\mu}[(a_1) + (a_2)]_{\mu\nu} = 0$$

$$q^{\mu}\widetilde{\Gamma}_{\mu}(q,r,-p) = D^{-1}(p) - D^{-1}(r) \quad \Rightarrow \quad q^{\mu}[(a_3) + (a_4)]_{\mu\nu} = 0$$

$$q^{\mu}\widetilde{\Gamma}^{mnrs}_{\mu\alpha\beta\gamma} = f^{mse}f^{ern}\Gamma_{\alpha\beta\gamma} + f^{mne}f^{esr}\Gamma_{\beta\gamma\alpha} + f^{mre}f^{ens}\Gamma_{\gamma\alpha\beta} \quad \Rightarrow \quad q^{\mu}[(a_5) + (a_6)]_{\mu\nu} = 0$$

A.C. A. and J.Papavassiliou, JHEP 0612, 012 (2006)
D. Binosi and J. Papavassiliou, Phys.Rev. D 77, 061702 (2008); JHEP 0811:063,2008.

Transversality of the first block



- Let us check the transversality of the first group of diagrams
- Considering first the diagram (a_1) contracting it with the external background gluon momentum q^ν

$$q^\nu\ \widetilde{\Pi}^{ab}_{\mu\nu}(q)\Big|_{(a_1)} = \frac{1}{2}\int_k \Gamma^{(0)amn}_{\mu\alpha\beta}\Delta^{\alpha\rho}_{mm'}(k)\left[q^\nu \widetilde{\Gamma}^{bm'n'}_{\nu\rho\sigma}\right]\Delta^{\beta\sigma}_{nn'}(k+q)$$
$$= \frac{1}{2}\int_k \Gamma^{(0)amn}_{\mu\alpha\beta}\Delta^{\alpha\rho}_{mm'}(k) g f^{bm'n'}\left[\Delta^{-1}_{\rho\sigma}(k+q) - \Delta^{-1}_{\rho\sigma}(k)\right]\Delta^{\beta\sigma}_{nn'}(k+q),$$

- Now using the conventional Feynman rule for the three-gluon vertex at tree-level, we obtain

$$\Gamma^{(0)}_{\mu\alpha\beta}(q,r,p) = g_{\alpha\beta}(r-p)_\mu + g_{\mu\beta}(p-q)_\alpha + g_{\mu\alpha}(q-r)_\beta$$

$$q^\nu\ \widetilde{\Pi}_{\mu\nu}(q)\Big|_{(a_1)} = \frac{1}{2}C_A g^2 \int_k \left[g_{\alpha\beta}(2k+q)_\mu + g_{\beta\mu}(-k-2q)_\alpha + g_{\mu\alpha}(q-k)_\beta\right] \times \left[\Delta^{\alpha\beta}(k) - \Delta^{\alpha\beta}(k+q)\right].$$

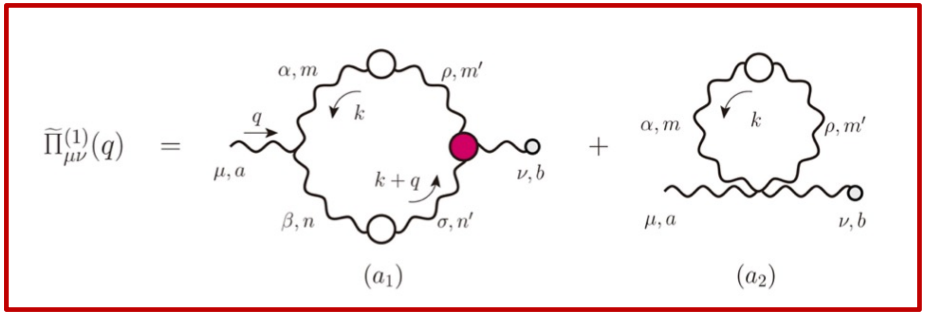
- Performing the contraction, we find

$$q^\nu\ \widetilde{\Pi}_{\mu\nu}(q)\Big|_{(a_1)} = \frac{1}{2}C_A g^2 \int_k \left\{(2k+q)_\mu\left[\Delta^\alpha_\alpha(k) - \Delta^\alpha_\alpha(k+q)\right] - (2k+q)_\alpha\left[\Delta^\alpha_\mu(k) - \Delta^\alpha_\mu(k+q)\right]\right\}$$

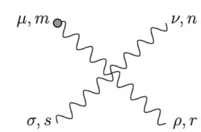
shift $k+q \rightarrow k$

- Then, we arrive at

$$q^\nu \, \widetilde{\Pi}_{\mu\nu}(q)\Big|_{(a_1)} = C_A g^2 \int_k \left[q_\mu \Delta^\alpha_\alpha(k) - q_\alpha \Delta^\alpha_\mu(k) \right].$$



BFM Feynman rule



$-ig^2\widetilde{\Gamma}^{mnrs\,(0)}_{\mu\nu\rho\sigma}$

$$\begin{aligned}\widetilde{\Gamma}^{mnrs\,(0)}_{\mu\nu\rho\sigma} = {} & f^{mse} f^{ern} (g_{\mu\rho} g_{\nu\sigma} - g_{\mu\nu} g_{\rho\sigma}) \\ & + f^{mne} f^{esr} (g_{\mu\sigma} g_{\nu\rho} - g_{\mu\rho} g_{\nu\sigma}) \\ & + f^{mre} f^{esn} (g_{\mu\sigma} g_{\nu\rho} - g_{\mu\nu} g_{\rho\sigma})\end{aligned}$$

- Now, contracting the diagram (a_2) with q^ν, we easily find

$$\begin{aligned} q^\nu \, \widetilde{\Pi}_{\mu\nu}(q)\Big|_{(a_2)} &= \frac{1}{2} q^\nu \int_k \widetilde{\Gamma}^{(0)}_{\mu\nu\alpha\beta} \Delta^{\alpha\beta}(k) \\ &= -C_A g^2 \int_k \left[q_\mu \Delta^\alpha_\alpha(k) - q_\alpha \Delta^\alpha_\mu(k) \right], \end{aligned}$$

- Therefore, the transversality of the first block of diagrams shown in the figure is proved,

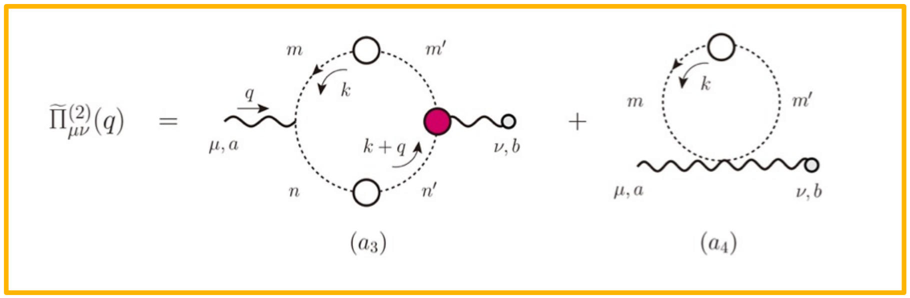
$$q^\nu \, \widetilde{\Pi}^{ab}_{\mu\nu}(q)\Big|_{(a_1)+(a_2)} = 0.$$

- In order to verify the transversality of the other two groups, one can follow the same procedure, using the appropriate WTIs

$$q^{\mu}\widetilde{\Gamma}_{\mu\alpha\beta}(q,r,p) = i\Delta^{-1}_{\alpha\beta}(r) - i\Delta^{-1}_{\alpha\beta}(p),$$

$$q^{\mu}\widetilde{\Gamma}_{\mu}(q,r,p) = iD^{-1}(r^2) - iD^{-1}(p^2),$$

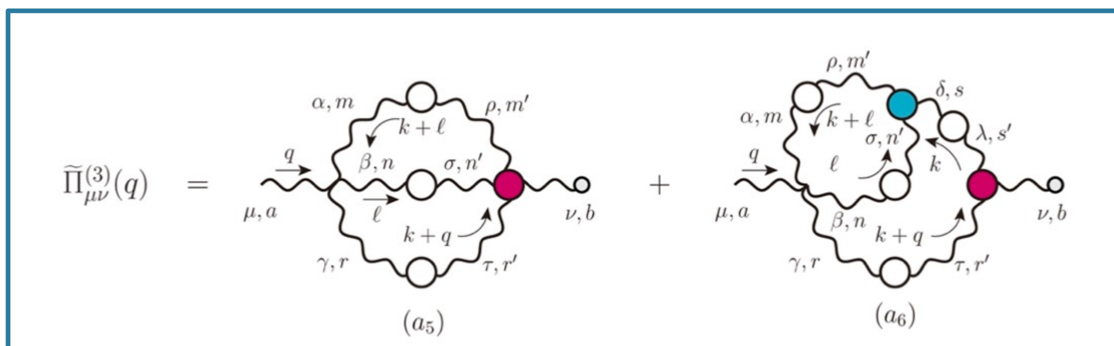
$$q^{\mu}\widetilde{\Gamma}^{mnrs}_{\mu\alpha\beta\gamma}(q,r,p,t) = f^{mse}f^{ern}\Gamma_{\alpha\beta\gamma}(r,p,q+t) + f^{mne}f^{esr}\Gamma_{\beta\gamma\alpha}(p,t,q+r)f^{mre}f^{ens}\Gamma_{\gamma\alpha\beta}(t,r,q+p)\,.$$



$$q^{\nu}\widetilde{\Pi}_{\mu\nu}(q)\Big|_{(a_3)+(a_4)} = 0$$

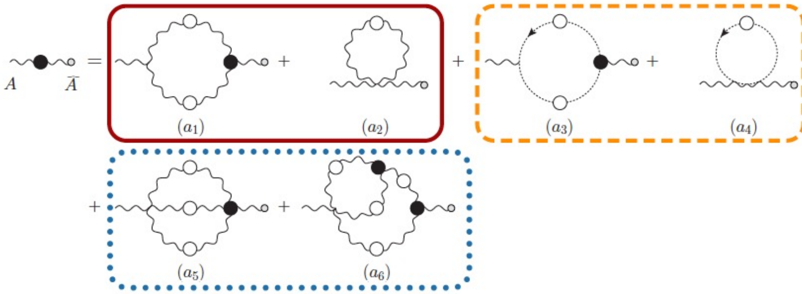
- ✓ To prove the transversality of second block (yellow) use the second WI.

$$q^{\nu}\widetilde{\Pi}_{\mu\nu}(q)\Big|_{(a_5)+(a_6)} = 0$$



- ✓ To verify the transversality of the last group (blue), one needs to use the first and last WIs.

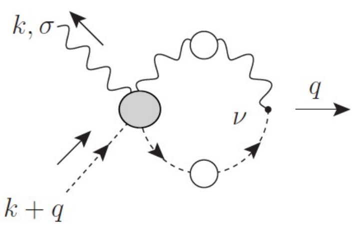
Converting the PT - BFM propagator



$$\Delta^{-1}(q^2)P_{\mu\nu}(q) = \frac{q^2 P_{\mu\nu}(q) + i\sum_{i=1}^{6}(a_i)_{\mu\nu}}{\underline{1+G(q^2)}},$$

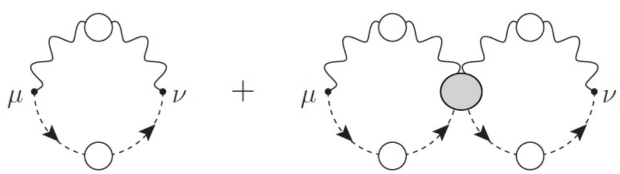
Auxiliary function

$H_{\sigma\nu}(k,q) \;=\; H^{(0)}_{\sigma\nu} \;+$



$$\begin{aligned}\Lambda_{\mu\nu}(q) &= -ig^2 C_A \int_k \Delta_\mu^\sigma(k) D(q-k) H_{\nu\sigma}(-q, q-k, k)\\ &= g_{\mu\nu} G(q^2) + \frac{q_\mu q_\nu}{q^2} L(q^2),\end{aligned}$$

$\Lambda_{\mu\nu}(q) \;=$

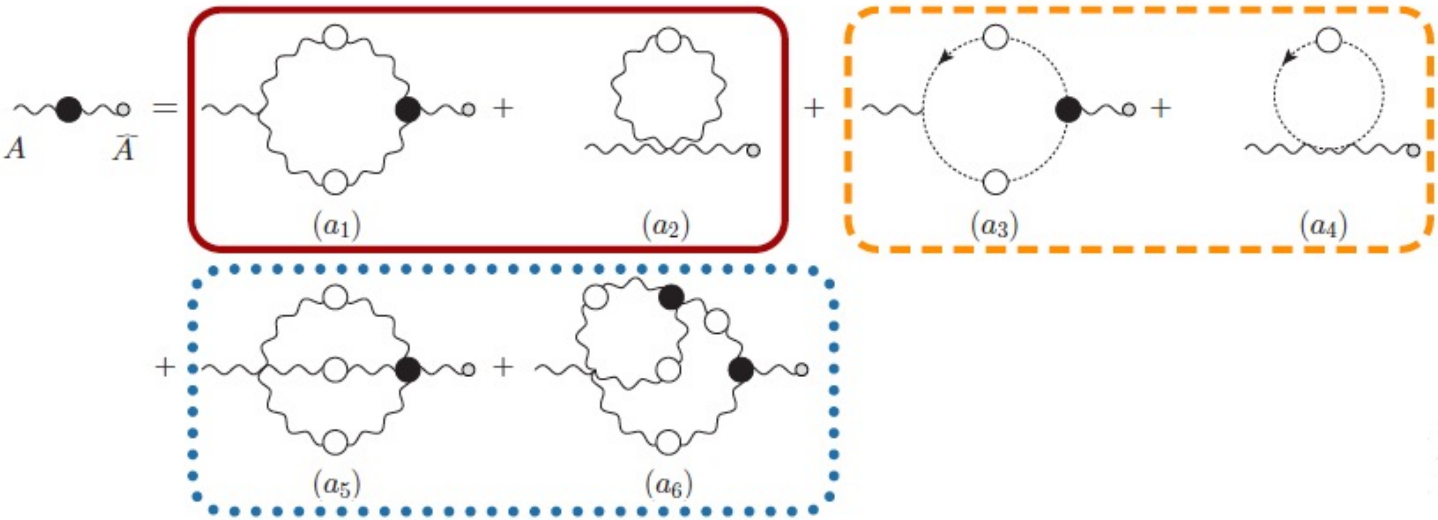


In Landau gauge the ghost dressing function satisfies

$$D(q^2) = \frac{F(q^2)}{q^2}$$

$$F^{-1}(q^2) = 1 + G(q^2) + L(q^2).$$

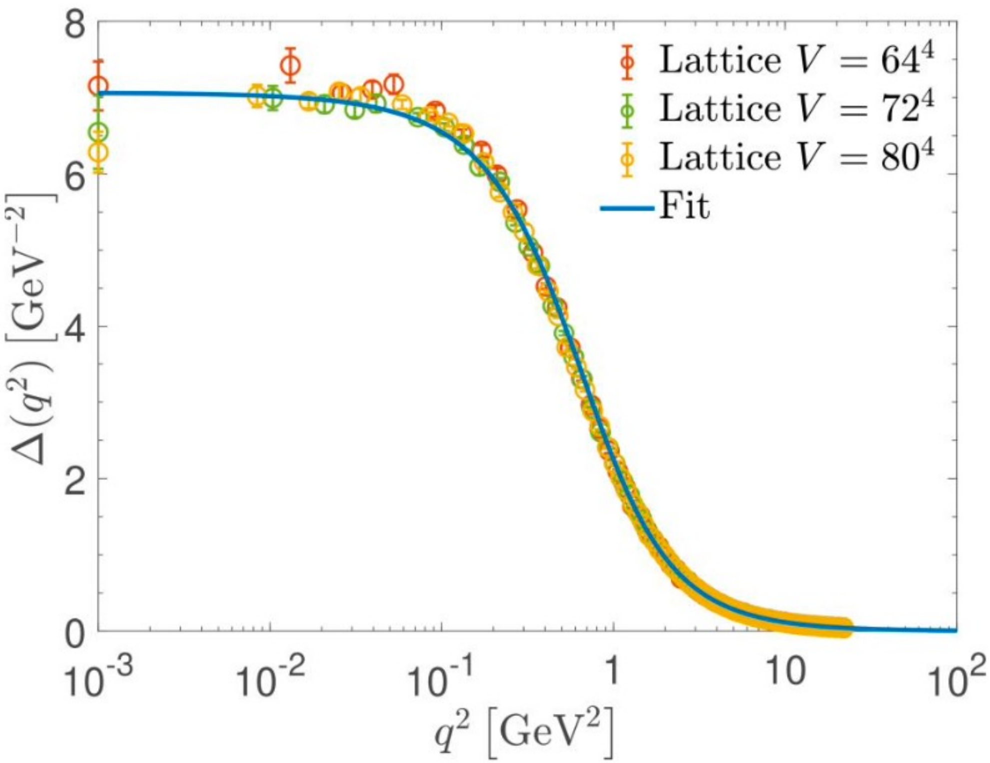
A.C. A., D. Binosi and J. Papavassiliou, JHEP 0911, 066 (2009)
A. C. A., D. Binosi, J. Papavassiliou and J. Rodriguez-Quintero, Phys. Rev. D 80, 085018 (2009)



Short Summary:

- ✓ We can truncate the SDE for the PT-BFM propagator without violating the transversality of the gluon self-energy, as long as we consider all diagrams within the chosen blocks.
- ✓ Note, however, that this fact does not imply that the contributions from the neglected groups are necessarily small.
- ✓ Nonetheless, being able to truncate the SDE while preserving, by construction, the symmetry of the theory is a great achievement!

Emergent mass scale in the gauge sector



Lattice data from:
I. L. Bogolubsky, et al , PoS LAT2007, 290 (2007).

- Saturation of the gluon propagator can be explained by the generation of a *gluon mass scale.*

- Natural parametrization:

$$\Delta^{-1}(q^2) = \underbrace{q^2 J(q^2)}_{\text{Kinetic term}} + \underbrace{m^2(q^2)}_{\text{running mass}}$$

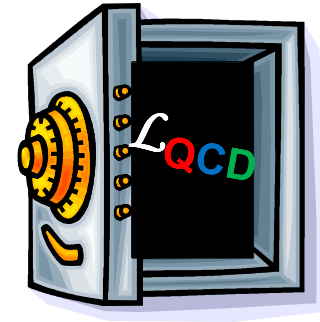
with

$$\Delta^{-1}(0) = m^2(0)$$

- A nonperturbative mechanism is needed.

J. M. Cornwall, Phys. Rev. D26, 1453 (1982).
ACA., D. Binosi and J. Papavassiliou., Phys.Rev. D78 (2008) 025010.

Gluon mass generation



- The dynamical gluon mass should be generated without modifying the QCD lagrangian

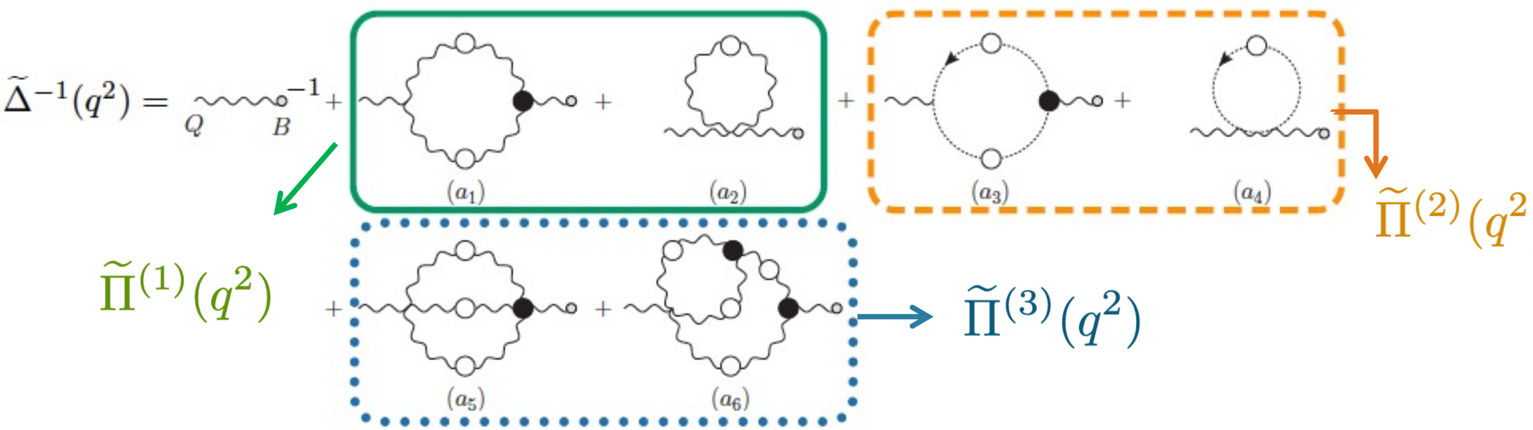
$$\mathcal{L}_{YM} = -\frac{1}{4} G_a^{\mu\nu} G^a_{\mu\nu} - \frac{1}{2\xi} (\partial^\mu A^a_\mu)^2 + \bar{c}^a (-\partial^\mu D^{ac}_\mu) c^c$$

where the gluonic field strength tensor

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - g f^{abc} A^b_\mu A^c_\nu$$

- A mass term ($m^2 A_\mu^2$) is forbidden by gauge invariance.
- The mechanism should not generate quadratic divergences → to renormalize them away you must add a mass term.

No mass without poles



$$\widetilde{\Delta}^{-1}(q^2) = q^2 + i\left[\widetilde{\Pi}^{(1)}(q^2) + \widetilde{\Pi}^{(2)}(q^2) + \widetilde{\Pi}^{(3)}(q^2)\right]$$

PT- BFM Ward identites + **Seagull cancellation** = $\widetilde{\Delta}^{-1}(0) = 0$

- *Seagull cancellation*

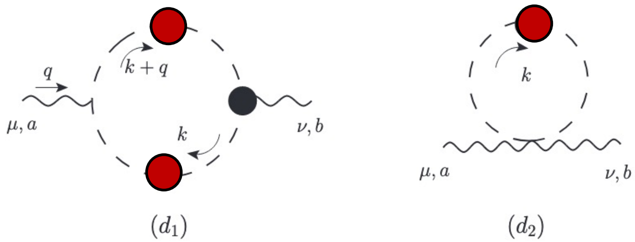
(valid in dimensional regularization)

$$\int_k k^2 \frac{\partial f(k^2)}{\partial k^2} + \frac{d}{2}\int_k f(k^2) = 0$$

ACA and J. Papavassiliou Phys. Rev. D 81, 034003 (2010)

Seagull identity the in Scalar QED

$$\Pi^{(1)}_{\mu\nu}(q) =$$



$$\Pi^{(1)}_{\mu\nu}(q) = (d_1)_{\mu\nu} + (d_2)_{\mu\nu},$$

$$\Pi^{(1)}_{\mu\nu}(q) = \left(g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}\right)\Pi^{(1)}(q^2).$$

$$(d_1)_{\mu\nu} = e^2 \int_k (2k+q)_\mu \mathcal{D}(k)\mathcal{D}(k+q)\Gamma_\nu(-q, k+q, -k),$$

$$(d_2)_{\mu\nu} = -2e^2 g_{\mu\nu} \int_k \mathcal{D}(k^2),$$

- Taking the limit q → 0, we have that $g^{\mu\nu}$ component

$$d_1 = \frac{2e^2}{d}\int_k k_\mu \mathcal{D}^2(k^2)\Gamma^\mu(0,k,-k),$$

$$d_2 = -2e^2 \int_k \mathcal{D}(k^2).$$

- Use the WIs

$$\Gamma_\mu(0,r,-r) = \frac{\partial}{\partial r^\mu}\mathcal{D}^{-1}(r)$$

$$\mathcal{D}^2(k^2)\Gamma^\mu(0,k,-k) = -\frac{\partial \mathcal{D}(k^2)}{\partial k^\mu},$$

- Thus, using that

$$k^\mu \frac{\partial f(k^2)}{\partial k^\mu} = 2k^2 \frac{\partial f(k^2)}{\partial k^2}.$$

we find

$$d_1 = -\frac{4e^2}{d}\int_k k^2 \frac{\partial \mathcal{D}(k^2)}{\partial k^2},$$

$$\Pi^{(1)}(0) = -\frac{4e^2}{d}\underbrace{\left[\int_k k^2 \frac{\partial \mathcal{D}(k^2)}{\partial k^2} + \frac{d}{2}\int_k \mathcal{D}(k^2)\right]}_{\text{seagull}}.$$

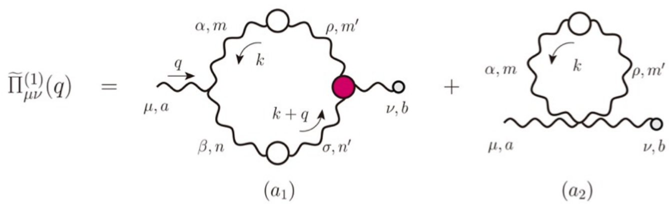
$$\Pi^{(1)}(0) = 0$$

$$\int_k k^2 \frac{\partial f(k^2)}{\partial k^2} + \frac{d}{2}\int_k f(k^2) = 0$$

ACA, D. Binosi, C. T. Figueiredo. and J. Papavassiliou, Phys. Rev. D94, no. 4, 045002 (2016).

Seagull identity in the PT-BFM

- Following exactly the same reasoning as in the scalar QED case, we have



$$\widetilde{\Pi}_{\mu\nu}(0) = \widetilde{\Pi}(0) g_{\mu\nu}$$

$$\widetilde{\Pi}^{(1)}(0) = a_1 + a_2,$$

$$\begin{aligned} a_1(0) &= \frac{1}{2} C_A g^2 \int_k \Gamma^{(0)}_{\mu\alpha\beta}(0,k,-k)\Delta^{\alpha\rho}(k)\Delta^{\beta\sigma}(k)\widetilde{\Gamma}^{\mu}_{\rho\sigma}(0,-k,k)\,, \\ a_2(0) &= -iC_A g^2 (d-1) \int_k \Delta^{\alpha}_{\alpha}(k)\,. \end{aligned}$$

- Use the WIs

$$\widetilde{\Gamma}_{\alpha\mu\nu}(0,r,-r) = -i\frac{\partial \Delta^{-1}_{\mu\nu}(r)}{\partial r^{\alpha}},$$

- Thus, we have that

$$\Delta^{\alpha\rho}(k)\Delta^{\beta\sigma}(k)\widetilde{\Gamma}^{\mu}_{\sigma\rho}(0,k,-k) = \frac{\partial \Delta^{\alpha\beta}(k)}{\partial k^{\mu}}.$$

$$a_1 = \frac{g^2 C_A}{2d} \int_k \Gamma^{(0)}_{\mu\alpha\beta}(0,k,-k)\frac{\partial \Delta^{\alpha\beta}(k)}{\partial k^{\mu}}.$$

- Using

$$k^{\mu}\frac{\partial f(k^2)}{\partial k^{\mu}} = 2k^2\frac{\partial f(k^2)}{\partial k^2}.$$

$$\Gamma^{(0)}_{\mu\alpha\beta}(0,k,-k) = 2k_{\mu}g_{\alpha\beta} - k_{\beta}g_{\alpha\mu} - k_{\alpha}g_{\beta\mu}.$$

- We find

$$a_1 = \frac{2(d-1)}{d} g^2 C_A \left[\int_k k^2 \frac{\partial \Delta(k^2)}{\partial k^2} + \frac{1}{2}\int_k \Delta(k^2)\right].$$

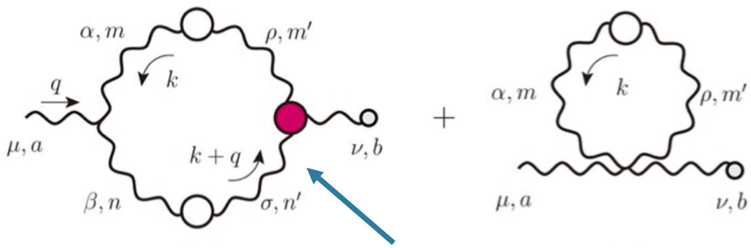
$$a_2 = g^2 C_A \frac{(d-1)^2}{d}\int_k \Delta(k^2).$$

$$\widetilde{\Pi}(0) = \frac{2(d-1)}{d} g^2 C_A \underbrace{\left[\int_k k^2 \frac{\partial \Delta(k^2)}{\partial k^2} + \frac{d}{2}\int_k \Delta(k^2)\right]}_{\text{seagull}} = 0,$$

$$\widetilde{\Pi}^{(1)}(0) = 0$$

Following the same steps, we can prove that

$\widetilde{\Pi}^{(1)}_{\mu\nu}(q) =$

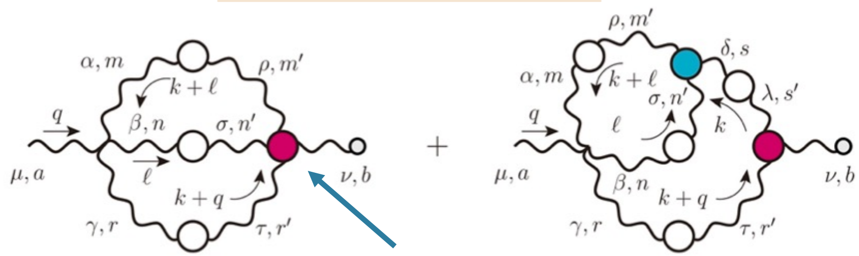


(a_1) (a_2)

$$\widetilde{\Gamma}_{\mu}(0,-p,p) = i\frac{\partial D^{-1}(p^2)}{\partial p^{\mu}}.$$

$$\widetilde{\Pi}^{(2)}(0) = 0$$

$\widetilde{\Pi}^{(3)}_{\mu\nu}(q) =$



(a_5) (a_6)

$$\widetilde{\Gamma}^{mnrs}_{\mu\alpha\beta\gamma}(0,r,p,-r-p) = \left(f^{mne}f^{esr}\frac{\partial}{\partial r^{\mu}} + f^{mre}f^{ens}\frac{\partial}{\partial p^{\mu}}\right)\Gamma_{\alpha\beta\gamma}(r,p,-r-p)\,;$$
$$\widetilde{\Gamma}^{mnrs}_{\mu\alpha\beta\gamma}(0,-r,-p,r+p) = -\left(f^{mne}f^{esr}\frac{\partial}{\partial r^{\mu}} + f^{mre}f^{ens}\frac{\partial}{\partial p^{\mu}}\right)\Gamma_{\alpha\beta\gamma}(-r,-p,r+p)\,.$$

$$\widetilde{\Pi}^{(3)}(0) = 0$$

$$\widetilde{\Delta}^{-1}(q^2) = q^2 + i\left[\widetilde{\Pi}^{(1)}(q^2) + \widetilde{\Pi}^{(2)}(q^2) + \widetilde{\Pi}^{(3)}(q^2)\right] \Rightarrow \widetilde{\Delta}^{-1}(0) = 0$$

The question is: How can one evade the seagull cancellation and get a gluon mass?

Answer: Introduce massless poles to trigger the Schwinger Mechanism

Schwinger Mechanism in QCD

J. S. Schwinger, Phys. Rev.125, 397 (1962);
Phys.Rev.128, 2425 (1962).

- Propagator in the Landau gauge:

$$\Delta_{\mu\nu} = -i\left[P_{\mu\nu}(q)\Delta(q^2)\right]$$

$$P_{\mu\nu}(q) = g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2}$$

- Vaccum polarization:

$$\Pi_{\mu\nu}(q) = P_{\mu\nu}(q)q^2\mathbf{\Pi}(q^2)$$

$$\Rightarrow \quad \Delta^{-1}(q^2) = q^2[1 + \mathbf{\Pi}(q^2)]$$

- If the vaccum polarization has a pole in $q^2 = 0$ with positive residue m^2, i.e.

$$\Pi(q^2) = \frac{m^2}{q^2}$$

- Then

$$\Delta^{-1}(q^2) = q^2 + m^2$$

$$\Delta^{-1}(0) = m^2$$

Dynamical gluon mass generation requires the existence of vertices containing poles of nonpertubative origin.

Gluon mass generation in a nutshell

- The gauge invariant generation of a gluon mass proceeds through the implementation of the *Schwinger mechanism.*

- It *requires the existence* of a *very special type of nonperturbative vertices:*

  ✓ Contain *massless poles* of nonpertubative origin → *evades the seagull cancellation* → make possible that $\Delta^{-1}(0) \neq 0$;

  ✓ They guarantee that the **STIs remain intact**;

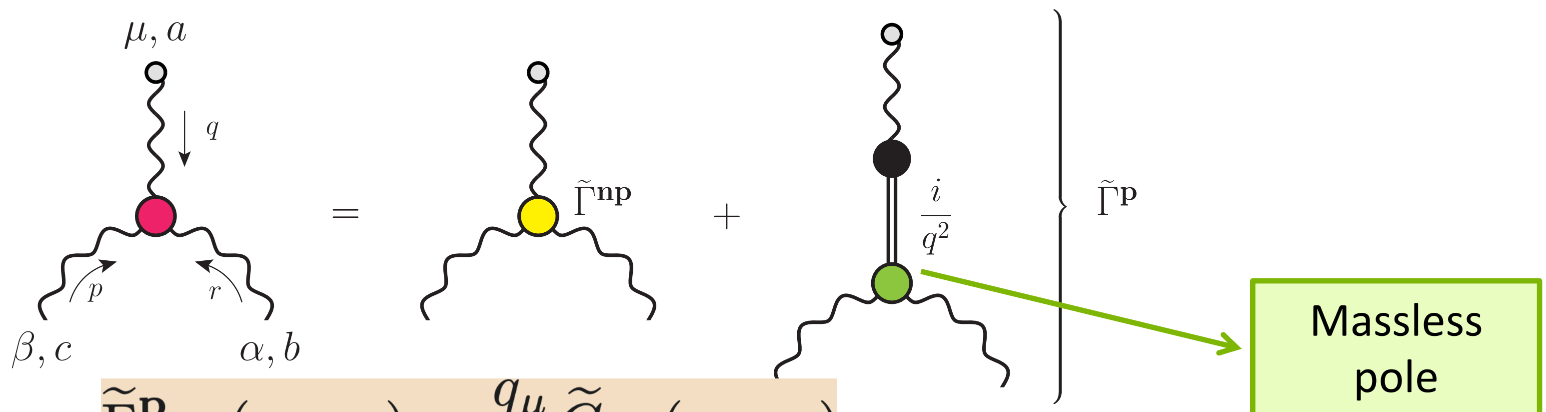
  ✓ They are completely *longitudinally coupled*, act as a composite Nambu-Goldstone bosons.

R. Jackiw and K. Johnson, Phys. Rev. D 8, 2386 (1973)
J. M. Cornwall, Phys. Rev. D26, 1453 (1982).
E. Eichten and F. Feinberg, Phys. Rev. D 10, 3254 (1974)

Vertices with massless poles

- To evade the previous result, one must relax one of the underlying assumptions
- In particular, the derivation of the WI hinges on the absence of poles $1/q^2$
- Therefore, let us introduce poles in $\widetilde{\Gamma}_{\mu\alpha\beta}$ (full three gluon vertex)

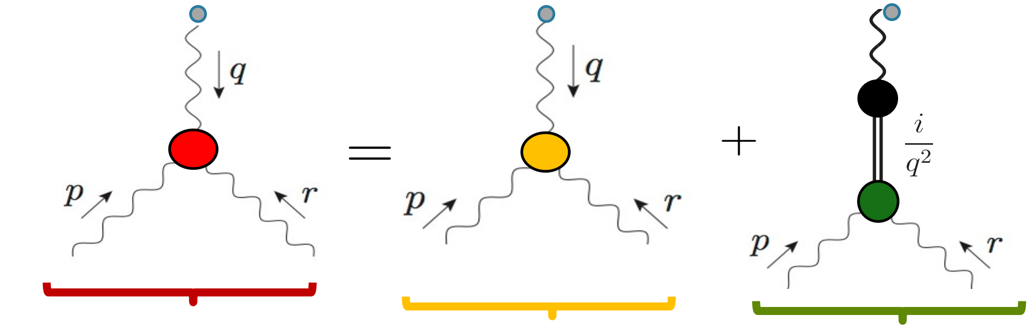
$$\widetilde{\Gamma}_{\mu\alpha\beta}(q,r,p) = \widetilde{\Gamma}^{\mathbf{np}}_{\mu\alpha\beta}(q,r,p) + \widetilde{\Gamma}^{\mathbf{p}}_{\mu\alpha\beta}(q,r,p)$$



$$\widetilde{\Gamma}^{\mathbf{p}}_{\mu\alpha\beta}(q,r,p) = \frac{q_\mu}{q^2}\widetilde{C}_{\alpha\beta}(q,r,p)$$

$$= \frac{q^\mu}{q^2}\left[\widetilde{C}_1\, g^{\alpha\beta} + \widetilde{C}_2\, q^\alpha q^\beta + \widetilde{C}_3\, q^\alpha r^\beta + \widetilde{C}_4\, r^\alpha q^\beta + \widetilde{C}_5\, r^\alpha r^\beta\right]$$

- Explicit implementation of the *Schwinger mechanism* in Yang-Mills theories



- The abelianized STI must be realized in part by means of a longitudinally coupled pole term

$$\widetilde{\Gamma}_{\mu\alpha\beta}(q,r,p) = \widetilde{\Gamma}^{\mathrm{np}}_{\mu\alpha\beta}(q,r,p) + \widetilde{\Gamma}^{\mathrm{p}}_{\mu\alpha\beta}(q,r,p)$$

$$\Delta^{-1}(q^2) = q^2 J(q^2) + m^2(q^2)$$

$$\mathrm{P}_{\alpha\mu}(q) = g_{\alpha\mu} - \frac{q_\mu q_\alpha}{q^2}$$

$$+\left\{\begin{array}{l} q^\alpha \widetilde{\Gamma}^{\mathrm{np}}_{\mu\alpha\beta}(q,r,p) = p^2 J(p^2) P_{\mu\nu}(p) - r^2 J(r^2) P_{\mu\nu}(r) \\ q^\alpha \widetilde{\Gamma}^{\mathrm{p}}_{\mu\alpha\beta}(q,r,p) = m^2(p^2) P_{\mu\nu}(p) - m^2(r^2) P_{\mu\nu}(r) \end{array}\right.$$

$$q^\alpha \widetilde{\Gamma}_{\mu\alpha\beta}(q,r,p) = \Delta^{-1}(p^2) P_{\mu\nu}(p) - \Delta^{-1}(r^2) P_{\mu\nu}(r)$$

Full Abelianized STI

- Pole part is longitudinal:

$$P^{\alpha}_{\alpha'}(q) P^{\mu}_{\mu'}(r) P^{\nu}_{\nu'}(p) \widetilde{\Gamma}^{\mathrm{p}}_{\mu\alpha\beta}(q,r,p) = 0$$

Drops out when embedded in a S-matrix element and also in transversely projected Green's functions

Ward identity in the presence of poles

$$\widetilde{\Gamma}_{\mu\alpha\beta}(q,r,p) = \widetilde{\Gamma}^{\mathbf{np}}_{\mu\alpha\beta}(q,r,p) + \frac{q_\mu}{q^2}\widetilde{C}_{\alpha\beta}(q,r,p)$$

$$\widetilde{\Gamma}^{\mathbf{p}}_{\mu\alpha\beta}(q,r,p) = \frac{q_\mu}{q^2}\widetilde{C}_{\alpha\beta}(q,r,p)$$

$$q^\mu\widetilde{\Gamma}_{\mu\alpha\beta}(q,r,p) = i\Delta^{-1}_{\alpha\beta}(r) - i\Delta^{-1}_{\alpha\beta}(p)$$

Same WTI identity !

$$q^\mu\widetilde{\Gamma}^{\mathbf{np}}_{\mu\alpha\beta}(q,r,p) + \widetilde{C}_{\alpha\beta}(q,r,p) = i\Delta^{-1}_{\alpha\beta}(r) - i\Delta^{-1}_{\alpha\beta}(p),$$

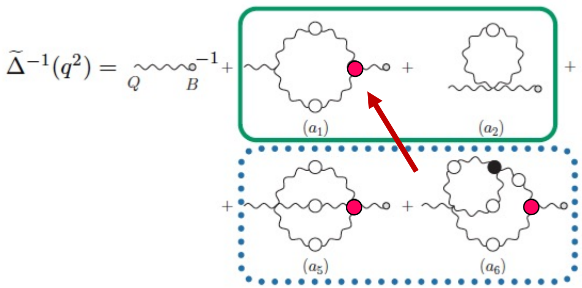
$$q^\mu\widetilde{\Gamma}^{\mathbf{np}}_{\mu\alpha\beta}(0,r,-r) + \widetilde{C}_{\alpha\beta}(0,r,-r) + q^\mu\left\{\frac{\partial}{\partial q^\mu}\widetilde{C}_{\alpha\beta}(q,r,p)\right\}_{q=0} = -iq^\mu\frac{\partial\Delta^{-1}_{\alpha\beta}(r)}{\partial r^\mu}$$

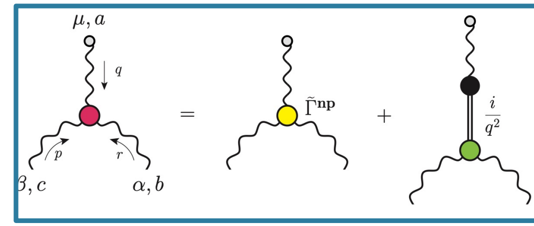
$$\widetilde{C}_{\alpha\beta}(0,r,-r) = 0$$

$$\widetilde{\Gamma}^{\mathbf{np}}_{\mu\alpha\beta}(0,r,-r) = -i\frac{\partial\Delta^{-1}_{\alpha\beta}(r)}{\partial r^\mu} - \left\{\frac{\partial}{\partial q^\mu}\widetilde{C}_{\alpha\beta}(q,r,p)\right\}_{q=0}$$

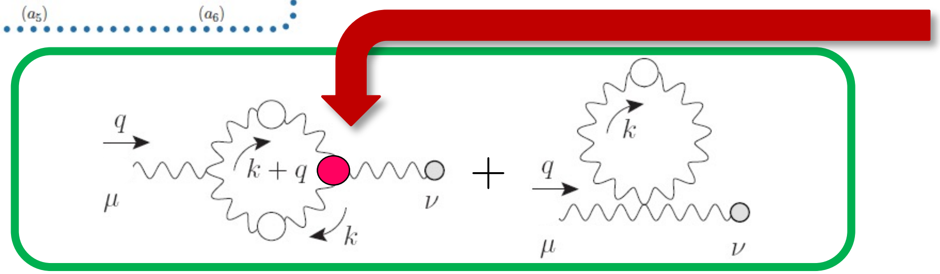
$$\widetilde{C}_{\alpha\beta}(q,r,p) = \widetilde{C}_{gl}(q,r,p)g_{\alpha\beta} + \cdots$$

Evading the seagull identity

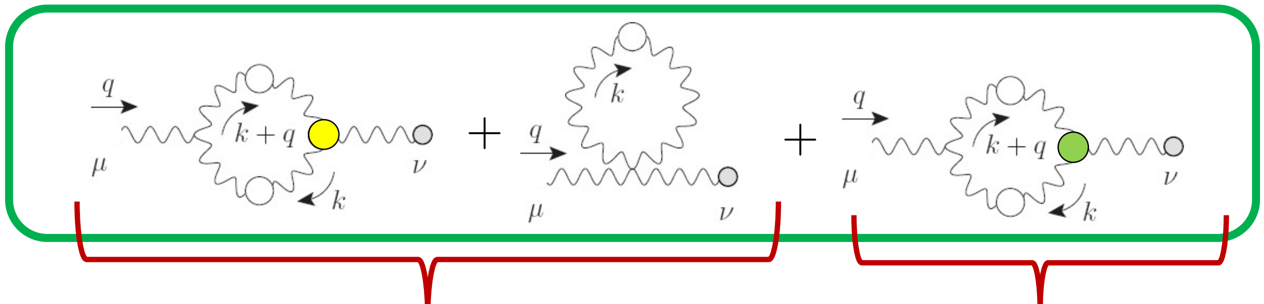




$$\widetilde{\Delta}^{-1}(0) = \lim_{q\to 0}$$



$$= \lim_{q\to 0}$$



Triggers seagull identity exactly as before

Vanishes identically

pole contribution

$\propto \widetilde{C}'_{gl}(k^2)$

$$\Delta^{-1}(0) \sim \int_k k^2 \Delta^2(k^2)\widetilde{C}'_{gl}(k^2)$$

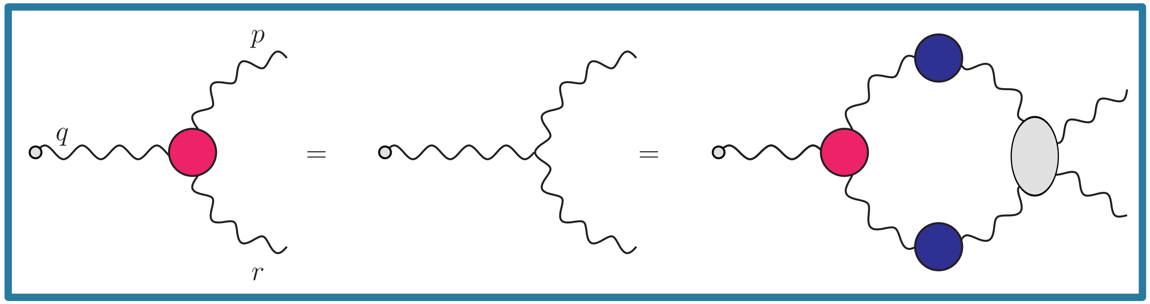
Effective gluon mass !

$$\widetilde{C}'_{gl}(k^2) = \lim_{q\to 0}\left\{\frac{\widetilde{C}_{gl}(q,k,-k-q)}{\partial(k+q)^2}\right\}$$

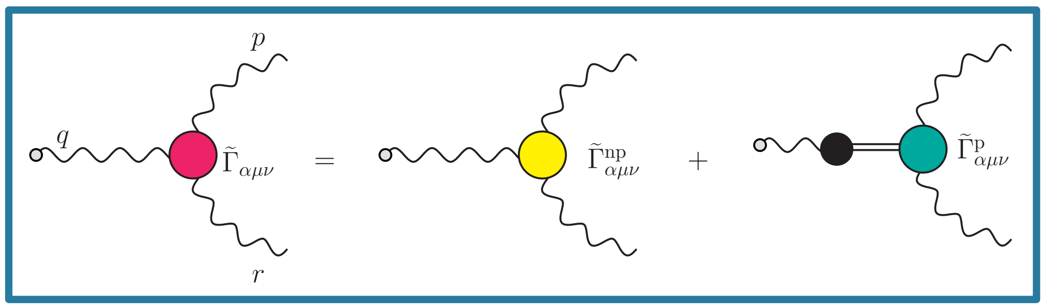
ACA, D.Binosi, C.T.Figueiredo and J.Papavassiliou, Phys. Rev. D 94, no. 4, 045002 (2016)

Dynamical equation for massless pole

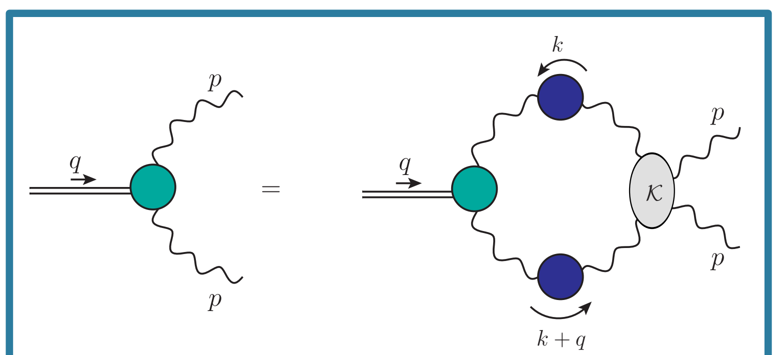
Bethe-Salpeter equation for the full vertex



Substitute:



$q \to 0$



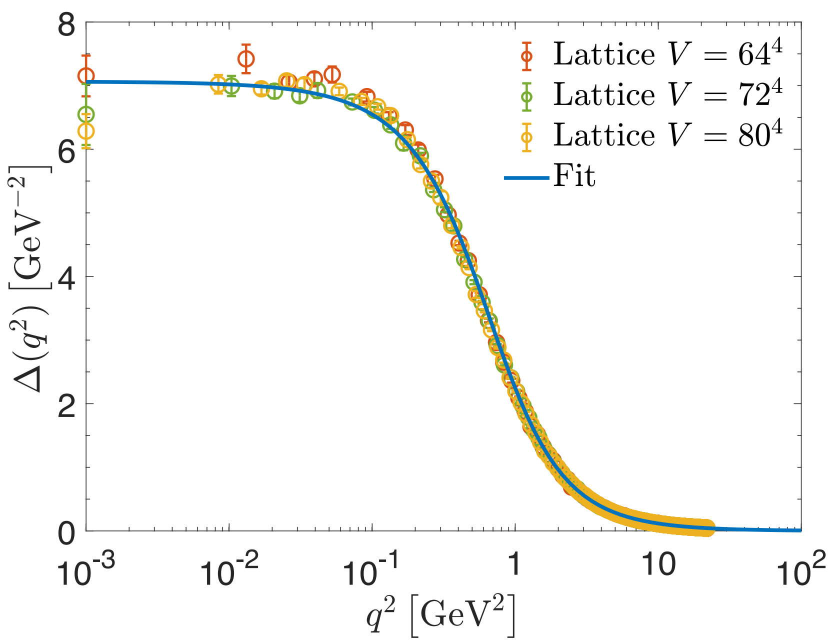
Describes the formation of the dynamical formation of massless pole

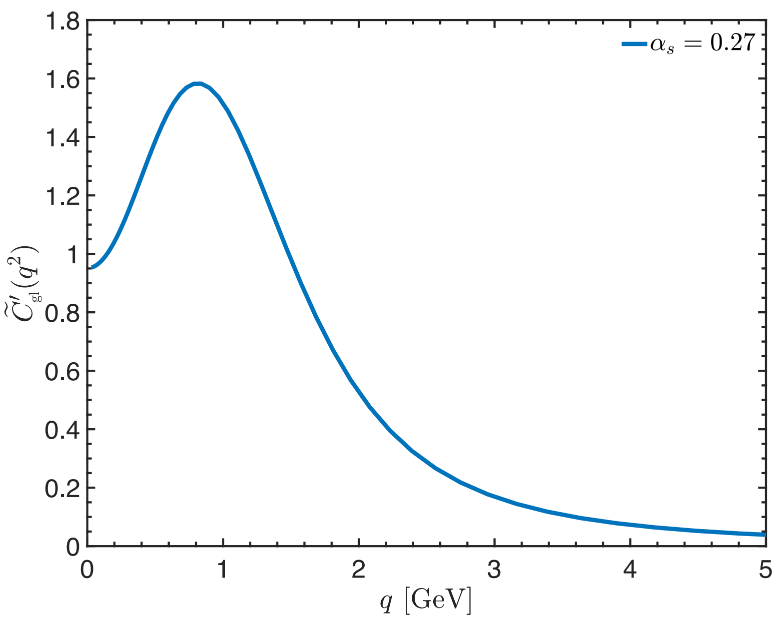
$$\widetilde{C}'_{gl}(q^2) = \frac{8\pi}{3} C_{\rm A} \alpha_s \int_k \Delta^2(k)\Delta(k+q)\mathcal{K}(q,k)\widetilde{C}'_{gl}(k^2)$$

$$\widetilde{C}'_{gl}(q^2) = \frac{8\pi}{3}\alpha_s C_{\rm A}\int_k \Delta^2(k)\Delta(k+q)\mathcal{K}(q,k)\widetilde{C}'_{gl}(k^2)$$

Homogeneous integral equation coupled with

$$\Delta^{-1}(0) = \lambda\int_k k^2\Delta^2(k^2)\widetilde{C}'_{gl}(k^2)$$





Relation with the gluon mass

$$\Delta^{-1}(q^2) = q^2 J(q^2) + m^2(q^2) \qquad \Delta_{\mu\nu} = -i\left[P_{\mu\nu}(q)\Delta(q^2)\right]$$

$$q^\mu \widetilde{\Gamma}^{\mathbf{np}}_{\mu\alpha\beta}(q,r,p) + \widetilde{C}_{\alpha\beta}(q,r,p) = i\Delta^{-1}_{\alpha\beta}(r) - i\Delta^{-1}_{\alpha\beta}(p)$$

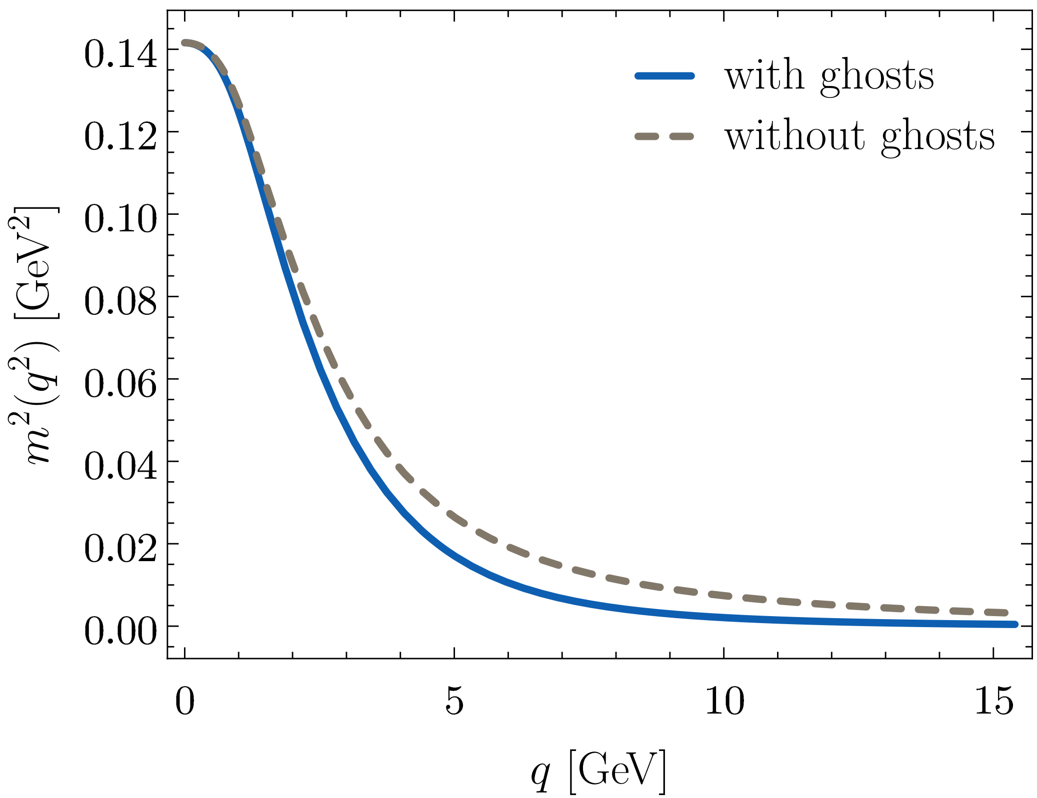
$$\widetilde{C}_{\alpha\beta}(q,r,p) = m^2(p^2)P_{\alpha\beta}(p) - m^2(r^2)P_{\alpha\beta}(r)$$

$$\widetilde{C}_{gl}(q,r,p) = m^2(r^2) - m^2(p^2) \quad \underset{q\to 0}{\Longrightarrow} \quad \widetilde{C}'_{gl}(r^2) = \frac{\mathrm{d}m^2(r^2)}{\mathrm{d}r^2}$$

$$m^2(q^2) = \Delta^{-1}(0) + \int_0^{q^2} \mathrm{d}y\, \widetilde{C}'_{gl}(y)$$

Dynamical gluon mass

$$m^2(q^2) = \Delta^{-1}(0) + \int_0^{q^2} \mathrm{d}y\, \widetilde{C}'_{gl}(y)$$



Including massless poles in the ghost-gluon vertex, one can see that the impact in the gluon mass is mild.

Power law behavior

$$m^2(q^2) = \frac{m_0^2(q^2)}{1 + (q^2/\mathcal{M}^2)^{1+p}}.$$

ACA, D. Binosi, C. T. Figueiredo., and J. Papavassiliou, Eur. Phys. J. C78, no. 3, 181 (2018).

Summary

- The fundamental issue of mass generation can be addressed in a self-consistent framework.

- The **quadratic divergences** that plague the study of dynamical mass generation can be shown to vanish by virtue of the **seagull identity**.

- Such identity, together with the WIs satisfied by the are responsible for the vanishing $\widetilde{\boldsymbol{\Delta}}^{-1}(\mathbf{0}) = \mathbf{0}$ in the absence of the poles.

- Therefore, to obtain massive solutions for the gluon propagator, we must require the **three gluon vertex** to contain **longitudinally coupled massless poles.**

- These poles are responsible for generating a pole in the gluon vacuum polarization, which triggers the **Schwinger mechanism** allowing for a **dynamical gluon mass.**