Helicity Evolution at Small x: The Single-Logarithmic Contribution

Yossathorn (Josh) Tawabutr
with Yuri Kovchegov and Andrey Tarasov

The Ohio State University

tawabutr.1@osu.edu

June 17, 2021

Based on: 2005.07285 and 2104.11765

Background

Proton helicity can be decomposed into spin and orbital angular momentum (OAM) of quarks and gluons [Jaffe and Manohar, 1990]

$$\frac{1}{2} = S_q + S_G + L_q + L_G \tag{1}$$

where

$$S_q(Q^2) = \frac{1}{2}\int_0^1 dx\, \Delta\Sigma(x, Q^2) = \frac{1}{2}\int_0^1 dx \sum_f \left[\Delta q_f(x, Q^2) + \Delta\overline{q}_f(x, Q^2)\right]. \tag{2}$$

Experiments have measured S_q but can only include $0 < x_{\text{min}} \le x \le 1$.

Objective: Find the contribution to S_q coming from $\Delta\Sigma$ as $x \to 0$.

Background

At small Bjorken-x, quark helicity distribution satisfies

$$\Delta\Sigma(x, Q^2) = \frac{N_c N_f}{2\pi^3} \int_{\Lambda^2/s}^{1} \frac{dz}{z} \int_{1/zs}^{1/zQ^2} \frac{dx_{10}^2}{x_{10}^2} \int d^2\underline{b}\; Q(\underline{x}_1, \underline{x}_0, z), \quad (3)$$

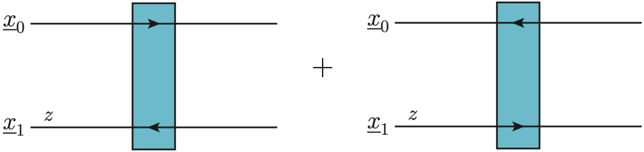
where $\underline{b} = \frac{\underline{x}_0 + \underline{x}_1}{2}$ and $\underline{x}_{10} = \underline{x}_1 - \underline{x}_0$.

Here, $Q(\underline{x}_1, \underline{x}_0, z) \equiv Q_{10}(z)$ is the quark (longitudinally) polarized dipole amplitude.

Quark Unpolarized Dipole Amplitude

$S_{10}(z)$ corresponds to a minus-moving quark dipole interacting with a plus-moving target proton, represented by the blue rectangle.

$S_{10}(z) =$



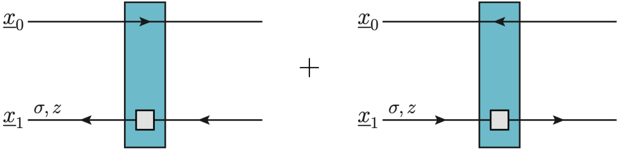
$$= \frac{1}{2N_c} \, \mathrm{Re} \left[\left\langle \mathcal{T} \; \mathrm{tr} \left[V_{\underline{0}} V_{\underline{1}}^{\dagger} \right] \right\rangle (z) + \left\langle \mathcal{T} \; \mathrm{tr} \left[V_{\underline{1}} V_{\underline{0}}^{\dagger} \right] \right\rangle (z) \right] . \tag{4}$$

- $V_{\underline{i}}$ is the fundamental Wilson's line at $\underline{x}_i$.
- The angle brackets average over target proton's wave function.

Quark Polarized Dipole Amplitude

Diagrammatically, $Q_{10}(z)$ corresponds to a quark dipole, one of which has helicity σ, interacting with a polarized target proton (blue rectangle).

$Q_{10}(z) =$



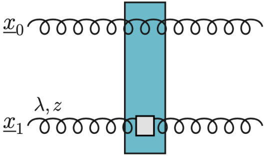
$$= \frac{zs}{2N_c}\,\mathrm{Re}\left[\left\langle \mathcal{T}\ \mathrm{tr}\left[V_{\underline{0}} V_{\underline{1}}^{\mathrm{pol}\,\dagger}\right]\right\rangle(z) + \left\langle \mathcal{T}\ \mathrm{tr}\left[V_{\underline{1}}^{\mathrm{pol}} V_{\underline{0}}^{\dagger}\right]\right\rangle(z)\right]. \quad (5)$$

- $V_{\underline{0}}$ is the fundamental unpolarized Wilson's line at $\underline{x}_0$.
- $V_{\underline{1}}^{\mathrm{pol}}$ is the fundamental *polarized Wilson's line* at $\underline{x}_1$.
- The angle brackets average over target proton's wave function.
- At Born level, $Q_{10}(z) \sim 1$ because the helicity-dependent tree-level cross-section $\sim \frac{1}{zs}$.

Gluon Polarized Dipole Amplitude

Similar to quark, consider a gluon dipole, one of which has helicity λ, interacting with a polarized target proton (blue rectangle).

$G_{10}(z) =$

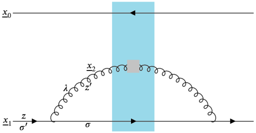


$$= \frac{zs}{2(N_c^2-1)} \, \mathrm{Re} \left[\left\langle \mathcal{T} \, \mathrm{tr} \left[U_{\underline{0}} U_{\underline{1}}^{\mathrm{pol}\,\dagger} \right] \right\rangle (z) + \left\langle \mathcal{T} \, \mathrm{tr} \left[U_{\underline{1}}^{\mathrm{pol}} U_{\underline{0}}^{\dagger} \right] \right\rangle (z) \right]. \quad (6)$$

- $U_{\underline{0}}$ is the adjoint unpolarized Wilson's line at $\underline{x}_0$.
- $U_{\underline{1}}^{\mathrm{pol}}$ is the adjoint *polarized Wilson's line* at $\underline{x}_1$.
- The angle brackets average over target proton's wave function.

Evolution

- The polarized dipole amplitudes obey integral equations resulting from quark/gluon splitting outside the target shockwave, e.g.



- To the first order in α_s, both dipole amplitudes evolve as

$$\alpha_s \left[\underbrace{\int \frac{dz'}{z'} \int \frac{dx_{21}^2}{x_{21}^2}}_{\text{Double-logarithmic}} + \underbrace{\int dz' \int \frac{dx_{32}^2}{x_{32}^2} + \int \frac{dz'}{z'} \int dx_{21}^2}_{\text{Single-logarithmic}} + \dots \right]$$

$$\times \text{ (dipole amplitudes)} .$$

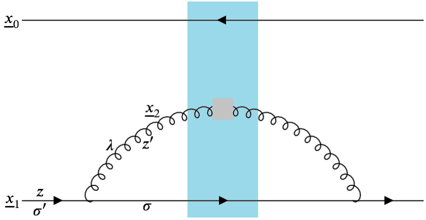
- The single-logarithmic (SLA) terms (resumming $\alpha_s \ln \frac{1}{x}$) are subleading to the double-logarithmic (DLA) term (resumming $\alpha_s \ln^2 \frac{1}{x}$).

Longitudinally Soft Parton Emission

- In a splitting in the limit $z' \ll z$, the longitudinal z'-integral is logarithmic, giving the evolution terms

$$\alpha_s \int \frac{dz'}{z'} \left[\underbrace{A \int \frac{dx_{21}^2}{x_{21}^2}}_{\text{DLA}} + \underbrace{B \int dx_{21}^2}_{\text{SLA}_\text{L}} \right] \text{(dipole amps)} \tag{7}$$

- SLA_L: single-logarithmic term with logarithmic longitudinal integral.



DLA Evolution Equations

With only DLA terms included, the evolution equations are [Kovchegov et al, 2016] [Kovchegov and Sievert, 2019]

$$\frac{1}{N_c}\left\langle\!\!\left\langle \mathrm{tr}\left[V_{\underline{0}}^{unp}\, V_{\underline{1}}^{pol\,\dagger}\right]\right\rangle\!\!\right\rangle(z) = \frac{1}{N_c}\left\langle\!\!\left\langle \mathrm{tr}\left[V_{\underline{0}}^{unp}\, V_{\underline{1}}^{pol\,\dagger}\right]\right\rangle\!\!\right\rangle_0(z) + \frac{\alpha_s}{2\pi^2}\int\limits_{z_i}^{z}\frac{dz'}{z'}\int\limits_{\rho'^2}\frac{d^2x_2}{x_{21}^2}$$
$$\times\Bigg\{\theta(x_{10}-x_{21})\,\frac{2}{N_c}\left\langle\!\!\left\langle \mathrm{tr}\left[t^b\, V_{\underline{0}}^{unp}\, t^a\, V_{\underline{1}}^{unp\,\dagger}\right] U_{\underline{2}}^{pol\,ba}\right\rangle\!\!\right\rangle(z') + \theta(x_{10}^2 z - x_{21}^2 z')\,\frac{1}{N_c}\left\langle\!\!\left\langle \mathrm{tr}\left[t^b\, V_{\underline{0}}^{unp}\, t^a\, V_{\underline{2}}^{pol\,\dagger}\right] U_{\underline{1}}^{unp\,ba}\right\rangle\!\!\right\rangle(z')$$
$$+\,\theta(x_{10}-x_{21})\,\frac{1}{N_c}\left[\left\langle\!\!\left\langle \mathrm{tr}\left[V_{\underline{0}}^{unp}\, V_{\underline{2}}^{unp\,\dagger}\right]\mathrm{tr}\left[V_{\underline{2}}^{unp}\, V_{\underline{1}}^{pol\,\dagger}\right]\right\rangle\!\!\right\rangle(z') - N_c\left\langle\!\!\left\langle \mathrm{tr}\left[V_{\underline{0}}^{unp}\, V_{\underline{1}}^{pol\,\dagger}\right]\right\rangle\!\!\right\rangle(z')\right]\Bigg\}.$$

$$\frac{1}{N_c^2-1}\left\langle\!\!\left\langle \mathrm{Tr}\left[U_{\underline{0}}^{unp}\, U_{\underline{1}}^{pol\,\dagger}\right]\right\rangle\!\!\right\rangle(z) = \frac{1}{N_c^2-1}\left\langle\!\!\left\langle \mathrm{Tr}\left[U_{\underline{0}}^{unp}\, U_{\underline{1}}^{pol\,\dagger}\right]\right\rangle\!\!\right\rangle_0(z) + \frac{\alpha_s}{2\pi^2}\int\limits_{z_i}^{z}\frac{dz'}{z'}\int\limits_{\rho'^2}\frac{d^2x_2}{x_{21}^2}$$
$$\times\Bigg\{\theta(x_{10}-x_{21})\,\frac{4}{N_c^2-1}\left\langle\!\!\left\langle \mathrm{Tr}\left[T^b\, U_{\underline{0}}^{unp}\, T^a\, U_{\underline{1}}^{unp\,\dagger}\right] U_{\underline{2}}^{pol\,ba}\right\rangle\!\!\right\rangle(z')$$
$$-\,\theta(x_{10}^2 z - x_{21}^2 z')\,\frac{N_f}{N_c^2-1}\left\langle\!\!\left\langle \mathrm{tr}\left[t^b\, V_{\underline{1}}^{unp}\, t^a\, V_{\underline{2}}^{pol\,\dagger}\right] U_{\underline{0}}^{unp\,ba} + \mathrm{tr}\left[t^b\, V_{\underline{2}}^{pol}\, t^a\, V_{\underline{1}}^{unp\,\dagger}\right] U_{\underline{0}}^{unp\,ba}\right\rangle\!\!\right\rangle(z')$$
$$+\theta(x_{10}-x_{21})\frac{2}{N_c^2-1}\left[\left\langle\!\!\left\langle \mathrm{Tr}\left[T^b\, U_{\underline{0}}^{unp}\, T^a\, U_{\underline{1}}^{pol\,\dagger}\right] U_{\underline{2}}^{unp\,ba}\right\rangle\!\!\right\rangle(z') - N_c\left\langle\!\!\left\langle \mathrm{Tr}\left[U_{\underline{0}}^{unp}\, U_{\underline{1}}^{pol\,\dagger}\right]\right\rangle\!\!\right\rangle(z')\right]\Bigg\}$$

where $\left\langle\!\left\langle \cdots \right\rangle\!\right\rangle = zs\left\langle \cdots \right\rangle$.

DLA Evolution Equations

- At large N_c and large $N_c \& N_f$, the equations become closed and linear [Kovchegov et al, 2016] [Kovchegov and Sievert, 2019].
- For example, at large N_c, the equations are:

$$G_{10}(z) = G_{10}^{(0)}(z) + \frac{\alpha_s N_c}{2\pi} \int\limits_{\frac{1}{s\, x_{10}^2}}^{z} \frac{dz'}{z'} \int\limits_{\frac{1}{z' s}}^{x_{10}^2} \frac{dx_{21}^2}{x_{21}^2} \left[\Gamma_{10,21}(z') + 3\, G_{21}(z') \right]$$

$$\Gamma_{10,21}(z') = \Gamma_{10,21}^{(0)}(z') + \frac{\alpha_s N_c}{2\pi} \int\limits_{\min\{\Lambda^2, \frac{1}{x_{10}^2}\}/s}^{z'} \frac{dz''}{z''} \int\limits_{\frac{1}{z'' s}}^{\min\{x_{10}^2, x_{21}^2 z'/z''\}} \frac{dx_{32}^2}{x_{32}^2} \left[\Gamma_{10,32}(z'') + 3\, G_{32}(z'') \right]$$

where Γ is an auxiliary function.

DLA Evolution Equations

These DLA equations have been analytically solved at large N_c [Kovchegov et al, 2017] and numerically solved at large N_c&N_f [Kovchegov and Tawabutr, 2020].

- At large N_c, the quark helicity PDF has the asymptotic form

$$\Delta\Sigma(x, Q^2) \sim (1/x)^{\alpha_h^q}. \tag{8}$$

- At large N_c&N_f, the asymptotic form displays oscillation pattern

$$\Delta\Sigma(x, Q^2) \sim (1/x)^{\alpha_h^q} \cos\left[\omega_q \ln(1/x) + \varphi_q\right]. \tag{9}$$

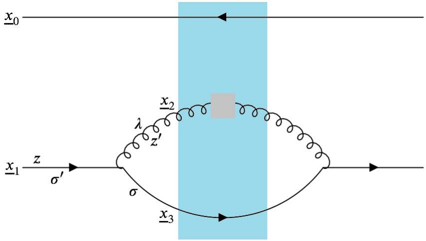
  In both cases, $\alpha_h^q \approx \frac{4}{\sqrt{3}}\sqrt{\frac{\alpha_s N_c}{2\pi}}$, and ω_q is small and increases with N_f.
- Phenomenological implications were studied in [Adamiak et al, 2021].

Longitudinally Hard Parton Emission

- In the limit $z' \sim z$, a parton splitting yields the SLA_T terms, i.e. single-logarithmic terms with logarithmic transverse integral

$$\alpha_s \int dz' \, \Delta P(z'/z) \int \frac{dx_{32}^2}{x_{32}^2} \, (\text{dipole amps}) \tag{10}$$

- Here, $\Delta P(z'/z)$ is the polarized DGLAP splitting function.
- Since $z' \sim (z - z') \sim z$, neither $\underline{x}_3$ nor $\underline{x}_2$ is close to $\underline{x}_1$, but we still have $x_{32} \ll x_{10}$.



SLA Evolution Equations

Including both double-log (DLA) and single-log (SLA) terms, we have

$$
\begin{aligned}
&\frac{1}{N_c}\Big\langle\!\!\Big\langle \mathrm{tr}\left[V_{\underline{0}}\, V_{\underline{1}}^{\mathrm{pol}\,\dagger}\right]\Big\rangle\!\!\Big\rangle(z_{\min}, z_{\mathrm{pol}}) = \frac{1}{N_c}\Big\langle\!\!\Big\langle \mathrm{tr}\left[V_{\underline{0}}\, V_{\underline{1}}^{\mathrm{pol}\,\dagger}\right]\Big\rangle\!\!\Big\rangle_0(z_{\mathrm{pol}}) + \frac{1}{2\pi^2}\int\limits_{\Lambda^2/s}^{z_{\min}}\frac{dz'}{z'}\int\limits_{1/(z's)} d^2x_2 \\
&\quad\times\Bigg[\left(\frac{\alpha_s(1/x_{21}^2)}{x_{21}^2}\,\theta(x_{10}^2 z_{\min} - x_{21}^2 z') - \alpha_s(\min\{1/x_{21}^2, 1/x_{20}^2\})\,\frac{\underline{x}_{21}\cdot\underline{x}_{20}}{x_{21}^2\, x_{20}^2}\,\theta(x_{10}^2 z_{\min} - \max\{x_{21}^2, x_{20}^2\} z')\right) \\
&\qquad\times\frac{2}{N_c}\Big\langle\!\!\Big\langle \mathrm{tr}\left[t^b\, V_{\underline{0}}\, t^a\, V_{\underline{1}}^{\dagger}\right] U_{\underline{2}}^{\mathrm{pol}\,ba}\Big\rangle\!\!\Big\rangle(z', z') \\
&\qquad+\frac{\alpha_s(1/x_{21}^2)}{x_{21}^2}\,\theta(x_{10}^2 z_{\min} - x_{21}^2 z')\,\frac{1}{N_c}\Big\langle\!\!\Big\langle \mathrm{tr}\left[t^b\, V_{\underline{0}}\, t^a\, V_{\underline{2}}^{\mathrm{pol}\,\dagger}\right] U_{\underline{1}}^{ba}\Big\rangle\!\!\Big\rangle(z', z')\Bigg] \\
&+\frac{1}{2\pi^2}\int\limits_{\Lambda^2/s}^{z_{\min}}\frac{dz'}{z'}\int\limits_{1/(z's)} d^2x_2\, K_{\mathrm{rcBK}}(\underline{x}_0, \underline{x}_1; \underline{x}_2)\,\theta(x_{10}^2 z_{\min} - x_{21}^2 z') \\
&\quad\times\frac{1}{N_c}\left[\Big\langle\!\!\Big\langle \mathrm{tr}\left[V_{\underline{0}}\, V_{\underline{2}}^{\dagger}\right]\mathrm{tr}\left[V_{\underline{2}}\, V_{\underline{1}}^{\mathrm{pol}\,\dagger}\right]\Big\rangle\!\!\Big\rangle(z', z_{\mathrm{pol}}) - N_c\Big\langle\!\!\Big\langle \mathrm{tr}\left[V_{\underline{0}}\, V_{\underline{1}}^{\mathrm{pol}\,\dagger}\right]\Big\rangle\!\!\Big\rangle(z', z_{\mathrm{pol}})\right] \\
&-\frac{1}{2\pi^2}\int\limits_0^{z_{\mathrm{pol}}}\frac{dz'}{z_{\mathrm{pol}}}\int\limits_{\frac{z_{\mathrm{pol}}}{z'(z_{\mathrm{pol}}-z')s}}\frac{d^2x_{32}}{x_{32}^2}\,\theta(x_{10}^2 z_{\min} z_{\mathrm{pol}} - x_{32}^2 z'(z_{\mathrm{pol}} - z'))\,\alpha_s\left(\frac{1}{x_{32}^2}\right) \\
&\quad\times\Bigg[\frac{1}{N_c}\Big\langle\!\!\Big\langle \mathrm{tr}\left[t^b\, V_{\underline{0}}\, t^a\, V^{\dagger}_{\underline{x}_1 - \frac{z'}{z_{\mathrm{pol}}}\underline{x}_{32}}\right] U^{\mathrm{pol}\,ba}_{\underline{x}_1 + \left(1-\frac{z'}{z_{\mathrm{pol}}}\right)\underline{x}_{32}}\Big\rangle\!\!\Big\rangle(\min\{z_{\min}, z', z_{\mathrm{pol}} - z'\}, z') \\
&\qquad+\frac{1}{N_c}\Big\langle\!\!\Big\langle \mathrm{tr}\left[t^b\, V_{\underline{0}}\, t^a\, V^{\mathrm{pol}\,\dagger}_{\underline{x}_1 + \left(1-\frac{z'}{z_{\mathrm{pol}}}\right)\underline{x}_{32}}\right] U^{ba}_{\underline{x}_1 - \frac{z'}{z_{\mathrm{pol}}}\underline{x}_{32}}\Big\rangle\!\!\Big\rangle(\min\{z_{\min}, z', z_{\mathrm{pol}} - z'\}, z')\Bigg] \\
&+\frac{C_F}{2\pi^2}\int\limits_0^{z_{\mathrm{pol}}}\frac{dz'}{z_{\mathrm{pol}}}\left(1 + \frac{z'}{z_{\mathrm{pol}}}\right)\int\limits_{\frac{z_{\mathrm{pol}}}{z'(z_{\mathrm{pol}}-z')s}}\frac{d^2x_{32}}{x_{32}^2}\,\theta(x_{10}^2 z_{\min} z_{\mathrm{pol}} - x_{32}^2 z'(z_{\mathrm{pol}} - z'))\,\alpha_s\left(\frac{1}{x_{32}^2}\right) \\
&\quad\times\frac{1}{N_c}\Big\langle\!\!\Big\langle \mathrm{tr}\left[V_{\underline{0}}\, V_{\underline{1}}^{\mathrm{pol}\,\dagger}\right]\Big\rangle\!\!\Big\rangle(\min\{z_{\min}, z', z_{\mathrm{pol}} - z'\}, z_{\mathrm{pol}}).
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{N_c^2-1}\Big\langle\!\!\Big\langle \mathrm{Tr}\left[U_{\underline{0}}\, U_{\underline{1}}^{\mathrm{pol}\,\dagger}\right]\Big\rangle\!\!\Big\rangle(z_{\min}, z_{\mathrm{pol}}) = \frac{1}{N_c^2-1}\Big\langle\!\!\Big\langle \mathrm{Tr}\left[U_{\underline{0}}\, U_{\underline{1}}^{\mathrm{pol}\,\dagger}\right]\Big\rangle\!\!\Big\rangle_0(z_{\mathrm{pol}}) + \frac{1}{2\pi^2}\int\limits_{\Lambda^2/s}^{z_{\min}}\frac{dz'}{z'}\int\limits_{1/(z's)} d^2x_2 \\
&\quad\times\Bigg[\left(\frac{\alpha_s(1/x_{21}^2)}{x_{21}^2}\,\theta(x_{10}^2 z_{\min} - x_{21}^2 z') - \alpha_s(\min\{1/x_{21}^2, 1/x_{20}^2\})\,\frac{\underline{x}_{21}\cdot\underline{x}_{20}}{x_{21}^2\, x_{20}^2}\,\theta(x_{10}^2 z_{\min} - \max\{x_{21}^2, x_{20}^2\} z')\right) \\
&\qquad\times\frac{4}{N_c^2-1}\Big\langle\!\!\Big\langle \mathrm{Tr}\left[T^b\, U_{\underline{0}}\, T^a\, U_{\underline{1}}^{\dagger}\right] U_{\underline{2}}^{\mathrm{pol}\,ba}\Big\rangle\!\!\Big\rangle(z', z') \\
&\qquad-\frac{\alpha_s(1/x_{21}^2)}{x_{21}^2}\,\theta(x_{10}^2 z_{\min} - x_{21}^2 z')\,\frac{N_f}{N_c^2-1}\Big\langle\!\!\Big\langle \mathrm{tr}\left[t^b\, V_{\underline{1}}\, t^a\, V_{\underline{2}}^{\mathrm{pol}\,\dagger}\right] U_{\underline{0}}^{ba} + \mathrm{tr}\left[t^b\, V_{\underline{2}}^{\mathrm{pol}}\, t^a\, V_{\underline{1}}^{\dagger}\right] U_{\underline{0}}^{ba}\Big\rangle\!\!\Big\rangle(z', z')\Bigg] \\
&+\frac{1}{\pi^2}\int\limits_{\Lambda^2/s}^{z_{\min}}\frac{dz'}{z'}\int\limits_{1/(z's)} d^2x_2\, K_{\mathrm{rcBK}}(\underline{x}_0, \underline{x}_1; \underline{x}_2)\,\theta(x_{10}^2 z_{\min} - x_{21}^2 z') \\
&\quad\times\frac{1}{N_c^2-1}\left[\Big\langle\!\!\Big\langle \mathrm{Tr}\left[T^b\, U_{\underline{0}}\, T^a\, U_{\underline{1}}^{\mathrm{pol}\,\dagger}\right] U_{\underline{2}}^{ba}\Big\rangle\!\!\Big\rangle(z', z_{\mathrm{pol}}) - N_c\Big\langle\!\!\Big\langle \mathrm{Tr}\left[U_{\underline{0}}\, U_{\underline{1}}^{\mathrm{pol}\,\dagger}\right]\Big\rangle\!\!\Big\rangle(z', z_{\mathrm{pol}})\right] \\
&-\frac{1}{2\pi^2}\int\limits_0^{z_{\mathrm{pol}}}\frac{dz'}{z_{\mathrm{pol}}}\int\limits_{\frac{z_{\mathrm{pol}}}{z'(z_{\mathrm{pol}}-z')s}}\frac{d^2x_{32}}{x_{32}^2}\,\theta(x_{10}^2 z_{\min} z_{\mathrm{pol}} - x_{32}^2 z'(z_{\mathrm{pol}} - z'))\,\alpha_s\left(\frac{1}{x_{32}^2}\right) \\
&\quad\times\frac{4}{N_c^2-1}\Big\langle\!\!\Big\langle \mathrm{Tr}\left[T^b\, U_{\underline{0}}\, T^a\, U^{\dagger}_{\underline{x}_1 - \frac{z'}{z_{\mathrm{pol}}}\underline{x}_{32}}\right] U^{\mathrm{pol}\,ba}_{\underline{x}_1 + \left(1-\frac{z'}{z_{\mathrm{pol}}}\right)\underline{x}_{32}}\Big\rangle\!\!\Big\rangle(\min\{z_{\min}, z', z_{\mathrm{pol}} - z'\}, z') \\
&+\frac{1}{\pi^2}\int\limits_0^{z_{\mathrm{pol}}}\frac{dz'}{z_{\mathrm{pol}}}\int\limits_{\frac{z_{\mathrm{pol}}}{z'(z_{\mathrm{pol}}-z')s}}\frac{d^2x_{32}}{x_{32}^2}\,\theta(x_{10}^2 z_{\min} z_{\mathrm{pol}} - x_{32}^2 z'(z_{\mathrm{pol}} - z'))\,\alpha_s\left(\frac{1}{x_{32}^2}\right)\frac{N_f}{N_c^2-1} \\
&\quad\times\Big\langle\!\!\Big\langle \mathrm{tr}\left[t^b\, V_{\underline{x}_1 - \frac{z'}{z_{\mathrm{pol}}}\underline{x}_{32}}\, t^a\, V^{\mathrm{pol}\,\dagger}_{\underline{x}_1 + \left(1-\frac{z'}{z_{\mathrm{pol}}}\right)\underline{x}_{32}}\right] U_{\underline{0}}^{ba} \\
&\qquad\qquad+\mathrm{tr}\left[t^b\, V^{\mathrm{pol}}_{\underline{x}_1 + \left(1-\frac{z'}{z_{\mathrm{pol}}}\right)\underline{x}_{32}}\, t^a\, V^{\dagger}_{\underline{x}_1 - \frac{z'}{z_{\mathrm{pol}}}\underline{x}_{32}}\right] U_{\underline{0}}^{ba}\Big\rangle\!\!\Big\rangle(\min\{z_{\min}, z', z_{\mathrm{pol}} - z'\}, z') \\
&+\frac{1}{2\pi^2}\int\limits_0^{z_{\mathrm{pol}}}\frac{dz'}{z_{\mathrm{pol}}}\int\limits_{\frac{z_{\mathrm{pol}}}{z'(z_{\mathrm{pol}}-z')s}}\frac{d^2x_{32}}{x_{32}^2}\left[N_c\left(2 - \frac{z'}{z_{\mathrm{pol}}} + \frac{z'^2}{z_{\mathrm{pol}}^2}\right) - \frac{N_f}{2}\left(\frac{z'^2}{z_{\mathrm{pol}}^2} + \left(1 - \frac{z'}{z_{\mathrm{pol}}}\right)^2\right)\right] \\
&\quad\times\theta\left(x_{10}^2 z_{\min} z_{\mathrm{pol}} - x_{32}^2 z'(z_{\mathrm{pol}} - z')\right)\alpha_s\left(\frac{1}{x_{32}^2}\right)\frac{1}{N_c^2-1}\Big\langle\!\!\Big\langle \mathrm{Tr}\left[U_{\underline{0}}\, U_{\underline{1}}^{\mathrm{pol}\,\dagger}\right]\Big\rangle\!\!\Big\rangle(\min\{z_{\min}, z', z_{\mathrm{pol}} - z'\}, z_{\mathrm{pol}}).
\end{aligned}
$$

- SLA_T terms are written in blue.
- The running coupling correction is SLA and also has to be included.

SLA Evolution Equations

Including both double-log (DLA) and single-log (SLA) terms, we have

$$
\begin{aligned}
&\frac{1}{N_c}\left\langle\!\left\langle \mathrm{tr}\left[V_{\underline{0}}\, V_{\underline{1}}^{\mathrm{pol}\,\dagger}\right] \right\rangle\!\right\rangle(z_{\min}, z_{\mathrm{pol}}) = \frac{1}{N_c}\left\langle\!\left\langle \mathrm{tr}\left[V_{\underline{0}}\, V_{\underline{1}}^{\mathrm{pol}\,\dagger}\right] \right\rangle\!\right\rangle_0(z_{\mathrm{pol}}) + \frac{1}{2\pi^2}\int\limits_{\Lambda^2/s}^{z_{\min}} \frac{dz'}{z'} \int\limits_{1/(z's)} d^2x_2 \\
&\quad\times\Bigg[\left(\frac{\alpha_s(1/x_{21}^2)}{x_{21}^2}\,\theta(x_{10}^2 z_{\min} - x_{21}^2 z') - \alpha_s(\min\{1/x_{21}^2, 1/x_{20}^2\})\,\frac{\underline{x}_{21}\cdot\underline{x}_{20}}{x_{21}^2\, x_{20}^2}\,\theta(x_{10}^2 z_{\min} - \max\{x_{21}^2, x_{20}^2\} z')\right) \\
&\qquad\times\frac{2}{N_c}\left\langle\!\left\langle \mathrm{tr}\left[t^b\, V_{\underline{0}}\, t^a\, V_{\underline{1}}^{\dagger}\right] U_{\underline{2}}^{\mathrm{pol}\,ba} \right\rangle\!\right\rangle(z', z') \\
&\qquad+\frac{\alpha_s(1/x_{21}^2)}{x_{21}^2}\,\theta(x_{10}^2 z_{\min} - x_{21}^2 z')\,\frac{1}{N_c}\left\langle\!\left\langle \mathrm{tr}\left[t^b\, V_{\underline{0}}\, t^a\, V_{\underline{2}}^{\mathrm{pol}\,\dagger}\right] U_{\underline{1}}^{ba} \right\rangle\!\right\rangle(z', z')\Bigg] \\
&+\frac{1}{2\pi^2}\int\limits_{\Lambda^2/s}^{z_{\min}} \frac{dz'}{z'} \int\limits_{1/(z's)} d^2x_2\, K_{\mathrm{rcBK}}(\underline{x}_0, \underline{x}_1; \underline{x}_2)\,\theta(x_{10}^2 z_{\min} - x_{21}^2 z') \\
&\quad\times\frac{1}{N_c}\left[\left\langle\!\left\langle \mathrm{tr}\left[V_{\underline{0}}\, V_{\underline{2}}^{\dagger}\right]\, \mathrm{tr}\left[V_{\underline{2}}\, V_{\underline{1}}^{\mathrm{pol}\,\dagger}\right] \right\rangle\!\right\rangle(z', z_{\mathrm{pol}}) - N_c\left\langle\!\left\langle \mathrm{tr}\left[V_{\underline{0}}\, V_{\underline{1}}^{\mathrm{pol}\,\dagger}\right] \right\rangle\!\right\rangle(z', z_{\mathrm{pol}})\right] \\
&-\frac{1}{2\pi^2}\int\limits_0^{z_{\mathrm{pol}}} \frac{dz'}{z_{\mathrm{pol}}} \int\limits_{\frac{z_{\mathrm{pol}}}{z'(z_{\mathrm{pol}}-z')s}} \frac{d^2x_{32}}{x_{32}^2}\,\theta(x_{10}^2 z_{\min} z_{\mathrm{pol}} - x_{32}^2 z'(z_{\mathrm{pol}} - z'))\;\alpha_s\left(\frac{1}{x_{32}^2}\right) \\
&\quad\times\Bigg[\frac{1}{N_c}\left\langle\!\left\langle \mathrm{tr}\left[t^b\, V_{\underline{0}}\, t^a\, V^{\dagger}_{\underline{x}_1 - \frac{z'}{z_{\mathrm{pol}}}\underline{x}_{32}}\right] U^{\mathrm{pol}\,ba}_{\underline{x}_1 + \left(1 - \frac{z'}{z_{\mathrm{pol}}}\right)\underline{x}_{32}} \right\rangle\!\right\rangle(\min\{z_{\min}, z', z_{\mathrm{pol}} - z'\}, z') \\
&\qquad+\frac{1}{N_c}\left\langle\!\left\langle \mathrm{tr}\left[t^b\, V_{\underline{0}}\, t^a\, V^{\mathrm{pol}\,\dagger}_{\underline{x}_1 + \left(1 - \frac{z'}{z_{\mathrm{pol}}}\right)\underline{x}_{32}}\right] U^{ba}_{\underline{x}_1 - \frac{z'}{z_{\mathrm{pol}}}\underline{x}_{32}} \right\rangle\!\right\rangle(\min\{z_{\min}, z', z_{\mathrm{pol}} - z'\}, z')\Bigg] \\
&+\frac{C_F}{2\pi^2}\int\limits_0^{z_{\mathrm{pol}}} \frac{dz'}{z_{\mathrm{pol}}}\left(1 + \frac{z'}{z_{\mathrm{pol}}}\right) \int\limits_{\frac{z_{\mathrm{pol}}}{z'(z_{\mathrm{pol}}-z')s}} \frac{d^2x_{32}}{x_{32}^2}\,\theta(x_{10}^2 z_{\min} z_{\mathrm{pol}} - x_{32}^2 z'(z_{\mathrm{pol}} - z'))\;\alpha_s\left(\frac{1}{x_{32}^2}\right) \\
&\quad\times\frac{1}{N_c}\left\langle\!\left\langle \mathrm{tr}\left[V_{\underline{0}}\, V_{\underline{1}}^{\mathrm{pol}\,\dagger}\right] \right\rangle\!\right\rangle(\min\{z_{\min}, z', z_{\mathrm{pol}} - z'\}, z_{\mathrm{pol}}).
\end{aligned}
$$

- SLA_T terms are written in blue.
- The running coupling correction is SLA and also has to be included.

SLA Evolution Equations

- At large N_c and large $N_c \& N_f$, the equations become closed but *non-linear* because the evolution of unpolarized dipoles are also SLA.
- For example, at large N_c, (half of) the equations are:

$$
\begin{aligned}
G_{10}(z_{\min}, z_{\mathrm{pol}}) &= G_{10}^{(0)}(z_{\mathrm{pol}}) + \frac{N_c}{\pi^2} \int\limits_{\Lambda^2/s}^{z_{\min}} \frac{dz'}{z'} \int\limits_{1/(z's)} d^2 x_2 \\
&\times \left(\frac{\alpha_s(1/x_{21}^2)}{x_{21}^2}\, \theta\left(x_{10}^2 z_{\min} - x_{21}^2 z'\right) - \alpha_s(\min\{1/x_{21}^2, 1/x_{20}^2\})\, \frac{\underline{x}_{21} \cdot \underline{x}_{20}}{x_{21}^2 x_{20}^2}\, \theta\left(x_{10}^2 z_{\min} - \max\left\{x_{21}^2, x_{20}^2\right\} z'\right) \right) \\
&\times \left[G_{21}(z', z')\, S_{20}(z') + \Gamma_{20,21}^{gen}(z', z')\, S_{21}(z') \right] \\
&+ \frac{N_c}{2\pi^2} \int\limits_{\Lambda^2/s}^{z_{\min}} \frac{dz'}{z'} \int\limits_{1/z's} d^2 x_2\, K_{\mathrm{rcBK}}(\underline{x}_0, \underline{x}_1; \underline{x}_2)\, \theta\left(x_{10}^2 z_{\min} - x_{21}^2 z'\right) \left[G_{21}(z', z_{\mathrm{pol}})\, S_{20}(z') - \Gamma_{10,21}^{gen}(z', z_{\mathrm{pol}}) \right] \\
&- \frac{N_c}{\pi^2} \int\limits_0^{z_{\mathrm{pol}}} \frac{dz'}{z_{\mathrm{pol}}} \int\limits_{\frac{z_{\mathrm{pol}}}{z'(z_{\mathrm{pol}} - z')s}} \frac{d^2 x_{32}}{x_{32}^2}\, \alpha_s\left(\frac{1}{x_{32}^2}\right) \theta(x_{10}^2 z_{\min} z_{\mathrm{pol}} - x_{32}^2 z'(z_{\mathrm{pol}} - z')) \\
&\qquad \times \left[G_{\underline{x}_1 + \left(1 - \frac{z'}{z_{\mathrm{pol}}}\right)\underline{x}_{32},\, \underline{x}_1 - \frac{z'}{z_{\mathrm{pol}}}\underline{x}_{32}}(z_{\min}, z')\, S_{10}(z_{\min}) + \Gamma_{10,32}(z_{\min}, z') \right] \\
&+ \frac{N_c}{2\pi^2} \int\limits_0^{z_{\mathrm{pol}}} \frac{dz'}{z_{\mathrm{pol}}} \left(2 - \frac{z'}{z_{\mathrm{pol}}} + \frac{z'^2}{z_{\mathrm{pol}}^2}\right) \int\limits_{\frac{z_{\mathrm{pol}}}{z'(z_{\mathrm{pol}} - z')s}} \frac{d^2 x_{32}}{x_{32}^2}\, \alpha_s\left(\frac{1}{x_{32}^2}\right) \theta(x_{10}^2 z_{\min} z_{\mathrm{pol}} - x_{32}^2 z'(z_{\mathrm{pol}} - z')) \\
&\times \Gamma_{10,32}(z_{\min}, z_{\mathrm{pol}}).
\end{aligned}
$$

Conclusion

- Quark's helicity contribution to proton spin follows evolution equations that contain leading DLA terms and subleading SLA terms.
- The SLA terms have been derived, with the effects of running coupling included, as the latter is also single-logarithmic.
- In the large-N_c and large-N_c&N_f limits, the equations are closed but non-linear, since the unpolarized evolution is single-logarithmic.
- Future work:
  - Connection to the polarized DGLAP evolution
  - Numerical solutions to the SLA evolution equations at large N_c and large N_c&N_f
  - Phenomenological implications

The End

References

R. L. Jaffe and A. Manohar (1990)
The g_1 problem: Deep inelastic electron scattering and the spin of the proton
Nucl. Phys. B 337, 509

Y. V. Kovchegov and H. Weigert (2007)
Triumvirate of Running Couplings in Small-x Evolution
Nucl. Phys. A 784, 188–226.

I. Balitsky (2007)
Quark contribution to the small-x evolution of color dipole
Phys. Rev. D 75, 014001.

Y. V. Kovchegov, D. Pitonyak and M. D. Sievert (2016)
Helicity Evolution at Small-x
JHEP 1601, 072; *JHEP* 1610, 148.

Y. V. Kovchegov, D. Pitonyak and M. D. Sievert (2017)
Small-x Asymptotics of the Quark Helicity Distribution: Analytic Results
Phys. Lett. B772, 136-140.

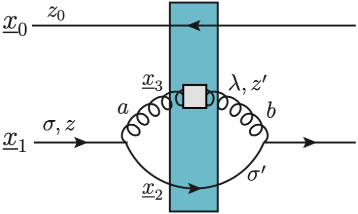
Y. V. Kovchegov and M. D. Sievert (2019)
Small-x Helicity Evolution: an Operator Treatment
Phys. Rev. D 5, 054032.

Y. V. Kovchegov and Y. Tawabutr (2020)
Helicity at Small x: Oscillations Generated by Bringing Back the Quarks
JHEP 2008, 014.

D. Adamiak, Y. Kovchegov, W. Melnitchouk, D. Pitonyak, N. Sato and M. Sievert (2021)
First analysis of world polarized DIS data with small-x helicity evolution
hep-ph/2102.06159.

Running Coupling

- For soft unpolarized parton emission, the strong coupling constant runs the same way as the unpolarized BK evolution [Kovchegov and Weigert, 2007] [Balitsky, 2007]
- For other splitting terms, i.e. soft polarized or hard parton emission, the strong coupling constant runs with the transverse size of the daughter dipole, e.g.



$$\sim \alpha_s(1/x_{32}^2)$$

With this prescription, there is no double counting with the SLA$_T$ terms.

Polarized Wilson Lines

$$V_{\underline{x}}^{pol} = \frac{igp_1^+}{s} \int\limits_{-\infty}^{\infty} dx^- \, V_{\underline{x}}[+\infty, x^-] \, F^{12}(x^-, \underline{x}) \, V_{\underline{x}}[x^-, -\infty]$$

$$- \frac{g^2 \, p_1^+}{s} \int\limits_{-\infty}^{\infty} dx_1^- \int\limits_{x_1^-}^{\infty} dx_2^- \, V_{\underline{x}}[+\infty, x_2^-] \, t^b \, \psi_\beta(x_2^-, \underline{x}) \, U_{\underline{x}}^{ba}[x_2^-, x_1^-] \left[\frac{1}{2} \gamma^+ \gamma^5\right]_{\alpha\beta} \bar{\psi}_\alpha(x_1^-, \underline{x}) \, t^a \, V_{\underline{x}}[x_1^-, -\infty].$$

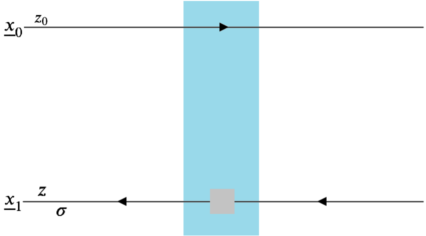
$$(U_{\underline{x}}^{pol})^{ab} = \frac{2i \, g \, p_1^+}{s} \int\limits_{-\infty}^{+\infty} dx^- \, \left(U_{\underline{x}}[+\infty, x^-] \, \mathcal{F}^{12}(x^+ = 0, x^-, \underline{x}) \, U_{\underline{x}}[x^-, -\infty]\right)^{ab}$$

$$- \frac{g^2 \, p_1^+}{s} \int\limits_{-\infty}^{\infty} dx_1^- \int\limits_{x_1^-}^{\infty} dx_2^- \, U_{\underline{x}}^{aa'}[+\infty, x_2^-] \, \bar{\psi}(x_2^-, \underline{x}) \, t^{a'} \, V_{\underline{x}}[x_2^-, x_1^-] \, \frac{1}{2} \gamma^+ \gamma_5 \, t^{b'} \, \psi(x_1^-, \underline{x}) \, U_{\underline{x}}^{b'b}[x_1^-, -\infty] + c.c..$$

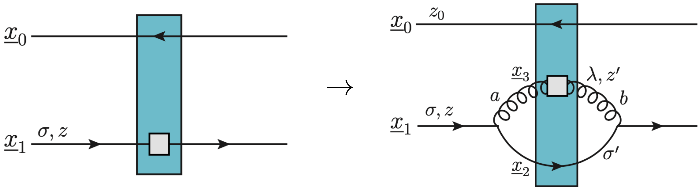
SLA Caveats

The polarized dipole amplitudes depend on two momentum fractions.

$$\left\langle \mathrm{tr}\left[V_{\underline{x}_0} V^{\dagger}_{\underline{x}_1}(\sigma)\right]\right\rangle(z) \quad \Rightarrow \quad \left\langle \mathrm{tr}\left[V_{\underline{x}_0} V^{\dagger}_{\underline{x}_1}(\sigma)\right]\right\rangle\Big(\underbrace{\min\{z, z_0\}}_{\text{minimum}}, \underbrace{z}_{\text{polarized}}\Big)$$



SLA Evolution Equations



Include not only the terms coming from the splitting of the polarized line.

- Since the evolutions of both polarized and unpolarized lines are SLA, they have to be included.
- The running coupling correction is SLA and also has to be included.

At SLA, the evolution of the dipole amplitude depends on momentum fractions of two lines – the polarized line and the softest-parton line.