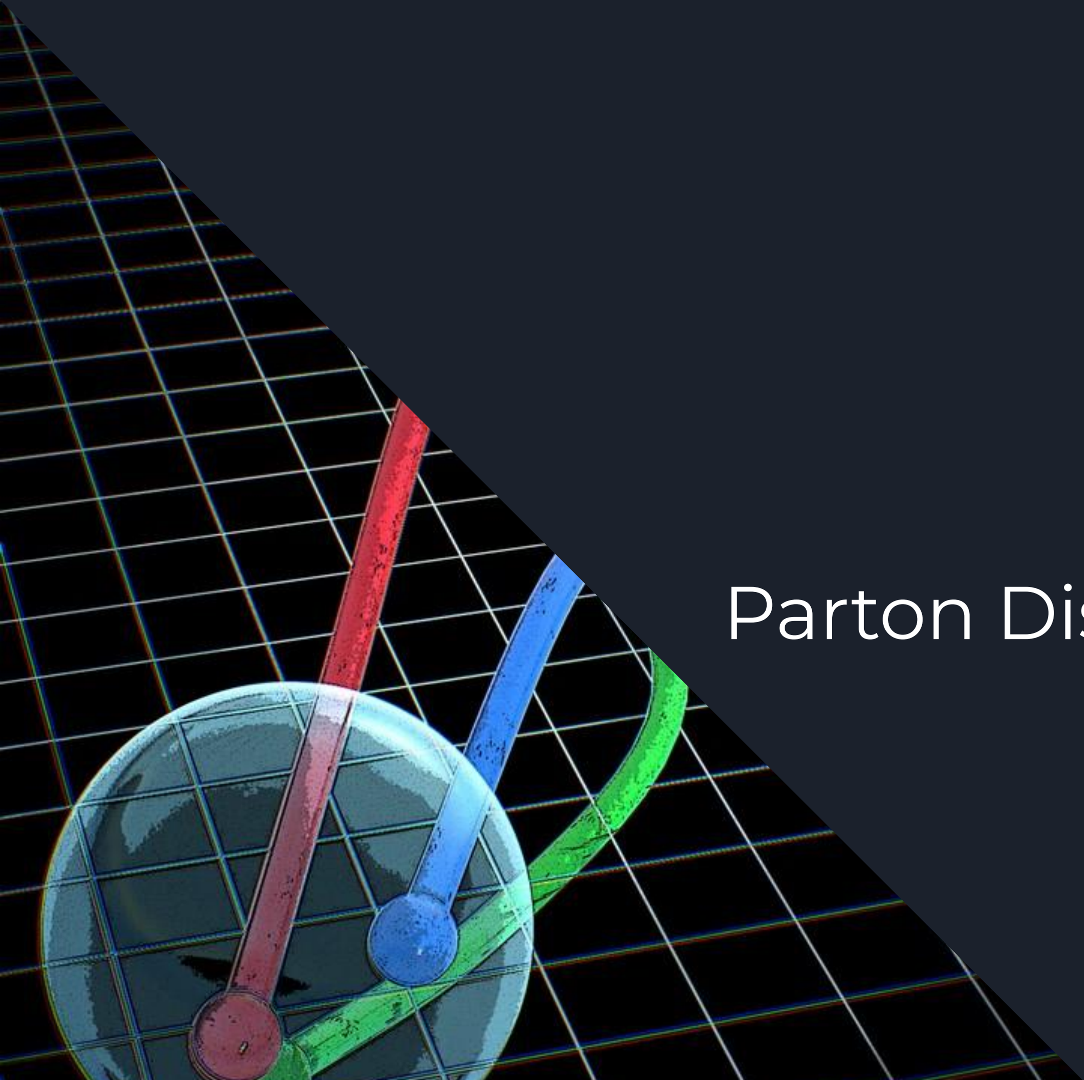


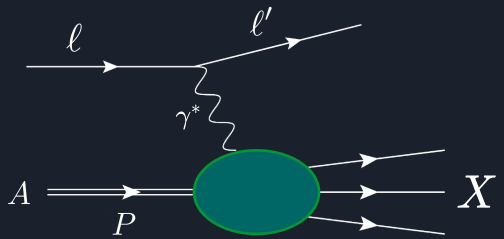
Lattice QCD and Parton Distribution Functions

36th Annual Hampton University Graduate Studies Program
(e-)HUGS
June 15th, 2021

Colin Egerer

Deep Inelastic Scattering (DIS) & Parton Distribution Functions (PDFs)

$\ell + A\,(P) \rightarrow \ell' + X$



Invariants

$$\begin{cases} q^\mu = \ell^\mu - \ell'^\mu \\ s = (P+\ell)^2 \\ W^2 = (P+q)^2 \end{cases}$$

$$Q^2 \equiv -q^2 \geq 0$$

$$x = \frac{Q^2}{2P\cdot q}$$

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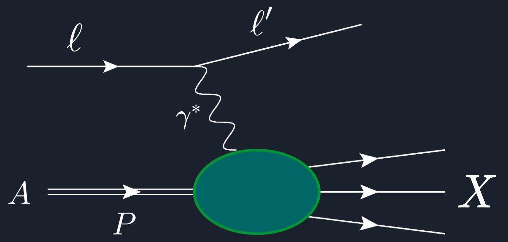

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Inclusive cross section in terms of leptonic/hadronic tensors

$$\frac{d\sigma_{\mathrm{DIS}}\left(x, Q^2, s\right)}{d^3\ell'} \propto \overbrace{L_{\mu\nu}}^{\text{Perturbative}}\underbrace{W^{\mu\nu}}_{\text{Non-perturbative}}$$

$$W^{\mu\nu}\left(q, P\right) = \frac{1}{4\pi}\int d^4z\; e^{iq\cdot z}\,\langle P, S|\, \mathcal{J}^\mu\left(z\right) \mathcal{J}^\nu\left(0\right) |P, S\rangle$$

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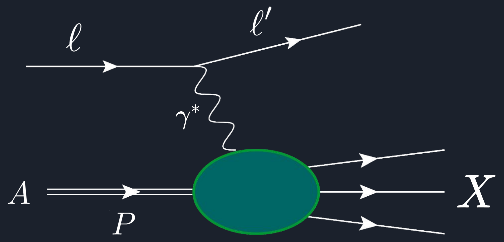

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Perturbative

Non-perturbative

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Lorentz decomposition into structure functions (SFs)

$$\hat{P}^\mu = P^\mu - \frac{P\cdot q}{q^2} q^\mu$$

$$W^{\mu\nu} = \left(-g^{\mu\nu} + \frac{q^\mu q^\nu}{q^2}\right) F_1\left(x, Q^2\right) + \frac{\hat{P}^\mu \hat{P}^\nu}{P\cdot q} F_2\left(x, Q^2\right)$$
$$+\, i\epsilon^{\mu\nu\alpha\beta}\frac{q_\alpha S_\beta}{P\cdot q} g_1\left(x, Q^2\right) + i\epsilon^{\mu\nu\alpha\beta}\frac{q_\alpha\left(S_\beta - P_\beta \frac{S\cdot q}{P\cdot q}\right)}{P\cdot q} g_2\left(x, Q^2\right) + \mathrm{P.V.}$$

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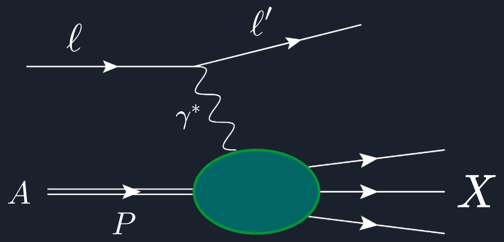

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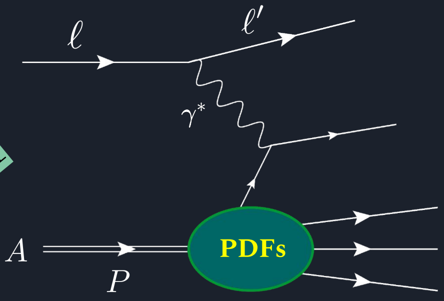
Parton Model



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- PDFs: number densities of partons with fraction ξ of hadron's P^+ momentum
  - probabilistic interpretation at leading-twist

$$k^+ = \xi P^+$$

Lattice Gauge Theory (QCD)

Numerically solve QCD using Monte Carlo methods

- quantitatively study strong-coupled regimes

QCD action given as input $S_{\mathrm{QCD}}\left[\psi, \overline{\psi}, G_\mu\right]$

- discretization (momentum cutoffs)
- path integral in Euclidean spacetime

$$\langle \hat{\mathcal{O}} \rangle_{\mathrm{E}} = \mathcal{Z}^{-1} \int \boxed{\mathcal{D}\psi \mathcal{D}\overline{\psi}} \boxed{\mathcal{D}U} \hat{\mathcal{O}} \left[\psi, \overline{\psi}, U\right] e^{-S^{\mathrm{E}}_{\mathrm{QCD}}[\psi, \overline{\psi}, U]}$$

$$\prod_{n \in \Lambda} \prod_{f, \alpha, c} \mathrm{d}\psi_{\mathrm{f}} \left(\mathrm{n}\right)^{\mathrm{c}}_{\alpha} \mathrm{d}\overline{\psi}_{\mathrm{f}} \left(\mathrm{n}\right)^{\mathrm{c}}_{\alpha} \qquad \prod_{n \in \Lambda} \prod_{\mu=1}^{4} \mathrm{d}\mathrm{U}_{\mu} \left(\mathrm{n}\right)$$

Compute observables non-perturbatively

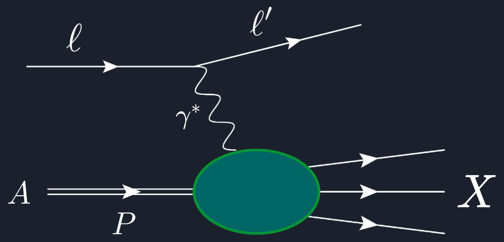
- i.e. correlation functions
  (averaged over gluon configurations)

  $C_{2\mathrm{pt}} \left(\vec{p}, t\right) = \langle 0 | \, h\left(\vec{p}, t\right) h^{\dagger} \left(0\right) | 0 \rangle$

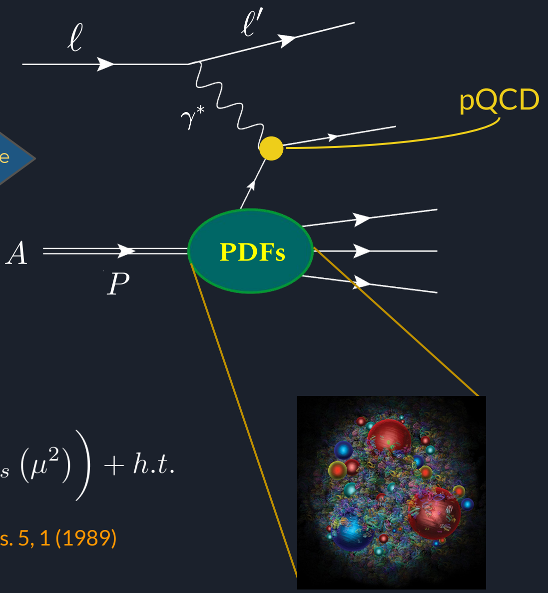
  $C_{3\mathrm{pt}} \left(\vec{p}, \vec{q}; t, \tau\right) = \langle 0 | \, h\left(\vec{p}, t\right) \mathcal{O}\left(\vec{q}, \tau\right) h^{\dagger} \left(0\right) | 0 \rangle$
- systematically improvable results

Inclusive Factorization

$\ell + A\,(P) \rightarrow \ell' + X$



Parton Picture



QCD factorization theorems relate cross sections (SFs) to PDFs

$$F_i\left(x, Q^2\right) = \sum_{a=q,\bar{q},g} f_{a/h}\left(x, \mu^2\right) \otimes H_i^a\left(x, \frac{Q^2}{\mu^2}, \alpha_s\left(\mu^2\right)\right) + h.t.$$

J. Collins, D. Soper, G. Sterman, Adv. Ser. Direct. High Energy Phys. 5, 1 (1989)

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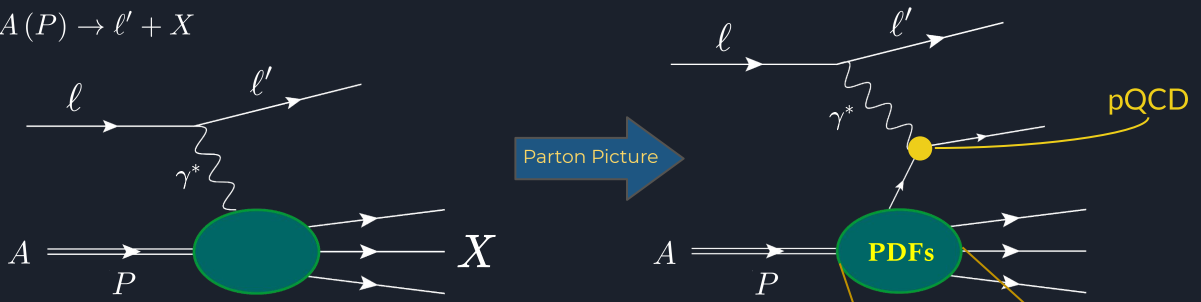

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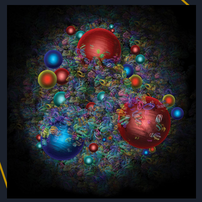

(Forward) integrated parton correlator

$$f_{q/h}^{[\gamma^+]}\left(x, \mu^2\right) = \frac{1}{2}\int \frac{dz^-}{2\pi} e^{ixP^+z^-} \langle h\,(p)|\, \overline{\psi}\left(\tfrac{z}{2}\right) \gamma^+ \Phi_{\hat{z}^-}^{(f)}\left(\left\{\tfrac{z}{2}, -\tfrac{z}{2}\right\}\right) \psi\left(-\tfrac{z}{2}\right) |h\,(p)\rangle$$

- input to cross section predictions (e.g. LHC)
- affect precision measurements of SM parameters
- focus of upcoming facilities (EIC)

J. Gao et al., Phys. Rept. 742 (2018) 1-121

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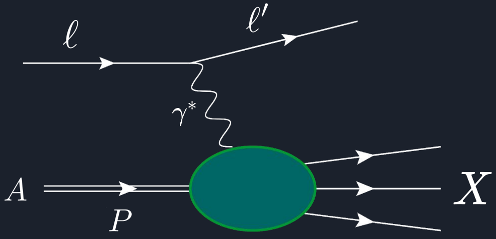

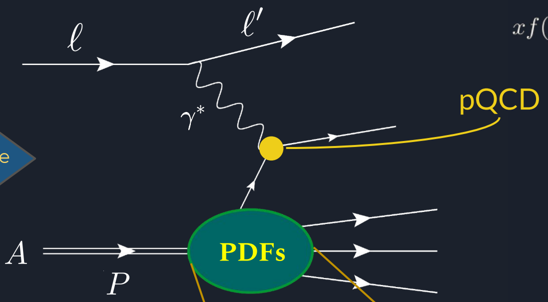

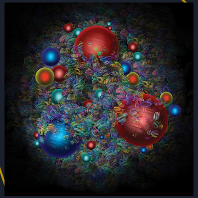

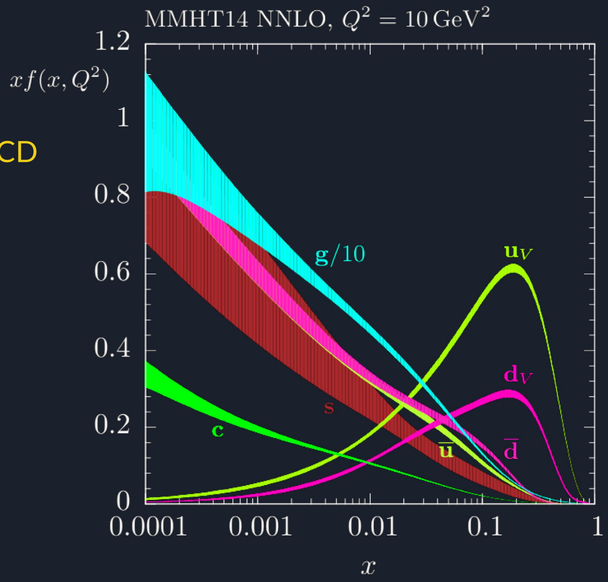

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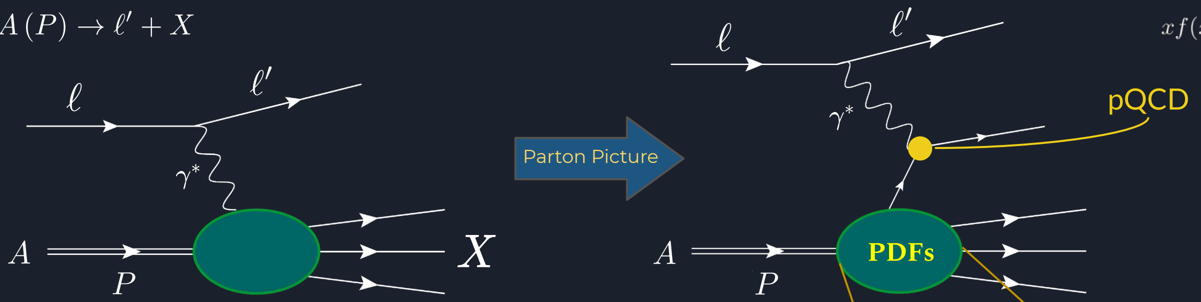

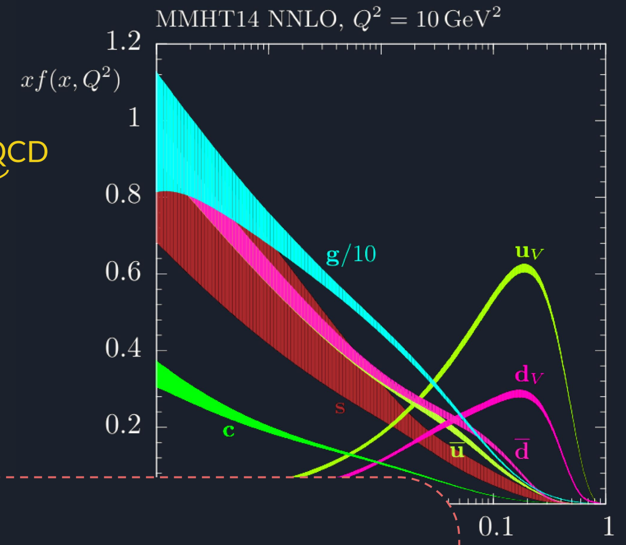

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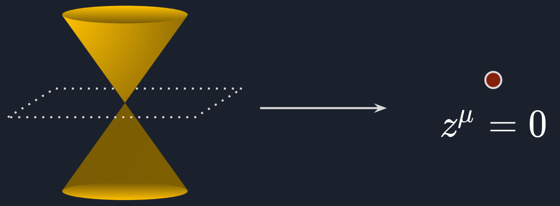

Collapse of light-cone in Euclidean metric of LQCD

Mellin Moments of PDFs

Consider integrated moments of forward parton distribution:

$$m_a^{(j)}\left(\mu^2\right) = \frac{1}{s_a}\int_{-1}^{1}\frac{dx}{x}x^j f_{a/h}^{[\Gamma]}\left(x,\mu^2\right)$$

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Moments provide crucial link with lattice calculable quantities...

$$m_{a,\gamma^+}^{(j)}\left(\mu^2\right) = s_a^{-1}\int\frac{dz^-}{4\pi}\int_{-\infty}^{\infty}dx\ \boxed{\left(\frac{-i}{p^+}\frac{\partial}{\partial z^-}\right)}^{j-1} e^{-ixp^+z^-}\left\langle h(p)\right|\overline{\psi}_a\left(z^-\right)\gamma^+\Phi_{\hat{z}^-}^{(f)}\left(\{z^-,0\}\right)\psi_a(0)\left|h(p)\right\rangle$$

$$x = \frac{k^+}{p^+} = \frac{-i}{p^+}\frac{\partial}{\partial z^-}$$

$$2\left(p^+\right)^j m_{a,\gamma^+}^{(j)}\left(\mu^2\right) = s_a^{-1}\left\langle h(p)\right|\overline{\psi}_a\gamma^+ iD^+\cdots iD^+\psi_a\left|h(p)\right\rangle$$

Mellin moments of PDFs are given in terms of *local* operators

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Mellin moments of PDFs are given in terms of *local* operators

Amenable to calculation in Lattice QCD

- limited by broken symmetry

$$O(4) \mapsto H(4)$$

E.g. G. Martinelli and C. T. Sachrajda, Phys. Lett. B196, 184 (1987)

Active Community Progress

Hadronic Tensor

K.F. Liu et al., PRL 72, 1790 (1994) & Phys. Rev. D59/62

$$W_{\mu\nu} = \frac{1}{2\pi} \mathfrak{Im} T_{\mu\nu}$$

Virtual Compton Amplitude [& OPE]

A.J. Chambers et al., Phys. Rev. Lett. 118

"OPE without OPE" - G. Martinelli

$$T_{\mu\nu} = \rho_{\lambda\lambda'} \int \mathrm{d}^4 z e^{iq\cdot z} \langle p, \lambda' | T \mathcal{J}_\mu (z) \mathcal{J} (0) | p, \lambda \rangle$$

Auxiliary Quark Methods

U. Aglietti et al., Phys. Lett. B441; W. Detmold & C.J.D. Lin, Phys. Rev. D73;
V. Braun & D. Mueller, Eur. Phys. J. C55

Quasi-Distributions [PDFs]

X. Ji, Phys. Rev. Lett. 110, 262002 (2013)

$$M^\alpha (z,p) = \langle h(p) | \overline{\psi} (z) \gamma^\alpha W (z, 0; A) \psi (0) | h(p) \rangle$$

Pseudo-PDFs

A. V. Radyushkin, Phys. Rev. D 96, 034025 (2017)

Lattice Cross Sections

Y. Q. Ma & J. W. Qiu; Phys. Rev. D 98, no. 7, 074021 (2018)
& Phys. Rev. Lett. 120, no. 2, 022003 (2018)

Coordinate-space Factorizable Matrix Elements

Consider a generic hadronic matrix element of time-local and space-like separated operators

$$M_n\left(p\cdot z, z^2\right) = \langle h\left(p\right)| \hat{\mathcal{T}}\{\mathcal{O}_n\left(z\right)\}\left|h\left(p\right)\right\rangle$$

$\nu \equiv p\cdot z$

V. Braun and D. Mueller, Eur.Phys.J.C 55 (2008) 349-361
B. L. Ioffe, Phys. Lett. 30B, 123 (1969)

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Unique traceless and symmetric rank-j tensor wrt. to hadron four-momentum

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$$\begin{aligned} M_n\left(\nu, z^2\right) &= 2\sum_{j=0}\sum_a C_n^{(j,a)}\left(z^2\mu^2,\alpha_s\right) z^{\mu_1}\cdots z^{\mu_j}\widetilde{m}_a^{(j)}\left(\mu^2\right)\left(p_{\mu_1}\cdots p_{\mu_j} - \mathrm{traces}\right) \\ &= 2\sum_{j=0}\sum_a C_n^{(j,a)}\left(z^2\mu^2,\alpha_s\right) \boxed{\widetilde{m}_a^{(j)}\left(\mu^2\right)} \nu^j + \mathcal{O}\left(z^2\Lambda^2_{\mathrm{QCD}}, z^2p^2\right) \end{aligned}$$

Leading contribution from smallest mass dimension - leading twist operators!

Ioffe-time dependence of coordinate space matrix element related to collinear momentum distributions!

$$M_n\left(\nu, z^2\right) = \sum_a \int_{-1}^{1}\frac{dx}{x} f_{a/h}\left(x,\mu^2\right) \sum_{j=1}\frac{2}{s_a} C_a^{(j,a)}\left(z^2\mu^2,\alpha_s\right)(x\nu)^j + \mathcal{O}\left(z^2\Lambda^2_{\mathrm{QCD}}, z^2p^2\right)$$

Y. Q. Ma and J. W. Qiu, Phys. Rev. Lett. 120 (2018) 2, 022003

A Working Coordinate-space Factorization

$$\boxed{\mathfrak{M}_n\left(\nu, z^2\right)} = \sum_a \int_{-1}^{1} \frac{dx}{x} \boxed{f_{a/h}\left(x, \mu^2\right)} \boxed{K_n^a\left(x\nu, z^2\mu^2, x^2p^2\right)} + \mathcal{O}\left(z^2\Lambda_{\rm QCD}^2, z^2p^2\right)$$

Ioffe-time dependent invariant amplitudes

PDFs

Hard Coefficients

Two flavors of Lattice calculable and factorizable coordinate space matrix elements

Two-current Correlators

Single-hadron matrix elements of renormalized non-local operators

$$M_{ij}^{[\mu\nu]}(p,z) = \langle h(p)| \mathcal{O}_{ij}^{[\mu\nu]}(z) |h(p)\rangle$$

$$\mathcal{O}_S(z) = z^4 Z_S^2 \{\bar\psi\psi\}(z)\{\bar\psi\psi\}(0)$$
$$\mathcal{O}_V(z) = z^2 Z_V^2 \{\bar\psi \not{z}\psi\}(z)\{\bar\psi\not{z}\psi\}(0)$$
$$\mathcal{O}_{\widetilde{V}\widetilde{A}}(z) = -\frac{z^4}{2} Z_V Z_A \{\bar\psi\gamma_\mu\psi\}(z)\{\bar\psi\gamma^\mu\gamma^5\psi\}(0)$$

$$p \sim \sqrt{s}$$

$$z^2 \sim \frac{1}{Q^2}$$

$$|\vec{z}| \ll \Lambda_{\rm QCD}^{-1}$$

Space-like non-local parton bilinears

Ioffe-time Pseudo-Distributions

$$M^\alpha(p,z) = \langle h(p)| \overline{\psi}(z)\gamma^\alpha \Phi_{\hat{z}}^{(f)}(\{z,0\})\psi(0)|h(p)\rangle$$

➢ Lorentz-invariant generalizations of PDFs onto space-like intervals

➢ Same starting point as quasi-PDFs

Y. Q. Ma and J. W. Qiu; Phys. Rev. D98 (2018) 7, 074021
& Phys. Rev. Lett. 120 (2018) 2, 022003

A. Radyushkin, Phys.Rev.D 96 (2017) 3, 034025

Tree-Level Perturbative Matching Kernel

Project matching relationship onto asymptotic and on-shell parton

$$\delta\left(1-x\right)\delta^{qa}$$

$$M_n^{q(0)}\left(\nu, z^2\right) = \sum_{a=q,\bar{q},g} \int_0^1 \frac{dx}{x} f_{a/h}^{q(0)}\left(x, \mu^2\right) K_n^a\left(x\nu, z^2\mu^2, x^2p^2\right)$$

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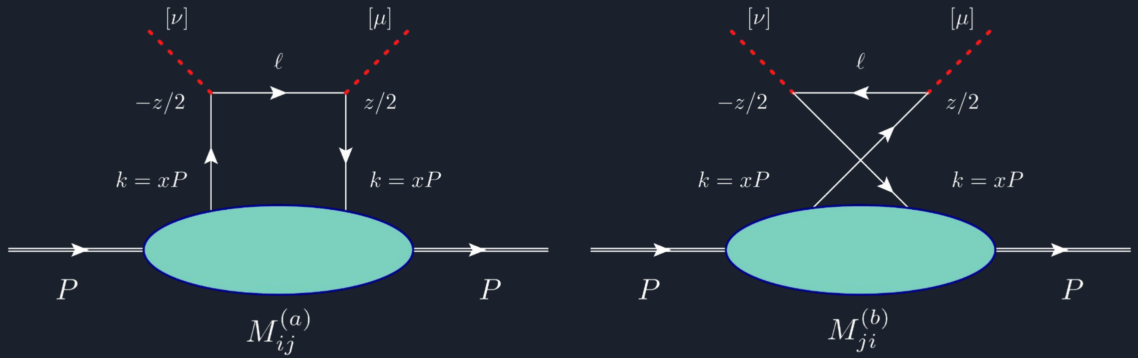
Consider a generic tensor operator:

$$M_{ij}^{\mu\nu}(p,z) = \langle h(p)|\, \mathcal{J}_i^{\mu}(z/2)\, \mathcal{J}_j^{\nu}(-z/2)\, |h(p)\rangle$$



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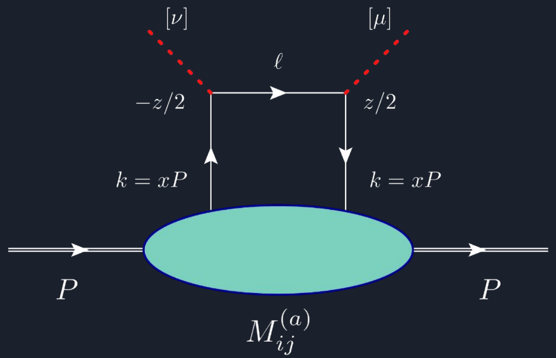
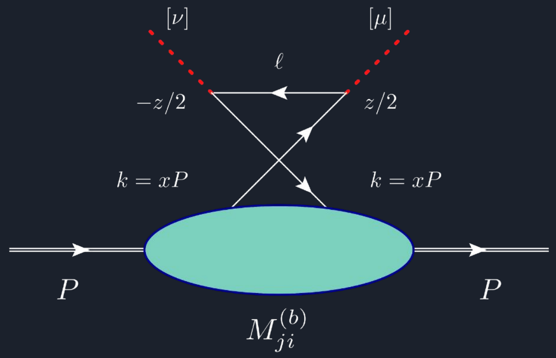
$$\begin{aligned} M_{ij}^{(a)} &= \frac{1}{2}\sum_s e^{ik\cdot z}\bar{u}_s\left(k\right)\Gamma_i^{\mu}\langle 0|\,\psi\left(z/2\right)\bar{\psi}\left(-z/2\right)\,|0\rangle\,\Gamma_j^{\nu}u_s\left(k\right) \\ &= \frac{1}{2}e^{ik\cdot z}k_{\alpha}\mathrm{Tr}\left[\gamma^{\alpha}\Gamma_i^{\mu}\gamma^{\beta}\Gamma_j^{\nu}\right]\int\frac{d^4\ell}{\left(2\pi\right)^4}\frac{i\ell_{\beta}}{\ell^2+i\epsilon}e^{-i\ell\cdot z} \\ &= \frac{i}{4\pi^2}\frac{k_{\alpha}z_{\beta}}{z^4}e^{ik\cdot z}\mathrm{Tr}\left[\gamma^{\alpha}\Gamma_i^{\mu}\gamma^{\beta}\Gamma_j^{\nu}\right] \end{aligned}$$





$$M_{ji}^{(b)} = -\frac{i}{4\pi^2}\frac{k_{\alpha}z_{\beta}}{z^4}e^{-ik\cdot z}\mathrm{Tr}\left[\gamma^{\alpha}\Gamma_j^{\nu}\gamma^{\beta}\Gamma_i^{\mu}\right]$$

Choice of Currents

Generic tree-level matching kernel

$$M_{ij}^{\mu\nu(0)}\left(p,z;\nu\right)=\frac{i}{4\pi^2}\frac{xp_\alpha z_\beta}{z^4}\left[e^{ixp\cdot z}\mathrm{Tr}\left(\gamma^\alpha\Gamma_i^\mu\gamma^\beta\Gamma_j^\nu\right)-e^{-ixp\cdot z}\mathrm{Tr}\left(\gamma^\alpha\Gamma_j^\nu\gamma^\beta\Gamma_i^\mu\right)\right]$$

$\mathcal{O}_n$	Prefactor	Γ_i	Γ_j	$M_{ij}^{\mu\nu}\left(p,z;\nu\right)$
$\mathcal{O}_S\left(z\right)$	z^4	$\mathbb{1}$	$\mathbb{1}$	$\frac{i}{\pi^2}x\nu\left(e^{ix\nu}-e^{-ix\nu}\right)$
$\mathcal{O}_V\left(z\right)$	z^2	$z_\mu\gamma^\mu$	$z_\nu\gamma^\nu$	$\frac{i}{\pi^2}x\nu\left(e^{ix\nu}-e^{-ix\nu}\right)$
$\mathcal{O}_{\widetilde{V}}\left(z\right)$	$-z^4/2$	$g_{\nu\mu}\gamma^\mu$	γ^ν	$\frac{i}{\pi^2}x\nu\left(e^{ix\nu}-e^{-ix\nu}\right)$
$\mathcal{O}_{V'}\left(z\right)$	z^2	$z_\mu\gamma^\mu$	$z_\nu\gamma^\nu$	$\frac{i}{\pi^2}x\nu e^{ix\nu}$
$\mathcal{O}_{\widetilde{V}\widetilde{A}}\left(z\right)$	$-z^4/2$	$g_{\nu\mu}\gamma^\mu$	$\gamma^\nu\gamma^5$	0
$\mathcal{O}_{VA}\left(z\right)$	z^4	γ^μ	$\gamma^\nu\gamma^5$	$\frac{1}{\pi^2}\epsilon^{\mu\nu\alpha\beta}xp_\alpha z_\beta\left(e^{ix\nu}+e^{-ix\nu}\right)$

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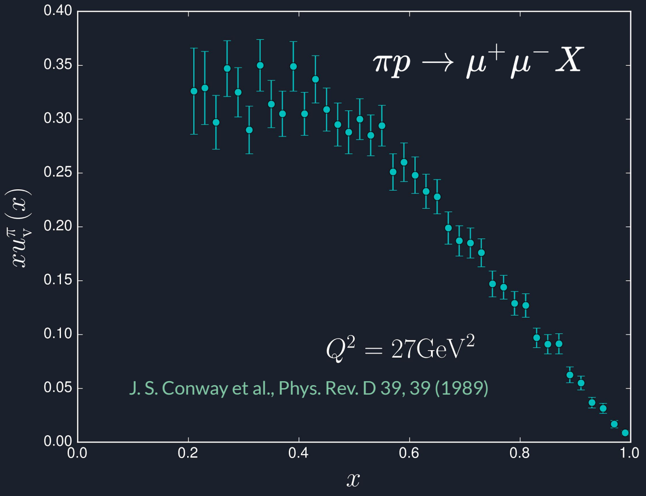
$\mathcal{O}_{\tilde{V}}(z)$	$-z^4/2$	$g_{\nu\mu}\gamma^\mu$	γ^ν	$\frac{i}{\pi^2}x\nu\left(e^{ix\nu} - e^{-ix\nu}\right)$
$\mathcal{O}_{V'}(z)$	z^2	$z_\mu\gamma^\mu$	$z_\nu\gamma^\nu$	$\frac{i}{\pi^2}x\nu e^{ix\nu}$
$\mathcal{O}_{\tilde{V}\tilde{A}}(z)$	$-z^4/2$	$g_{\nu\mu}\gamma^\mu$	$\gamma^\nu\gamma^5$	0
$\mathcal{O}_{VA}(z)$	z^4	γ^μ	$\gamma^\nu\gamma^5$	$\frac{1}{\pi^2}\epsilon^{\mu\nu\alpha\beta}xp_\alpha z_\beta\left(e^{ix\nu} + e^{-ix\nu}\right)$

A puzzle in the pion...



- pQCD [Phys. Rev. Lett. 42, 940] $\beta \simeq 2$
- NLO [Phys. Rev. C 72, 065203] $\beta \simeq 1.3$
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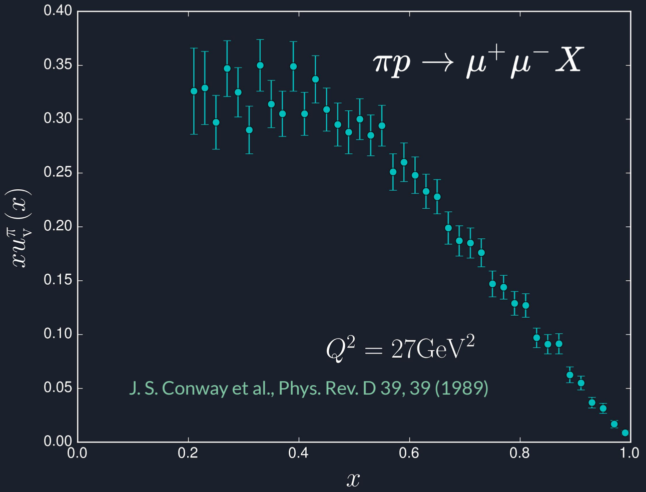
$$\Downarrow$$

Simplest non-vanishing CP-even current combination at tree-level is a vector-axial combination

$\mathcal{J}_V^\mu=\bar{\psi}\gamma^\mu\psi$ $\mathcal{J}_A^\nu=\bar{\psi}\gamma^\nu\gamma^5\psi$

A puzzle in the pion...



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Rely on parity, time-reversal invariance of QCD to construct appropriate CP-even combination

$$\langle p|\,\mathcal{O}_{ij}^{\mu\nu}\left(z\right)|p\rangle=\langle p|\,(\hat{\mathcal{P}}\hat{\mathcal{T}})\mathcal{O}_{ij}^{\mu\nu}\left(z\right)^\dagger(\hat{\mathcal{P}}\hat{\mathcal{T}})^{-1}\,|p\rangle$$

$$(\hat{\mathcal{P}}\hat{\mathcal{T}})\mathcal{J}_V^\mu\left(z\right)(\hat{\mathcal{P}}\hat{\mathcal{T}})^{-1}=\mathcal{J}_V^\mu\left(-z\right)$$

$$(\hat{\mathcal{P}}\hat{\mathcal{T}})\mathcal{J}_A^\mu\left(z\right)(\hat{\mathcal{P}}\hat{\mathcal{T}})^{-1}=-\mathcal{J}_A^\mu\left(-z\right)$$

Vector-Axial Currents

Antisymmetric combination of vector/axial currents

$$M^{\mu\nu}_{VA}(p,z) + M^{\mu\nu}_{AV}(p,z) \equiv \langle \pi(p) | \left[\mathcal{O}^{\mu\nu}_{VA}(z) + \mathcal{O}^{\mu\nu}_{AV}(z) \right] | \pi(p) \rangle$$

pseudo-structure functions

Lorentz-invariance

$$\sigma^{\mu\nu}_{VA} \equiv \frac{z^4}{2}\left[M^{\mu\nu}_{VA}(p,z) + M^{\mu\nu}_{AV}(p,z)\right] = \frac{\epsilon^{\mu\nu\alpha\beta}p_\alpha z_\beta}{\nu} T_1\left(\nu, z^2\right) + \frac{\left(p^\mu z^\nu - z^\mu p^\nu\right)}{\nu} T_2\left(\nu, z^2\right)$$

- tree-level matching result: $\sigma^{\mu\nu(0)}_{VA} = \frac{2}{\pi^2}\epsilon^{\mu\nu\alpha\beta} x p_\alpha z_\beta \cos(x\nu)$

Tree-level matching kernels

$$K_1^{q(0)}\left(x\nu, z^2\right) = \frac{2\nu x}{\pi^2}\cos(x\nu)$$
$$K_2^{q(0)}\left(x\nu, z^2\right) = 0$$

NLO

Judicious selection of momenta, separations, Dirac indices obviates full tensor contractions

$$p^\alpha = (E, \mathbf{0}_\perp, p_z)$$
$$z^\alpha = (0, \mathbf{0}_\perp, z_3)$$

$$T_1\left(\nu, z^2\right) = \frac{\nu}{p_0 z_3}\sigma^{12}_{VA}$$

R. Sufian, **C.E.**, J. Karpie, et al., Phys. Rev. D 102 (2020) 5, 054508

Many Wick Contractions

- Mass degenerate light quarks - expensive Wick contraction topology

$$\mathcal{J}_\Gamma = \bar{\ell}\Gamma\ell$$

$$C_{4\text{pt}}\left(\vec{p}, z; T, \tau\right) = \langle \Pi\left(-\vec{p}, T\right) \mathcal{J}_\Gamma^\dagger\left(\vec{z} + \vec{z}_0, \tau\right) \mathcal{J}_{\Gamma'}\left(\vec{z}_0, \tau\right) \overline{\Pi}\left(\vec{p}, 0\right)\rangle$$

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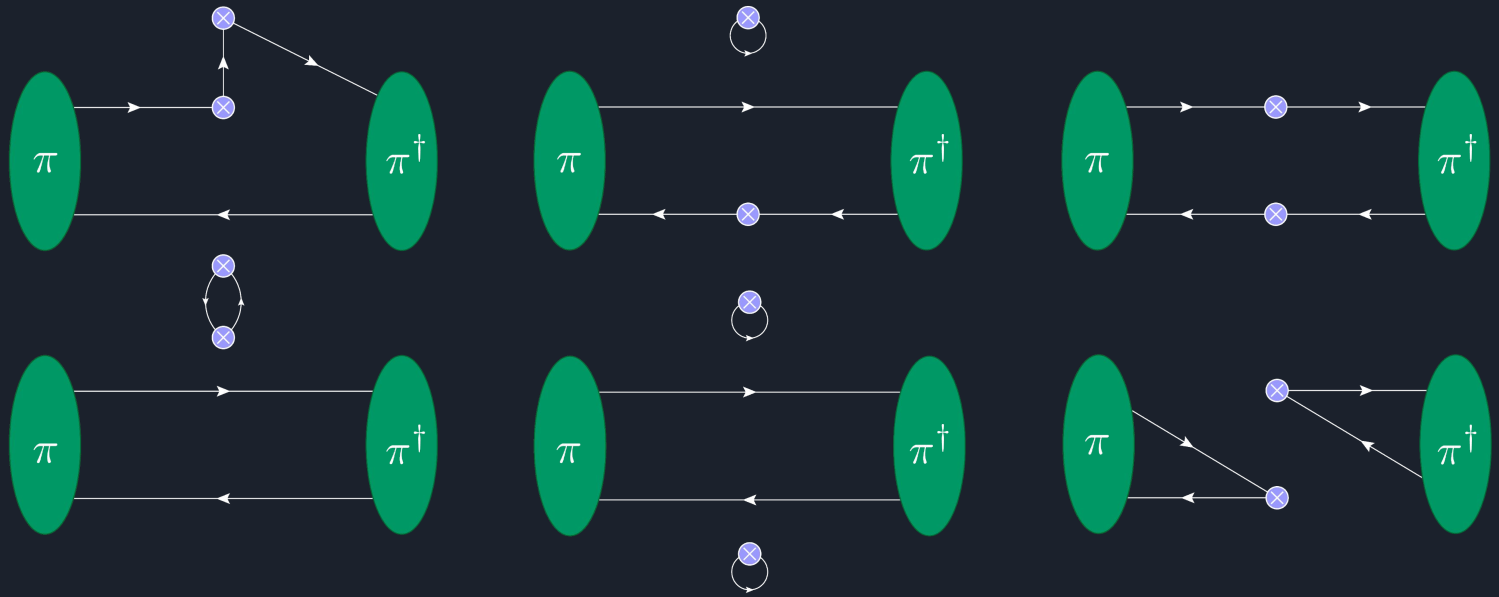


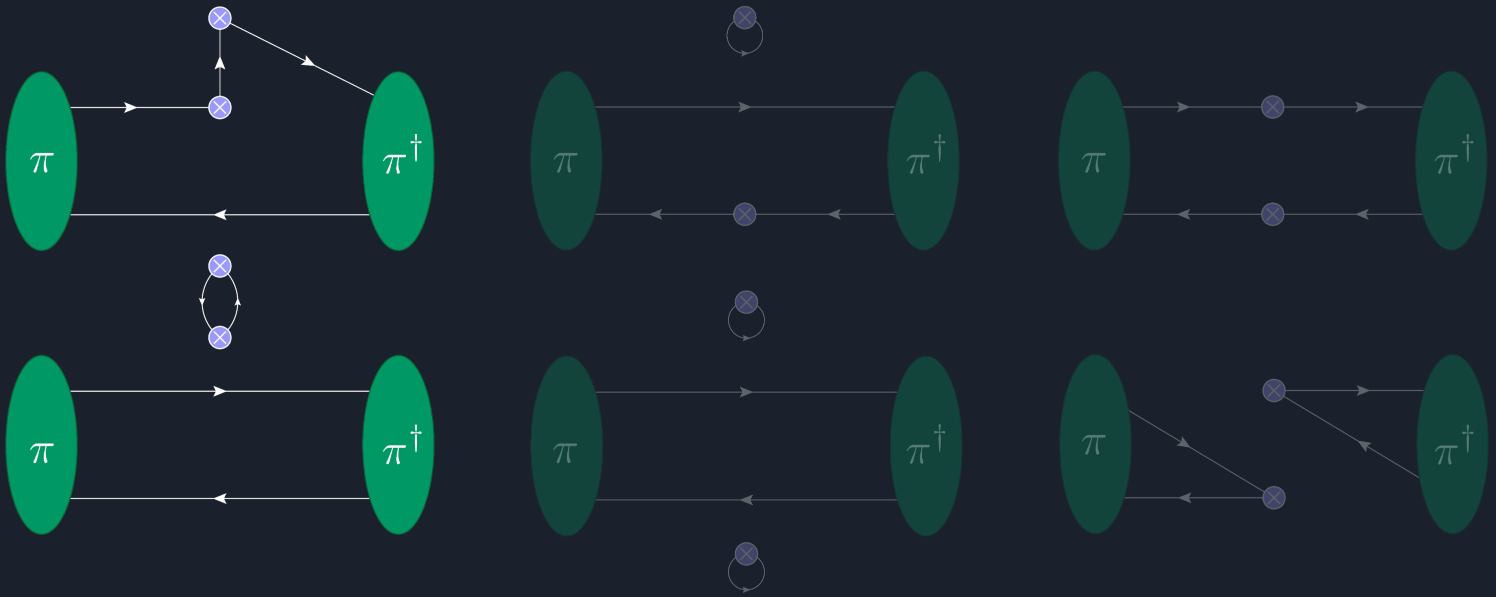
Many Wick Contractions

- Mass degenerate light quarks - expensive Wick contraction topology
- Heavy-light flavor changing currents
  - fewer contractions/saturate phase space

    W. Detmold and C.J.D. Lin, Phys.Rev.D 73 (2006) 014501

$$C_{4\text{pt}}\left(\vec{p}, z; T, \tau\right) = \left\langle \Pi\left(-\vec{p}, T\right) \mathcal{J}_{\Gamma}^{\dagger}\left(\vec{z}+\vec{z}_0, \tau\right) \mathcal{J}_{\Gamma'}\left(\vec{z}_0, \tau\right) \overline{\Pi}\left(\vec{p}, 0\right)\right\rangle$$

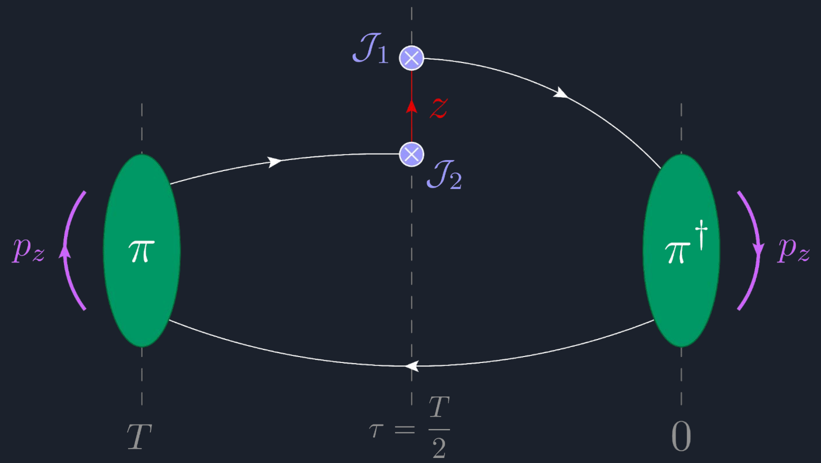
$$\mathcal{J}_{\Gamma} = \{\bar{\ell}\Gamma Q, \bar{Q}\Gamma\ell\}$$



First Numerical Implementation - The Pion

Solving QCD for properties of a free, isolated pion

- less computationally demanding than baryons
- straightforward realization in mesonic systems
- auxiliary mass set to strange quark



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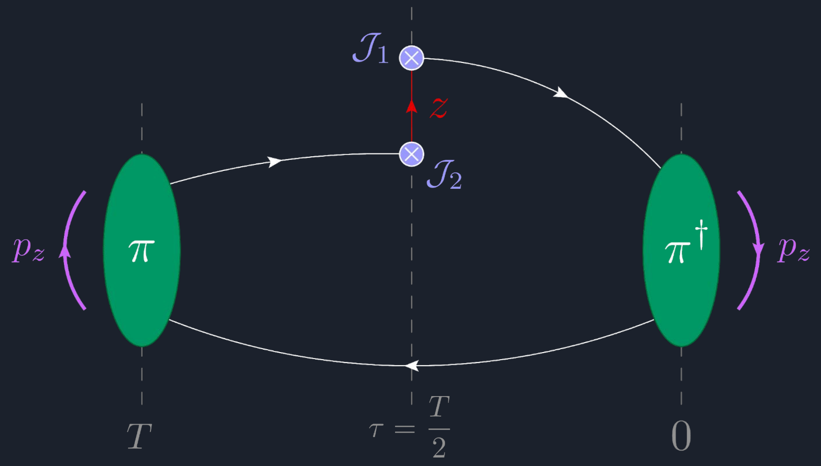

Two-current factorization requires currents at fixed spatial separations

→ modified sequential operator method needed

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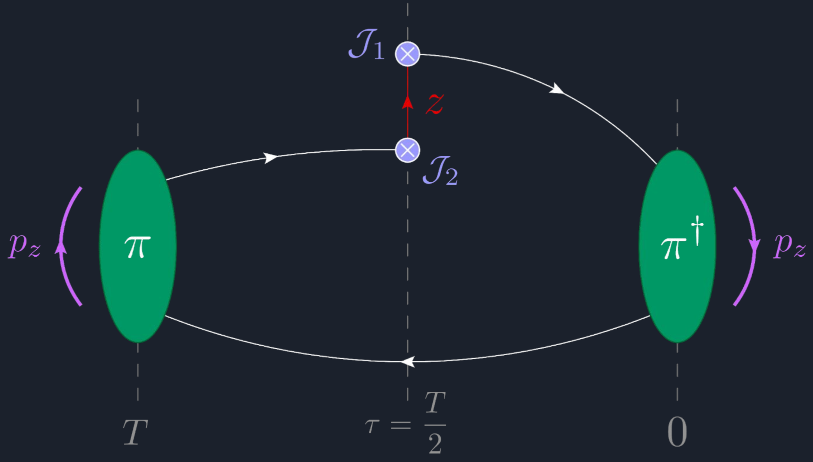

Two-current factorization requires currents at fixed spatial separations

→ modified sequential operator method needed

1 **Inversion from first current**

$$\sum_{z',t'} D_q(z,t;z',t')\, G_q(z',t';z_0,\tau) = \delta(z-z_0)\,\delta(t-\tau)$$

2 **Inversion from source to sink**

...following spatial/momentum smearing and momentum projection

3 **Inversion to second current**

Tie together with auxiliary quark propagator

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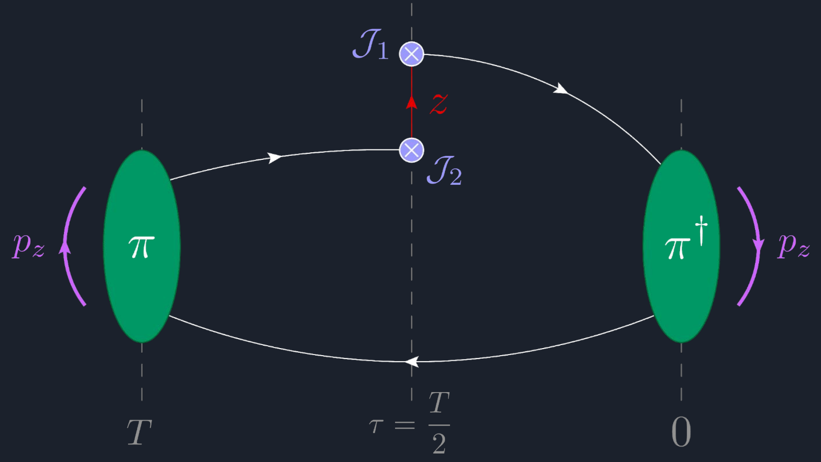

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2 **Inversion from source to sink**

...following spatial/momentum smearing and momentum projection

$$\sum_{z',t'} D_q\left(z,t;z',t'\right) H_q^p\left(z',t';z_0,\tau\right) = e^{-iz\cdot p}\sum_{y',y''} S\left[z;y'\right]\gamma^5 S\left[y';y''\right] G_q\left(y'',t;z_0,\tau\right)\delta\left(t\right)$$

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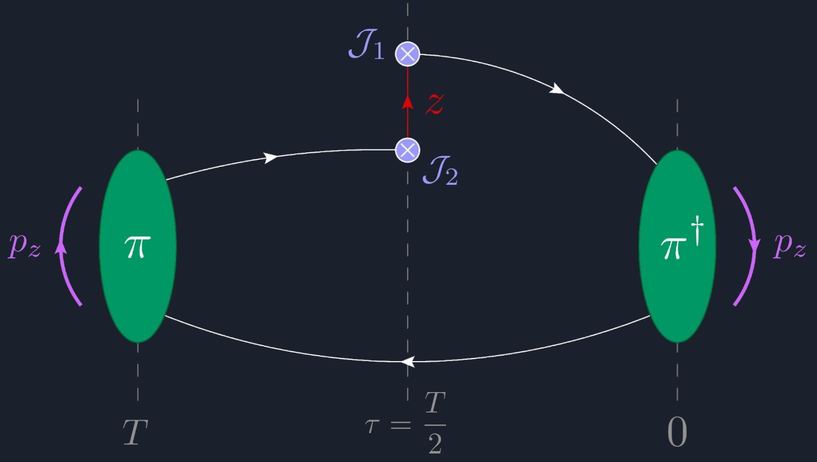

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$$\sum_{z',t'} D_q\left(z,t;z',t'\right) I_q^p\left(z',t';z_0,\tau\right) = e^{iz\cdot p}\sum_{y',y''} S\left[z;y'\right]\gamma^5 S\left[y';y''\right] H_q\left(y'',t;z_0,\tau\right)\delta\left(t-T\right)$$

Excited States

All lattice calculations must contend with contamination from unwanted states

- interpolators that best reflect properties of desired state

$$\langle 0 | \hat{\mathcal{O}} \left(\vec{p} \right) | h \left(\vec{p} \right) \rangle \gg \langle 0 | \hat{\mathcal{O}} \left(\vec{p} \right) | h' \left(\vec{p} \right) \rangle$$

- broken rotational symmetry

$$O\left(3\right) \mapsto O_h^{[D]}$$

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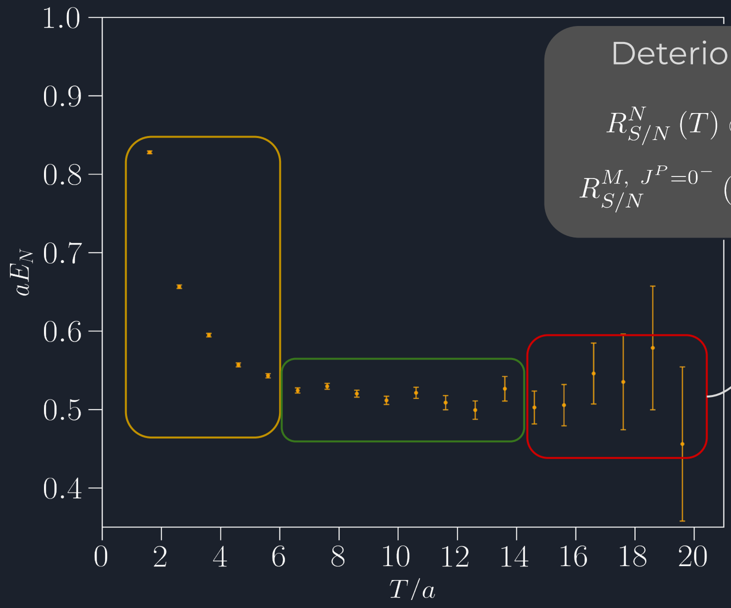

Deterioration of signal...

$$R^N_{S/N}\,(T) \propto e^{-\left(E_N(\vec{p})-\frac{3}{2}m_\pi\right)T}$$

$$R^{M,\ J^P=0^-}_{S/N}\,(T) \propto e^{-(E_M(\vec{p})-m_M)T}$$

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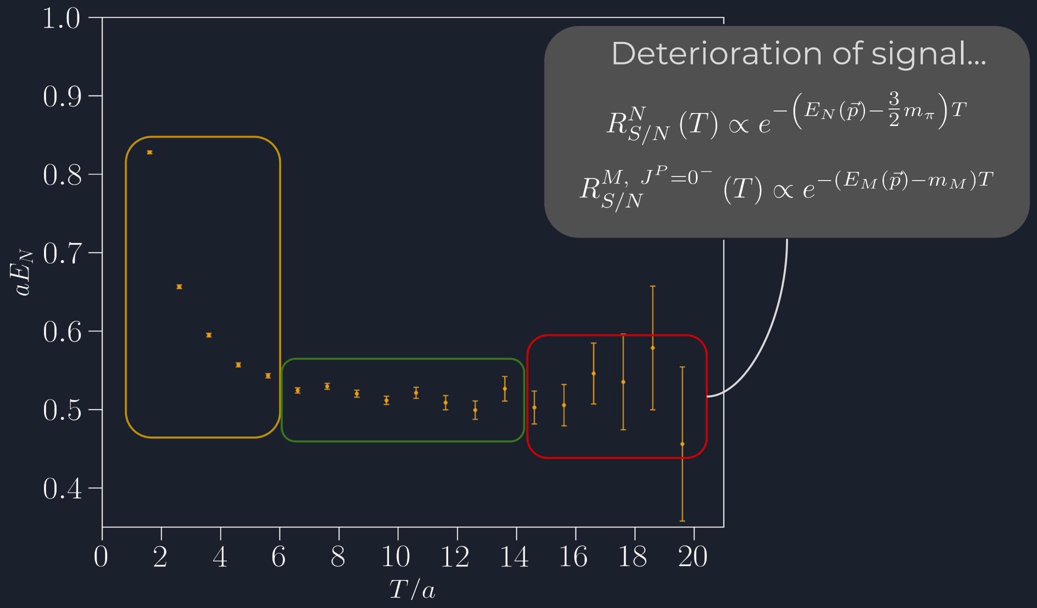
Key demand:
state of interest saturate correlation function at early Euclidean times

$$C_{2\mathrm{pt}}\,(\vec{p},T) = \sum_n \frac{|Z_n|^2}{2E_n\,(\vec{p})} e^{-E_n(\vec{p})T}$$

Spatial smearing* to increase overlap with ground-state

$$\hat{q}\,(\vec{x},T) = \sum_{\vec{y}} S\,[U]\,(\vec{x},\vec{y})\,q\,(\vec{y},T)$$

$$\left[1+\sigma\nabla^2\,(T;U)\right]^{n_\sigma}$$



C.R. Allton et al. (UKQCD), Phys. Rev. D47, 5128 (1993)

Excited States

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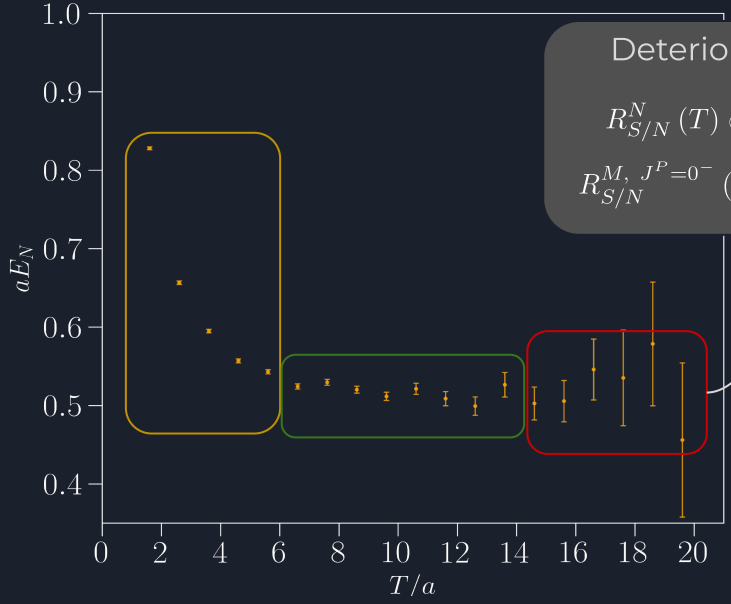

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Variational improvement: B. Blossier, et al., JHEP 04, 094 (2009)

➔ exploit operator redundancy (GEVP)

$$C_{ij}\,(t') = \langle \mathcal{O}_i\,(t')\,\mathcal{O}_j^\dagger\,(0)\rangle$$

$$C\,(t)\underbrace{\boxed{v_{\mathbf{n}}\,(t,t_0)}}_{\text{operator weights}} = \underbrace{\boxed{\lambda_{\mathbf{n}}\,(t,t_0)}}_{\text{principal correlator}}C\,(t_0)\,v_{\mathbf{n}}\,(t,t_0)$$

Interpolator Construction

ID	a (fm)	m_π (MeV)	$L^3 \times N_t$	N_{cfg}
$a127m413$	$0.127(2)$	$413(4)$	$24^3 \times 64$	2124
$a127m413L$	$0.127(2)$	$413(5)$	$32^3 \times 96$	490
$a94m358$	$0.094(1)$	$358(3)$	$32^3 \times 64$	417
$a94m278$	$0.094(1)$	$278(4)$	$32^3 \times 64$	503

In addition to two-current (four-point) correlator, spectral information is needed

$$Z_n = \langle 0 | \Pi_{\vec{p}} | n \rangle$$

$$C^{\pi}_{\mathrm{2pt}}(\vec{p}, T) = \langle \Pi_{\vec{p}}(T)\, \overline{\Pi}_{\vec{p}}(0) \rangle = \sum_n \frac{|Z_n|^2}{2E_n(\vec{p})} e^{-E_n(\vec{p})T}$$

$\bar{q}\gamma^5 q$

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Maintain short-distance factorization

$$\mathcal{O}\left(z^2\Lambda^2_{\text{QCD}}, z^2p^2\right)$$

➢ high-momenta needed!

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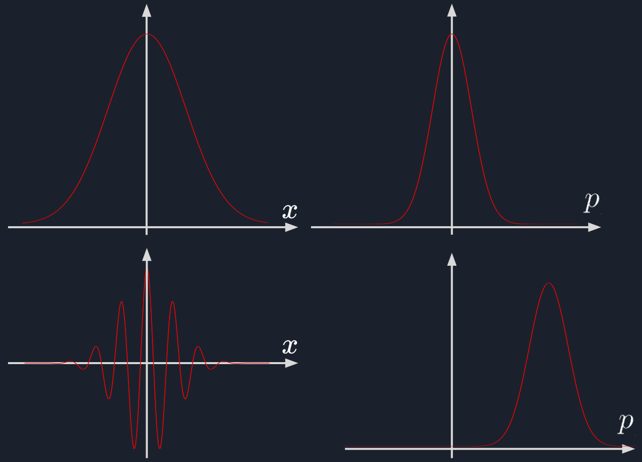
$\bar{q}\gamma^5 q$

Maintain short-distance factorization

$$\mathcal{O}\left(z^2\Lambda^2_{\rm QCD}, z^2p^2\right)$$

➢ high-momenta needed!

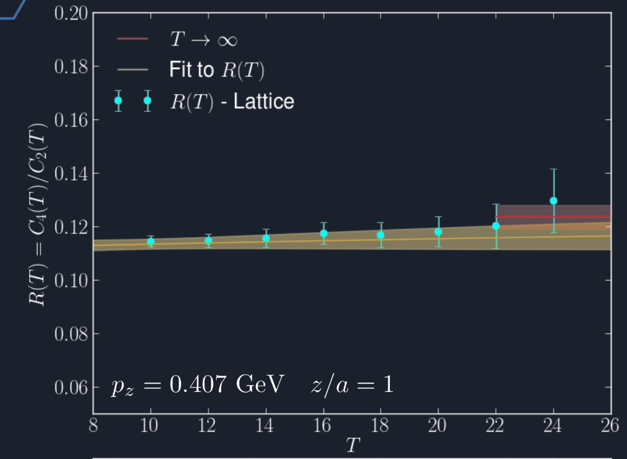
Poor overlap of spatially-smeared interpolators onto boosted states G. S. Bali et al., Phys. Rev. D93, 094515 (2016)

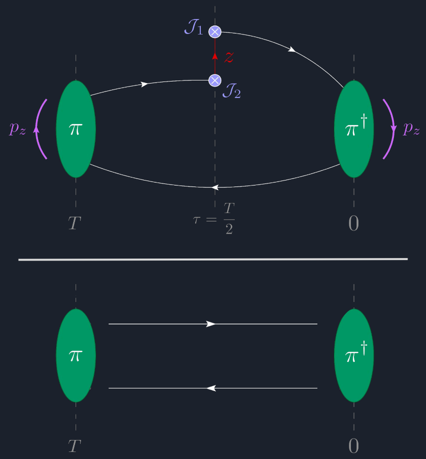


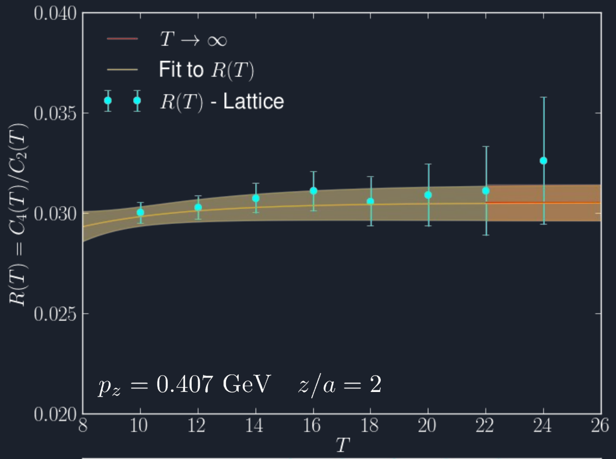
Resolution: "**momentum smearing**" $\tilde{U}_\mu[x] = e^{i\frac{2\pi}{L}\zeta d_\mu} U_\mu[x]$

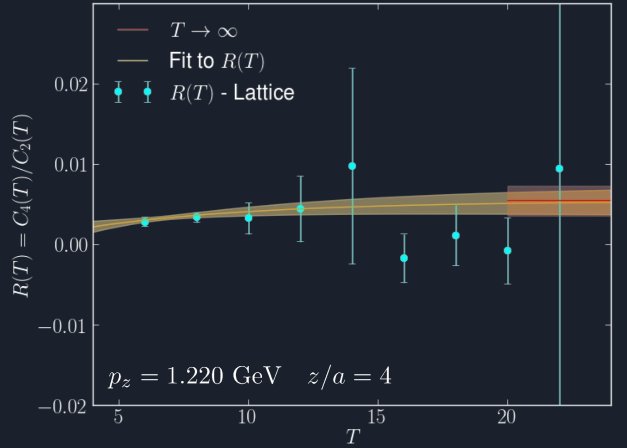
$$\Pi_{\vec{p}}(T) = \sum_{\vec{x}} e^{i\vec{p}\cdot\vec{x}} \bar{\tilde{q}}(\vec{x}, T)\gamma^5 \tilde{q}(\vec{x}, T)$$

Select Two-Current Matrix Elements

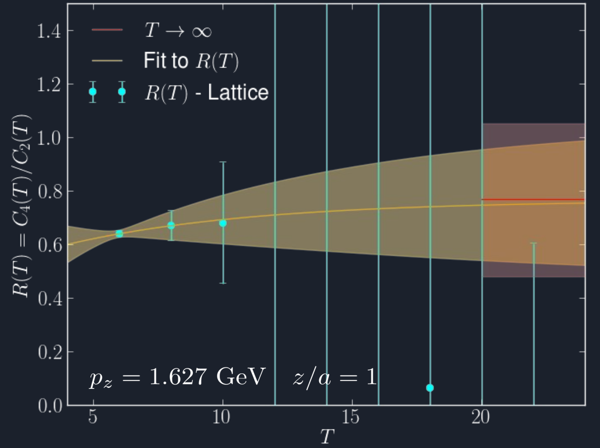




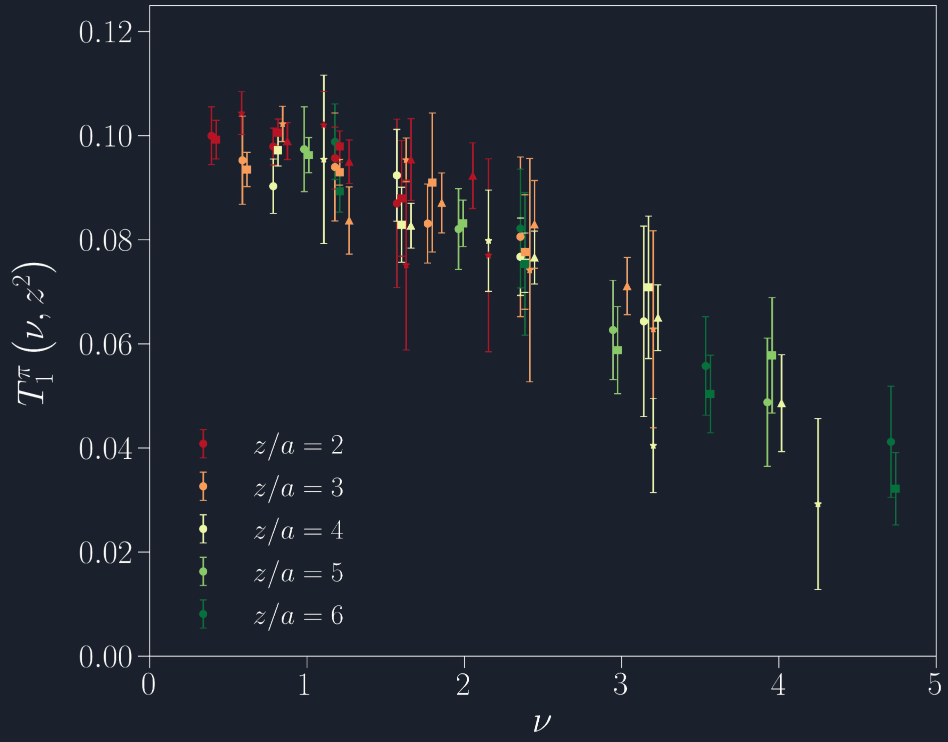




$$\frac{C_{4\mathrm{pt}}(T)}{C_{2\mathrm{pt}}(T)} = A + Be^{-\Delta E_{eff} T}$$



Pion Ioffe-time Pseudo-Structure Function



R. Sufian, **C.E.**, J. Karpie, et al., Phys. Rev. D 102 (2020) 5, 054508
R. Sufian, J. Karpie, **C.E.**, et al., Phys. Rev. D 99 (2019) 7, 074507

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- Data included for
  - S/N ratios of unity or larger
  - $z/a = 1$ neglected (contact terms)
- Lack of a volume average
  - shorter Euclidean separations to maintain reasonable signal
- questionable resolution of excited-state contamination
  - restricted to midpoint in time series
  - high-momenta S/N ratios degrade rapidly
    - matrix element constrained by few measurements
    - excited-states likely uncontrolled

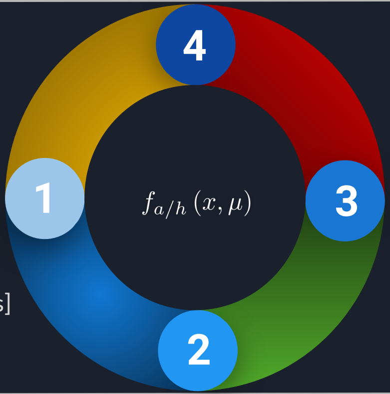
From Lattice Matrix Elements to PDFs

Matrix Element

- two-currents
- pseudo-ITD

Renormalization

- Trivial factor [currents]
- Reduced distribution



Inverse Problem

A grossly ill-posed convolutional relationship connecting lattice data to desired structure function

Evolution/Matching

Coordinate-space factorization; perturbative matching kernels

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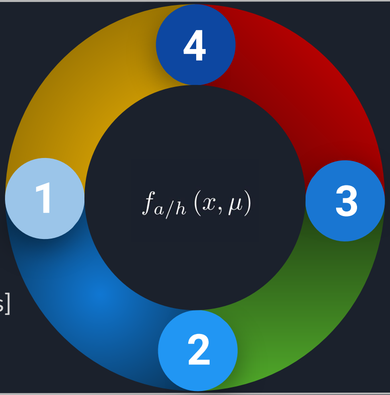

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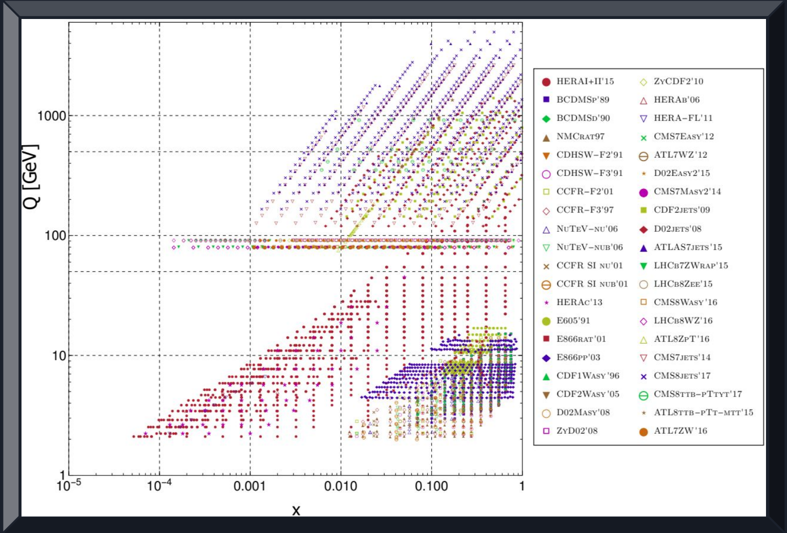
A serious systematic that must be confronted

Analogous challenge faced by global fitting community!



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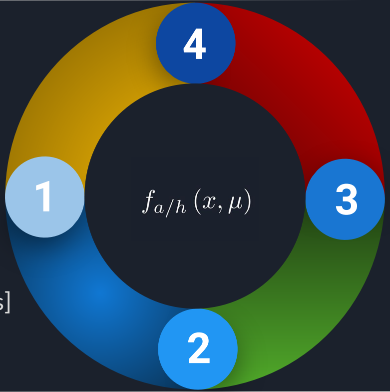

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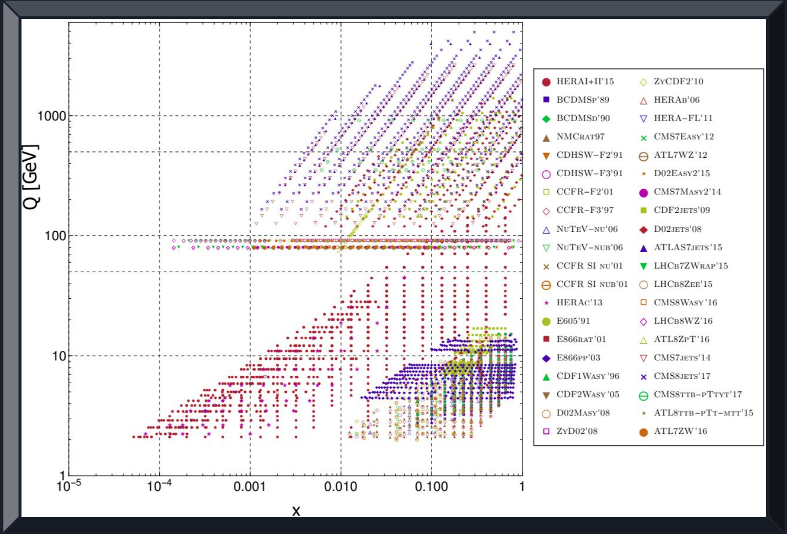


How to Proceed?

A) Parametric fits

$$x f_{a/h}\left(x, Q_0^2\right) = A x^{\alpha}(1-x)^{\beta} P(x)$$

$$\rightarrow 1+\sum_{k=1}^{4} c_k T_k^{\mathrm{Ch}}\left(1-2\sqrt{x}\right) \quad \text{MMHT}$$

$$\rightarrow 1+\gamma\sqrt{x}+\delta x \quad \text{CJ/MSTW}$$

$$\rightarrow \sum_{k=1}^{4} c_k b_k^4\left(\sqrt{x}\right) \quad \text{CT}$$

B) Advanced reconstructions

a) Bayesian reconstruction, Backus-Gilbert, Maximum Entropy, etc

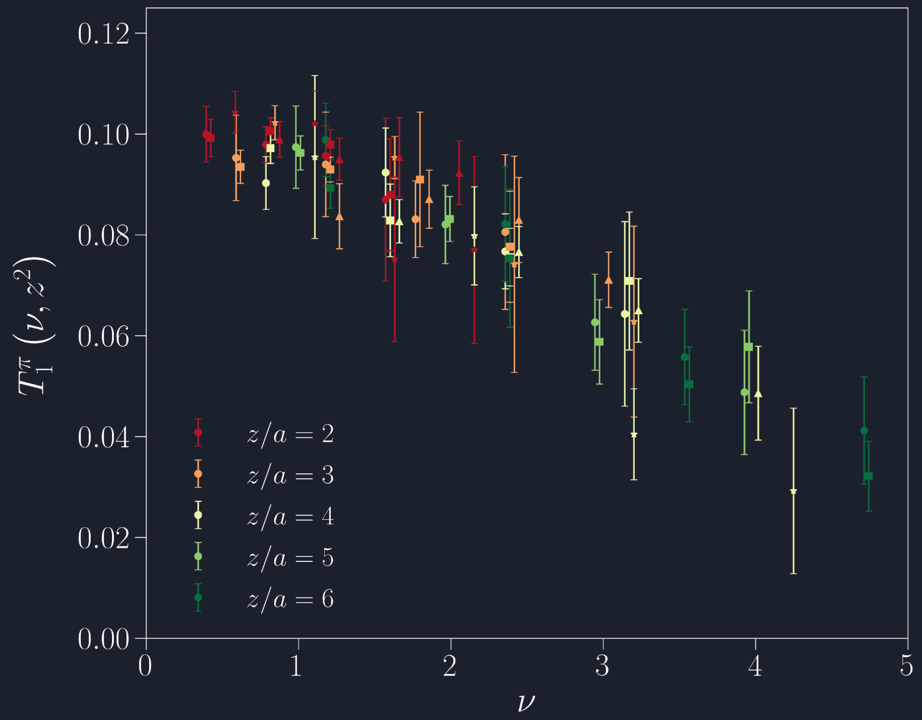
b) novel deep-learning methods

J. Karpie, et al., JHEP 04, 057 (2019)
L. D. Debbio, et al., arXiv: 2010.03996 [hep-ph]
K. Cichy, L. D. Debbio, T. Giani, JHEP 10 (2019) 137

Conformal Fit of Pseudo-Structure Function

Pseudo-structure function is analytic in Ioffe-time, but is otherwise unknown

- exploit a (model independent) conformal mapping - *z-expansion*
- supplement with corrections



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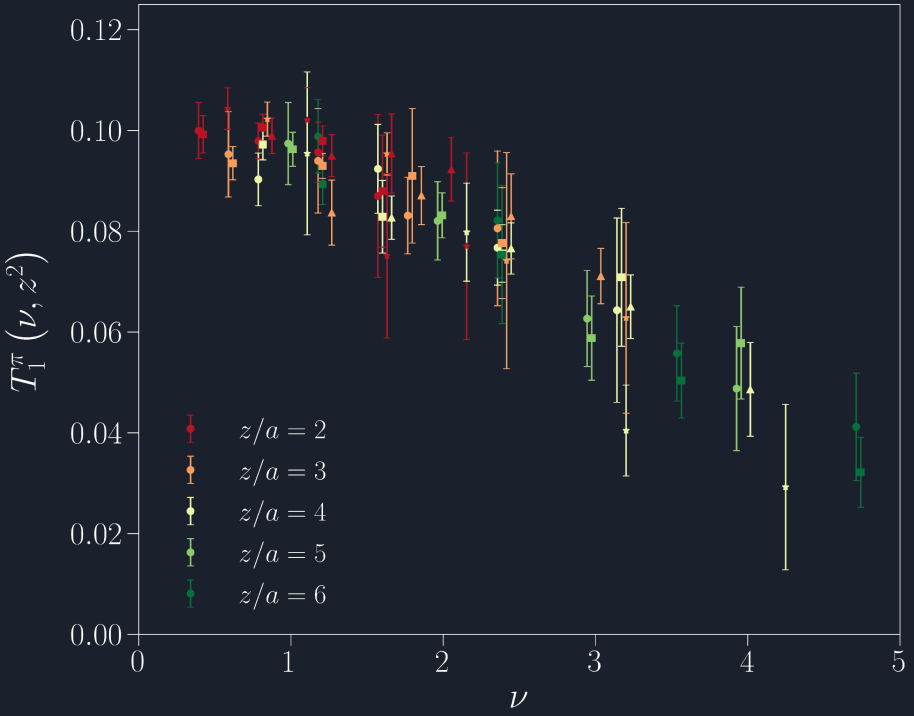
$$T_1^\pi\left(\nu, z^2\right) = \sum_{k=0}^{k_{\max}=4} \lambda_k \rho^k + b_1\left(m_\pi - m_\pi^{\text{phys}}\right) + b_2 a + b_3 z^2 + b_4 a^2 p^2 + b_5 e^{-m_\pi(L-z)}$$

Currents H.T. High-momentum

$$\rho = \frac{\sqrt{\nu_{\text{cut}}+\nu} - \sqrt{\nu_{\text{cut}}}}{\sqrt{\nu_{\text{cut}}+\nu} + \sqrt{\nu_{\text{cut}}}}$$



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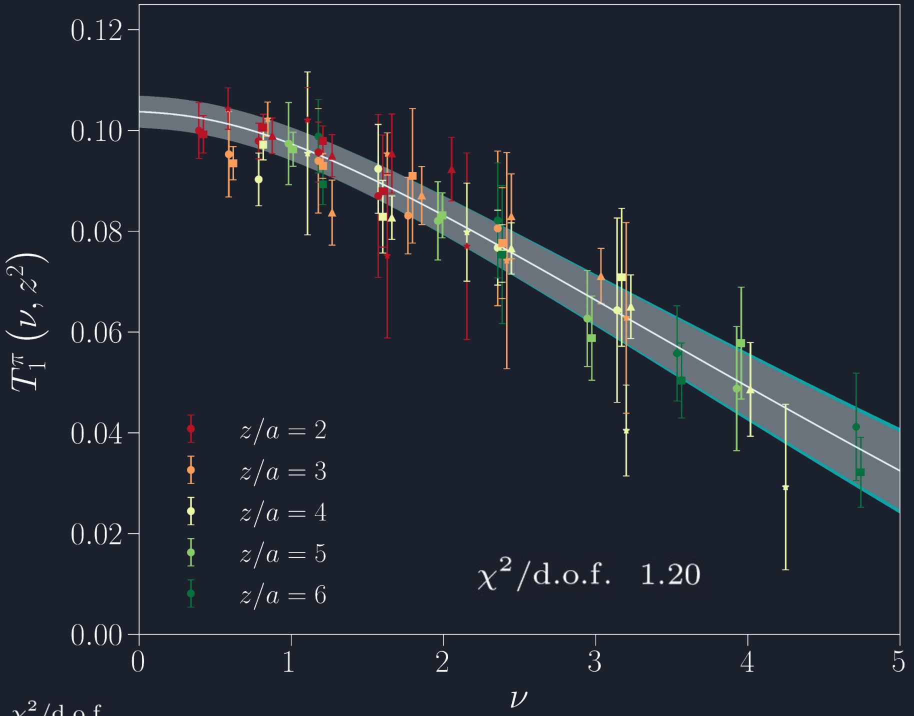
Correction	b_i	ρ_0	ρ_1	ρ_2	ρ_3	ρ_4	$\chi^2/\mathrm{d.o.f.}$
$(m_\pi - m_\pi^{\mathrm{phys}})$	0.174(96)	0.104(3)	−0.006(3)	−0.029(9)	−0.907(404)	0.124(136)	
a	0.083(43)	...	...	...	...	...	
z^2	0.0004(7)	...	...	...	...	...	
a^2p^2	0.007(8)	...	...	...	...	...	
$e^{-m_\pi(L-z)}$	0.102(51)	...	...	...	...	...	
a^2	−0.049(34)	0.104(3)	−0.006(3)	−0.028(9)	−0.901(391)	0.124(135)	1.26
$(m_\pi^2 - m_{\mathrm{phys},\pi}^2)$	0.15(12)	0.104(3)	−0.006(3)	−0.029(10)	−0.926(388)	0.118(132)	1.18
$Le^{-m_\pi(L-\xi)}$	0.007(3)	0.104(3)	−0.006(3)	−0.028(10)	−0.915(402)	0.121(136)	1.22
$\sqrt{L}e^{-m_\pi(L-\xi)}$	0.026(14)	0.104(3)	−0.006(3)	−0.029(10)	−0.914(403)	0.121(136)	1.21



➔ Alternate corrections are subleading
➔ Expansion unaffected

Extraction of Pion Valence Quark PDF

$$T_1^{\pi}(\nu)_{\text{fit}} = \int_0^1 dx\ K_1^{\text{NLO}}\left(x\nu, z^2\mu^2\right) f_{q_{\text{v}}/\pi}\left(x, \mu^2\right)$$

#1

Information Content

Physical Ioffe-time Distribution (ITD) cannot carry more information than original discrete data

- 30 equally-spaced and correlated slices
- mean/covariance, and 200 Gaussian distributed pseudo-data samples

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Convolution

Numerically perform convolution, fitting to bootstrap samples of pseudo-data

- two- and three-parameter phenomenological fits

$$N_{\text{v}} x^{\alpha} (1-x)^{\beta}$$

$$N_{\text{v}} x^{\alpha} (1-x)^{\beta} (1+\delta x)$$

$$\alpha \leq 0 \quad \beta \leq 4$$

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#3

Matching

Lack of observed scale dependence

- assignment in NLO matching kernel

$$z = 2a = 0.188 \text{ fm}$$

- initial scale

$$\mu_0 = 2 \text{ GeV}$$
$$\alpha_s\left(2 \text{ GeV}\right) \simeq 0.303$$

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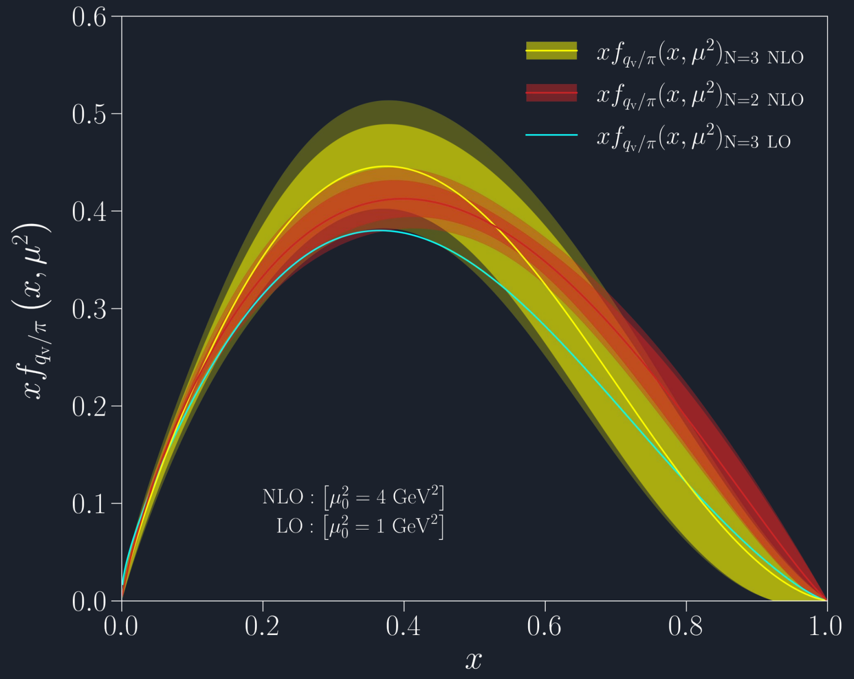

$N_{\rm param}$	α	β	δ	χ_r^2
2	$-0.17(7)_{\rm stat}(2)_{\rm sys}$	$1.24(22)_{\rm stat}(7)_{\rm sys}$	–	1.41
3	$-0.22(11)_{\rm stat}(3)_{\rm sys}$	$2.12(56)_{\rm stat}(14)_{\rm sys}$	$4.28(1.73)_{\rm stat}(25)_{\rm sys}$	1.29
3	$-0.34(31)_{\rm stat}$	$1.93(68)_{\rm stat}$	$3.05(2.50)_{\rm stat}$	2.2

Comparison with Experiment

$$T_1^{\pi}(\nu)_{\text{fit}} = \int_0^1 dx\ K_1^{\text{NLO}}\left(x\nu, z^2\mu^2\right) f_{q_v/\pi}\left(x, \mu^2\right)$$

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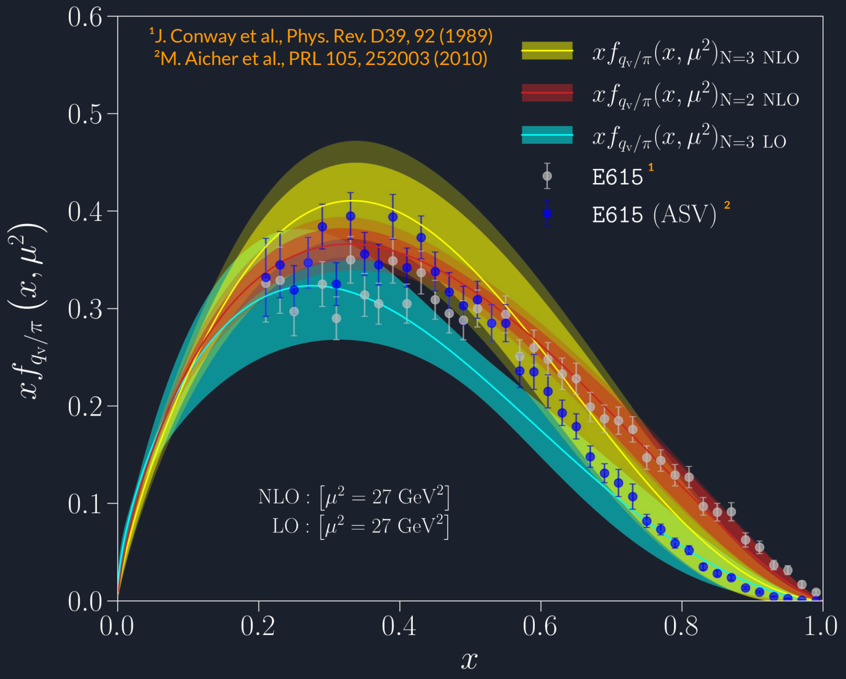

- ➢ Broadly consistent w/ experiment
  - ○ 3-parameter NLO fit structurally similar to ASV reanalysis (soft-gluons in partonic cross section)
- ➢ Importance of NLO kernel
- ➢ Higher Ioffe-times to discriminate functional forms

Matrix Elements of Non-Local Parton Bilinears

A matrix element of a distinct character

$$M^{\alpha}\left(p,z\right)=\langle h\left(p\right)|\,\overline{\psi}\left(z\right)\gamma^{\alpha}\Phi_{\hat{z}}^{(f)}\left(\{z,0\}\right)\psi\left(0\right)|h\left(p\right)\rangle=2p^{\alpha}\mathcal{M}\left(\nu,z^{2}\right)+2z^{\alpha}\mathcal{N}\left(\nu,z^{2}\right)$$

$$\nu\equiv p\cdot z$$

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A

Light-cone PDF

Unpolarized leading-twist PDF
defined in terms of $k^-, \mathbf{k}_\perp$
integrated parton correlator

$$p^\alpha = \left(p^+, \frac{m_h^2}{2p^+}, \mathbf{0}_\perp\right)$$

$$z^\alpha = (0, z^-, \mathbf{0}_\perp) \qquad \alpha = +$$

Ioffe-time Distribution (ITD)

$$\mathcal{M}(p^+z^-,0)_{\mu^2} \equiv Q(\nu,\mu^2) = \int_{-1}^{1} dx\; e^{i\nu x} f_{q/h}(x,\mu^2)$$

V. Braun et al., Phys.Rev.D 51 (1995) 6036-6051

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V. Braun et al., Phys.Rev.D 51 (1995) 6036-6051

Pseudo-PDF

Generalization of light-cone PDFs onto space-like intervals; Lorentz covariant parton momentum fraction

Frame amenable to calculation in Lattice QCD

$$p^\alpha = (E, \mathbf{0}_\perp, p_z)$$

$$z^\alpha = (0, \mathbf{0}_\perp, z_3) \quad \alpha = 0$$

Ioffe-time Pseudo-distribution (pseudo-ITD)

$$\mathcal{M}(p_z z_3, z_3^2) = \int_{-1}^{1} dx\ e^{i\nu x}\mathcal{P}(x, z_3^2)$$

A. Radyushkin, Phys.Rev.D 96 (2017) 3, 034025

More on Pseudo-ITDs

Pseudo-ITD has support
only on canonical interval

A. Radyushkin, Phys.Lett.B 767 (2017) 314-320

$$\mathcal{M}\left(p_z z_3, z_3^2\right) = \int_{-1}^{1} dx\ e^{i\nu x} \mathcal{P}\left(x, z_3^2\right)$$

$$\Downarrow$$

FT along constant z_3

$$\mathcal{P}\left(x, z_3^2\right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\nu\ e^{-i\nu x} \boxed{\mathcal{M}\left(\nu, z_3^2\right)}$$

pseudo-PDF

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A. Radyushkin, Phys.Lett.B 767 (2017) 314-320

v. Quasi-PDFs

$$\tilde{f}_{a/h}\left(\tilde{x}, p_z, \tilde{\mu}^2\right) = \int \frac{dz_3}{4\pi} e^{-i\tilde{x}p_z z_3} M_4\left(p_z, z_3\right)$$

$$\mathcal{M}\left(p_z z_3, z_3^2\right) = \int_{-1}^{1} dx\; e^{i\nu x} \mathcal{P}\left(x, z_3^2\right)$$

$$\Downarrow \quad \text{FT along constant } z_3$$

$$\mathcal{P}\left(x, z_3^2\right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\nu\; e^{-i\nu x} \boxed{\mathcal{M}\left(\nu, z_3^2\right)}$$

pseudo-PDF

Space-like Wilson line acquires additional UV divergences

A. Polyakov, Nucl.Phys.B 164 (1980) 171-188

$$Z_{\text{link}}\left(z_3, a\right) \simeq e^{-A|z_3|/a}$$

- Ioffe-time independent

V.S. Dotsenko, S.N. Vergeles, Nucl.Phys.B 169 (1980) 527-546

- Multiplicatively renormalizable

T. Ishikawa et al., Phys.Rev.D 96 (2017) 9, 094019
X. Ji et al., Phys.Rev.Lett. 120 (2018) 11, 112001
J. Green et al., Phys.Rev.Lett. 121 (2018) 2, 022004

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A. Radyushkin, Phys.Lett.B 767 (2017) 314-320

v. Quasi-PDFs

$$\tilde{f}_{a/h}\left(\tilde{x}, p_z, \tilde{\mu}^2\right) = \int \frac{dz_3}{4\pi} e^{-i\tilde{x}p_z z_3} M_4\left(p_z, z_3\right)$$

$$\mathcal{M}\left(p_z z_3, z_3^2\right) = \int_{-1}^{1} dx\; e^{i\nu x} \mathcal{P}\left(x, z_3^2\right)$$

$\Downarrow$ FT along constant z_3

$$\mathcal{P}\left(x, z_3^2\right) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\nu\; e^{-i\nu x} \mathcal{M}\left(\nu, z_3^2\right)$$

pseudo-PDF

$\ln z_3^2$ contributions generate perturbative evolution of collinear PDFs

Space-like Wilson line acquires additional UV divergences

A. Polyakov, Nucl.Phys.B 164 (1980) 171-188

$$Z_{\text{link}}\left(z_3, a\right) \simeq e^{-A|z_3|/a}$$

- Ioffe-time independent

  V.S. Dotsenko, S.N. Vergeles, Nucl.Phys.B 169 (1980) 527-546

- Multiplicatively renormalizable

T. Ishikawa et al., Phys.Rev.D 96 (2017) 9, 094019
X. Ji et al., Phys.Rev.Lett. 120 (2018) 11, 112001
J. Green et al., Phys.Rev.Lett. 121 (2018) 2, 022004

$$\mathfrak{M}\left(\nu, z^2\right) = \left\{\delta\left(1-u\right) - \frac{\alpha_s C_F}{2\pi} \int_0^1 du \left[\ln\left(\frac{e^{2\gamma_E+1} z^2 \mu^2}{4}\right) B\left(u\right) + L\left(u\right)\right]\right\} \mathcal{Q}\left(u\nu, \mu^2\right) + \mathcal{O}\left(z^2 \Lambda_{\rm QCD}^2\right)$$

Nucleon Pseudo-ITDs - Numerical Study

Distillation - chosen spatial smearing scheme

M. Peardon et al., Phys. Rev. D80, 054506 (2009)

- low-rank approximation of a gauge-covariant smearing kernel

$$J_{\sigma,n_\sigma} = e^{\sigma\nabla^2} = \sum_\lambda e^{-\sigma\lambda} |\lambda\rangle\langle\lambda|$$

$$\square\left(\vec{x},\vec{y};t\right)_{ab} = \sum_{k=1}^{R_{\mathcal{D}}} \xi_a^{(k)}\left(\vec{x},t\right) \xi_b^{(k)\dagger}\left(\vec{y},t\right)$$

Nucleon Pseudo-ITDs - Numerical Study

$$C_{mn}(t) = \sum_{\vec{x},\vec{y}} \langle 0 | \mathcal{O}_m(t,\vec{x}) \, \mathcal{O}_n^\dagger(0,\vec{y}) | 0 \rangle$$

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$$\equiv \mathrm{Tr}\left[\Phi_m\left(t\right)\otimes\tau\left(t,0\right)\tau\left(t,0\right)\tau\left(t,0\right)\otimes\Phi_n\left(0\right)\right]$$

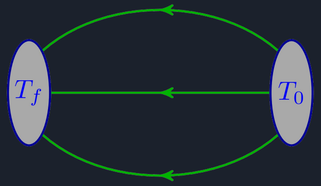
"Perambulators"

$$\tau_{\alpha\beta}^{kl}\left(t_f,t_0\right) = \xi^{(k)\dagger}\left(t_f\right)M_{\alpha\beta}^{-1}\left(t_f,t_0\right)\xi^{(l)}\left(t_0\right)$$

$$\Phi_{\mu\nu\sigma}^{(i,j,k)}\left(t\right) = \epsilon^{abc}\left(\mathcal{D}_1\xi^{(i)}\right)^a\left(\mathcal{D}_2\xi^{(j)}\right)^b\left(\mathcal{D}_3\xi^{(k)}\right)^c\left(t\right)S_{\mu\nu\sigma}$$

"Elementals"

Irrep. projection



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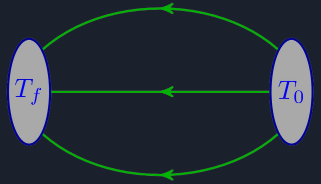
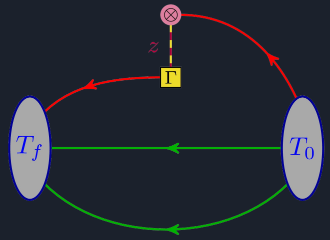


Distillation induces *expensive* generalized perambulator ("genprop")

$$\Xi_{\alpha\beta}^{(l,k)}(T_f,T_0;\tau,z_3) = \sum_{z_3} \xi^{(l)\dagger}(T_f)\,D_{\alpha\sigma}^{-1}(T_f,\tau)\boxed{\left[\gamma^4\right]_{\sigma\rho}}\boxed{\Phi_{\hat{z}_3}^{(f)}(\{z_3,0\})}\,D_{\rho\beta}^{-1}(\tau,T_0)\,\xi^{(k)}(T_0)$$

Space-like Wilson line

Unpolarized PDFs



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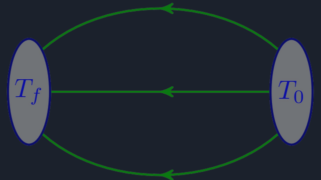

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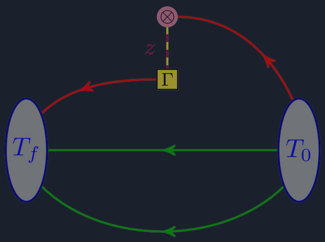
Unpolarized PDFs



JLab/WM/LANL 2+1 Flavor Isotropic Lattices

ID	a (fm)	m_π (MeV)	$L^3 \times N_t$	$N_{\rm cfg}$	$N_{\rm srcs}$	$N_{\rm vec}$
$a094m358$	0.094(1)	358(3)	$32^3 \times 64$	349	4	64

Parameters/Statistics

t_{sep}/a	p_z $\left(\times\frac{2\pi}{L}\right)$	z/a
$4, 6, \cdots, 14$	$0, \pm1, \cdots, \pm6$	$0, \pm1, \cdots, \pm12, \cdots$
$0.38, \cdots 1.32$ fm	$0, 0.411, \cdots, 2.47$ GeV	$0, 0.094, \cdots, 1.13$ fm

Obtaining the Ioffe-time Pseudo-Distribution

Correlation functions needed:

$$C_2\left(p_z, T\right) = \langle \mathcal{N}\left(-p_z, T\right) \overline{\mathcal{N}}\left(p_z, 0\right)\rangle = \sum_n \left|\mathcal{A}_n\right|^2 e^{-E_n T}$$

$$\begin{aligned} C_3\left(p_z, T, \tau; z_3\right) &= \sum_{z_3} \langle \mathcal{N}\left(-p_z, T\right) \mathring{\mathcal{O}}^{[\gamma_4]}_{\mathrm{WL}}\left(z_3, \tau\right) \overline{\mathcal{N}}\left(p_z, 0\right)\rangle \\ &= \sum_{n, n', z_3} \langle \mathcal{N} | n' \rangle \langle n | \overline{\mathcal{N}} \rangle \langle n' | \mathring{\mathcal{O}}^{[\gamma_4]}_{\mathrm{WL}}\left(z_3, \tau\right) | n \rangle e^{-E_{n'}(T-\tau)} e^{-E_n T} \end{aligned}$$

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$$\begin{aligned}C_3\left(p_z,T,\tau;z_3\right)&=\sum_{z_3}\langle\mathcal{N}\left(-p_z,T\right)\mathring{\mathcal{O}}^{[\gamma_4]}_{\mathrm{WL}}\left(z_3,\tau\right)\overline{\mathcal{N}}\left(p_z,0\right)\rangle\\&=\sum_{n,n',z_3}\langle\mathcal{N}|n'\rangle\langle n|\overline{\mathcal{N}}\rangle\langle n'|\,\mathring{\mathcal{O}}^{[\gamma_4]}_{\mathrm{WL}}\left(z_3,\tau\right)|n\rangle\,e^{-E_{n'}(T-\tau)}e^{-E_nT}\end{aligned}$$

Like two-currents, pseudo-distributions require a short-distance factorization

➢ high momenta!

C. Egerer, et al., *Distillation at High Momentum*, Phys. Rev. D 103 (2021) 3, 034502

Union of Distillation and Momentum Smearing ideas

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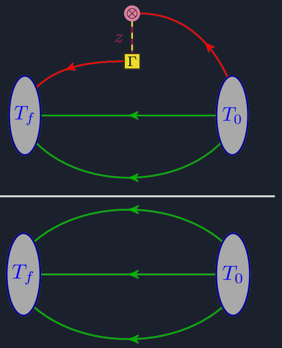
Matrix element from ratio of correlators



$$R\left(p_z,z_3;T\right)=\sum_{\tau/a=1}^{T-1}\frac{C_3\left(p_z,T,\tau;z_3\right)}{C_2\left(p_z,T\right)}$$

L. Maiani et al., Nucl. Phys. B293 (1987)
C. Bouchard et al., Phys. Rev. D 96, no. 1, 014504 (2017)

$$R_{\mathrm{fit}}\left(p_z,z_3;T\right)=\mathcal{A}+M_4\left(p_z,z_3\right)T+\mathcal{O}\left(e^{-\Delta ET}\right)$$

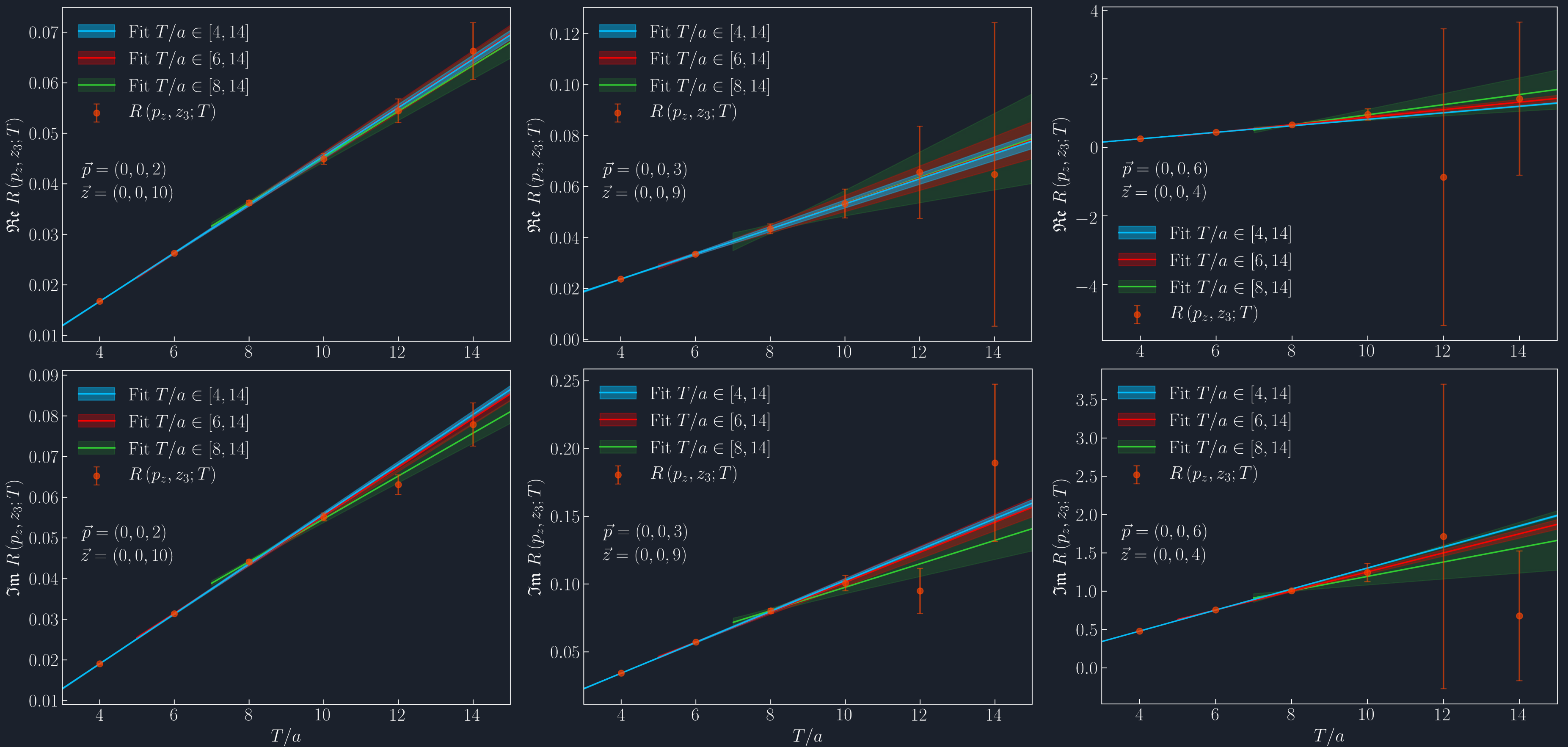
Construct *Reduced Distribution* (reduced pseudo-ITD)

K. Orginos, et al., Phys. Rev. D96, 094503 (2017)

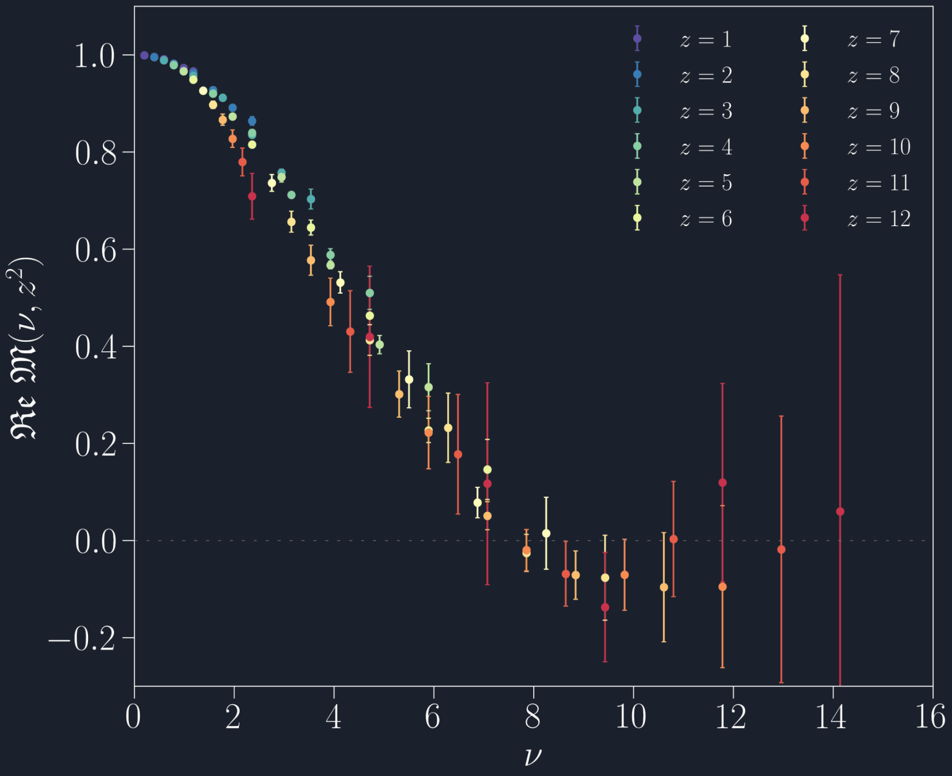
$$\begin{aligned}\mathfrak{M}\left(\nu,z_3^2\right)&\equiv\frac{\mathcal{M}_p\left(\nu,z_3^2\right)}{\mathcal{M}_p\left(0,z_3^2\right)}\\&=\underbrace{\left(\frac{\mathcal{M}_p\left(\nu,z_3^2\right)}{\mathcal{M}_p\left(\nu,0\right)|_{z_3=0}}\right)\times\left(\frac{\mathcal{M}_p\left(0,0\right)|_{p=0,z_3=0}}{\mathcal{M}_p\left(0,z_3^2\right)|_{p=0}}\right)}_{\texttt{RGI!}}\end{aligned}$$

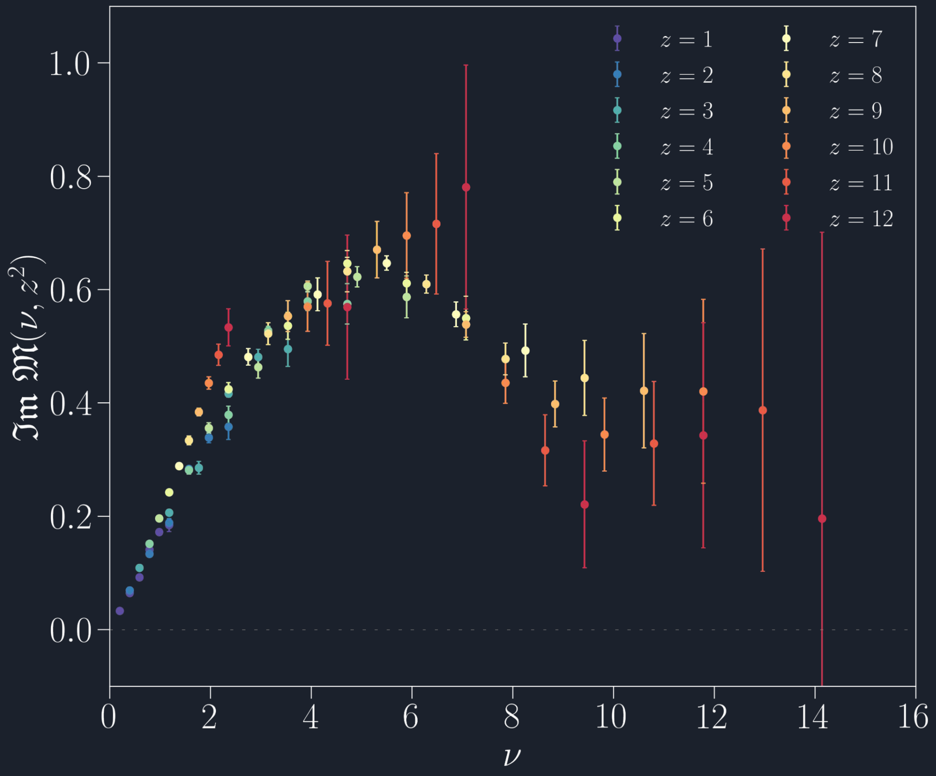
Local vector current in zero sep. limit
(not conserved)

Selected Matrix Element Extractions



Unpolarized Ioffe-time Pseudo-Distribution





Evolution and Scheme Conversion

Matching reduced pseudo-ITD to ITD requires a continuous description

$$B(u) = \left(\frac{1+u^2}{1-u}\right)_+ \qquad L(u) = \left[4\frac{\ln(1-u)}{1-u} - 2(1-u)\right]_+$$

$$\mathcal{Q}(\nu,\mu^2) = \mathfrak{M}(\nu,z^2) + \frac{\alpha_s C_F}{2\pi}\int_0^1 du\left[\ln\left(\frac{e^{2\gamma_E+1}z^2\mu^2}{4}\right)\boxed{B(u)} + \boxed{L(u)}\right]\mathfrak{M}(u\nu,z^2)$$

T. Izubuchi, et al., Phys.Rev. D98 (2018) no.5, 056004
A. Radyushkin, Phys.Lett. B781 (2018) 433-442
A. Radyushkin, Phys. Rev. D 98 (2018) no.1, 014019
J.-H. Zhang, et al., Phys.Rev. D97 (2018) no.7, 074508

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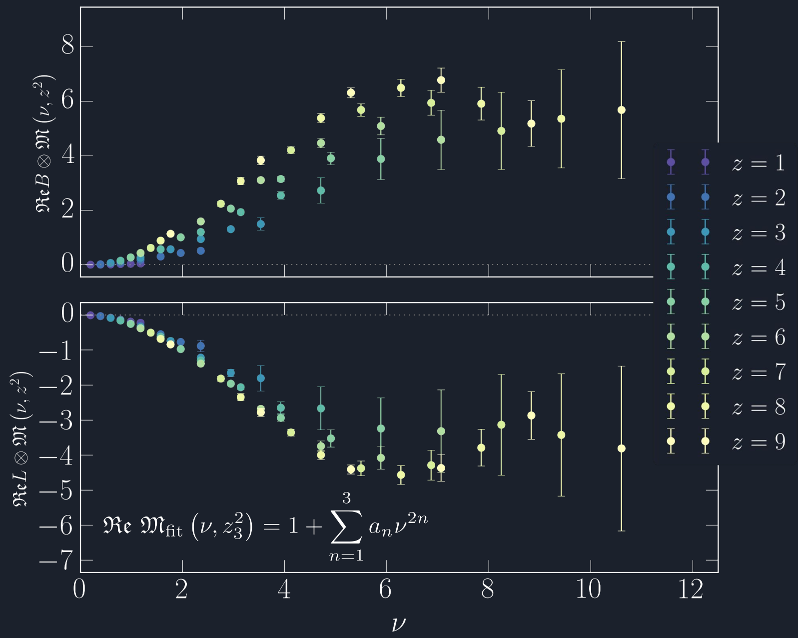
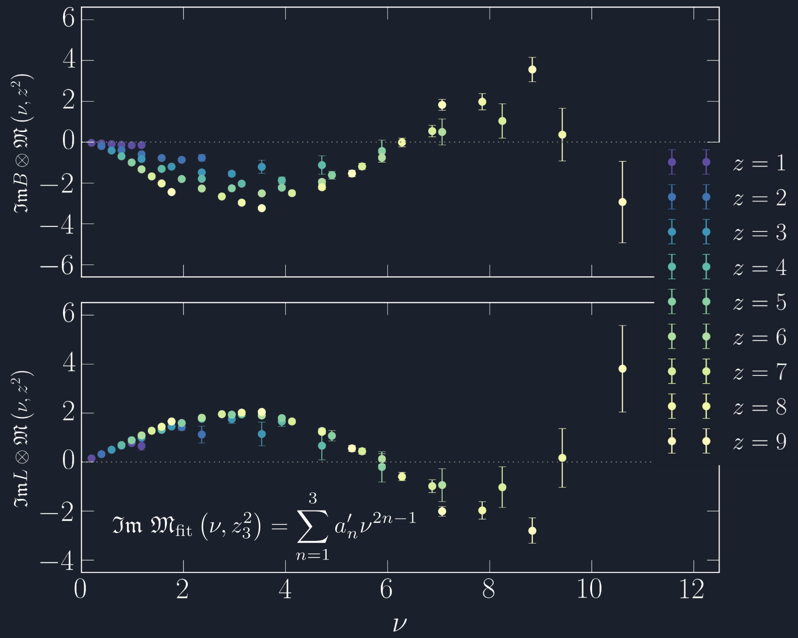
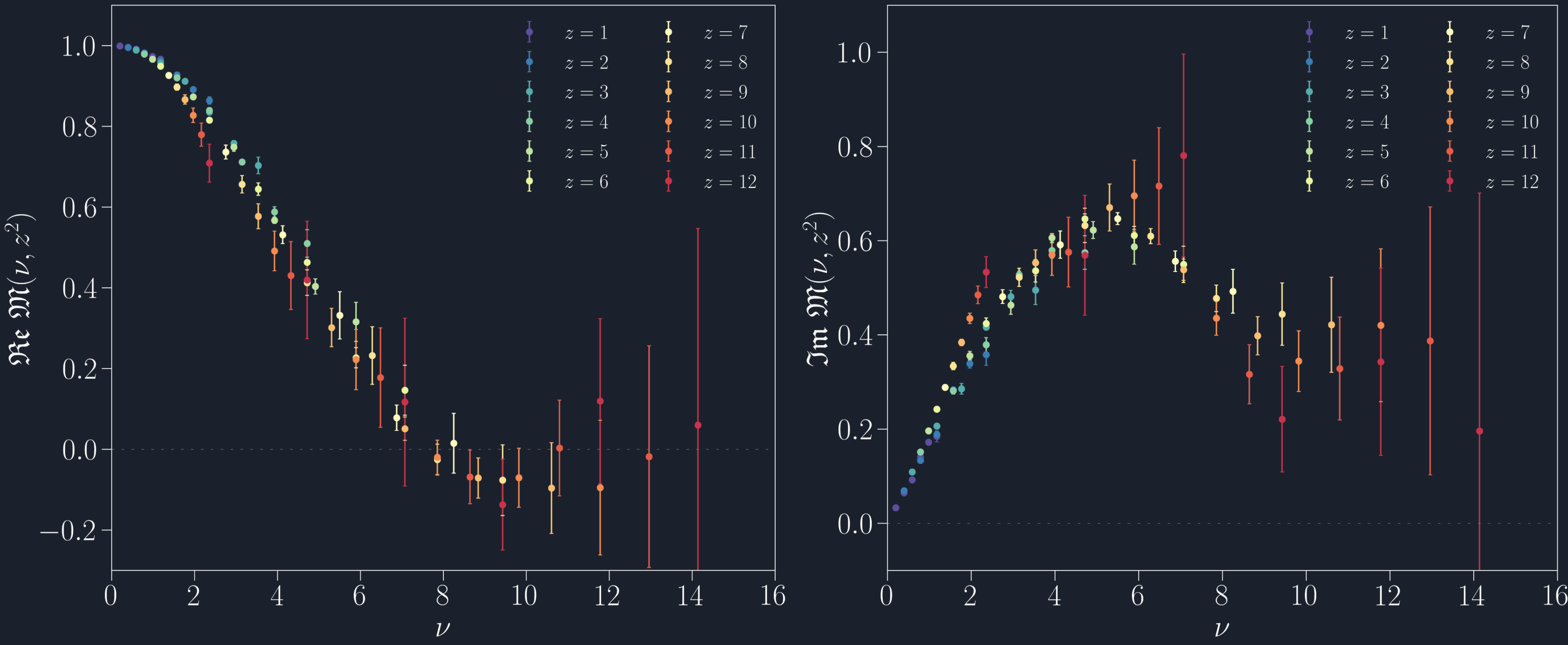




Unpolarized Ioffe-time Pseudo-Distribution



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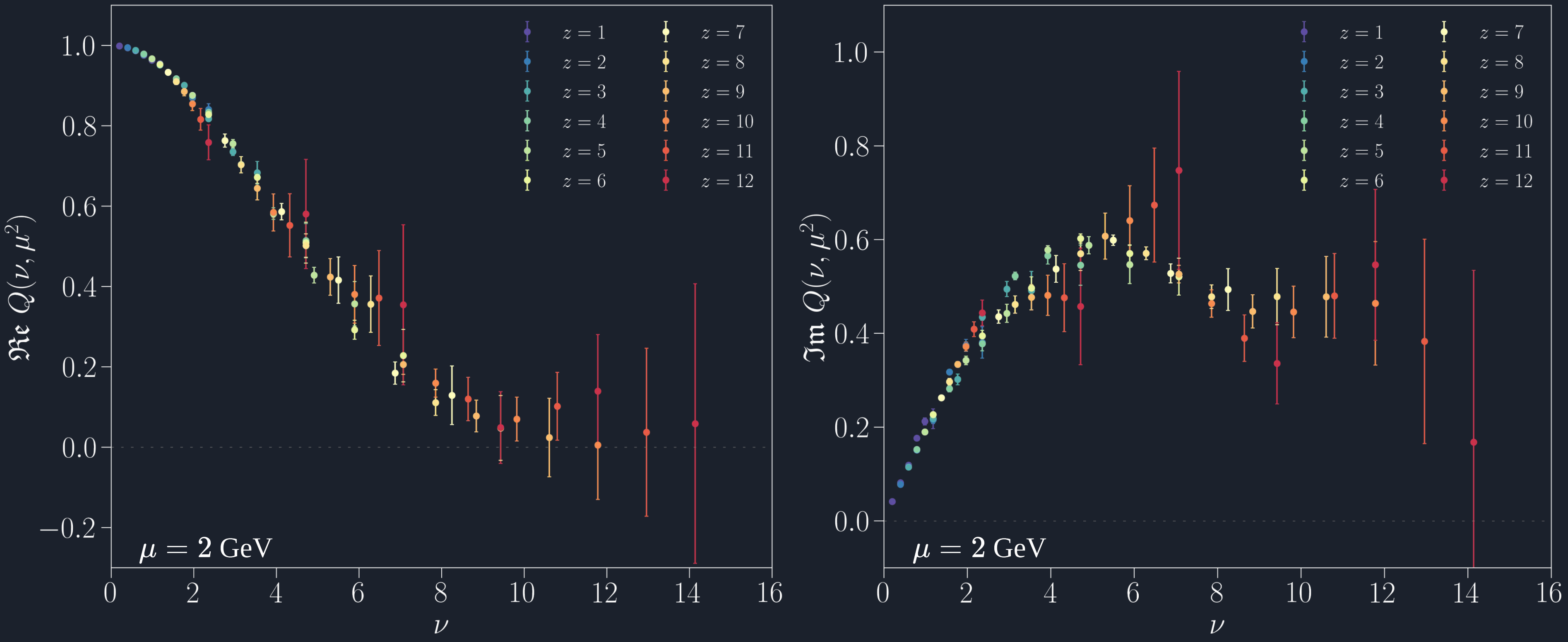
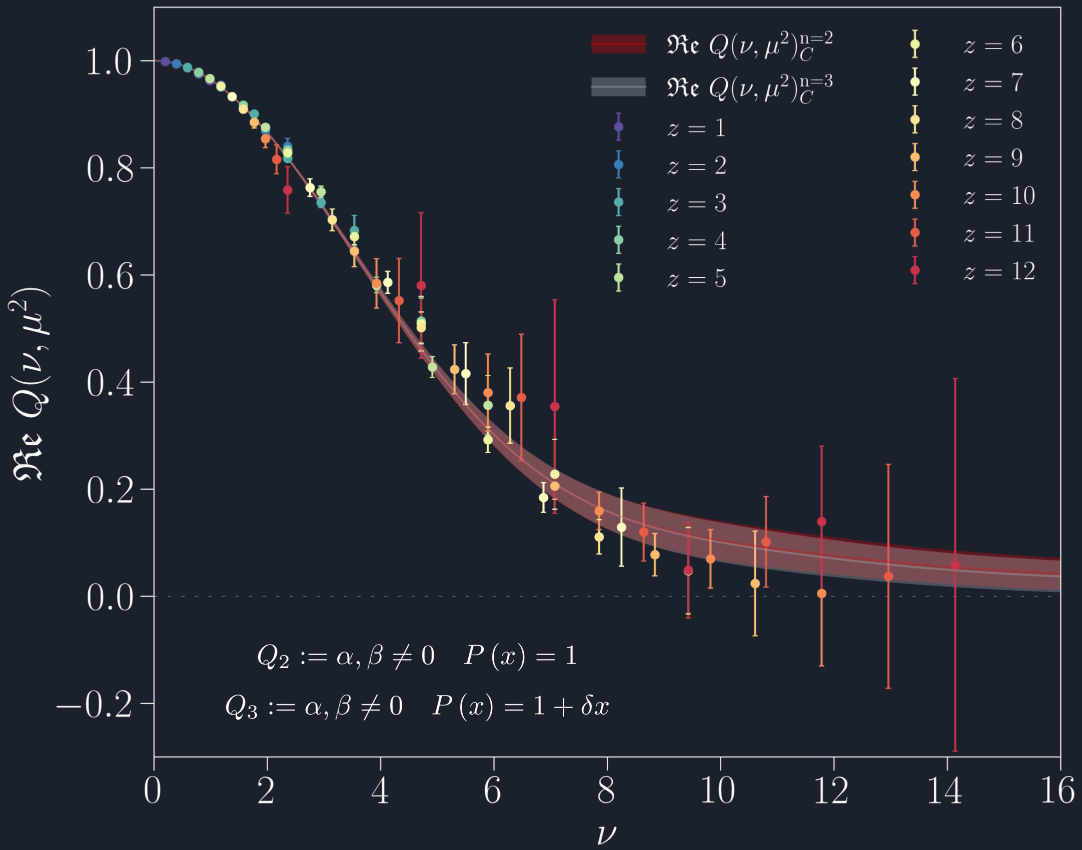


PDFs from Ioffe-time Distribution Fits



$$\mathfrak{Re}\ Q\left(\nu,\mu^2\right) = \int_0^1 dx \cos\left(\nu x\right) q_{\mathrm{v}}\left(x,\mu^2\right)$$

Ill-posed ITD - PDF relation

A) Supply extra physically motivated information

B) Parametric fits (model bias - i.e. functional forms & at what stage)

$$q_{\mathrm{v}}\left(x\right) = N_{\mathrm{v}} x^{\alpha}\left(1-x\right)^{\beta} P\left(x\right)$$

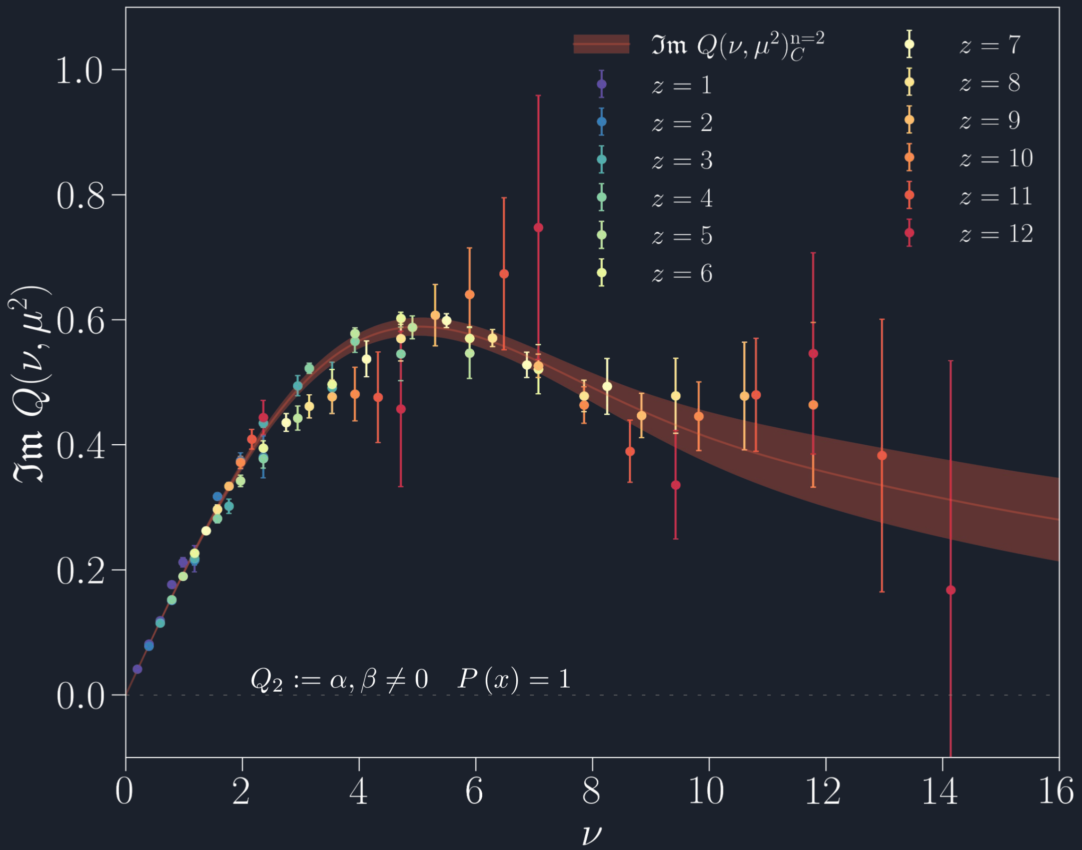
C) Smooth function to connect nominal behavior

$$P\left(x\right) = 1 + \sum_k \lambda_k x^{(k+1)/2}$$

$$N_{\mathrm{v}} = B\left(\alpha+1,\beta+1\right) + \sum_k \lambda_k B\left(\alpha+1+\frac{k+1}{2},\beta+1\right)$$

D) Least-squares fit to matched ITD

PDFs from Ioffe-time Distribution Fits



$$\mathfrak{Im}\ Q\left(\nu,\mu^2\right) = \int_0^1 dx \sin\left(\nu x\right) q_+\left(x,\mu^2\right)$$

Ill-posed ITD - PDF relation

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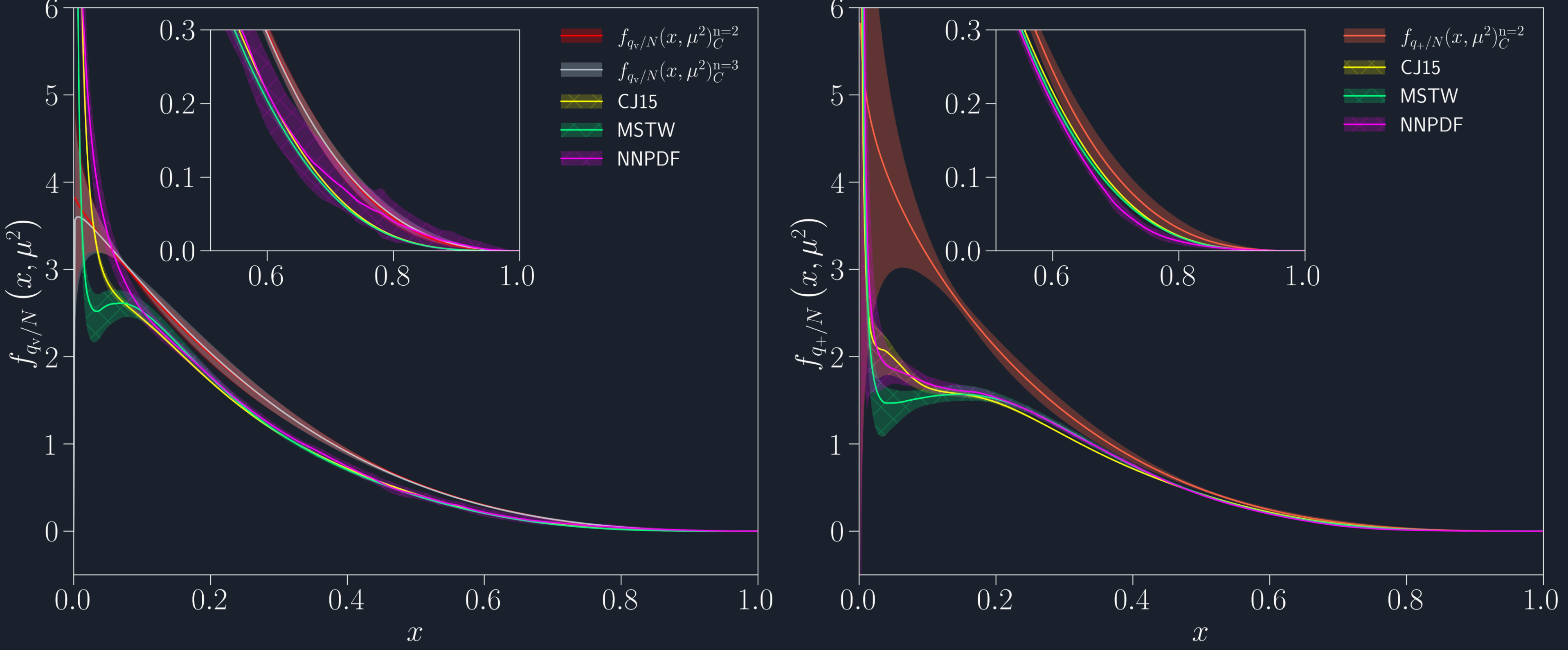
PDFs and Phenomenological Comparison



Summary and Outlook

Hadronic structure accessible from certain lattice calculable matrix elements

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Nucleon valence (plus) quark PDF

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Nucleon valence (plus) quark PDF

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Each faces an inverse problem

- (most common) regularization through parametric forms
- (more sophisticated) parameterize systematic effects simultaneously

C. E., R. Edwards, C. Kallidonis, et al., *Towards High-Precision Parton Distributions From Lattice QCD via Distillation*, To Appear Soon.

Thank You!

Distillation

M. Peardon et al., Phys. Rev. D80, 054506 (2009)

Low-rank and *non-iterative* approximation of a gauge-covariant smearing kernel

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Wick contract *distilled* (smeared) fields

$$C_{mn}\left(t\right) = \sum_{\vec{x},\vec{y}} \langle 0|\,\mathcal{O}_m\left(t,\vec{x}\right)\mathcal{O}_n^\dagger\left(0,\vec{y}\right)|0\rangle$$

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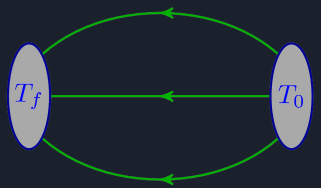


"Perambulators"

$$\tau_{\alpha\beta}^{kl}\left(t_f,t_0\right) = \xi^{(k)\dagger}\left(t_f\right)M_{\alpha\beta}^{-1}\left(t_f,t_0\right)\xi^{(l)}\left(t_0\right)$$

$$\Phi_{\mu\nu\sigma}^{(i,j,k)}\left(t\right) = \epsilon^{abc}\left(\mathcal{D}_1\xi^{(i)}\right)^a\left(\mathcal{D}_2\xi^{(j)}\right)^b\left(\mathcal{D}_3\xi^{(k)}\right)^c\left(t\right)S_{\mu\nu\sigma}$$

"Elementals"

Irrep. projection

Distillation

M. Peardon et al., Phys. Rev. D80, 054506 (2009)

Low-rank and *non-iterative* approximation of a gauge-covariant smearing kernel

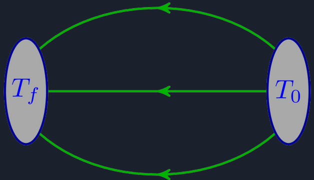
$$J_{\sigma,n_\sigma} = e^{\sigma\nabla^2} = \sum_\lambda e^{-\sigma\lambda}\,|\lambda\rangle\langle\lambda|$$

$$\square\left(\vec{x},\vec{y};t\right)_{ab} = \sum_{k=1}^{R_{\mathcal{D}}} \xi_a^{(k)}\left(\vec{x},t\right)\xi_b^{(k)\dagger}\left(\vec{y},t\right)$$

Wick contract *distilled* (smeared) fields

$$C_{mn}\left(t\right) = \sum_{\vec{x},\vec{y}} \langle 0|\,\mathcal{O}_m\left(t,\vec{x}\right)\mathcal{O}_n^\dagger\left(0,\vec{y}\right)|0\rangle$$

$$\equiv \mathrm{Tr}\left[\Phi_m\left(t\right)\otimes\tau\left(t,0\right)\tau\left(t,0\right)\tau\left(t,0\right)\otimes\Phi_n\left(0\right)\right]$$



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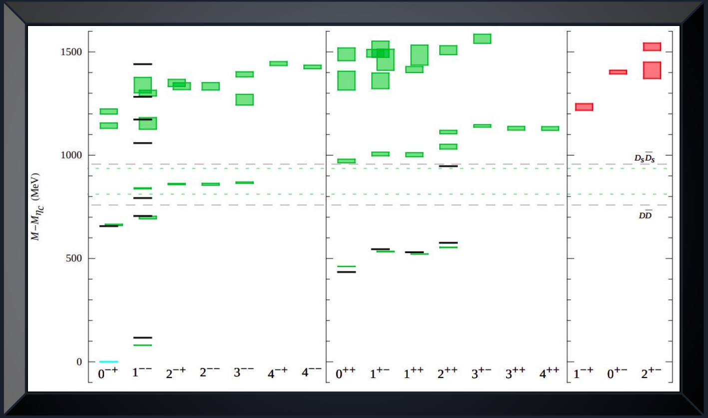

L. Liu, et. al., JHEP 07, (2012) 126

Admits efficient implementation of variational method

➔ low-lying meson spectrum/exotic hadrons

R. Briceno et al., Phys.Rev.D 97 (2018) 5, 054513
J. Dudek et. al., Phys.Rev.D 88 (2013) 9, 094505
J. Dudek et al., Phys.Rev.D 87 (2013) 3, 034505
J. Dudek, et. al., Phys. Rev.D83, 111502 (2011)
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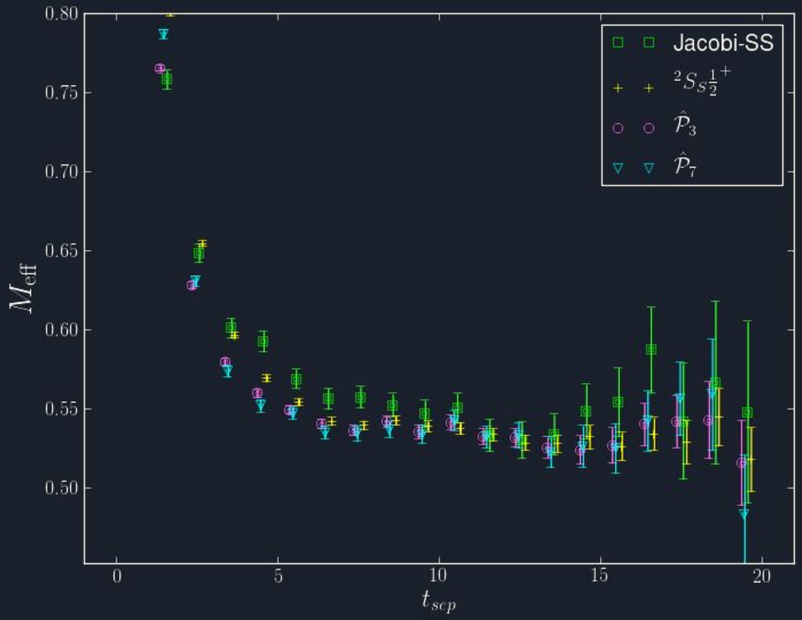
Distillation and Momentum Smearing

Distillation affords improvement over conventional smearing kernels



C.E., D. Richards, F. Winter, Phys. Rev. D 99 (2019) 3, 034506

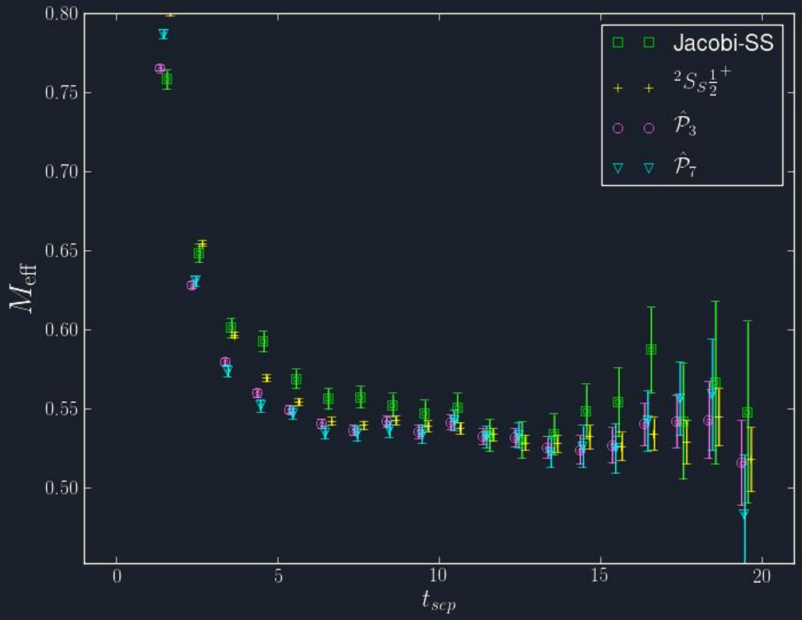
Distillation and Momentum Smearing

Boost to
high-momenta

Distillation affords improvement over conventional smearing kernels

Limited utility in structure studies without momentum smearing idea

Dense spectrum
Further broken symmetries



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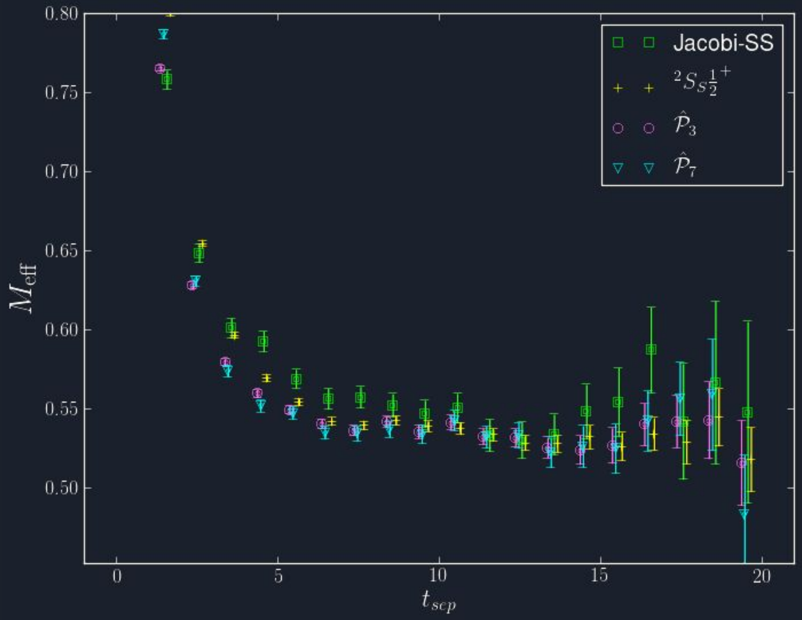

C.E., D. Richards, F. Winter, Phys. Rev. D 99 (2019) 3, 034506

Requirements of any modification

1. Preserve symmetries of lattice & resultant little groups
2. Minimize number of additional eigenvector bases

$$\tilde{\xi}_a^{(k)}\left(\vec{z},t\right) = e^{i\vec{\zeta}\cdot\vec{z}}\xi_a^{(k)}\left(\vec{z},t\right)$$

$$\vec{\zeta} = \frac{2\pi}{L}\hat{z}$$

$$\vec{\zeta} = 2\cdot\frac{2\pi}{L}\hat{z}$$

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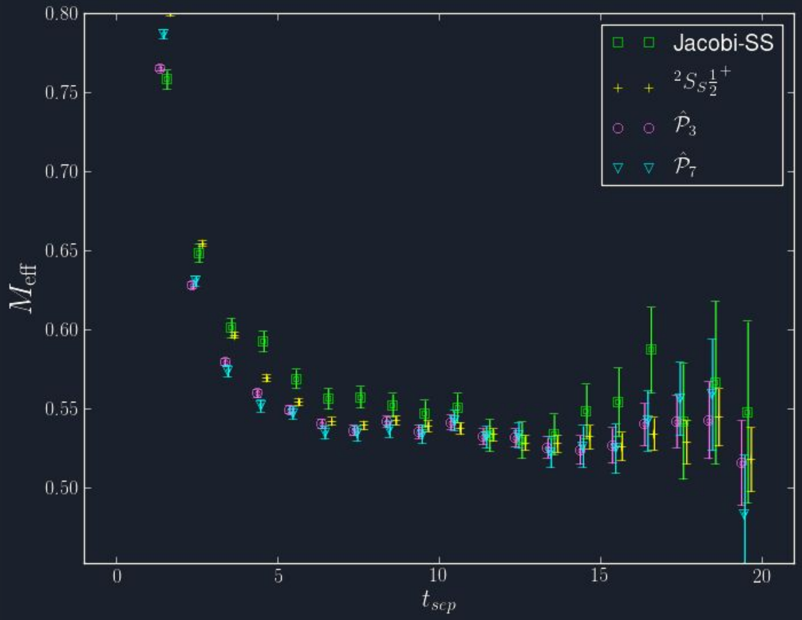

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C. Egerer, et al., *Distillation at High Momentum*, Phys. Rev. D 103 (2021) 3, 034502

Union of Distillation and Momentum is feasible

Phasing/GEVP improves boosted overlaps

Regularization via Orthogonal Polynomials

Arise as solutions to many differential equations

- span a space of functions (e.g Fourier series)
- phenomenology MMHT/CT

  L.A. Harland-Lang et al., Eur. Phys. J. C75, 204 (2015)
- distribution amplitudes

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Jacobi (hypergeometric) polynomials

$$P_n^{(\alpha,\beta)}(z) = \frac{\Gamma(\alpha+n+1)}{n!\Gamma(\alpha+\beta+n+1)}\sum_{j=0}^{n}\binom{n}{j}\frac{\Gamma(\alpha+\beta+n+j+1)}{\Gamma(\alpha+j+1)}\left(\frac{z-1}{2}\right)^j$$

$z \in [-1,1]$ Interval

$(1-z)^\alpha (1+z)^\beta$ Metric

$\alpha, \beta > -1$ Validity

$$\int_{-1}^{1} dz\,(1-z)^\alpha (1+z)^\beta P_n^{(\alpha,\beta)}(z) P_m^{(\alpha,\beta)}(z) = \delta_{n,m} h_n(\alpha,\beta)$$

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A convenient change of variables: $z \mapsto 1-2x$

$$\Omega_n^{(\alpha,\beta)}(x) = \sum_{j=0}^{n} \underbrace{\frac{\Gamma(\alpha+n+1)}{n!\Gamma(\alpha+\beta+n+1)} \binom{n}{j} \frac{(-1)^j\,\Gamma(\alpha+\beta+n+j+1)}{\Gamma(\alpha+j+1)}}_{\omega_{n,j}^{(\alpha,\beta)}} x^j$$

$x \in [0,1]$ — Interval

$x^\alpha (1-x)^\beta$ — Metric

$\alpha, \beta > -1$ — Validity

Flexibility of PDF functional form captured without bias via $\{\Omega_n^{(\alpha,\beta)}\}$

$$f_{q/h}(x) = x^\alpha (1-x)^\beta \sum_{n=0}^{\infty} C_{q,n}^{(\alpha,\beta)} \Omega_n^{(\alpha,\beta)}(x)$$

Regularization via Orthogonal Polynomials

Model-independent expansion must be truncated

- bias introduced
  - ideally bias less than pheno. forms
- study this bias:
  - fix truncation orders, find optimum $\{\alpha, \beta\}$
  - fix basis $\{\alpha, \beta\}$, optimize truncation

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Establish direct matching between reduced pseudo-ITD and PDF

➢ avoid bias/artifacts in evolution/matching (e.g. polynomial fit of pseudo-ITD)

$$\sigma_n^{(\alpha,\beta)}\left(\nu, z^2\mu^2\right) = \int_0^1 dx\, \mathcal{K}_{\mathrm{v}}\left(x\nu, z^2\mu^2\right) x^\alpha \left(1-x\right)^\beta \Omega_n^{(\alpha,\beta)}\left(x\right)$$

$$\eta_n^{(\alpha,\beta)}\left(\nu, z^2\mu^2\right) = \int_0^1 dx\, \mathcal{K}_{+}\left(x\nu, z^2\mu^2\right) x^\alpha \left(1-x\right)^\beta \Omega_n^{(\alpha,\beta)}\left(x\right)$$

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$$\sigma_n^{(\alpha,\beta)}\left(\nu, z^2\mu^2\right) = \sum_{j=0}^{n}\sum_{k=0}^{\infty} \frac{(-1)^k}{(2k)!} c_{2k}\left(z^2\mu^2\right) \omega_{n,j}^{(\alpha,\beta)} B\left(\alpha+2k+j+1, \beta+1\right)\nu^{2k}$$

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$$c_n\left(z^2\mu^2\right) = 1 - \frac{\alpha_s C_F}{2\pi}\left[\gamma_n \ln\left(\frac{e^{2\gamma_E+1}}{4} z^2\mu^2\right) + d_n\right]$$

Leading moments of the Altarelli-Parisi kernel

Leading moments of the scheme matching kernel

J. Karpie, K. Orginos, S. Zafeiropoulos, JHEP 11, 178 (2018)

Parameterization of Systematic Effects

Reduced pseudo-ITD expansion in Jacobi polynomials

$$\mathfrak{Re}\,\mathfrak{M}^{lt}\left(\nu, z^2\right) = \sum_{n=0}^{\infty} \sigma_n^{(\alpha,\beta)}\left(\nu, z^2\mu^2\right) C_{\mathrm{v},n}^{lt\,(\alpha,\beta)}$$

$$\mathfrak{Im}\,\mathfrak{M}^{lt}\left(\nu, z^2\right) = \sum_{n=0}^{\infty} \eta_n^{(\alpha,\beta)}\left(\nu, z^2\mu^2\right) C_{+,n}^{lt\,(\alpha,\beta)}$$

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- subject to systematic corrections
  - discretization effects
  - higher-twist effects

Selection $\{\alpha, \beta\}$ is merely a choice of basis

- any contaminating effects describable by same basis

$$\mathfrak{Re}\ \mathfrak{M}^{corr}\left(\nu, z^2\right) = \boxed{\kappa_{corr}} \sum_{n=1}^{\infty} \sigma_{0,n}^{(\alpha,\beta)} C_{\mathrm{v},n}^{corr\ (\alpha,\beta)}$$

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$\mathcal{O}\left(a/z\right)$ $\mathcal{O}\left(z^2\Lambda_{\mathrm{QCD}}^2\right)^n$

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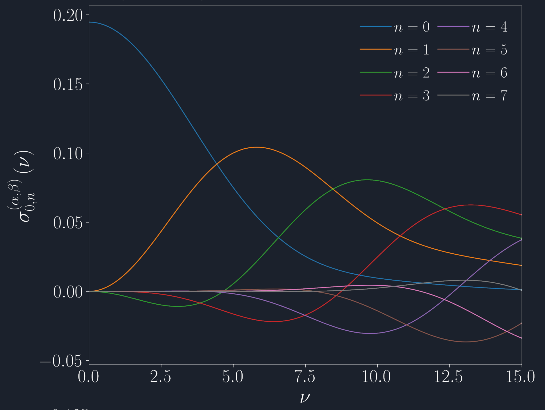

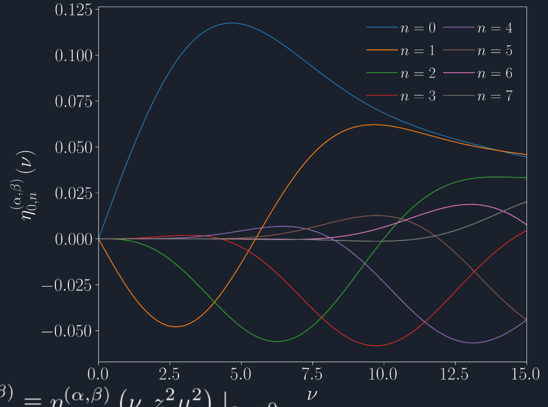

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Jacobi Polynomial Parameterization with Corrections

$$\mathfrak{Re}\,\mathfrak{M}_{\text{fit}}\left(\nu, z^2\right) = \sum_{n=0}^{\infty} \sigma_n^{(\alpha,\beta)}\left(\nu, z^2\mu^2\right) C_{\text{v},n}^{lt\,(\alpha,\beta)} + \left(\frac{a}{z}\right) \sum_{n=1}^{\infty} \sigma_{0,n}^{(\alpha,\beta)}(\nu)\, C_{\text{v},n}^{az\,(\alpha,\beta)}$$
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Discretization

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Discretization

Twist-4

- leading correction to factorization

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Jacobi Polynomial Parameterization with Corrections

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Twist-6

- ➢ large effect at large Ioffe-times
- ➢ likely beyond data

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Strategy of parametric fits with Jacobi polynomials

1. fix order of truncation, search for optimal expansion coefficients
2. establish polynomial hierarchy
   a. preference given to low-order polynomials
   b. restrict x-space contaminating distributions to be sub-leading to leading-twist PDF
   c. Bayesian priors (gaussian)
3. separability of non-linear optimization

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$$+ z^2\Lambda_{\text{QCD}}^2 \sum_{n=1}^{\infty} \sigma_{0,n}^{(\alpha,\beta)}\left(\nu\right) C_{\text{v},n}^{t4\,(\alpha,\beta)} + z^4\Lambda_{\text{QCD}}^4 \sum_{n=1}^{\infty} \sigma_{0,n}^{(\alpha,\beta)}\left(\nu\right) C_{\text{v},n}^{t6\,(\alpha,\beta)}$$

$$\mathfrak{Im}\,\mathfrak{M}_{\text{fit}}\left(\nu, z^2\right) = \sum_{n=0}^{\infty} \eta_n^{(\alpha,\beta)}\left(\nu, z^2\mu^2\right) C_{+,n}^{lt\,(\alpha,\beta)} + \left(\frac{a}{z}\right) \sum_{n=0}^{\infty} \eta_{0,n}^{(\alpha,\beta)}\left(\nu\right) C_{+,n}^{az\,(\alpha,\beta)}$$
$$+ z^2\Lambda_{\text{QCD}}^2 \sum_{n=0}^{\infty} \eta_{0,n}^{(\alpha,\beta)}\left(\nu\right) C_{+,n}^{t4\,(\alpha,\beta)} + z^4\Lambda_{\text{QCD}}^4 \sum_{n=0}^{\infty} \eta_{0,n}^{(\alpha,\beta)}\left(\nu\right) C_{+,n}^{t6\,(\alpha,\beta)}$$

Strategy of parametric fits with Jacobi polynomials

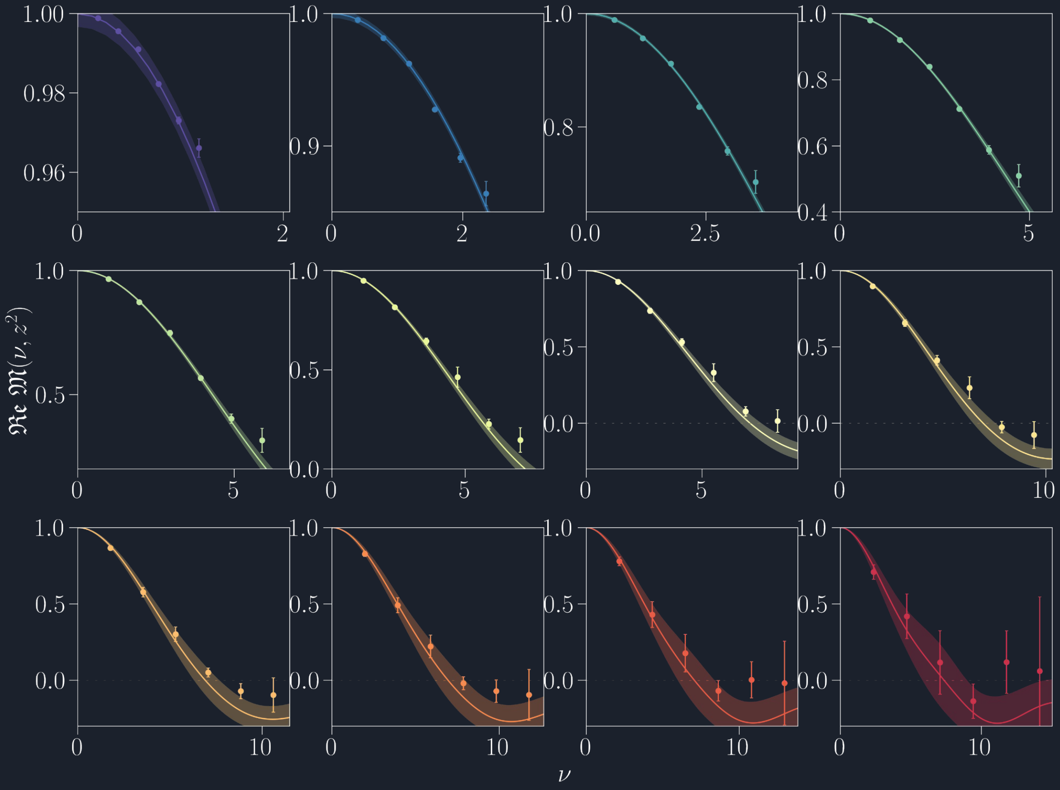
1. fix order of truncation, search for optimal expansion coefficients
2. establish polynomial hierarchy
   a. preference given to low-order polynomials
   b. restrict x-space contaminating distributions to be sub-leading to leading-twist PDF
   c. Bayesian priors (gaussian)
3. separability of non-linear optimization

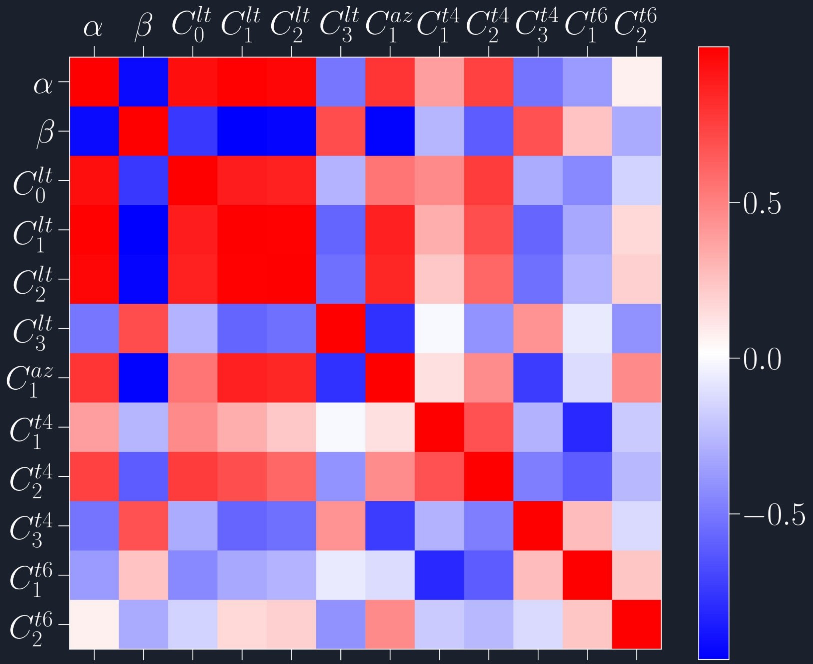
$$\Downarrow$$

Jacobi polynomial basis are only non-linear terms
Separable non-linear optimization $\longrightarrow$ variable projection

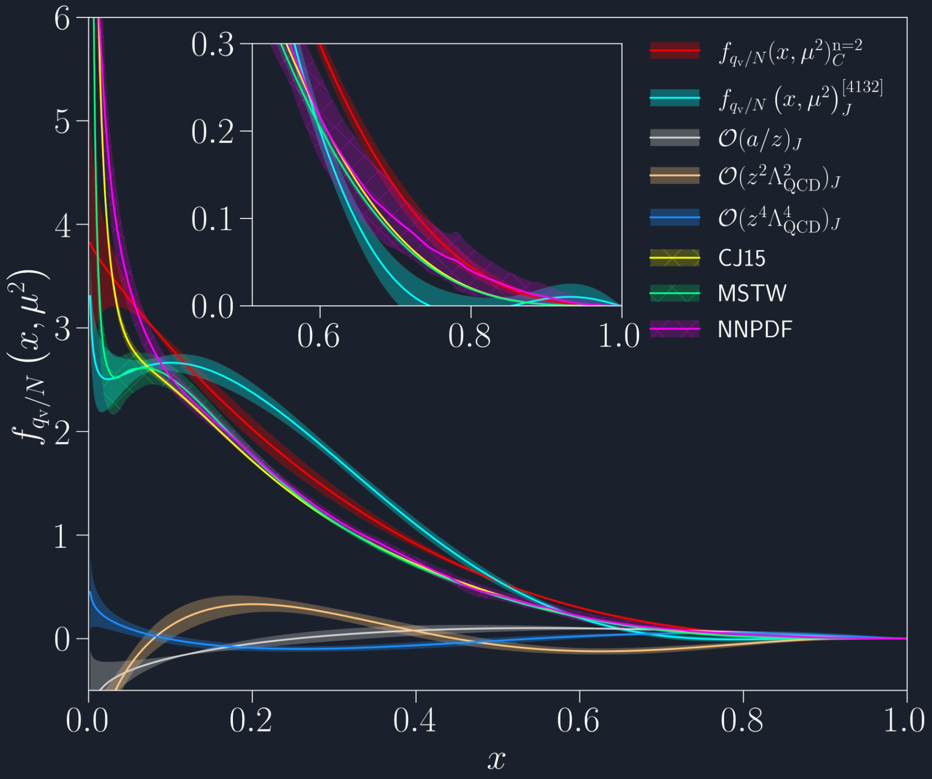
G. Golub and V. Pereyra, SIAM Journal on Numerical Analysis 10, 413 (1973)

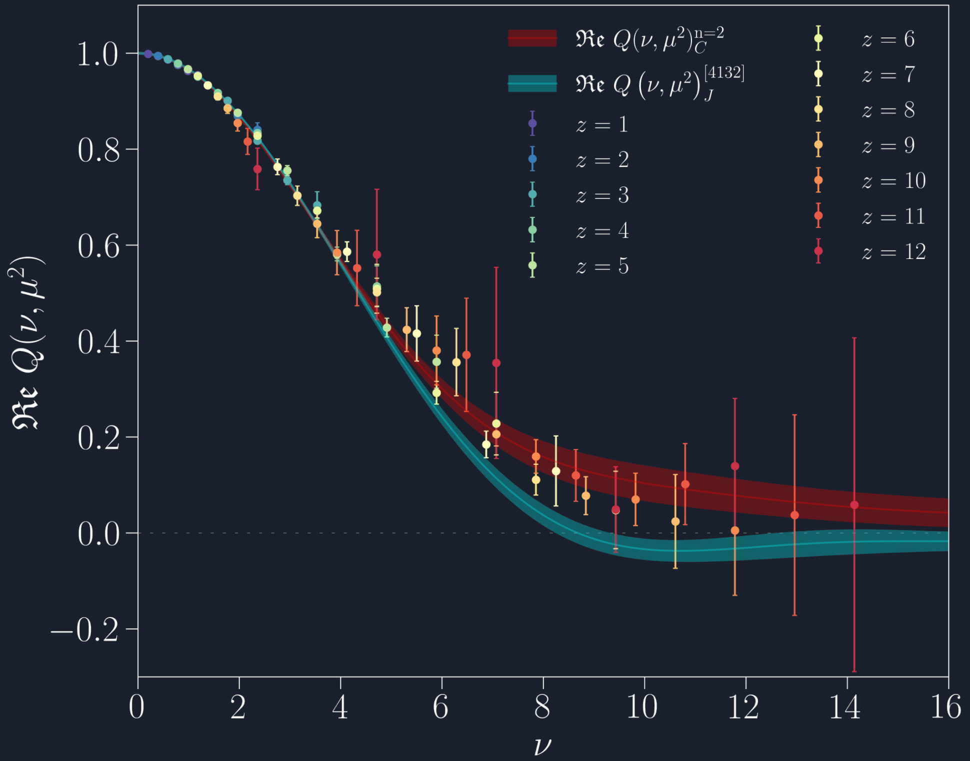
Optimal Fit for Valence PDF



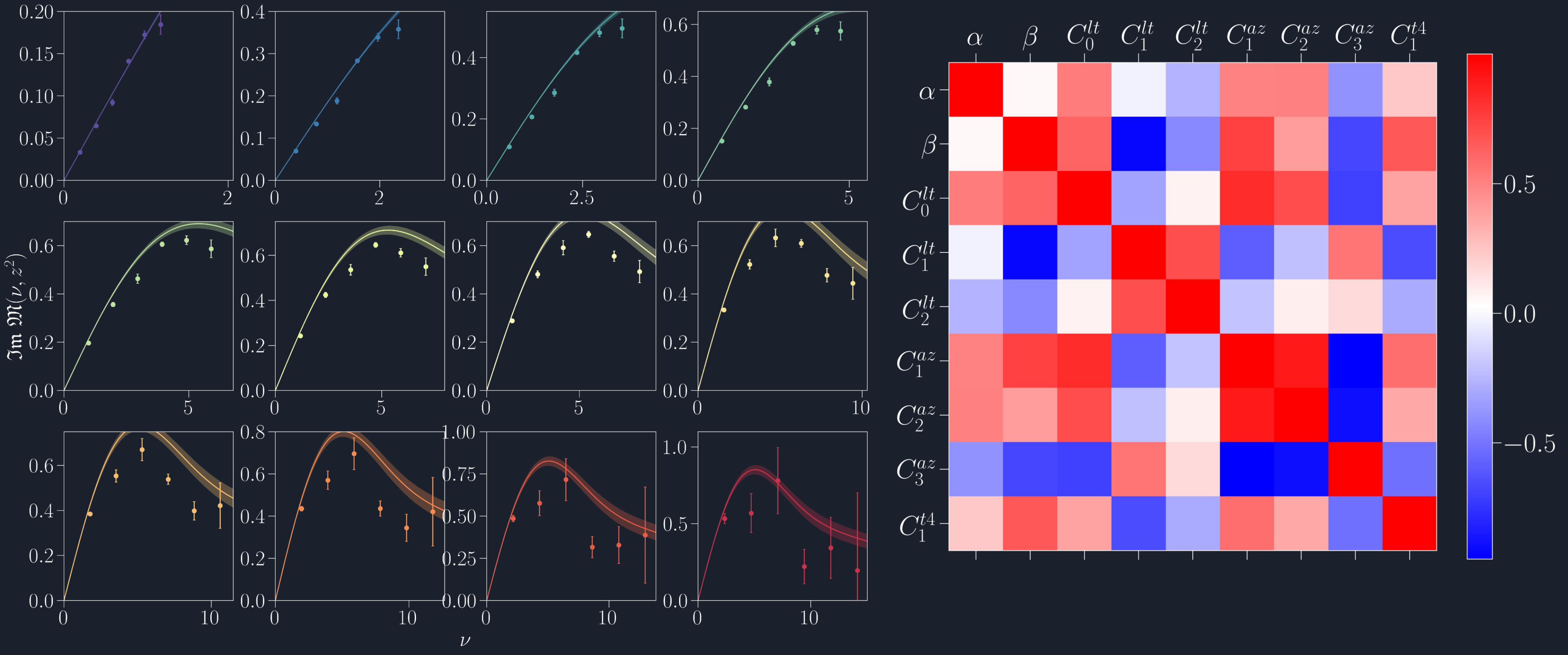


Valence Quark PDF and Leading-Twist ITD

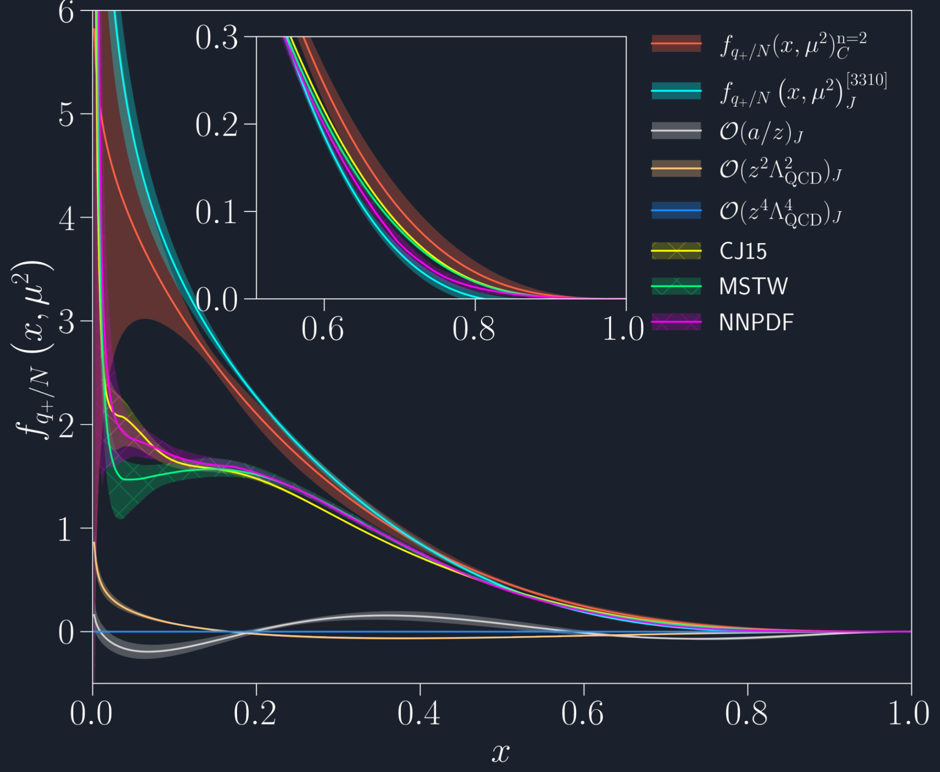


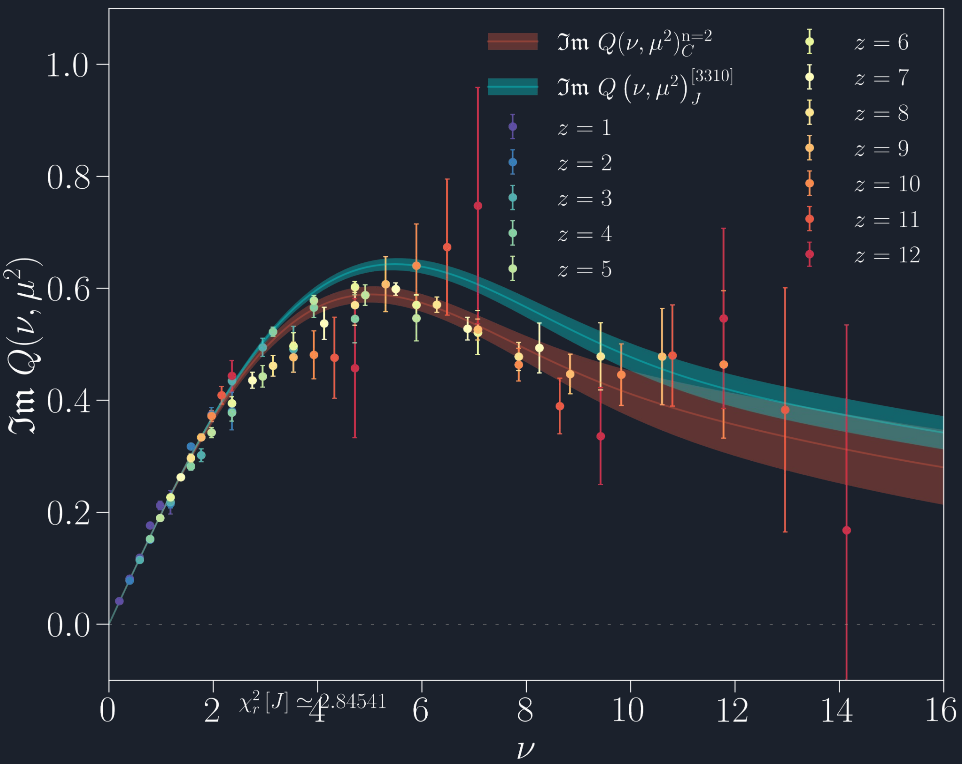


Optimal Fit for Plus PDF



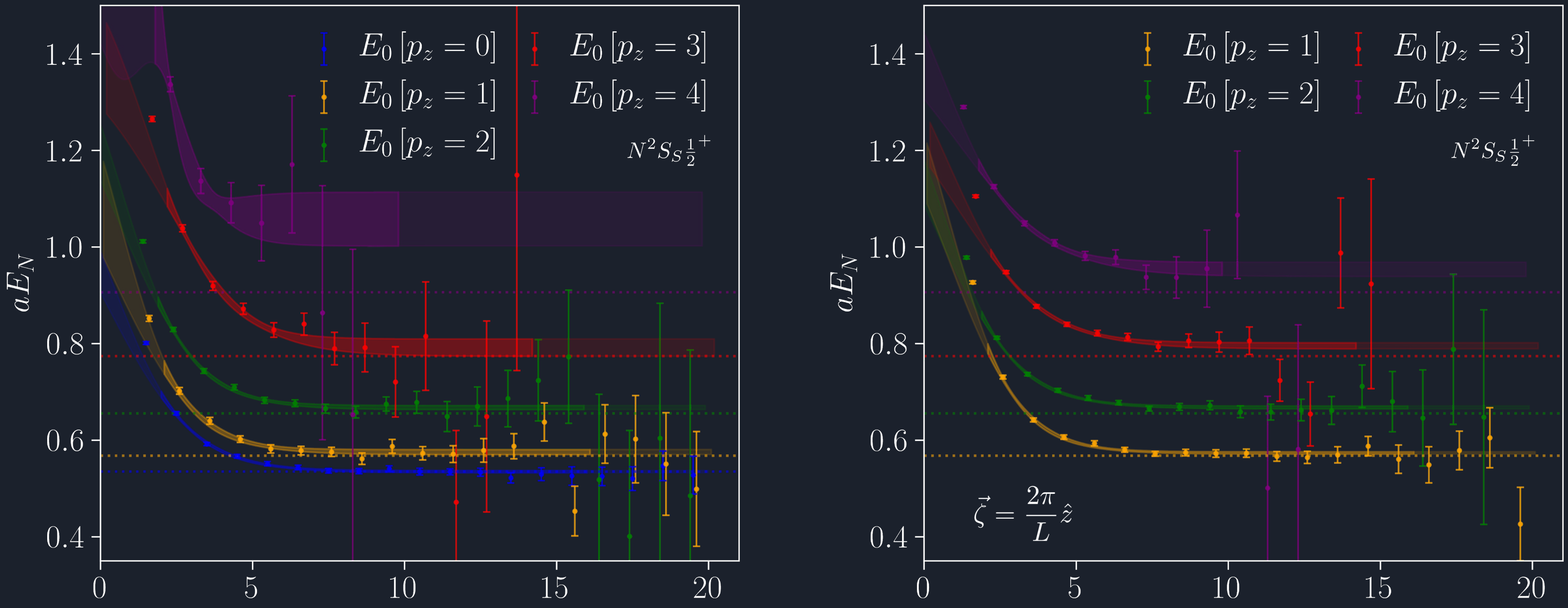
Plus Quark PDF and Leading-Twist ITD





Effective Energies

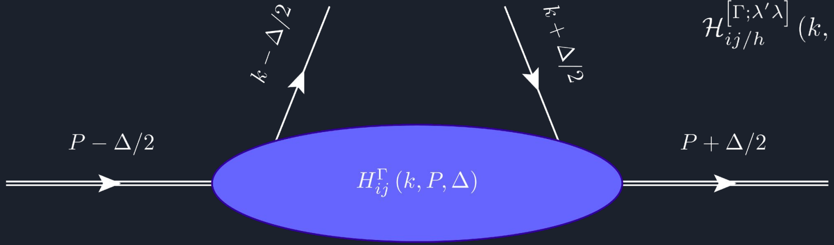
$$E_{\text{eff}}^{[\vec{p}]}(T) = \frac{1}{\delta T}\ln\left(\frac{C_2(\vec{p},T)}{C_2(\vec{p},T+\delta T)}\right)$$



$$C_2(p_z,T) = \langle N(-p_z,T)\,\overline{N}(p_z,0)\rangle = \sum_n |\mathcal{A}_n|^2 e^{-E_nT}$$

Hadron Structure - Parton Correlations

Soft dynamics of a composite hadron given generically in terms of parton correlations



$$\mathcal{H}^{[\Gamma;\lambda'\lambda]}_{ij/h}(k,P,\Delta)=\int\frac{d^4z}{(2\pi)^4}e^{ik\cdot z}\left\langle h\left(P+\tfrac{\Delta}{2},\lambda'\right)\right|\overline{\psi}_i\left(\tfrac{z}{2}\right)\Gamma\underbrace{\Phi^{(f)}\left(\left\{\tfrac{z}{2},-\tfrac{z}{2}\right\}\right)}_{\mathcal{P}\,\exp\left(ig\int_{-z/2}^{z/2}\mathrm{d}\eta^{\nu}A^{c}_{\nu}(\eta)\,t_c\right)}\psi_j\left(-\tfrac{z}{2}\right)\left|h\left(P-\tfrac{\Delta}{2},\lambda\right)\right\rangle$$

Accessed experimentally in collider or fixed-target experiments

- ➢ kinematics:

$$P^{\mu}=\left(P^{+},\frac{M_h^2}{2P^{+}},\mathbf{0}_T\right)\qquad k^{\mu}=\left(xP^{+},k^{-},\mathbf{k}_T\right)$$

Integrated parton distributions

- ➢ k^{-}- integration: partons restricted at $z^{+}=0$
- ➢ $\mathbf{k}_{\perp}$- integration: *collinear* distributions
  - ○ light-like separations $z^{-}\neq 0$
  - ○ μ^2 scale dependence (RG equations)

Quantify numerous aspects of a hadron's non-perturbative structure

- ◆ collinear/transverse parton momentum distributions
- ◆ coordinate/impact parameter distributions
- ◆ OAM/spin contributions
- ◆ mechanical properties - shears/pressures

LCS - PDF NLO Matching

$$K_{\mathrm{NLO}}\left(\nu, z^2, \mu^2\right) = \frac{1}{\pi^2}\frac{\epsilon^{12\alpha\beta} z_\alpha p_\beta}{\nu}\left[K^{(0)}(\nu) + \frac{\alpha_s C_F}{2\pi}\left\{\boxed{K^{(1,0)}(\nu)} + \boxed{K^{(1,1)}(\nu)\ln\left[-z^2\mu^2 e^{2\gamma_E}/4\right]}\right\}\right]$$

Scheme/op. dep.

Scale dep. of LCS

$$K^{(0)}(\nu) = 2\nu\cos\nu$$

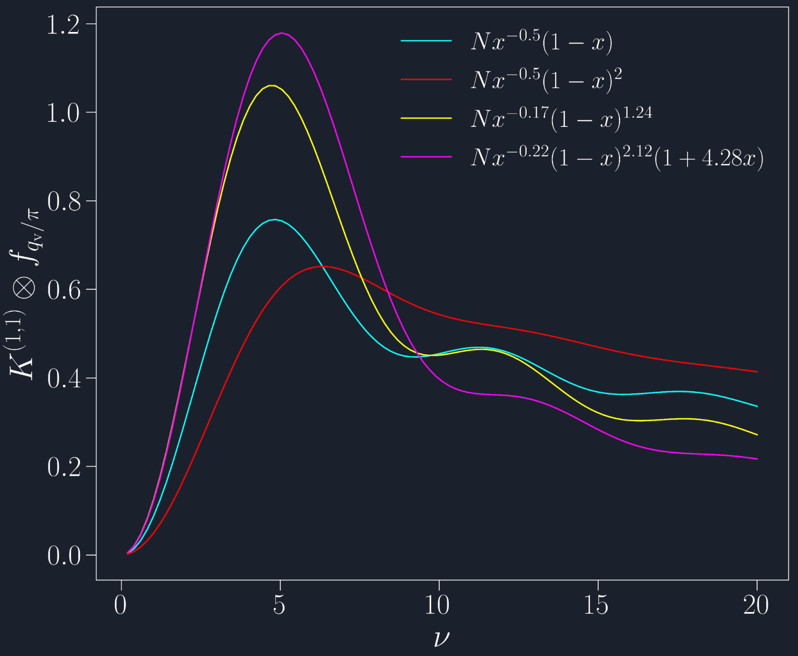
$$\boxed{K^{(1,0)}(\nu)} = 2\nu\int_0^1 \mathrm{dy}\cos(\mathrm{y}\nu)\left[\frac{1}{2}\delta(1-\mathrm{y}) - \left(\frac{2\ln(1-\mathrm{y})}{1-\mathrm{y}} - \frac{\mathrm{y}^2 - 3\mathrm{y} + 1}{1-\mathrm{y}}\right)_+\right]$$

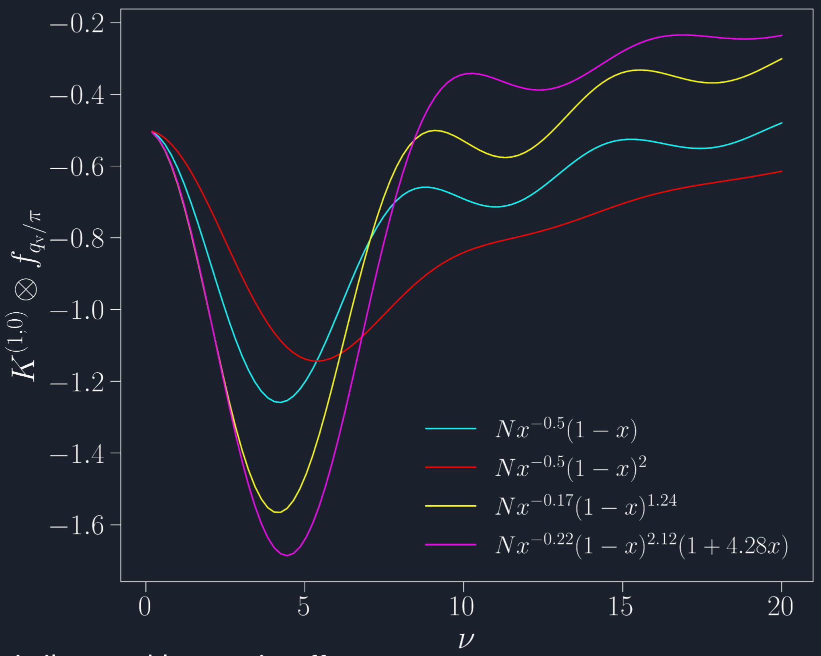
$$\boxed{K^{(1,1)}(\nu)} = -2\nu\int_0^1 \mathrm{dy}\cos(\mathrm{y}\nu)\left(\frac{1+\mathrm{y}^2}{1-\mathrm{y}}\right)_+$$

Stability of Two-Current NLO Matching Kernel

Convolutions represent difference between PDF and two-current LCS matrix elements

$$K^{(1,i)}(\nu) \otimes f_{q_v/\pi}(\mu^2) = \int_0^1 \frac{dx}{x\nu} K^{(1,i)}(x\nu)\, f_{q_v/\pi}(x,\mu^2)$$





Structurally similar; roughly opposite effect

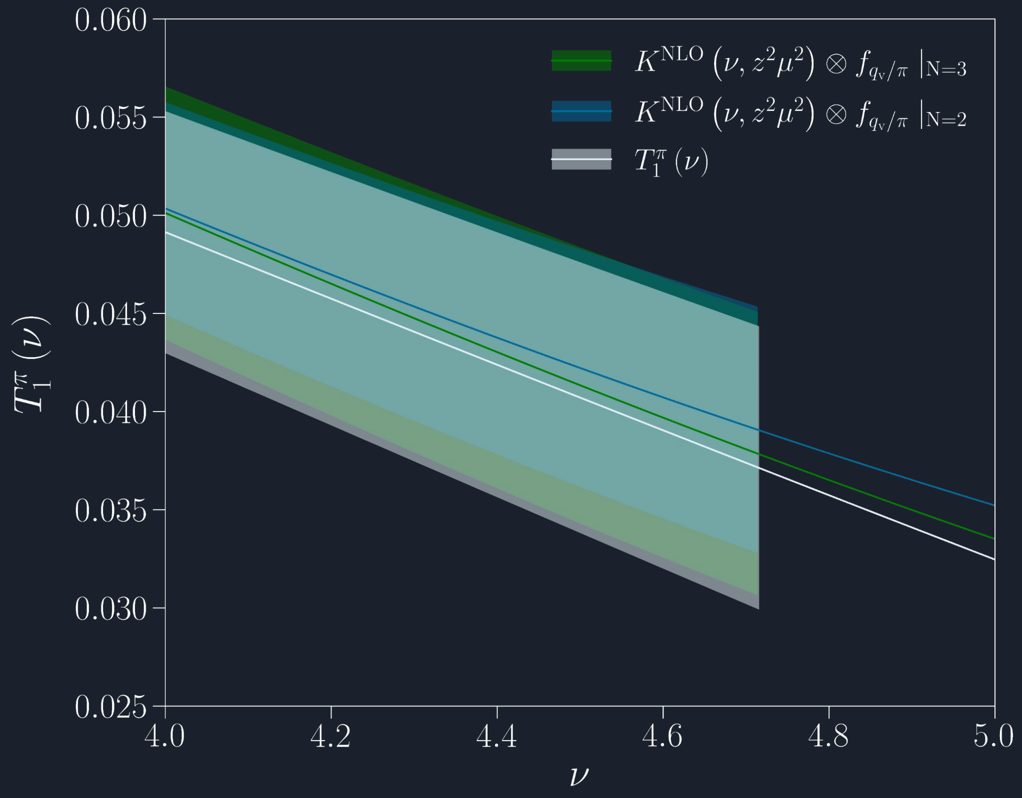
- ➢ most significant for largest Ioffe-times numerically accessible

Coordinate space matching is mild and numerically stable

- ➢ no large logarithmic corrections or divergent behavior

Entire NLO kernel is $\mathcal{O}(\alpha_s)$ over entire range of Ioffe-time

Areas for Improvement: Two-Current LCSs



Nucleon Interpolators with Distillation

Generic light-quark nucleon interpolator smeared with distillation

- derivatives to extend angular momenta

$$\mathcal{O}_i(t) = \epsilon^{abc}\left(\mathcal{D}_1\Box u\right)_a^\alpha \left(\mathcal{D}_2\Box d\right)_b^\beta \left(\mathcal{D}_3\Box u\right)_c^\gamma (t)\, S_i^{\alpha\beta\gamma}$$

Dirac structure/covariant derivatives

Discretized continuum-like interpolators of definite permutational symmetries

$$\mathcal{O}_B = \left(\mathcal{F}_{\mathcal{P}(\mathrm{F})} \otimes \mathcal{S}_{\mathcal{P}(\mathrm{S})} \otimes \mathcal{D}_{\mathcal{P}(\mathrm{D})}\right)\{q_1 q_2 q_3\}$$

$$\left(N_M \otimes (\tfrac{1}{2}^+)^1_M \otimes D^{[2]}_{L=1,A}\right)^{J^P=\frac{1}{2}^+} \equiv N\,{}^2P_A\,\tfrac{1}{2}^+ \qquad N\,{}^{(2S+1)}L_{\mathcal{P}}\,J^P$$

(Generally) Continuum spins reducible under octahedral group

Canonical subductions

➢ spinors/derivatives combined into object of definite J^P

$$\mathcal{O}^{\{J\}}_{n\Lambda,r} = \sum_m S^{J,m}_{n\Lambda,r}\ \mathcal{O}^{\{J,m\}}$$

R. Edwards, et. al., Phys. Rev. D84, 074508 (2011)
J. Dudek and R. Edwards, Phys. Rev. D85, 054016 (2012)

Helicity subductions

C. Thomas, et al., Phys. Rev. D85, 014507 (2012)
C. Thomas, private communication

➢ boost breaks O_h^D symmetry to little groups

$$\left[\mathbb{O}^{J^P,\lambda}(\vec{p})\right]^\dagger = \sum_m \mathcal{D}^{(J)}_{m,\lambda}(R)\left[O^{J^P,m}(\vec{p})\right]^\dagger$$

➢ subduce into little groups

$$\left[\mathbb{O}^{J^P,|\lambda|}_{\Lambda,\mu}(\vec{p})\right]^\dagger = \sum_{\hat{\lambda}=\pm|\lambda|} S^{\tilde{\eta},\hat{\lambda}}_{\Lambda,\mu}\left[\mathbb{O}^{J^P,\hat{\lambda}}(\vec{p})\right]^\dagger$$

Variational Method

- Exploit redundancy of interpolators in a symmetry channel
- Optimal linear combination to project onto $|\mathbf{n}\rangle$

$$C(t)\, v_{\mathbf{n}}(t, t_0) = \lambda_{\mathbf{n}}(t, t_0)\, C(t_0)\, v_{\mathbf{n}}(t, t_0)$$

$$C_{ij}(t') = \langle \mathcal{O}_i(t')\, \mathcal{O}_j^\dagger(0) \rangle$$

$$v_{\mathbf{n}'}^\dagger C(t_0)\, v_{\mathbf{n}} = \delta_{\mathbf{n}',\mathbf{n}}$$

- Fixed t_0 and solved for $t > t_0$
- Solutions yield (organized by $|\lambda_n(t, t_0)|$)
  - "Principal correlator" $\lambda_{\mathbf{n}}(t, t_0) \sim e^{-E_{\mathbf{n}}(t-t_0)}$
  - Interpolator weights $\mathcal{O}_{\mathbf{n}}^{\mathrm{opt}\,\dagger} = \sum_i v_{\mathbf{n}}^i(t, t_0)\, \mathcal{O}_i^\dagger$