lattice QCD and the hadron spectrum

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hadron spectrum collaboration
hadspec.org

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Jefferson Lab
Thomas Jefferson National Accelerator Facility

contents

meson spectroscopy *"illustrating the problem"*

resonances, scattering, elastic phase-shifts

lattice QCD *"introducing the tool"*

discrete spectrum, finite volume, computing the spectrum

elastic scattering *"solving the simplest problem"*

lattice QCD phase-shift results

coupled-channel scattering *"a more realistic situation"*

mapping the discrete spectrum to the ***t***-matrix

lattice QCD calculation results

the complex energy plane *"well-defined quantities"*

rigorously determining resonances

recent pedagogic review

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Scattering processes and resonances from lattice QCD

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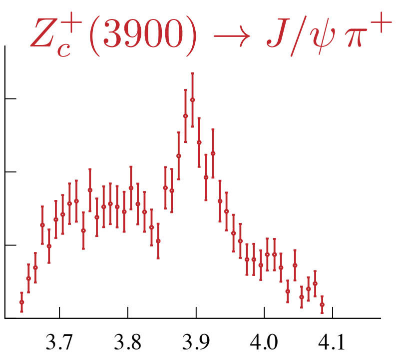
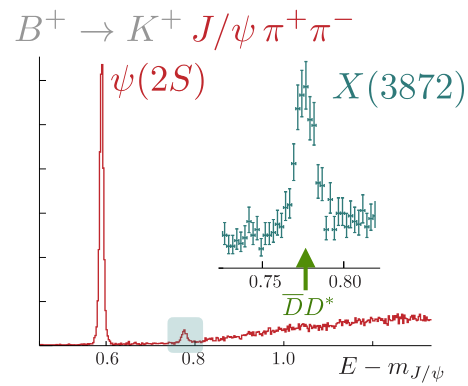
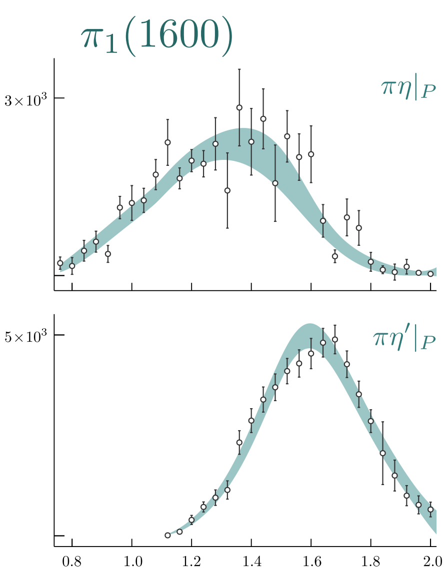
contents

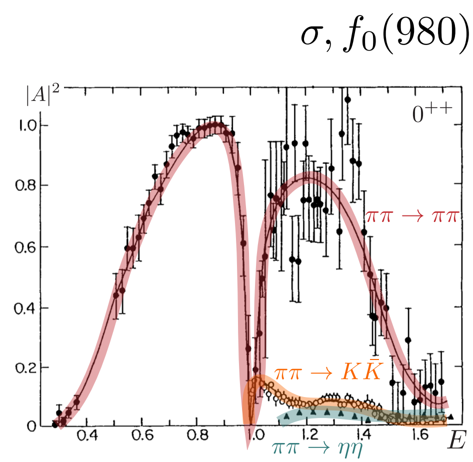
meson spectroscopy

experimental signals









'theory' ?

$q\bar{q}$ mesons
glueballs
hybrids
tetraquarks
molecules
...

these are **'pictures'**

the theory is QCD

... how do we bridge the gap ?

I'll try to show you over the next five lectures

meson spectroscopy — ‘rigorously’

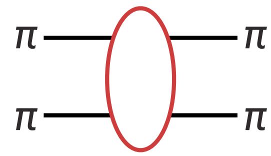
want to study excited hadrons as they really are — **rapidly decaying resonances**

same dynamics that binds them also causes their decay

we need to compute **scattering amplitudes** and see if they resonate

start with the simplest case: **elastic scattering** …

e.g.



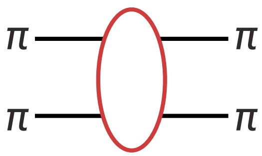
amplitude

$t(E, \cos\theta)$

differential cross-section

$$\frac{d\sigma}{d\Omega} \propto \big|t(E, \cos\theta)\big|^2$$

elastic partial-waves & unitarity



elastic scattering amplitude
can be expanded in partial-waves

$$t(E,\cos\theta) = \sum_\ell (2\ell+1)\, \boxed{t_\ell(E)}\, P_\ell(\cos\theta)$$

partial-wave amplitude

resonances appear in a single partial-wave

ℓ	0	1	2	...
J^P	0^+	1^-	2^+	

conservation of probability
a.k.a **elastic unitarity**

$$\mathrm{Im}\, t_\ell(E) = \rho(E)\big|t_\ell(E)\big|^2 \quad \text{or} \quad \mathrm{Im}\,\frac{1}{t_\ell(E)} = -\rho(E)$$

‘phase-space’ $\rho(E) = \dfrac{2k(E)}{E}$

c.m. momentum $k(E) = \frac{1}{2}\sqrt{E^2 - 4m^2}$

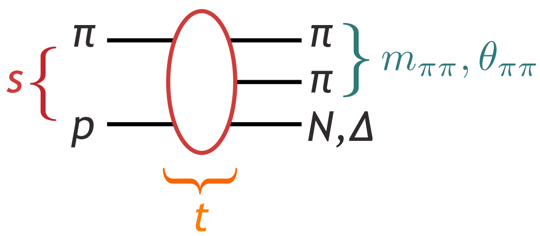
can parameterise elastic scattering
in terms of a single real parameter

$$t_\ell(E) = \frac{1}{\rho(E)}\, e^{i\delta_\ell(E)} \sin \boxed{\delta_\ell(E)}$$

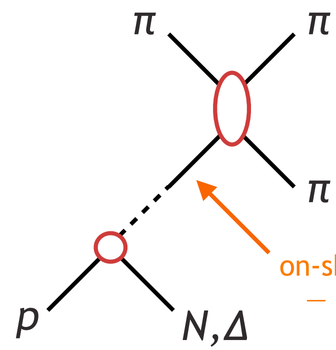
‘phase-shift’

the “simplest” case: $\pi\pi$ elastic scattering

extract from **charged pion beams** on **nucleon targets**



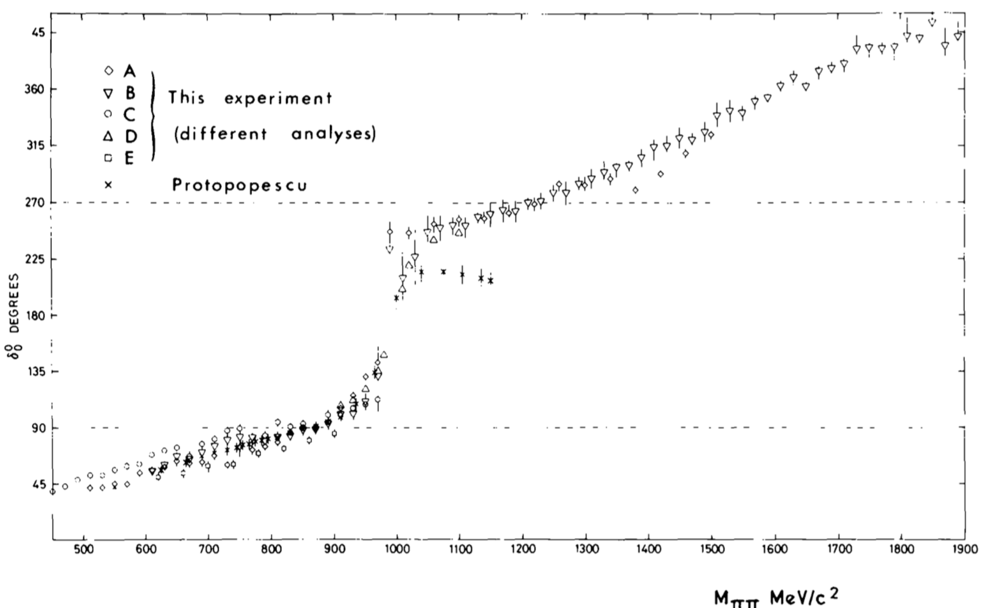
physical scattering
has negative t



on-shell π exchange
— extrapolate to $t = m_\pi^2$

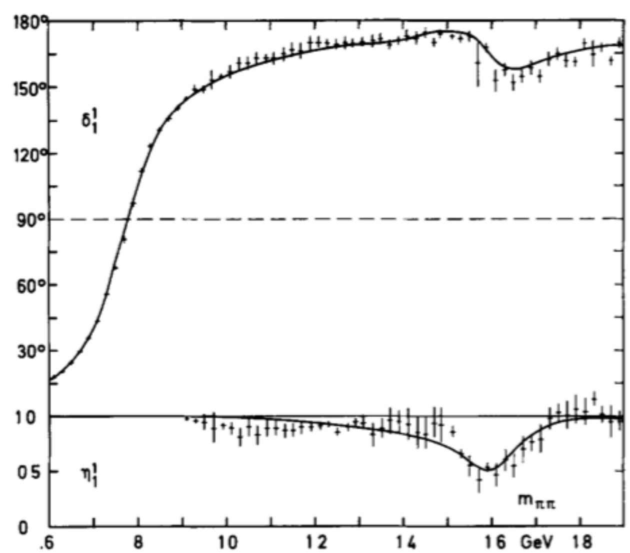
$\pi\pi$ partial-waves
— project $P_\ell(\cos\theta_{\pi\pi})$

isospin=0



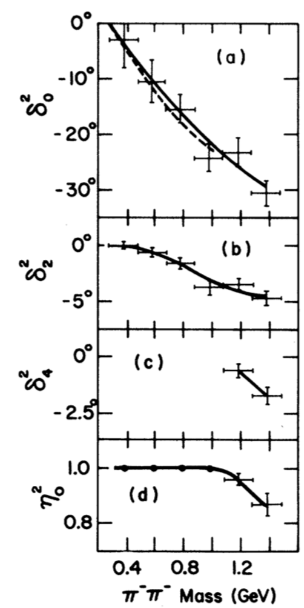
Grayer 1974

isospin=1



Hyams 1973

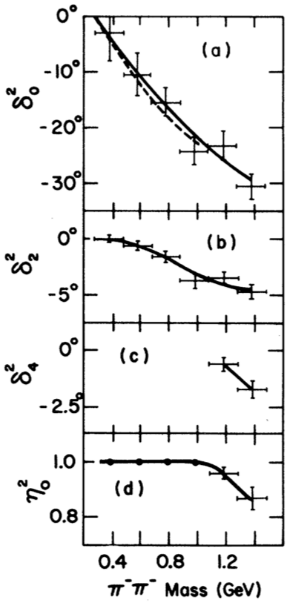
isospin=2



Cohen 1972

the “simplest” case: $\pi\pi$ elastic scattering

isospin=2



ℓ=0, *a.k.a* “*S*-wave”

ℓ=2, *a.k.a* “*D*-wave”

ℓ=4, *a.k.a* “*G*-wave”

inelasticity
– indicates other final states accessible

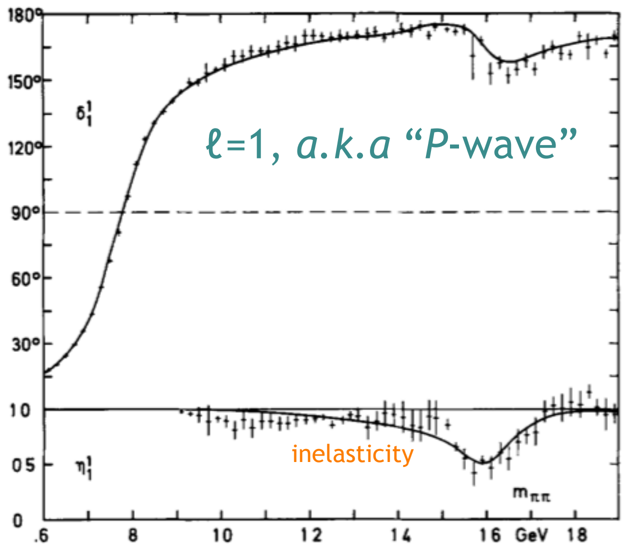
$$t_\ell(E) = \frac{\eta_\ell(E) e^{2i\delta_\ell(E)} - 1}{2i\rho(E)}$$

isospin=2 phase-shift is **small** and **negative**

at low energies $|\delta_0| \gg |\delta_2| \gg |\delta_4| \ldots$

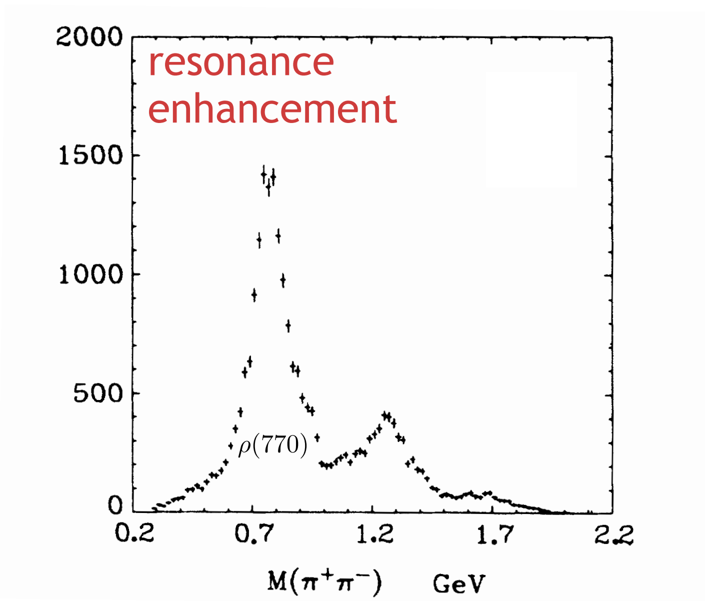
the "simplest" case: $\pi\pi$ elastic scattering

isospin=1



isospin=1 phase-shift **rises rapidly through 90°** near 770 MeV

gives rise to a 'bump' in the cross-section



this is the famous **ρ** resonance

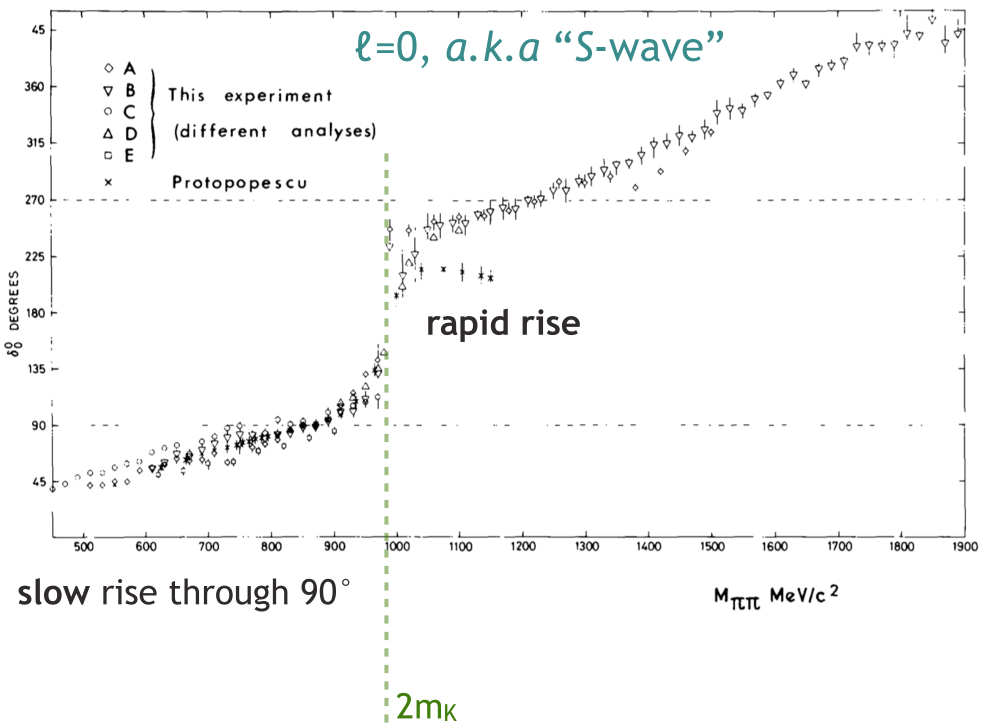
ρ(770) $I^G(J^{PC}) = 1^+(1^{--})$

Mass $m = 775.26 \pm 0.25$ MeV
Full width $\Gamma = 149.1 \pm 0.8$ MeV

ρ(770) DECAY MODES	Fraction (Γ_i/Γ)	p (MeV/c)
$\pi\pi$	$\sim$ 100 %	363

the “simplest” case: $\pi\pi$ elastic scattering

isospin=0



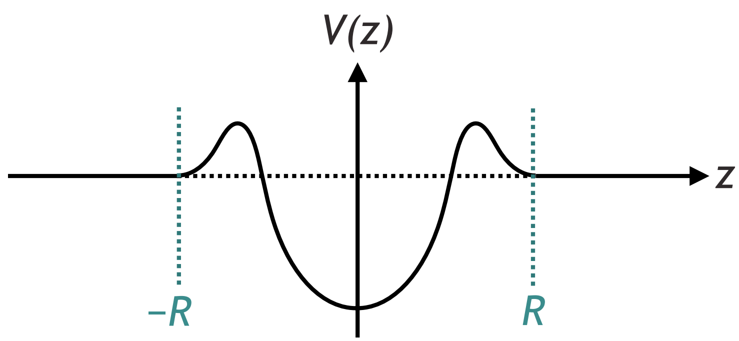
interpretation of isospin=0
phase-shift is **interesting** !

will come back to it later ...

this $\pi\pi$ system is **relativistic** scattering of **composite particles**
— we are going to need **quantum field theory**
— but first let’s orient ourselves with good old fashioned quantum mechanics ...

most easily illustrated considering **one-dimensional non-relativistic quantum mechanics**

imagine two identical bosons separated by a distance *z* interacting through a finite-range potential *V(z)*



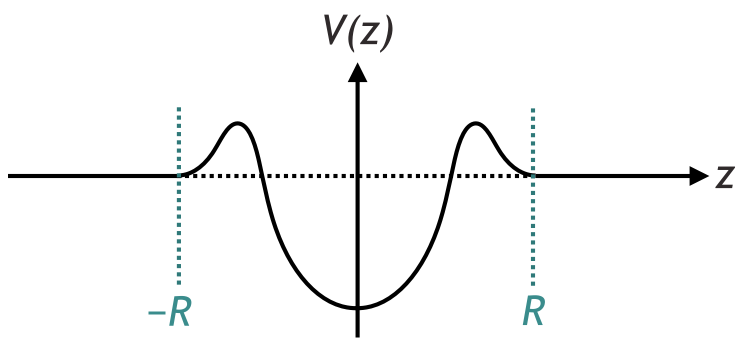
solve the Schrödinger equation

$$-\frac{1}{m}\frac{d^2\psi}{dz^2} + V(z)\psi(z) = E\psi(z)$$

scattering

most easily illustrated considering one-dimensional non-relativistic quantum mechanics

imagine two identical bosons separated by a distance *z* interacting through a finite-range potential *V(z)*



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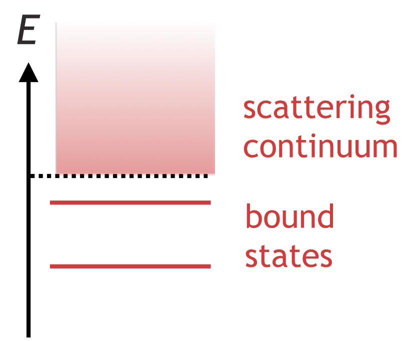
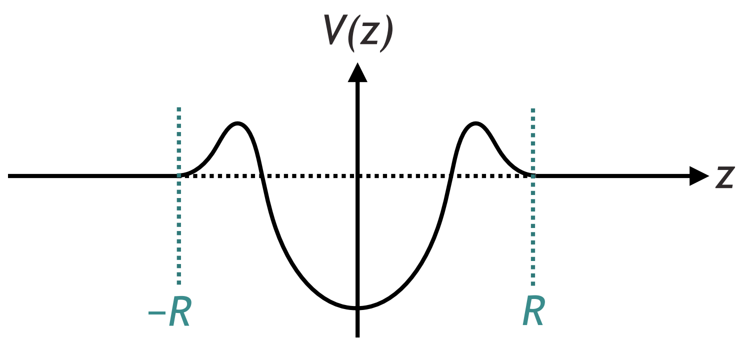


$$\psi(|z| > R) \sim \cos\big(\, p\,|z| + \delta(p)\,\big)$$

scattering

most easily illustrated considering one-dimensional non-relativistic quantum mechanics

imagine two identical bosons separated by a distance *z* interacting through a finite-range potential *V(z)*



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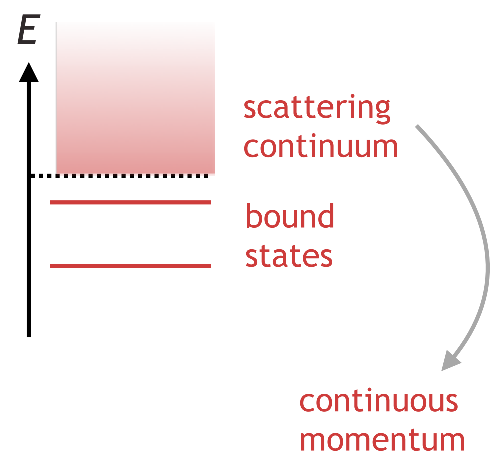

$$\psi(|z| > R) \sim \cos\big(p|z| + \delta(p)\big)$$

most easily illustrated considering one-dimensional non-relativistic quantum mechanics

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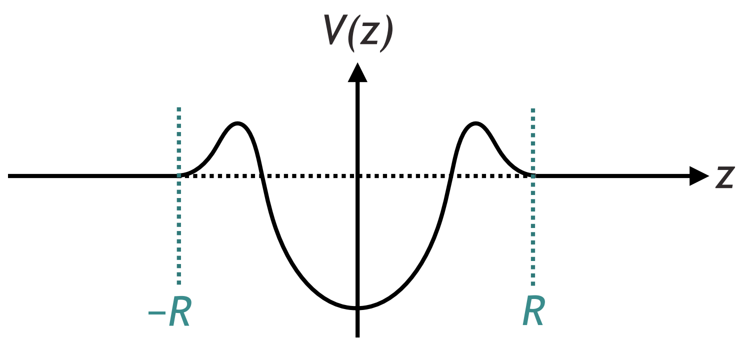
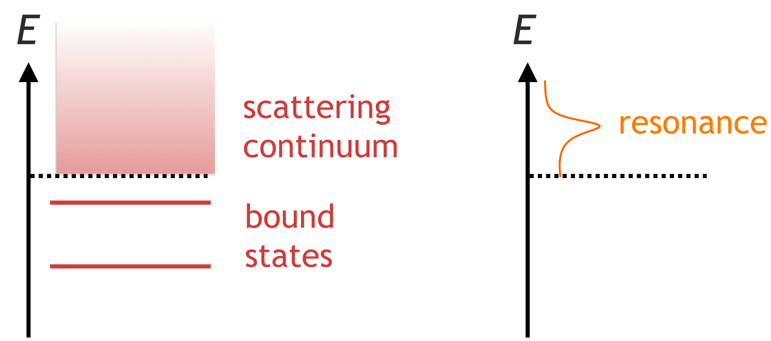


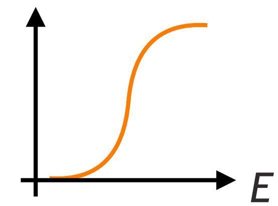
solve the Schrödinger equation

$$-\frac{1}{m}\frac{d^2\psi}{dz^2} + V(z)\psi(z) = E\psi(z)$$



$$\psi(|z| > R) \sim \cos\big(p\,|z| + \delta(p)\big)$$

phase-shift



scattering

$$\psi(|z|>R) \sim \cos\left(p\,|z| + \delta(p)\right)$$

phase-shift

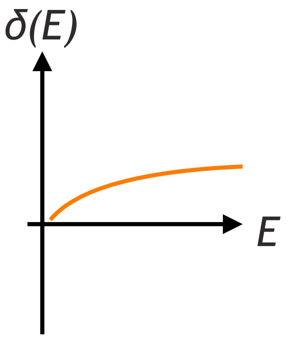
generally, consider S-matrix

$$|\text{in}\rangle = S\,|\text{out}\rangle$$

elastic scattering

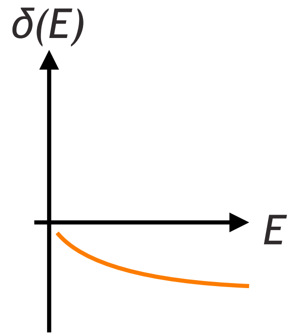
$$S(E) = e^{2i\delta(E)}$$

e.g.



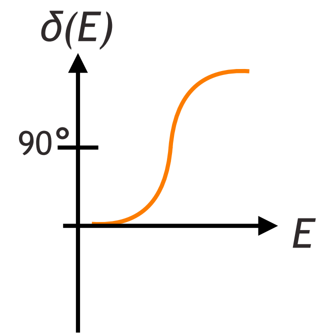
‘weak’ attraction

c.f. $\pi\pi$ isospin=0
at low energy ?



‘weak’ repulsion

c.f. $\pi\pi$ isospin=2 ?



resonance

c.f. $\pi\pi$ isospin=1 ?

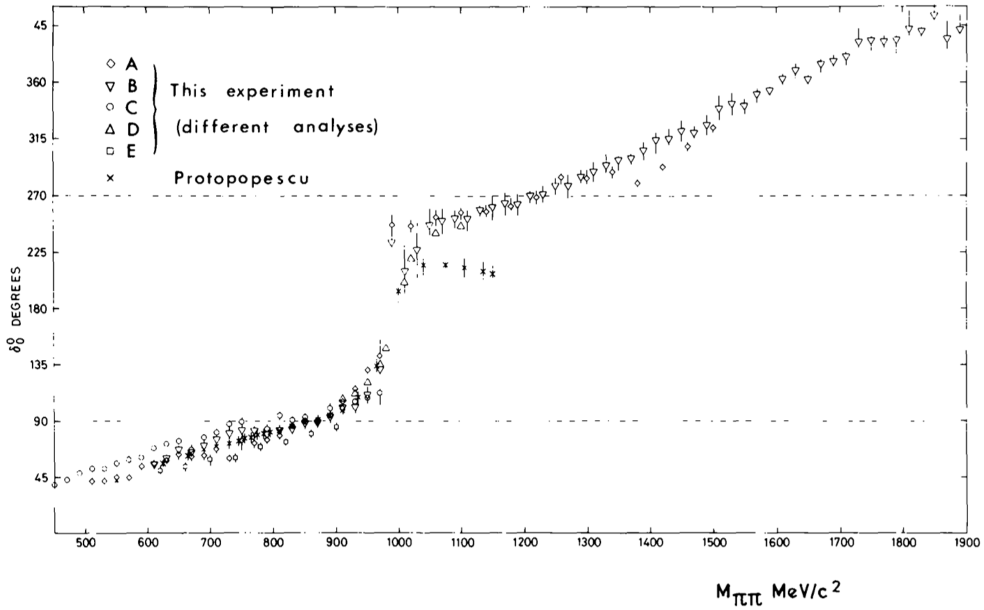
scattering in three-dimensions:

brings in the concept of the angular momentum ‘barrier’ $\sim \dfrac{\ell(\ell+1)}{r^2}$ which causes $\delta_\ell(p) \sim p^{2\ell+1}$

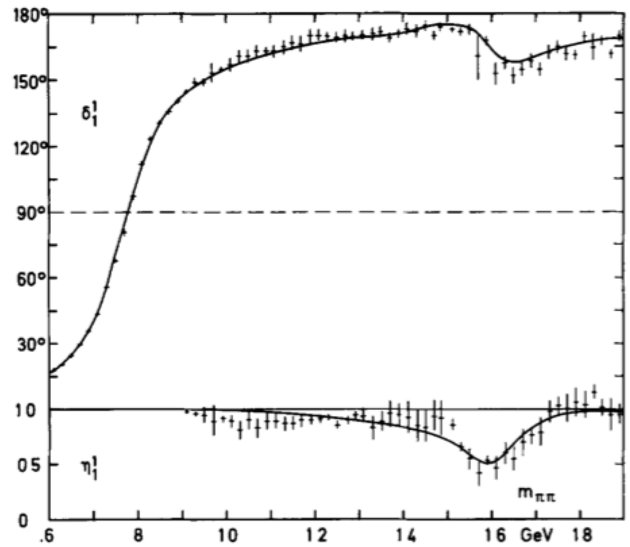
suppresses higher partial-waves at low energies

the "simplest" case: $\pi\pi$ elastic scattering

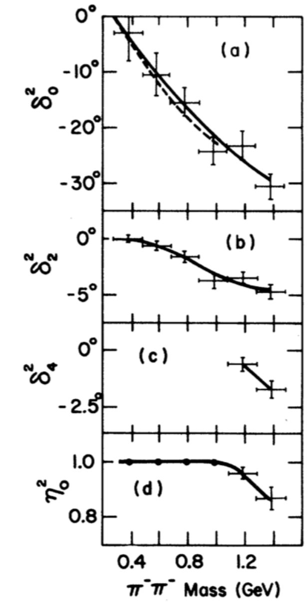
isospin=0



isospin=1



isospin=2



a first target: can a **first-principles QCD** calculation lead to these kinds of behaviour ?

a next target: can we understand these behaviours in terms of **resonances** ?

an ultimate target: can we understand the **quark-gluon make-up** of these resonances ?

contents

how should we approach a
quantum field theory
so that it is possible to compute even when **non-perturbative** ?

(**not** a truncated power series in small α_s)

once again, step back to one-dim quantum mechanics …

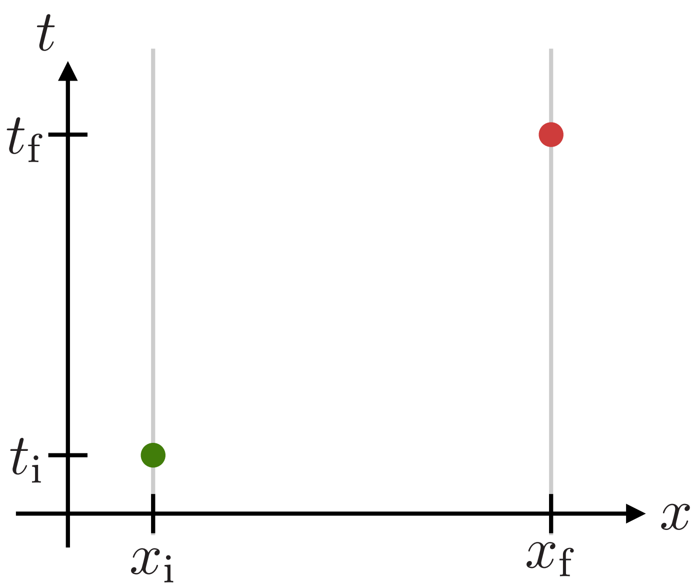
e.g. a free particle moving between a

definite initial position (x_i, t_i)

and a

definite final position (x_f, t_f)

space-time diagram



path integrals in quantum mechanics

e.g. a free particle moving between a

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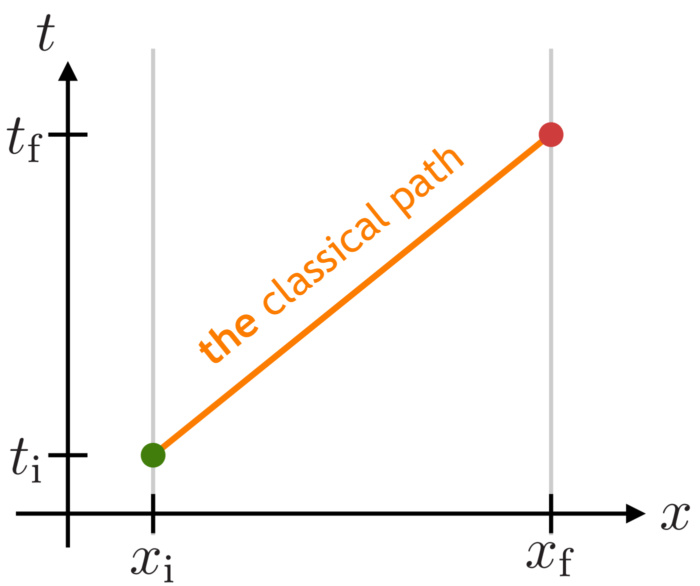
and a

definite final position (x_f, t_f)

space-time diagram



the **unique** classical path is the path of **minimum action**

the *action* $$S\big[x(t)\big] = \int_{t_\mathrm{i}}^{t_\mathrm{f}} dt\, L(x, \dot{x})$$

$$L_\mathrm{free} = \tfrac{1}{2} m \dot{x}^2$$

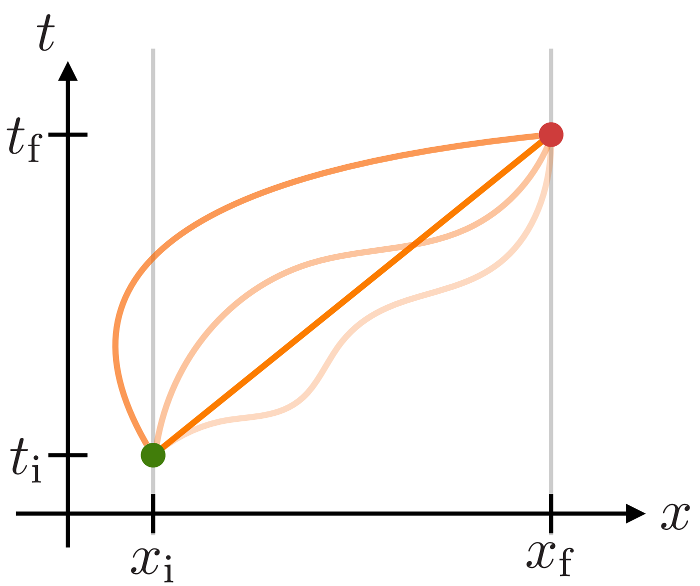
e.g. a free particle moving between a

definite initial position (x_i, t_i)

and a

definite final position (x_f, t_f)

space-time diagram



quantum mechanical amplitude

$$\langle x_{\rm f} | e^{-i\hat{H}(t_{\rm f}-t_{\rm i})} | x_{\rm i} \rangle$$

$$= \int \mathcal{D}x \, e^{-iS[x(t)]}$$

path integrals in quantum mechanics

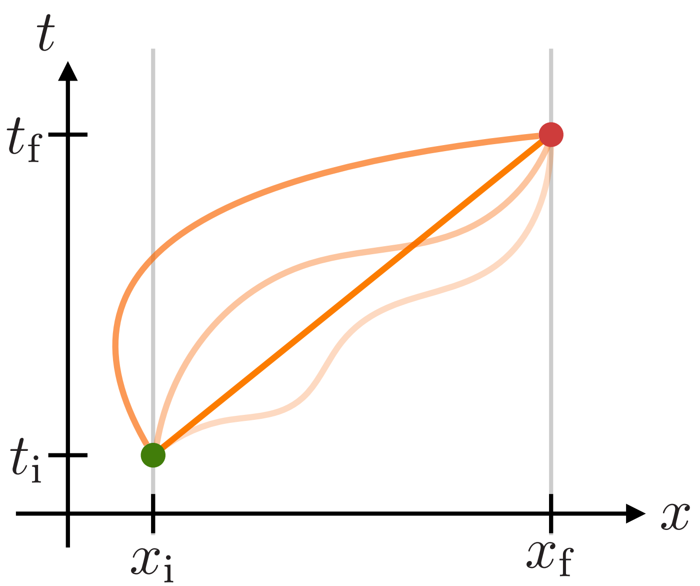
e.g. a free particle moving between a

definite initial position (x_i, t_i)

and a

definite final position (x_f, t_f)

space-time diagram



quantum mechanical amplitude

$$\left\langle x_\mathrm{f} \middle| e^{-i\hat{H}(t_\mathrm{f}-t_\mathrm{i})} \middle| x_\mathrm{i} \right\rangle$$

$$= \int \mathcal{D}x \, e^{-iS[x(t)]}$$

"sum" over all paths

weighted by a phase set by the action

and conventional quantum mechanics follows ...

path integrals in quantum field theory

consider a real scalar field theory

$\varphi(\mathbf{x}, t)$

$$\mathcal{L} = \tfrac{1}{2}\partial_\mu\varphi\,\partial^\mu\varphi - \tfrac{1}{2}m^2\varphi^2 + V\big[\varphi\big]$$

can define a path integral

$$Z = \int\mathcal{D}\varphi(x)\, e^{-iS[\varphi(x)]}$$

where the action is $S\big[\varphi(x)\big] = \int d^4x\,\mathcal{L}\big[\varphi(x)\big]$

"sum" over all
field configurations

correlation functions can be expressed similarly

e.g. relationship between the field value at one space-time point
and the value at another space-time point

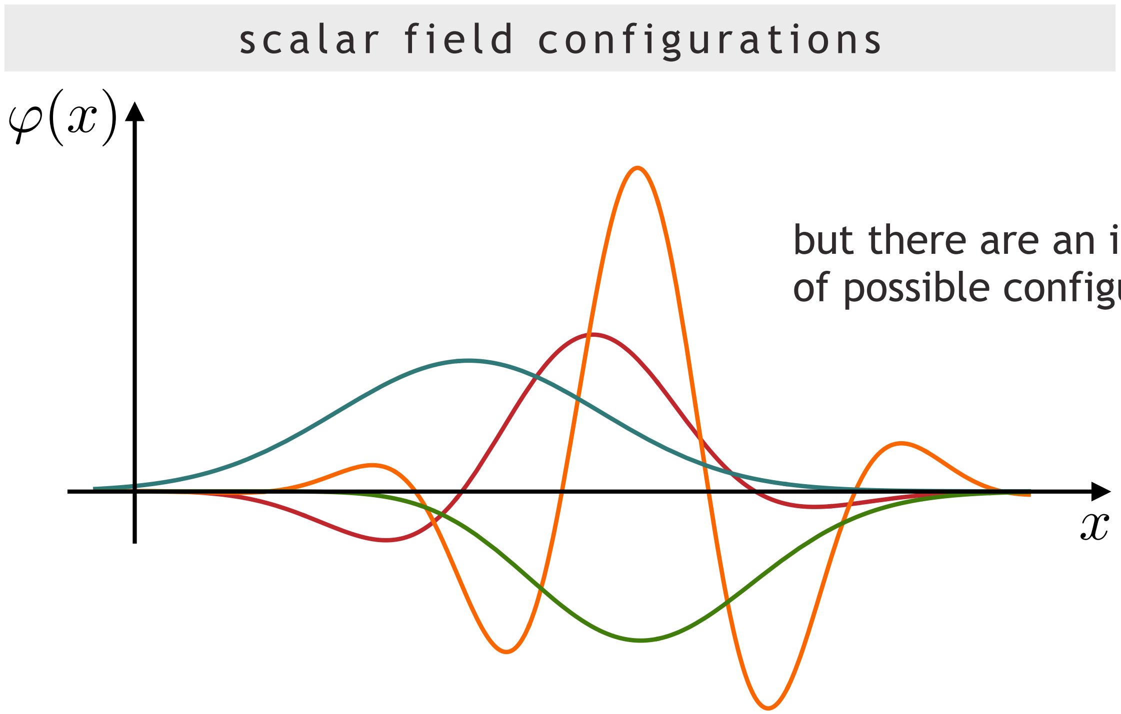
the spin vectors in a ferromagnet is a nice classical example

$$\langle 0\big|\hat{\varphi}(y)\,\hat{\varphi}(z)\big|0\rangle = \frac{1}{Z}\int\mathcal{D}\varphi(x)\;\varphi(y)\,\varphi(z)\;e^{-iS[\varphi(x)]}$$

but practically how does one 'do' the integral $\int\mathcal{D}\varphi(x)$?

lattice quantum field theory

go to one dimension for simplicity of illustration



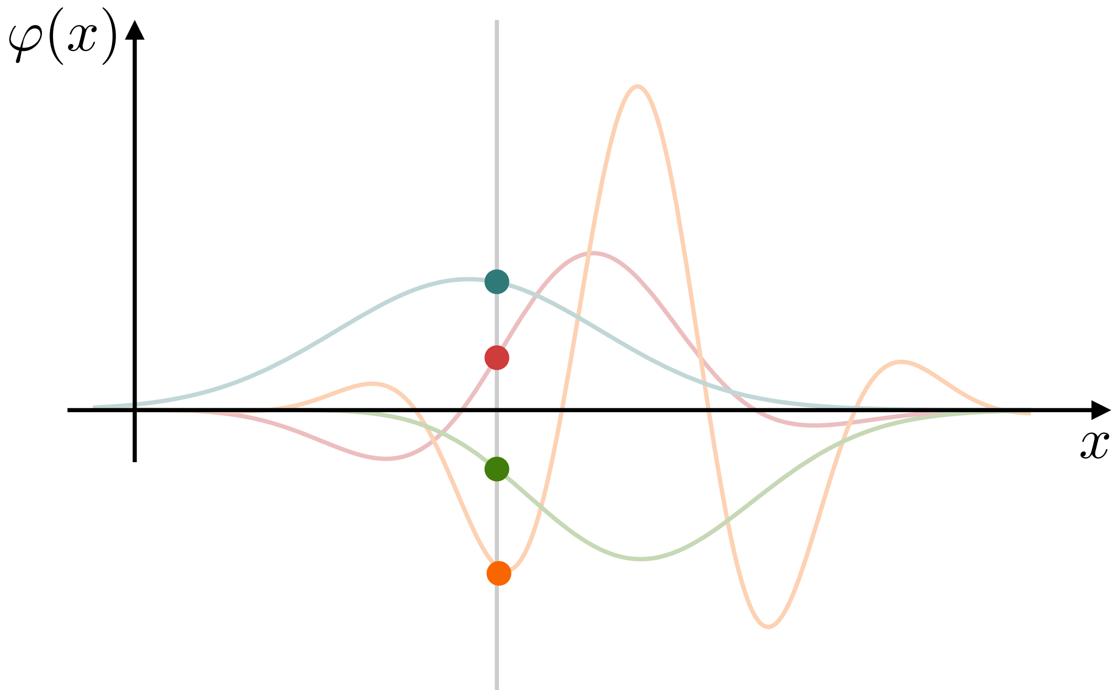
but there are an infinite number of possible configurations …

discretize the space

$$\int \mathcal{D}\varphi(x) = \prod_x \int d\varphi_x = \int d\varphi_1 \int d\varphi_2 \int d\varphi_3 \cdots$$

an integral over all the values the field can take at x_2

scalar field configurations

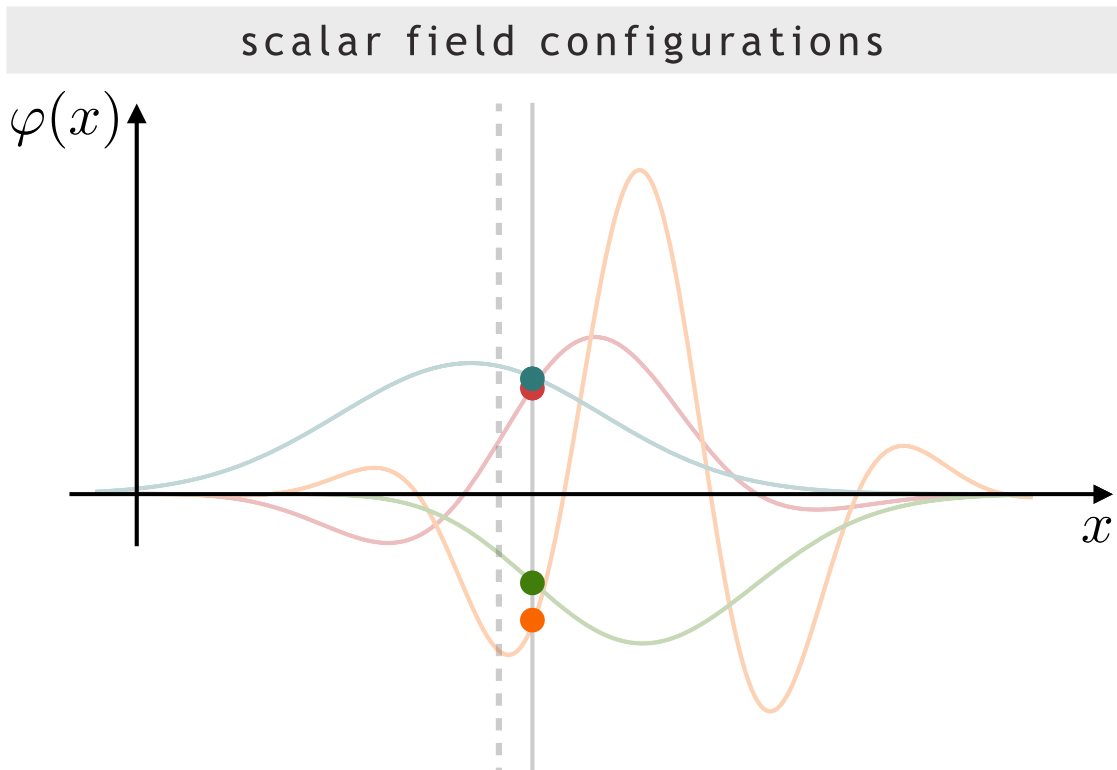


lattice quantum field theory

discretize the space

$$\int \mathcal{D}\varphi(x) = \prod_x \int d\varphi_x = \int d\varphi_1 \int d\varphi_2 \int d\varphi_3 \cdots$$

an integral over all the values the field can take at x_3

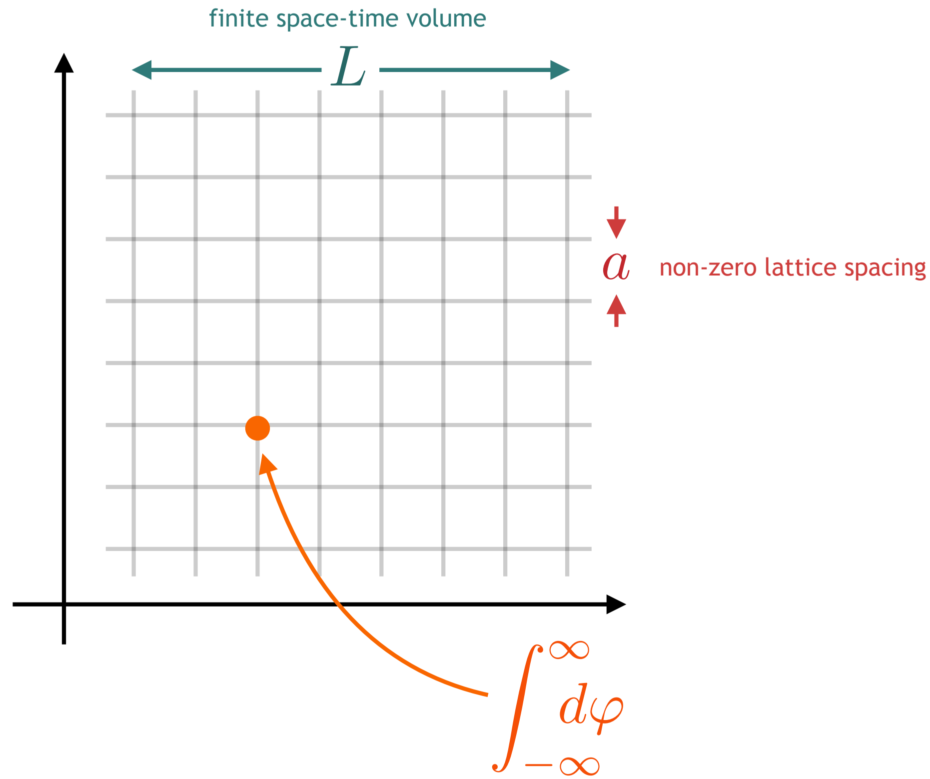


lattice quantum field theory

approach generally is to use a (hyper)cubic grid

$$\int \mathcal{D}\varphi(x) = \prod_x \int d\varphi_x$$

space-time grid



hiding it here,
but boundary conditions
are important

euclidean lattice quantum field theory

even with the grid, still not practical: $Z = \int \mathcal{D}\varphi(x)\, e^{-iS[\varphi(x)]}$

a **phase** is not ideal for averaging

make a variable transform $t \to -it$ then $-iS = -i \int d^3x\, dt\, \mathcal{L} \to -\int d^3x\, dt\, \mathcal{L}_\mathrm{E} = -S_\mathrm{E}$

euclidean path integral

$$Z_\mathrm{E} = \int \mathcal{D}\varphi(x)\, e^{-S_\mathrm{E}[\varphi(x)]}$$

a bounded real number
⇝ a probability ?

computational approach

euclidean path integral

$$Z_{\mathrm{E}} = \int \mathcal{D}\varphi(x)\, e^{-S_{\mathrm{E}}[\varphi(x)]}$$

probability for a field configuration $\varphi(x)$

⇒ **importance sampled Monte Carlo generation of field configurations**

obtain an **ensemble** of configurations $\{\varphi_x\}_{i=1\ldots N}$

[value of the field at each point on the grid]

computing observables

for some observable (vacuum matrix element)

$$\langle 0 | \, O[\hat{\varphi}] \, | 0 \rangle = \int \mathcal{D}\varphi \, O[\varphi] \, e^{-S_{\mathrm{E}}[\varphi]}$$

can now be estimated as an **average over the ensemble**

$$\langle 0 | \, O[\hat{\varphi}] \, | 0 \rangle \approx \overline{O} = \frac{1}{N} \sum_{i=1}^{N} O[\varphi^{(i)}]$$

plus get an **uncertainty estimate** from the variance

$$\sigma(O) = \sqrt{\tfrac{1}{N(N-1)} \sum_{i=1}^{N} \left(O[\varphi^{(i)}] - \overline{O} \right)^2}$$

ensemble mean and error

$$\langle 0 | \, O[\hat{\varphi}] \, | 0 \rangle \approx \overline{O} \pm \sigma(O)$$

what about the theory we require: **quantum chromo dynamics** ?

quantum chromodynamics

gauge field theory of quarks (fermions) and gluons (vector gauge fields) with SU(3) 'color' symmetry

qcd lagrangian

$$\mathcal{L} = \bar{\psi}\big(i\gamma^\mu D_\mu - m\big)\psi - \tfrac{1}{2}\mathrm{tr}\big(F_{\mu\nu}F^{\mu\nu}\big)$$

gauge covariant derivative

$$D_\mu = \partial_\mu + igA_\mu$$

field strength tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig\big[A_\mu, A_\nu\big]$$

qcd fields

color index $i = 1\ldots3$

quark field $\psi^i_\alpha(x)$

Dirac spin index $\alpha = 1\ldots4$

traceless matrix in color

gluon field $A^{ij}_\mu(x)$

Lorentz vector

$$= \sum_{a=1\ldots8} A^a_\mu(x)\, t^a_{ij}$$

expansion in SU(3) generators

$$t^a = \tfrac{1}{2}\lambda^a$$

$$\big[t^a, t^b\big] = i f^{abc} t^c$$

$$\mathrm{tr}\big(t^a t^b\big) = \tfrac{1}{2}\delta^{ab}$$

quantum chromodynamics

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$$\mathcal{L} = \bar{\psi}\big(i\gamma^\mu D_\mu - m\big)\psi - \tfrac{1}{2}\mathrm{tr}\big(F_{\mu\nu}F^{\mu\nu}\big)$$

gauge covariant derivative

$$D_\mu = \partial_\mu + igA_\mu$$

field strength tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + ig\big[A_\mu, A_\nu\big]$$

qcd ingredients

relativistic fermions

$$\bar{\psi}\big(i\gamma^\mu \partial_\mu - m\big)\psi$$

color vector current

$$g\,\big(\bar{\psi}\gamma^\mu t^a \psi\big)\, A^a_\mu$$

massless gluons

$$\big(\partial_\mu A_\nu - \partial_\mu A_\nu\big)^2$$

gluon self-interactions

$$g[A, A]\partial A,\ g^2\big([A, A]\big)^2$$

transforming to Euclidean space-time
required for Monte-Carlo sampling

$$(x_{\rm M})^0 = -i(x_{\rm E})_4 \qquad (x_{\rm M})^i = (x_{\rm E})_i$$

$$(\gamma_{\rm M})^0 = (\gamma_{\rm E})_4 \qquad (\gamma_{\rm M})^i = i(\gamma_{\rm E})_i$$

$$(A_{\rm M})^0 = i(A_{\rm E})_4 \qquad (A_{\rm M})^i = -(A_{\rm E})_i$$

euclidean qcd action

(entirely in Euclidean variables)

$$S_{\rm E} = \int d^4x \Big[\overline{\psi}(\gamma_\mu D_\mu + m)\psi + \tfrac{1}{4} F^a_{\mu\nu} F^a_{\mu\nu} \Big]$$

we'd like to discretize on a hypercubic grid

quark fields take values
on the sites of the lattice

$$\psi^i_\alpha(x_\mu = a n_\mu)$$

with one wrinkle
still to be dealt with

but what shall we do with the **gluon fields** ... ?

parallel transporters and gauge invariance

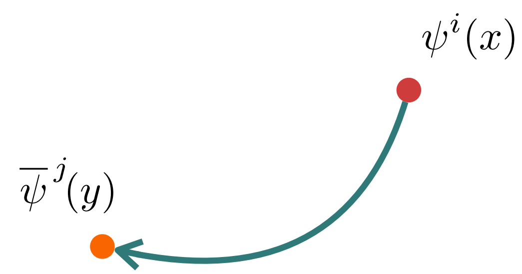
in the continuum theory,
consider a quark field pair separated by some distance

$q\bar{q}$ pair

the combination $\overline{\psi}^j(y)\,\delta_{ji}\,\psi^i(x)$ is not **gauge-invariant**

we can perform **different** local gauge transformations at locations ***x*** and ***y***

$q\bar{q}$ pair with Wilson line



a **gauge-invariant** combination is $\overline{\psi}^j(y)\left[e^{ig\int_x^y dz_\mu A^\mu(z)}\right]_{ji}\psi^i(x)$

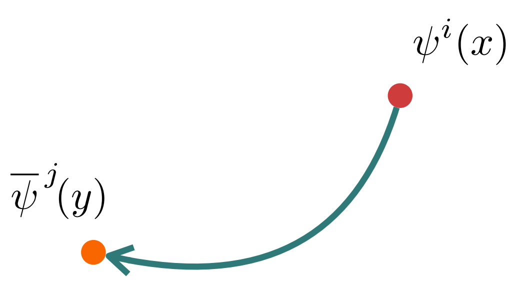
a **‘Wilson line’** transports the color

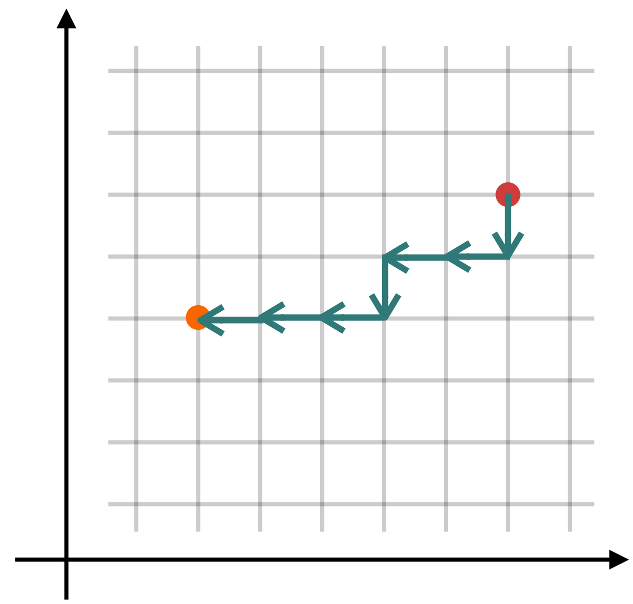
gauge 'links'

on a lattice, make hops to neighboring sites

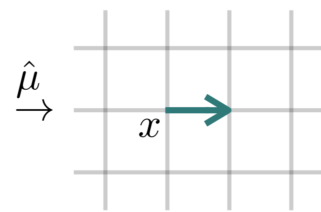
space-time grid

$q\bar{q}$ pair with Wilson line





shortest path between neighboring sites = a **'link'**



$$\left[e^{igaA^\mu(x)}\right]_{ji}$$

$$U_\mu(x) = e^{igaA^\mu(x)}$$

SU(3) matrix on each **link** of the lattice

lattice QCD action

can construct a gauge-invariant **finite-difference** – approximation to a derivative ?

$$\overline{\psi}(x)\,\gamma_\mu\,U_\mu(x)\,\psi(x+\hat{\mu}a) - \overline{\psi}(x)\,\gamma_\mu\,U^\dagger_\mu(x-\hat{\mu}a)\,\psi(x-\hat{\mu}a)$$

c.f. $\frac{1}{2a}\big(f(x+a)-f(x-a)\big) \xrightarrow{a\to 0} \frac{df}{dx} + O(a^2)$

$$\xrightarrow{a\to 0} 2a\,\overline{\psi}\,\gamma_\mu\big(\partial_\mu + igA_\mu\big)\,\psi + \dots$$

and using constructions like these we can build **discretized actions**

e.g. $\int d^4x\,\overline{\psi}\big(\gamma_\mu D_\mu + m\big)\psi \quad \rightsquigarrow \quad \overline{\psi}^{\,i\alpha}_x\, M^{i\alpha,j\beta}_{x,y}[U]\,\psi^{j\beta}_y$

Dirac matrix

matrix in
color, spin, spacetime

N.B. **large** matrix, but **sparse**

e.g. for a
$24^3\times128$ lattice,
~ 21M×21M

most of the
elements
are zero

(100 Pb !!!)

integrating out the quarks

a gauge-field 'configuration' is simple — it's an SU(3) matrix on each link

but what about a quark-field configuration? **fermion fields anticommute ⇒ Grassmann variables**

actually we can do the quark field integration exactly in the path integral:

$$\int \mathcal{D}\psi\mathcal{D}\overline{\psi}\mathcal{D}U\, e^{-S_{\mathrm{E}}[\psi,\overline{\psi},U]} = \int \mathcal{D}U\, e^{-S^{\mathrm{g}}_{\mathrm{E}}[U]} \int \mathcal{D}\psi\mathcal{D}\overline{\psi}\, e^{-\overline{\psi}M[U]\psi}$$

$$= \det M[U]$$

$$= \int \mathcal{D}U \, \det M[U]\, e^{-S^{\mathrm{g}}_{\mathrm{E}}[U]}$$

interpret as the probability for configuration $U_\mu(x)$

a (gauge-variant) correlation function

$$\langle 0 | \hat{\psi}_x^{i\alpha} \, \hat{\overline{\psi}}_y^{j\beta} | 0 \rangle = \int \mathcal{D}\psi \mathcal{D}\overline{\psi} \mathcal{D}U \, \psi_x^{i\alpha} \, \overline{\psi}_y^{j\beta} \, e^{-S_\mathrm{E}[\psi, \overline{\psi}, U]}$$

correlation between
quark at *x*, color *i*, spin *α*
and
quark at *y*, color *j*, spin *β*

$$= \int \mathcal{D}U \, e^{-S_\mathrm{E}^\mathrm{g}[U]} \int \mathcal{D}\psi \mathcal{D}\overline{\psi} \; \psi_x^{i\alpha} \overline{\psi}_y^{j\beta} \, e^{-\overline{\psi} M[U] \psi}$$

$$= \int \mathcal{D}U \left[M^{-1}[U] \right]_{x,y}^{i\alpha, j\beta} \det M[U] \, e^{-S_\mathrm{E}^\mathrm{g}[U]}$$

c.f. Wick's theorem

probability

$$= \sum_{\{U\}} \left[M^{-1}[U] \right]_{x,y}^{i\alpha, j\beta}$$

compute
'quark propagator'
on each
configuration

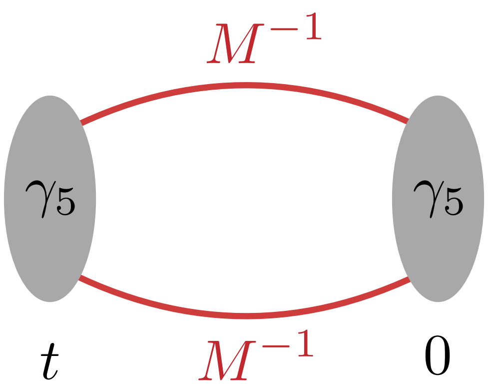
a meson correlation function

$$\langle 0 | \sum_{\vec{x}} \left(\overline{\psi}\gamma_5\psi\right)_{\vec{x},t} \left(\overline{\psi}\gamma_5\psi\right)_{\vec{0},0} | 0 \rangle$$

$\overline{\psi}\gamma_5\psi$ pseudoscalar quantum numbers

$\sum_{\vec{x}} f(\vec{x}) = \sum_{\vec{x}} e^{-i\vec{p}\cdot\vec{x}} f(\vec{x})\Big|_{\vec{p}=\vec{0}}$ projection into zero momentum

$$= -\sum_{\{U\}} \sum_{\vec{x}} \mathrm{tr}\Big(\big[M^{-1}[U]\big]_{\vec{0}0,\vec{x}t}\gamma_5 \big[M^{-1}[U]\big]_{\vec{x}t,\vec{0}0}\gamma_5 \Big)$$



point – all propagator

$$\big[M[U]\big]_{\vec{y}t',\vec{x}t}\, \chi_{\vec{x}t} = \delta_{\vec{y},\vec{0}}\, \delta_{t',0}$$

sparse matrix — point source

$$\chi_{\vec{x}t} = \big[M^{-1}[U]\big]_{\vec{x}t,\vec{0}0}$$

point-all propagator

solving a sparse linear system: $\mathbf{A} \cdot x = b$

e.g. for a
$24^3 \times 128$ lattice,
~ 21M×12
(few Gb)

the lattice QCD workflow

select a discretization

'tune' the parameters

generate 100s of gauge-field configurations — **serious parallel supercomputing**

compute quark propagators — **serious computing**
GPUs very useful

'contract' into correlation functions — **capacity computing**
'bookkeeping' / memory management

•
•
•

PHYSICS ?

lattice systematics (much simplified)

finite lattice spacing

acts as a UV cutoff $\Lambda \sim \frac{1}{a}$

appears as a scale $\hat{m} = am$

discretization errors $X(a) = X(0) + a\,\delta X_1 + \ldots$

extrapolate $a \to 0$

finite lattice volume

need $L \gg \frac{1}{m_\pi}$

impacts multi-hadron systems
in an interesting way

carefully understand QFT in a finite volume

discretization choice

all should agree in the $a \to 0$ limit

impact at finite a depends on observable

choose a discretization appropriate to your quantities

quark mass choice

many calculations done with $m_{u,d} > m_{u,d}^{\mathrm{phys}}$

use quark mass as a tool to understand QCD