

# HUGS: Introduction to QCD

- **Lec. 1: Fundamentals of QCD**
- **Lec. 2: Matching observed hadrons to quarks and gluons**
- **Lec. 3: QCD for cross sections with identified hadrons**
- **Lec. 4: QCD for cross sections with polarized beam(s)**

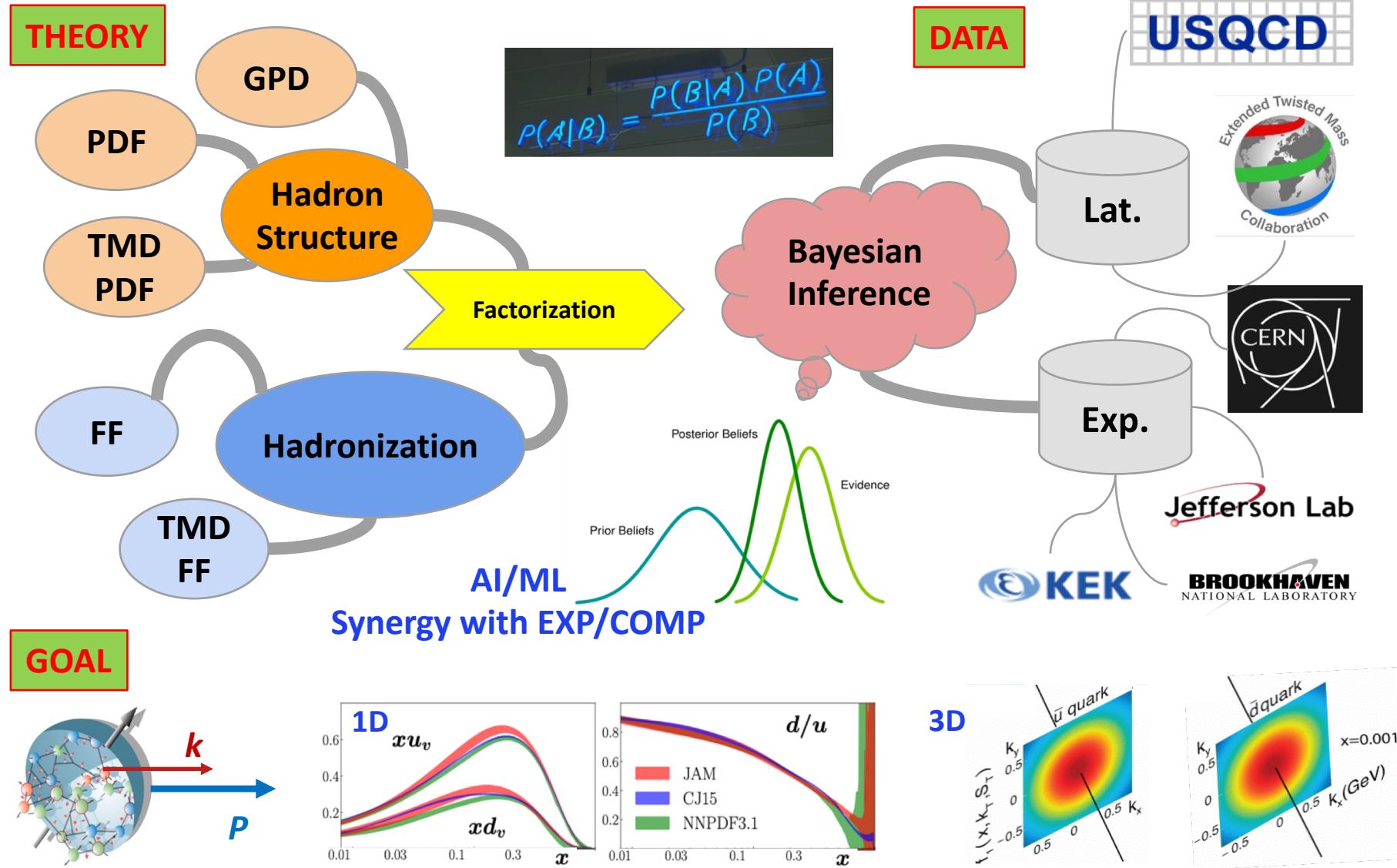
*HUGS 2020 was canceled due to COVID-19  
HUGS 2021 will be the first virtual school*

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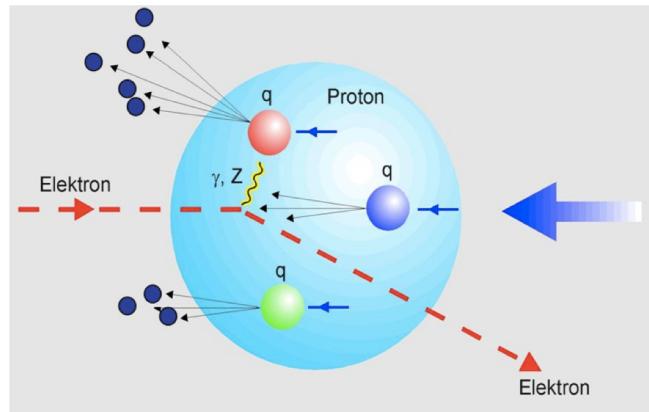


# Observables with identified hadrons – Phenomenology

☐ Need QCD global analyses of all data on factorizable cross sections!



# Inclusive lepton-hadron DIS – one hadron



$$\sigma_{\ell p \rightarrow \ell' X(\text{everything})}^{\text{DIS}}$$

**Identified initial-state hadron-proton!**

- DIS cross section is infrared divergent, and nonperturbative!

$$\sigma_{\ell p \rightarrow \ell' X(\text{everything})}^{\text{DIS}} \propto \begin{array}{c} \text{Feynman diagram showing a virtual photon (q) interacting with a proton (P) via a quark loop, followed by a muon (mu) exchange.} \\ + \end{array} \begin{array}{c} \text{Feynman diagram showing a virtual photon (q) interacting with a proton (P) via a quark loop, followed by a muon (mu) exchange, with a gluon (k_I) exchange between the quark loop and the proton.} \\ + \end{array} \begin{array}{c} \text{Feynman diagram showing a virtual photon (q) interacting with a proton (P) via a quark loop, followed by a muon (mu) exchange, with two gluon exchanges (k_I, k_2) between the quark loop and the proton.} \\ + \dots \end{array}$$

- QCD factorization (approximation!)

$$\sigma_{\ell p \rightarrow \ell' X(\text{everything})}^{\text{DIS}} = \begin{array}{c} \text{Feynman diagram showing an electron (e-) interacting with a proton (P) via a quark (q) exchange, with a virtual photon (q) exchange between the electron and the proton.} \\ \times \end{array}$$

**Physical Observable**

**Controllable Probe**

**Color entanglement Approximation**

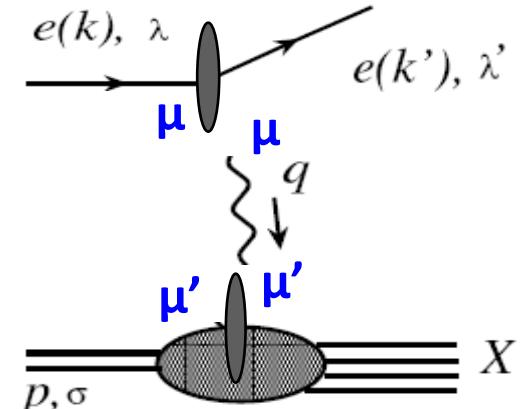
$$\begin{array}{c} \text{Feynman diagram showing a virtual photon (q) interacting with a proton (P) via a quark (q) exchange, with a gluon (g) exchange between the quark loop and the proton.} \\ \otimes \end{array} \begin{array}{c} \text{Two overlapping ellipses representing color clouds (xp, b_T, k_T) and (qd, k_T). Quantum probabilities are shown for each quark exchange.} \\ + O\left(\frac{1}{QR}\right) \end{array}$$

**Quantum Probabilities Structure**

# Inclusive lepton-hadron DIS – one hadron

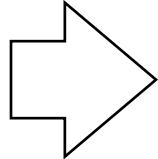
## □ Scattering amplitude:

$$\begin{aligned} M(\lambda, \lambda'; \sigma, q) &= \bar{u}_{\lambda'}(k')[-ie\gamma_\mu]u_\lambda(k) \\ &\ast \left( \frac{i}{q^2} \right) (-g^{\mu\mu'}) \\ &\ast \langle X | eJ_{\mu'}^{em}(0) | p, \sigma \rangle \end{aligned}$$



## □ Cross section:

$$d\sigma^{\text{DIS}} = \frac{1}{2s} \left( \frac{1}{2} \right)^2 \sum_X \sum_{\lambda, \lambda', \sigma} |M(\lambda, \lambda'; \sigma, q)|^2 \left[ \prod_{i=1}^X \frac{d^3 l_i}{(2\pi)^3 2E_i} \right] \frac{d^3 k'}{(2\pi)^3 2E'} (2\pi)^4 \delta^4 \left( \sum_{i=1}^X l_i + k' - p - k \right)$$



$$E, \frac{d\sigma^{\text{DIS}}}{d^3 k'} = \frac{1}{2s} \left( \frac{1}{Q^2} \right)^2 L^{\mu\nu}(k, k') W_{\mu\nu}(q, p)$$

## □ Leptonic tensor:

– known from QED

$$L^{\mu\nu}(k, k') = \frac{e^2}{2\pi^2} (k^\mu k'^\nu + k^\nu k'^\mu - k \cdot k' g^{\mu\nu})$$

# DIS structure functions

## □ Hadronic tensor:

$$W_{\mu\nu}(q, p, S) = \frac{1}{4\pi} \int d^4z e^{iq\cdot z} \langle p, S | J_\mu^\dagger(z) J_\nu(0) | p, S \rangle$$

## □ Symmetries:

- ❖ Parity invariance (EM current)  $\rightarrow W_{\mu\nu} = W_{\nu\mu}$  symmetric for spin avg.
- ❖ Time-reversal invariance  $\rightarrow W_{\mu\nu} = W_{\mu\nu}^*$  real
- ❖ Current conservation  $\rightarrow q^\mu W_{\mu\nu} = q^\nu W_{\mu\nu} = 0$

$$\begin{aligned} W_{\mu\nu} = & - \left( g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) F_1(x_B, Q^2) + \frac{1}{p \cdot q} \left( p_\mu - q_\mu \frac{p \cdot q}{q^2} \right) \left( p_\nu - q_\nu \frac{p \cdot q}{q^2} \right) F_2(x_B, Q^2) \\ & + i M_p \epsilon^{\mu\nu\rho\sigma} q_\rho \left[ \frac{S_\sigma}{p \cdot q} g_1(x_B, Q^2) + \frac{(p \cdot q) S_\sigma - (S \cdot q) p_\sigma}{(p \cdot q)^2} g_2(x_B, Q^2) \right] \end{aligned}$$
$$Q^2 = -q^2$$
$$x_B = \frac{Q^2}{2p \cdot q}$$

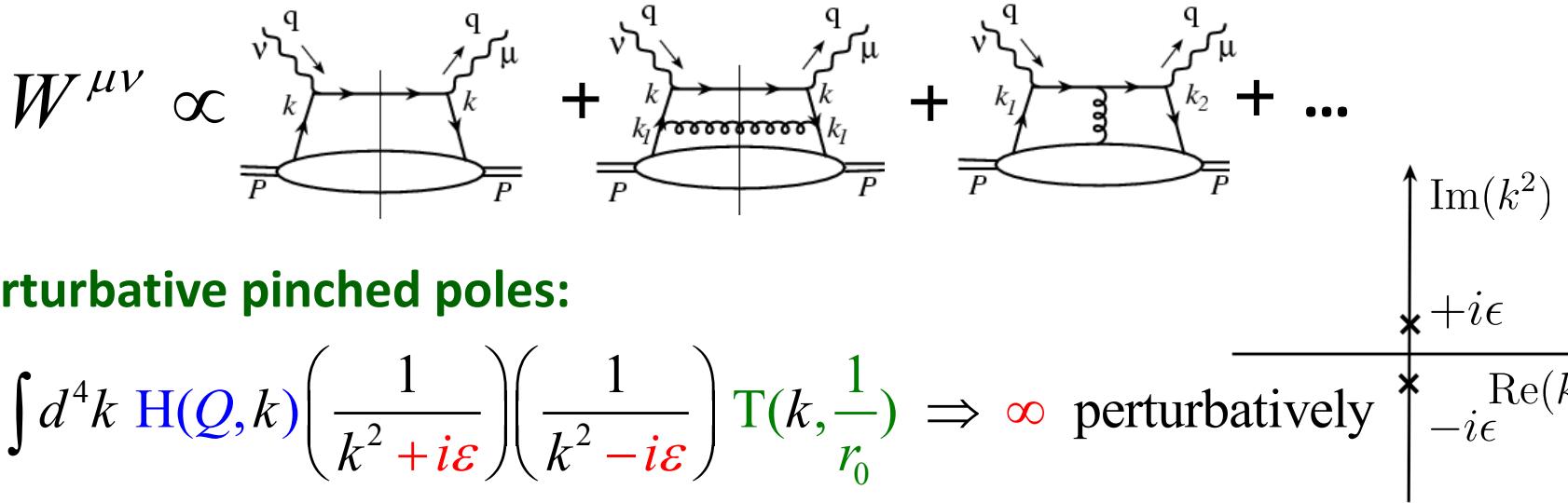
## □ Structure functions – infrared sensitive:

$$F_1(x_B, Q^2), F_2(x_B, Q^2), g_1(x_B, Q^2), g_2(x_B, Q^2)$$

No QCD parton dynamics used in above derivation!

# Long-lived parton states

## □ Feynman diagram representation of the hadronic tensor:



## □ Perturbative pinched poles:

$$\int d^4k \ H(Q, k) \left( \frac{1}{k^2 + i\epsilon} \right) \left( \frac{1}{k^2 - i\epsilon} \right) T(k, \frac{1}{r_0}) \Rightarrow \infty \text{ perturbatively}$$

## □ Perturbative factorization:

$$k^\mu = x p^\mu + \frac{k^2 + k_T^2}{2xp \cdot n} n^\mu + k_T^\mu$$

## Light-cone coordinate:

$$v^\mu = (v^+, v^-, v^\perp), v^\pm = \frac{1}{\sqrt{2}}(v^0 \pm v^3)$$

$$\int \frac{dx}{x} d^2 k_T \ H(Q, k^2 = 0) \int dk^2 \left( \frac{1}{k^2 + i\epsilon} \right) \left( \frac{1}{k^2 - i\epsilon} \right) T(k, \frac{1}{r_0}) + \mathcal{O} \left( \frac{\langle k^2 \rangle}{Q^2} \right)$$

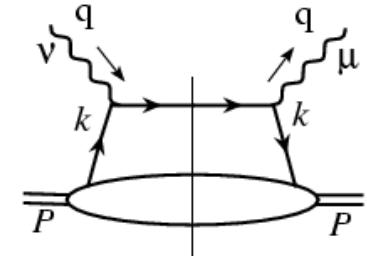
Short-distance

Nonperturbative matrix element

# Collinear factorization – further approximation

□ Collinear approximation, if  $Q \sim xp \cdot n \gg k_T, \sqrt{k^2}$

– Lowest order:  $\delta((k+q)^2) = \frac{1}{2P \cdot q} \delta(x - \xi) = \frac{1}{2P \cdot q} \delta\left(x - \frac{k^+}{P^+}\right)$



$$\begin{aligned} W_{\gamma^* p}^{\mu\nu} &= \sum_f \int \frac{d^4 k}{(2\pi)^4} \sum_{ij} (\gamma^\mu \gamma \cdot (k+q) \gamma^\nu)_{ij} (2\pi) \delta((k+q)^2) \int d^4 y e^{iky} \langle p | \bar{\psi}_j(0) \psi_i(y) | p \rangle + \dots \\ &\equiv \sum_f \int \frac{d^4 k}{(2\pi)^4} \text{Tr} \left[ \mathcal{H}_{\gamma^* f}^{\mu\nu}(Q, k) \mathcal{F}_{f/p}(k, p) \right] + \dots \\ &\approx \sum_f \int dx \text{Tr} \left[ \mathcal{H}_{\gamma^* f}^{\mu\nu}(Q, k \approx xp) \int \frac{d^4 k}{(2\pi)^4} \delta(x - \frac{k \cdot n}{p \cdot n}) \mathcal{F}_{f/p}(k, p) \right] + \mathcal{O}(\frac{\langle k^2 \rangle}{Q^2}, \frac{\langle k_T^2 \rangle}{Q^2}) + \dots \\ &\approx \sum_f \int \frac{dx}{x} \text{Tr} \left[ \mathcal{H}_{\gamma^* f}^{\mu\nu}(Q, xp) \frac{1}{2} \gamma \cdot (xp) \right] \int \frac{d^4 k}{(2\pi)^4} \delta(x - \frac{k \cdot n}{p \cdot n}) \text{Tr} \left[ \frac{\gamma \cdot n}{2p \cdot n} \mathcal{F}_{f/p}(k, p) \right] + \dots \\ &\approx \sum_f \int \frac{dx}{x} \widehat{W}_{\gamma^* f}^{\mu\nu}(x, Q^2/\mu^2) \phi_{f/p}(x, \mu^2) + \dots \end{aligned}$$

$$\int \frac{dx}{x} \quad \frac{\gamma \cdot n}{2p \cdot n} \delta\left(x - \frac{k \cdot n}{p \cdot n}\right) \frac{d^4 k}{(2\pi)^4}$$

$$\approx \left[ \text{Diagram with } k = xp \right] + \mathcal{O}\left(\frac{k_T^2}{Q^2}\right) \otimes \left[ \text{Diagram with } \otimes \text{ and } +\text{UVCT}(\mu) \right]$$

$$\frac{1}{2} \gamma \cdot (xp)$$

$$\widehat{W}_{\gamma^* f}^{\mu\nu}(x, Q^2/\mu^2) = \text{Tr} \left[ \mathcal{H}_{\gamma^* f}^{\mu\nu}(Q, xp) \frac{1}{2} \gamma \cdot (xp) \right]$$

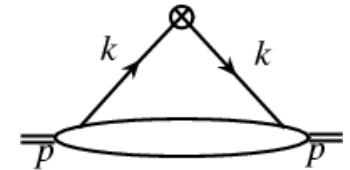
# Parton distribution functions (PDFs)

## □ PDFs as matrix elements of two parton fields:

- combine the amplitude & its complex-conjugate

$$\phi_{q/h}(x, \mu^2) = \int \frac{p^+ dy^-}{2\pi} e^{ixp^+y^-} \langle h(p) | \bar{\psi}_q(0) \frac{\gamma^+}{2p^+} \psi_q(y^-) | h(p) \rangle$$

$|h(p)\rangle$  can be a hadron, or a nucleus, or a parton state!



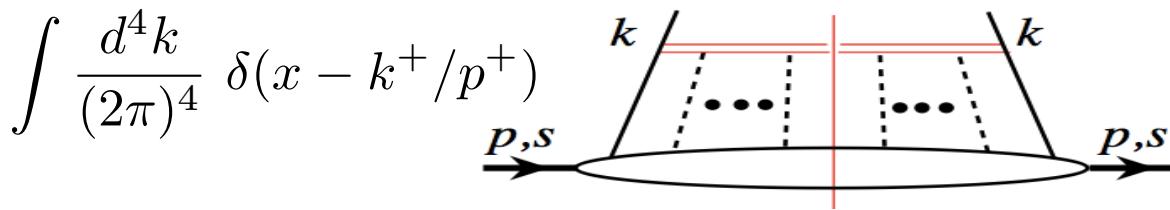
But, it is NOT gauge invariant!

$$\psi(x) \rightarrow e^{i\alpha_a(x)t_a} \psi(x) \quad \bar{\psi}(x) \rightarrow \bar{\psi}(x) e^{-i\alpha_a(x)t_a}$$

- need a gauge link:

$$\phi_{q/h}(x, \mu^2) = \int \frac{dy^-}{2\pi} e^{ixp^+y^-} \langle h(p) | \bar{\psi}_q(0) \left[ \mathcal{P} e^{-ig \int_0^{y^-} d\eta^- A^+(\eta^-)} \right] \frac{\gamma^+}{2p^+} \psi_q(y^-) | h(p) \rangle \mathcal{Z}_{\mathcal{O}}(\mu^2)$$

- corresponding diagram in momentum space:



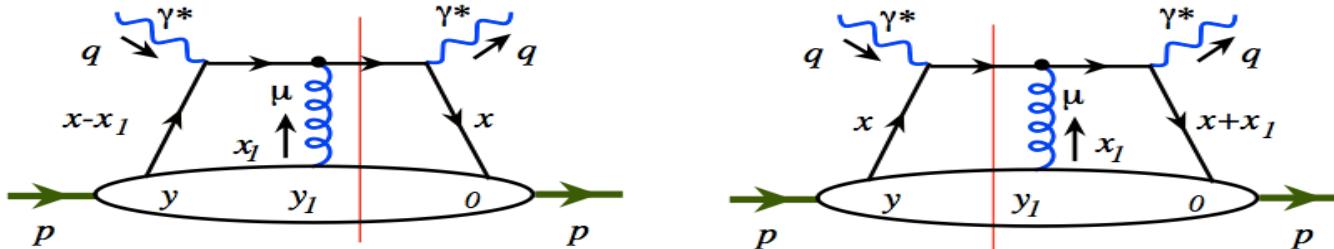
+ UVCT( $\mu^2$ )

$\mu$ -dependence

Universality – process independence – predictive power

# Gauge link – 1<sup>st</sup> order in coupling “g”

## □ Longitudinal gluon:



## □ Left diagram:

$$\begin{aligned} & \int dx_1 \left[ \int \frac{p^+ dy_1^-}{2\pi} e^{ix_1 p^+ (y_1^- - y^-)} n \cdot A^a(y_1^-) \right] \mathcal{M}(-igt^a) \frac{\gamma \cdot p}{p^+} \frac{i((x - x_1 - x_B) \gamma \cdot p + (Q^2/2x_B p^+) \gamma \cdot n)}{(x - x_1 - x_B) Q^2/x_B + i\epsilon} \\ &= g \int \frac{dy_1^-}{2\pi} n \cdot A^a(y_1^-) t^a \left[ \int dx_1 \frac{1}{-x_1 + i\epsilon} e^{ix_1 p^+ (y_1^- - y^-)} \right] \mathcal{M} = -ig \int_{y^-}^{\infty} dy_1^- n \cdot A(y_1^-) \mathcal{M} \end{aligned}$$

## □ Right diagram:

$$\begin{aligned} & \int dx_1 \left[ \int \frac{p^+ dy_1^-}{2\pi} e^{ix_1 p^+ y_1^-} n \cdot A^a(y_1^-) \right] \frac{-i((x + x_1 - x_B) \gamma \cdot p + (Q^2/2x_B p^+) \gamma \cdot n)}{(x + x_1 - x_B) Q^2/x_B - i\epsilon} (+igt^a) \frac{\gamma \cdot p}{p^+} \mathcal{M} \\ &= g \int \frac{dy_1^-}{2\pi} n \cdot A^a(y_1^-) t^a \left[ \int dx_1 \frac{1}{x_1 - i\epsilon} e^{ix_1 p^+ y_1^-} \right] \mathcal{M} = ig \int_0^{\infty} dy_1^- n \cdot A(y_1^-) \mathcal{M} \end{aligned}$$

## □ Total contribution:

$$-ig \left[ \int_0^{\infty} - \int_{y^-}^{\infty} \right] dy_1^- n \cdot A(y_1^-) \mathcal{M}_{\text{LO}}$$

O(g)-term of  
the gauge link!

# QCD high order corrections

## □ NLO partonic diagram to structure functions:

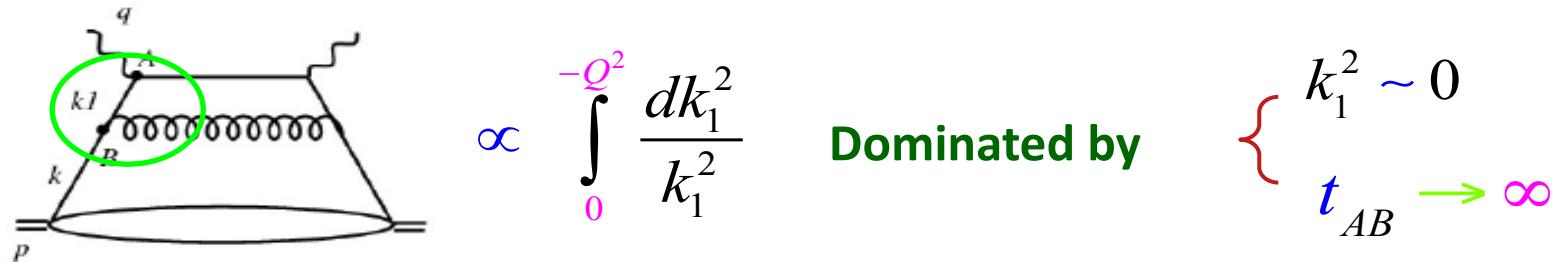
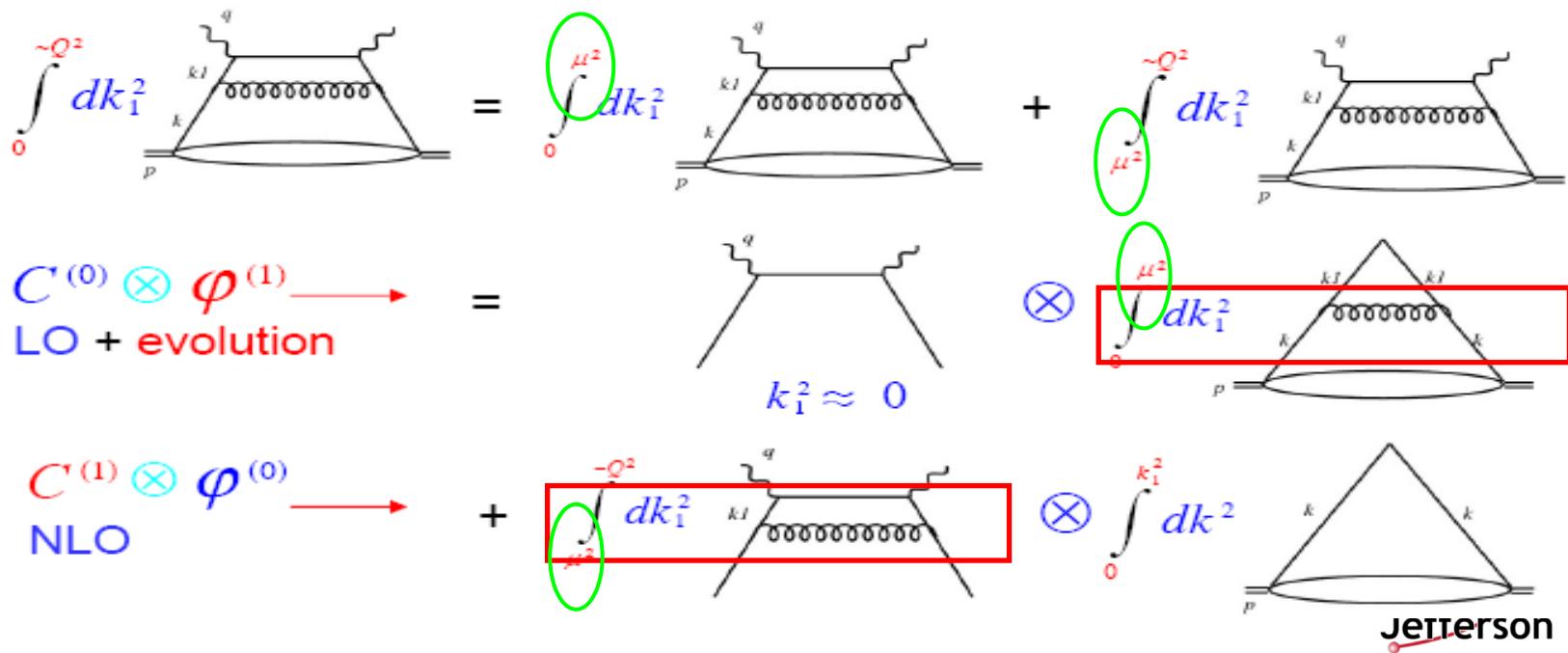


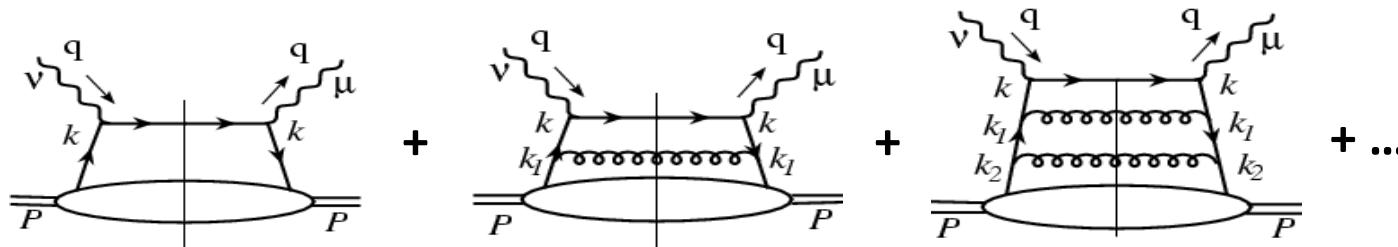
Diagram has both long- and short-distance physics

## □ Factorization, separation of short- from long-distance:

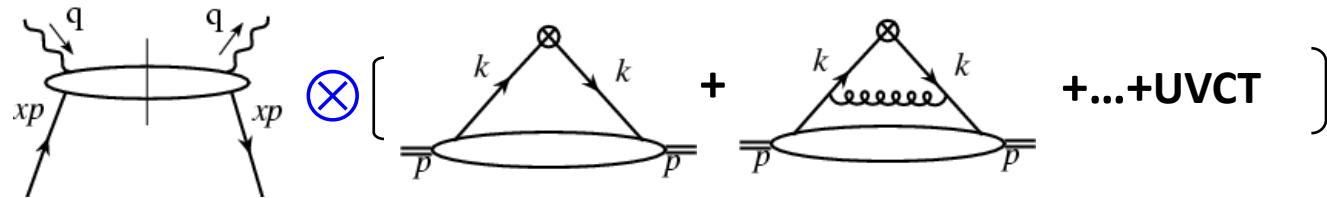


# QCD leading power factorization

## □ QCD corrections: pinch singularities in



## □ Logarithmic contributions into parton distributions:



$$\rightarrow F_2(x_B, Q^2) = \sum_f C_f \left( \frac{x_B}{x}, \frac{Q^2}{\mu_F^2}, \alpha_s \right) \otimes \varphi_f(x, \mu_F^2) + O\left(\frac{\Lambda_{\text{QCD}}^2}{Q^2}\right)$$

## □ Factorization scale: $\mu_F^2$

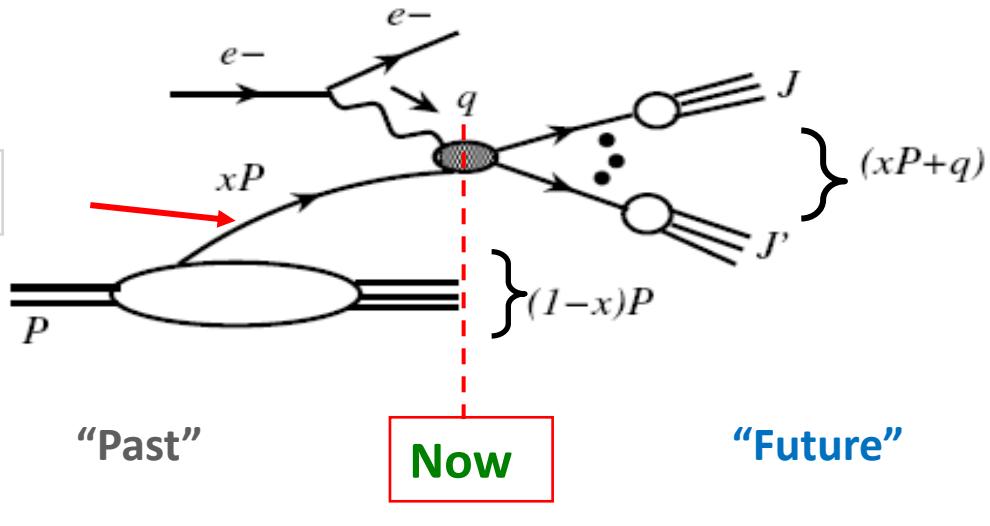
→ To separate the collinear from non-collinear contribution

Recall: renormalization scale to separate local from non-local contribution

# Picture of factorization for DIS

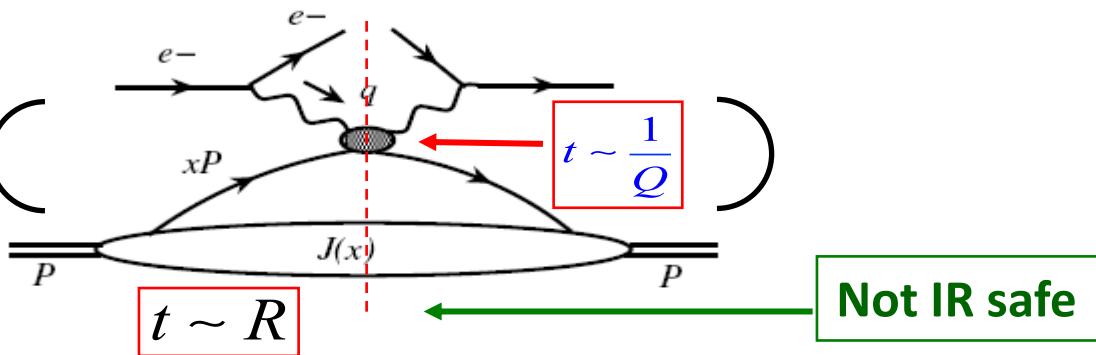
## □ Time evolution:

Long-lived parton state



## □ Unitarity – summing over all hard jets:

$$\sigma_{\text{tot}}^{\text{DIS}} \propto \text{Im} \left( \frac{1}{Q^2} \right)$$



Interaction between the “past” and “now” are suppressed!

# How to calculate the perturbative parts?

## □ Use DIS structure function $F_2$ as an example:

$$F_{2h}(x_B, Q^2) = \sum_{q,f} C_{q/f} \left( \frac{x_B}{x}, \frac{Q^2}{\mu^2}, \alpha_s \right) \otimes \varphi_{f/h}(x, \mu^2) + O\left(\frac{\Lambda_{\text{QCD}}^2}{Q^2}\right)$$

❖ Apply the factorized formula to parton states:  $h \rightarrow q$

**Feynman diagrams** →  $F_{2q}(x_B, Q^2) = \sum_{q,f} C_{q/f} \left( \frac{x_B}{x}, \frac{Q^2}{\mu^2}, \alpha_s \right) \otimes \varphi_{f/q}(x, \mu^2)$  ← **Feynman diagrams**

❖ Express both SFs and PDFs in terms of powers of  $\alpha_s$ :

**0<sup>th</sup> order:**  $F_{2q}^{(0)}(x_B, Q^2) = C_q^{(0)}(x_B/x, Q^2/\mu^2) \otimes \varphi_{q/q}^{(0)}(x, \mu^2)$

$C_q^{(0)}(x) = F_{2q}^{(0)}(x)$        $\varphi_{q/q}^{(0)}(x) = \delta_{qq} \delta(1-x)$

**1<sup>th</sup> order:**  $F_{2q}^{(1)}(x_B, Q^2) = C_q^{(1)}(x_B/x, Q^2/\mu^2) \otimes \varphi_{q/q}^{(0)}(x, \mu^2)$

$$+ C_q^{(0)}(x_B/x, Q^2/\mu^2) \otimes \varphi_{q/q}^{(1)}(x, \mu^2)$$

$C_q^{(1)}(x, Q^2/\mu^2) = F_{2q}^{(1)}(x, Q^2) - F_{2q}^{(0)}(x, Q^2) \otimes \varphi_{q/q}^{(1)}(x, \mu^2)$

# PDFs of a parton

## □ Change the state without changing the operator:

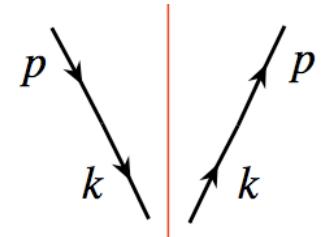
$$\phi_{q/h}(x, \mu^2) = \int \frac{dy^-}{2\pi} e^{ixp^+y^-} \langle h(p) | \bar{\psi}_q(0) \frac{\gamma^+}{2} U_{[0,y^-]}^n \psi_2(y^-) | h(p) \rangle$$

$|h(p)\rangle \Rightarrow |\text{parton}(p)\rangle$    $\phi_{f/q}(x, \mu^2)$  – given by Feynman diagrams

## □ Lowest order quark distribution:

✧ From the operator definition:

$$\begin{aligned} \phi_{q'/q}^{(0)}(x) &= \delta_{qq'} \int \frac{d^4k}{(2\pi)^4} \text{Tr} \left[ \left( \frac{1}{2} \gamma \cdot p \right) \left( \frac{\gamma^+}{2p^+} \right) \right] \delta \left( x - \frac{k^+}{p^+} \right) (2\pi)^4 \delta^4(p - k) \\ &= \delta_{qq'} \delta(1 - x) \end{aligned}$$

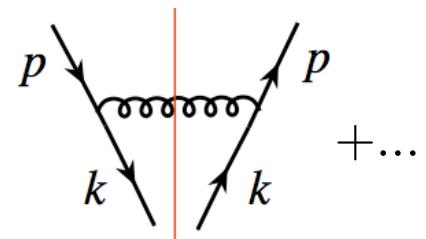


## □ Leading order in $\alpha_s$ quark distribution:

✧ Expand to  $(g_s)^2$  – logarithmic divergent:

$$\phi_{q/q}^{(1)}(x) = C_F \frac{\alpha_s}{2\pi} \int \frac{dk_T^2}{k_T^2} \left[ \frac{1+x^2}{(1-x)_+} + \frac{3}{2} \delta(1-x) \right] + \text{UVCT}$$

UV and CO divergence



# Partonic cross sections

## □ Projection operators for SFs:

$$W_{\mu\nu} = - \left( g_{\mu\nu} - \frac{q_\mu q_\nu}{q^2} \right) F_1(x, Q^2) + \frac{1}{p \cdot q} \left( p_\mu - q_\mu \frac{p \cdot q}{q^2} \right) \left( p_\nu - q_\nu \frac{p \cdot q}{q^2} \right) F_2(x, Q^2)$$

$$F_1(x, Q^2) = \frac{1}{2} \left( -g^{\mu\nu} + \frac{4x^2}{Q^2} p^\mu p^\nu \right) W_{\mu\nu}(x, Q^2)$$

$$F_2(x, Q^2) = x \left( -g^{\mu\nu} + \frac{12x^2}{Q^2} p^\mu p^\nu \right) W_{\mu\nu}(x, Q^2)$$

## □ 0<sup>th</sup> order:

$$\begin{aligned} F_{2q}^{(0)}(x) &= x g^{\mu\nu} W_{\mu\nu,q}^{(0)} = x g^{\mu\nu} \left[ \frac{1}{4\pi} \text{Diagram} \right] \\ &= \left( x g^{\mu\nu} \right) \frac{e_q^2}{4\pi} \text{Tr} \left[ \frac{1}{2} \gamma \cdot p \gamma_\mu \gamma \cdot (p+q) \gamma_\nu \right] 2\pi \delta((p+q)^2) \\ &= e_q^2 x \delta(1-x) \end{aligned}$$

$$C_q^{(0)}(x) = e_q^2 x \delta(1-x)$$

# NLO coefficient function – complete example

$$C_q^{(1)}(x, Q^2 / \mu^2) = F_{2q}^{(1)}(x, Q^2) - F_{2q}^{(0)}(x, Q^2) \otimes \varphi_{q/q}^{(1)}(x, \mu^2)$$

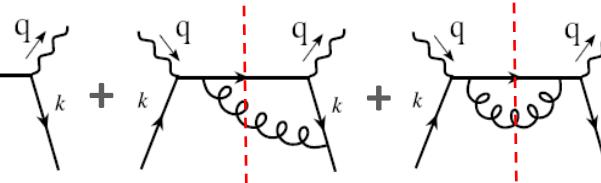
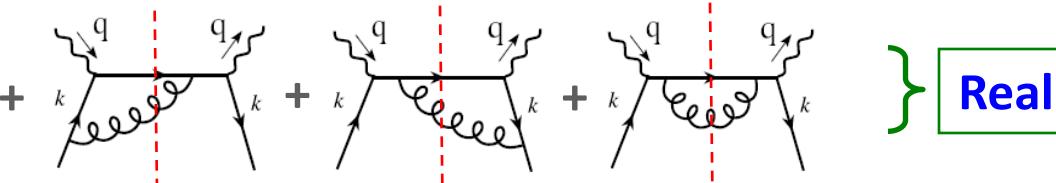
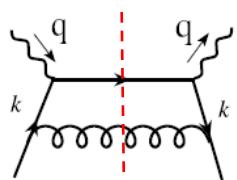
□ Projection operators in n-dimension:

$$g_{\mu\nu} g^{\mu\nu} = n \equiv 4 - 2\varepsilon$$

$$(1-\varepsilon) F_2 = x \left( -g^{\mu\nu} + (3-2\varepsilon) \frac{4x^2}{Q^2} p^\mu p^\nu \right) W_{\mu\nu}$$

□ Feynman diagrams:

$$W_{\mu\nu,q}^{(1)}$$



Real

Virtual

{

+

+ c.c.

+

+ c.c.

+ UV CT

□ Calculation:

$$-g^{\mu\nu} W_{\mu\nu,q}^{(1)} \text{ and } p^\mu p^\nu W_{\mu\nu,q}^{(1)}$$

# Contribution from the trace of $W_{\mu\nu}$

□ Lowest order in n-dimension:

$$-g^{\mu\nu}W_{\mu\nu,q}^{(0)} = e_q^2(1-\varepsilon)\delta(1-x)$$

□ NLO virtual contribution:

$$-g^{\mu\nu}W_{\mu\nu,q}^{(1)V} = e_q^2(1-\varepsilon)\delta(1-x)$$

$$*\left(-\frac{\alpha_s}{\pi}\right) \textcolor{blue}{C_F} \left[ \frac{4\pi\mu^2}{Q^2} \right]^\varepsilon \frac{\Gamma(1+\varepsilon)\Gamma^2(1-\varepsilon)}{\Gamma(1-2\varepsilon)} \left[ \frac{1}{\varepsilon^2} + \frac{3}{2} \frac{1}{\varepsilon} + 4 \right]$$

□ NLO real contribution:

$$-g^{\mu\nu}W_{\mu\nu,q}^{(1)R} = e_q^2(1-\varepsilon) \textcolor{blue}{C_F} \left(-\frac{\alpha_s}{2\pi}\right) \left[ \frac{4\pi\mu^2}{Q^2} \right]^\varepsilon \frac{\Gamma(1+\varepsilon)}{\Gamma(1-2\varepsilon)}$$

$$*\left\{-\frac{1-\varepsilon}{\varepsilon} \left[ 1-x + \left(\frac{2x}{1-x}\right) \left(\frac{1}{1-2\varepsilon}\right) \right] + \frac{1-\varepsilon}{2(1-2\varepsilon)(1-x)} + \frac{2\varepsilon}{1-2\varepsilon}\right\}$$

□ The “+” distribution:

$$\left(\frac{1}{1-x}\right)^{1+\varepsilon} = -\frac{1}{\varepsilon} \delta(1-x) + \frac{1}{(1-x)_+} + \varepsilon \left( \frac{\ell n(1-x)}{1-x} \right)_+ + O(\varepsilon^2)$$

$$\int_z^1 dx \frac{f(x)}{(1-x)_+} \equiv \int_z^1 dx \frac{f(x)-f(1)}{1-x} + \ell n(1-z) f(1)$$

□ One loop contribution to the trace of  $W_{\mu\nu}$ :

$$\begin{aligned} -g^{\mu\nu} W_{\mu\nu,q}^{(1)} &= e_q^2 (1-\varepsilon) \left( \frac{\alpha_s}{2\pi} \right) \left\{ -\frac{1}{\varepsilon} \textcolor{magenta}{P}_{qq}(x) + \textcolor{magenta}{P}_{qq}(x) \ell n \left( \frac{Q^2}{\mu^2 (4\pi e^{-\gamma_E})} \right) \right. \\ &\quad + \textcolor{blue}{C}_F \left[ \left( 1+x^2 \right) \left( \frac{\ell n(1-x)}{1-x} \right)_+ - \frac{3}{2} \left( \frac{1}{1-x} \right)_+ - \frac{1+x^2}{1-x} \ell n(x) \right. \\ &\quad \left. \left. + 3-x - \left( \frac{9}{2} + \frac{\pi^2}{3} \right) \delta(1-x) \right] \right\} \end{aligned}$$

□ Splitting function:

$$\textcolor{magenta}{P}_{qq}(x) = \textcolor{blue}{C}_F \left[ \frac{1+x^2}{(1-x)_+} + \frac{3}{2} \delta(1-x) \right]$$

□ One loop contribution to  $p^\mu p^\nu W_{\mu\nu}$ :

$$p^\mu p^\nu W_{\mu\nu,q}^{(1)V} = 0 \quad p^\mu p^\nu W_{\mu\nu,q}^{(1)R} = e_q^2 \textcolor{blue}{C}_F \frac{\alpha_s}{2\pi} \frac{Q^2}{4x}$$

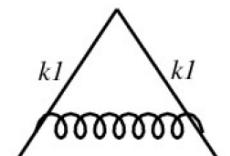
□ One loop contribution to  $F_2$  of a quark:

$$\begin{aligned} F_{2q}^{(1)}(x, Q^2) &= e_q^2 x \frac{\alpha_s}{2\pi} \left\{ \left( -\frac{1}{\varepsilon} \right)_{\text{CO}} \textcolor{magenta}{P}_{qq}(x) \left( 1 + \varepsilon \ln(4\pi e^{-\gamma_E}) \right) + \textcolor{magenta}{P}_{qq}(x) \ln \left( \frac{Q^2}{\mu^2} \right) \right. \\ &\quad \left. + \textcolor{blue}{C}_F \left[ (1+x^2) \left( \frac{\ln(1-x)}{1-x} \right)_+ - \frac{3}{2} \left( \frac{1}{1-x} \right)_+ - \frac{1+x^2}{1-x} \ln(x) + 3 + 2x - \left( \frac{9}{2} + \frac{\pi^2}{3} \right) \delta(1-x) \right] \right\} \\ &\Rightarrow \infty \quad \text{as } \varepsilon \rightarrow 0 \end{aligned}$$

□ One loop contribution to quark PDF of a quark:

$$\varphi_{q/q}^{(1)}(x, \textcolor{blue}{\mu^2}) = \left( \frac{\alpha_s}{2\pi} \right) \textcolor{magenta}{P}_{qq}(x) \left\{ \left( \frac{1}{\varepsilon} \right)_{\text{UV}} + \left( -\frac{1}{\varepsilon} \right)_{\text{CO}} \right\} + \text{UV-CT}$$

– in the dimensional regularization



**Different UV-CT = different factorization scheme!**

# Evolution

- Physical cross sections should not depend on the factorization scale

$$\mu_F^2 \frac{d}{d\mu_F^2} F_2(x_B, Q^2) = 0$$

$$F_2(x_B, Q^2) = \sum_f C_f(x_B/x, Q^2/\mu_F^2, \alpha_s) \phi_f(x, \mu_F^2)$$

→ Evolution (differential-integral) equation for PDFs

$$\sum_f \left[ \mu_F^2 \frac{d}{d\mu_F^2} C_f \left( \frac{x_B}{x}, \frac{Q^2}{\mu_F^2}, \alpha_s \right) \right] \otimes \varphi_f(x, \mu_F^2) + \sum_f C_f \left( \frac{x_B}{x}, \frac{Q^2}{\mu_F^2}, \alpha_s \right) \otimes \mu_F^2 \frac{d}{d\mu_F^2} \varphi_f(x, \mu_F^2) = 0$$

- PDFs and coefficient functions share the same logarithms

PDFs:  $\log(\mu_F^2 / \mu_0^2)$  or  $\log(\mu_F^2 / \Lambda_{\text{QCD}}^2)$

Coefficient functions:  $\log(Q^2 / \mu_F^2)$  or  $\log(Q^2 / \mu^2)$

→ DGLAP evolution equation:

$$\mu_F^2 \frac{\partial}{\partial \mu_F^2} \varphi_i(x, \mu_F^2) = \sum_j P_{i/j} \left( \frac{x}{x'}, \alpha_s \right) \otimes \varphi_j(x', \mu_F^2)$$

# Calculation of evolution kernels

## ❑ Evolution kernels are process independent

- ❖ Parton distribution functions are universal
- ❖ Could be derived in many different ways

## ❑ Extract from calculating parton PDFs' scale dependence

$$Q^2 \frac{d}{dQ^2} q_i(x, Q^2) = \frac{\alpha_s}{2\pi} \int_x^1 \frac{dx_1}{x_1} q_i(x_1, Q^2) \gamma_{qq} \left( \frac{x}{x_1} \right) - \frac{\alpha_s}{2\pi} q_i(x, Q^2) \int_0^1 dz \gamma_{qq}(z)$$

Collins, Qiu, 1989

Change

“Gain”

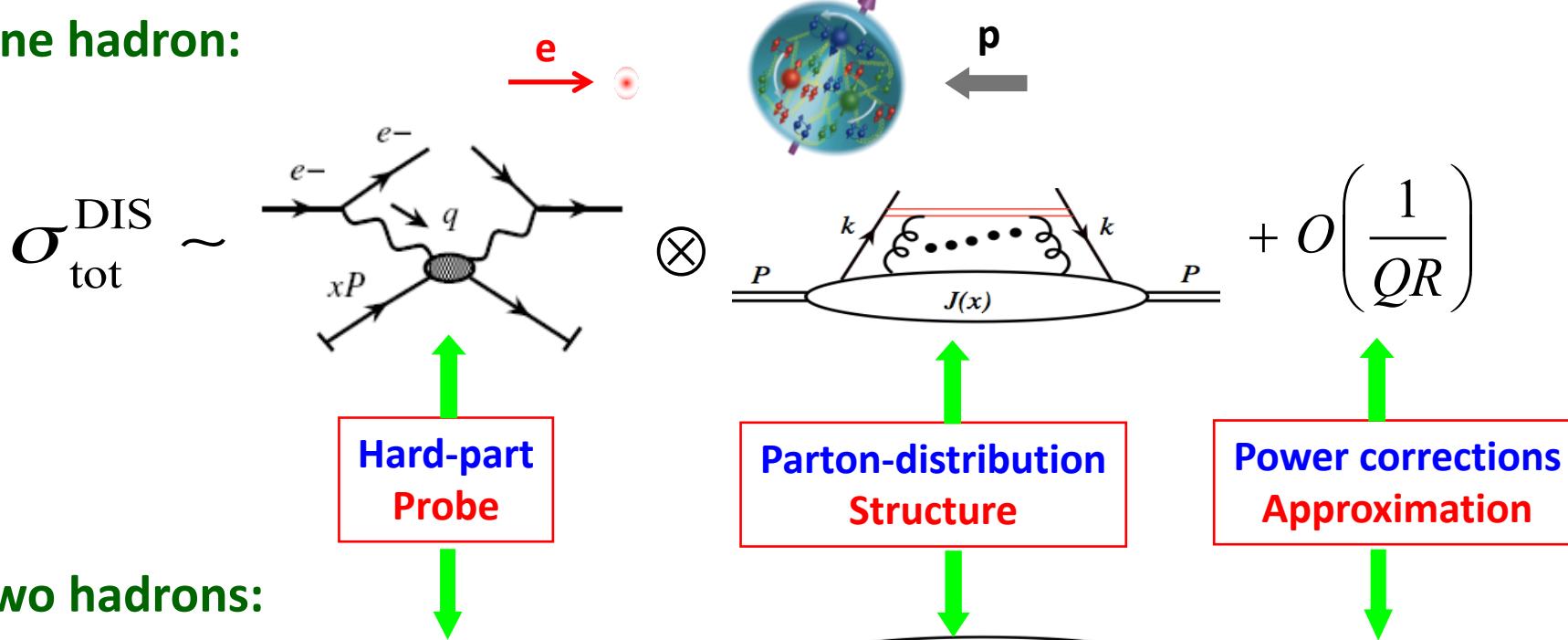
“Loss”

- ❖ Same is true for gluon evolution, and mixing flavor terms

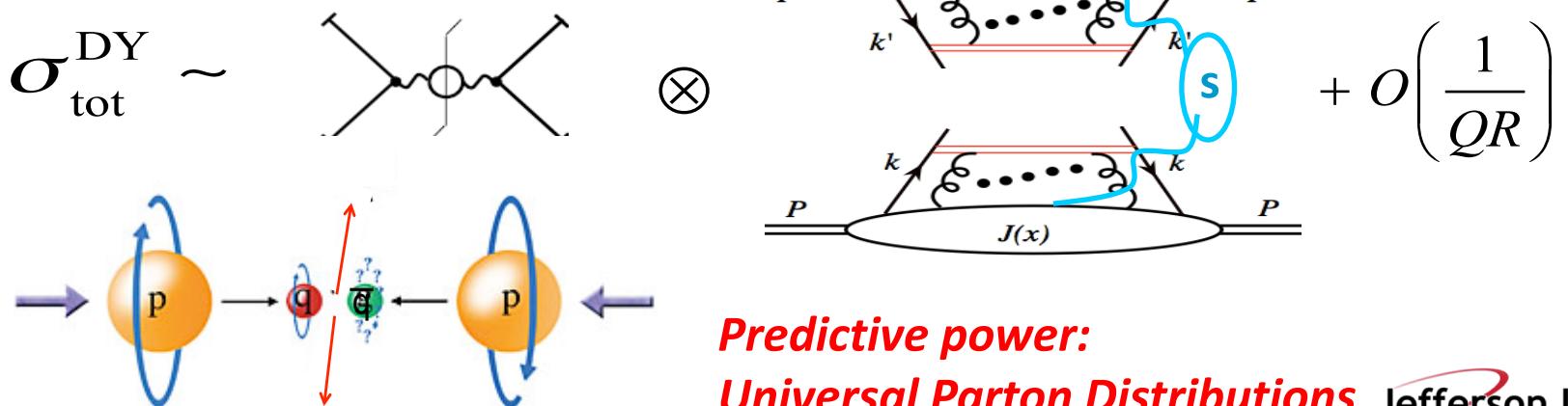
## ❑ One can also extract the kernels from the CO divergence of partonic cross sections

# From one hadron to two hadrons

## □ One hadron:



## □ Two hadrons:



**Predictive power:**

**Universal Parton Distributions**

Jefferson Lab

# Drell-Yan process – two hadrons

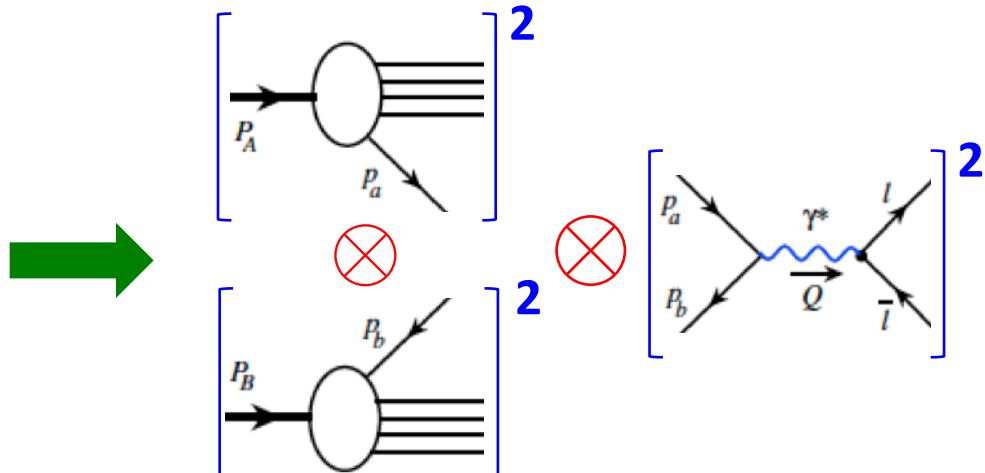
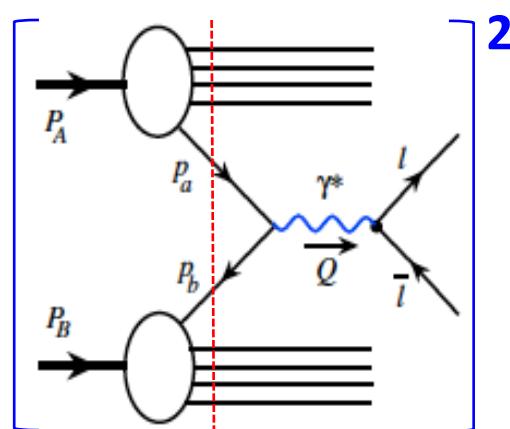
S.D. Drell and T.-M. Yan  
Phys. Rev. Lett. 25, 316 (1970)

## □ Drell-Yan mechanism:

$$A(P_A) + B(P_B) \rightarrow \gamma^*(q) [\rightarrow l\bar{l}(q)] + X \quad \text{with} \quad q^2 \equiv Q^2 \gg \Lambda_{\text{QCD}}^2 \sim 1/\text{fm}^2$$

Lepton pair – from decay of a virtual photon, or in general,  
a massive boson, e.g., W, Z, H<sup>0</sup>, ... (called Drell-Yan like processes)

## □ Original Drell-Yan formula:



$$\frac{d\sigma_{A+B \rightarrow l\bar{l}+X}}{dQ^2 dy} = \frac{4\pi\alpha_{em}^2}{3Q^4} \sum_{p,\bar{p}} x_A \phi_{p/A}(x_A) x_B \phi_{\bar{p}/B}(x_B)$$

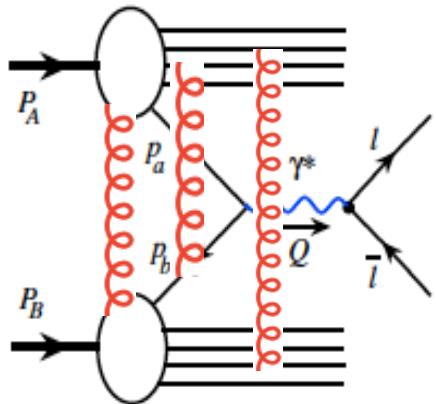
No color yet!

Rapidity:  $y = \frac{1}{2} \ln(x_A/x_B)$

$$x_A = \frac{Q}{\sqrt{S}} e^y \quad x_B = \frac{Q}{\sqrt{S}} e^{-y}$$

# Drell-Yan process in QCD – factorization

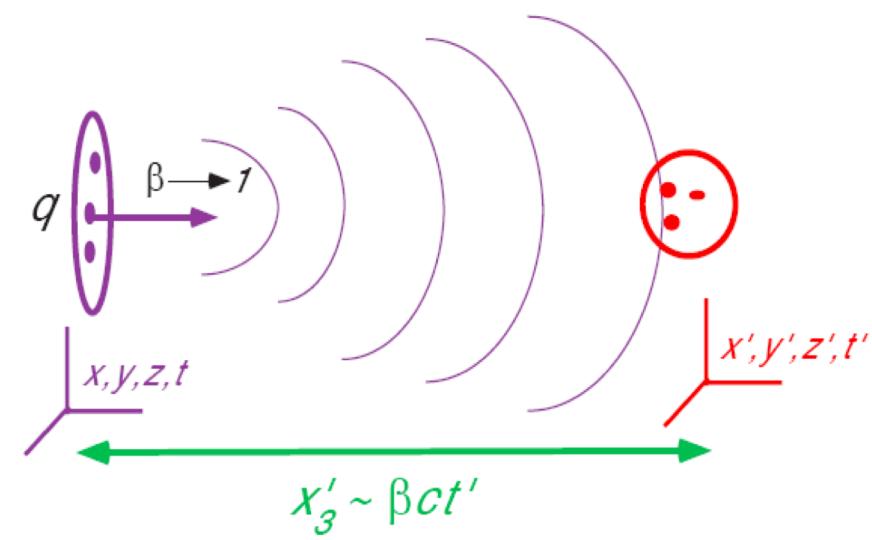
## □ Beyond the lowest order:



- ❖ Soft-gluon interaction takes place all the time
- ❖ Long-range gluon interaction before the hard collision

→ Break the Universality of PDFs  
Loss the predictive power

## □ Factorization – power suppression of soft gluon interaction:



x-Frame

$$A^-(x) = \frac{e}{|\vec{x}|}$$

$$E_3(x) = \frac{e}{|\vec{x}|^2}$$

$x'$ -Frame

$$A'^-(x') = \frac{e\gamma(1+\beta)}{(x_T^2 + \gamma^2\Delta^2)^{1/2}}$$

$\Rightarrow 1$  “not contracted!”

$$E_3(x') = \frac{-e\gamma\Delta}{(x_T^2 + \gamma^2\Delta^2)^{3/2}}$$

$\Rightarrow \frac{1}{\gamma^2}$  “strongly contracted!”

# Factorized Drell-Yan cross section

## □ TMD factorization ( $q_\perp \ll Q$ ):

$$\frac{d\sigma_{AB}}{d^4 q} = \sigma_0 \int d^2 k_{a\perp} d^2 k_{b\perp} d^2 k_{s\perp} \delta^2(q_\perp - k_{a\perp} - k_{b\perp} - k_{s\perp}) \mathcal{F}_{a/A}(x_A, k_{a\perp}) \mathcal{F}_{b/B}(x_B, k_{b\perp}) \mathcal{S}(k_{s\perp})$$
$$+ \mathcal{O}(q_\perp/Q) \quad x_A = \frac{Q}{\sqrt{s}} e^y \quad x_B = \frac{Q}{\sqrt{s}} e^{-y}$$

The soft factor,  $\mathcal{S}$ , is universal, could be absorbed into the definition of TMD parton distribution

## □ Collinear factorization ( $q_\perp \sim Q$ ):

$$\frac{d\sigma_{AB}}{d^4 q} = \int dx_a f_{a/A}(x_a, \mu) \int dx_b f_{b/B}(x_b, \mu) \frac{d\hat{\sigma}_{ab}}{d^4 q}(x_a, x_b, \alpha_s(\mu), \mu) + \mathcal{O}(1/Q)$$

## □ Spin dependence:

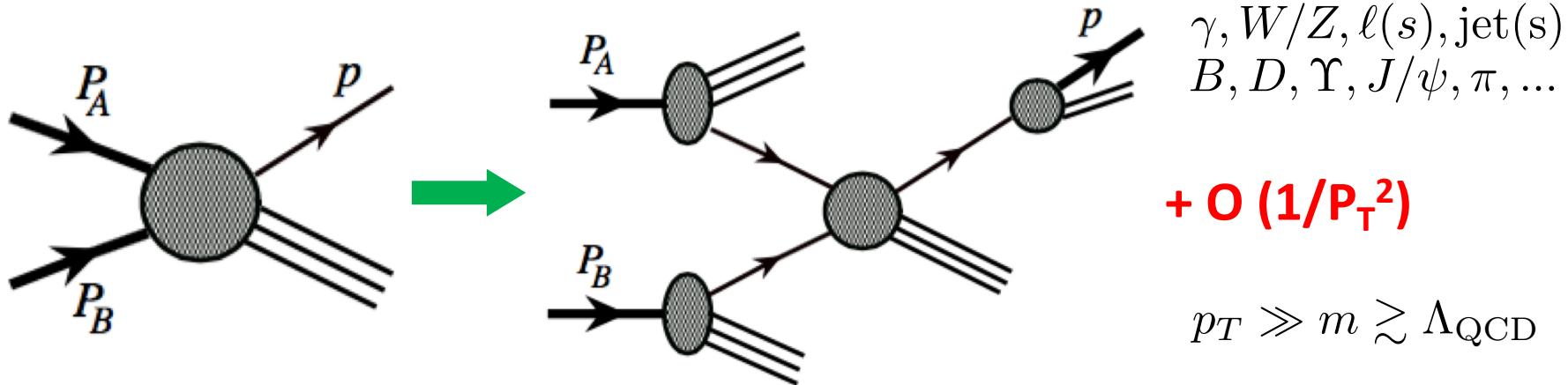
The factorization arguments are independent of the spin states of the colliding hadrons

→ same formula with polarized PDFs for  $\gamma^*, W/Z, H^0\dots$

# Factorization for more than two hadrons

## □ Factorization for high $p_T$ single hadron:

Nayak, Qiu, Sterman, 2006



$$\frac{d\sigma_{AB \rightarrow C+X}(p_A, p_B, p)}{dy dp_T^2} = \sum_{a,b,c} \phi_{A \rightarrow a}(x, \mu_F^2) \otimes \phi_{B \rightarrow b}(x', \mu_F^2) \\ \otimes \frac{d\hat{\sigma}_{ab \rightarrow c+X}(x, x', z, y, p_T^2 \mu_F^2)}{dy dp_T^2} \otimes D_{c \rightarrow C}(z, \mu_F^2)$$

✧ Fragmentation function:

$$D_{c \rightarrow C}(z, \mu_F^2)$$

✧ Choice of the scales:

$$\mu_{\text{Fac}}^2 \approx \mu_{\text{ren}}^2 \approx p_T^2$$

To minimize the size of logs in the coefficient functions

# Global QCD analysis – Testing QCD

## □ Factorization for observables with identified hadrons:

### ✧ One-hadron (DIS):

$$F_2(x_B, Q^2) = \sum_f C_f(x_B/x, \mu^2/Q^2) \otimes f(x, \mu^2)$$

### ✧ Two-hadrons (DY, Jets, W/Z, ...):

$$\frac{d\sigma}{dydp_T^2} = \sum_{ff'} f(x) \otimes \frac{d\hat{\sigma}_{ff'}}{dydp_T^2} \otimes f'(x')$$

*Testing QCD:*

### ✧ DGLAP Evolution:

*Universal PDFs for all cross sections?*

$$\frac{\partial f(x, \mu^2)}{\partial \ln \mu^2} = \sum_{f'} P_{ff'}(x/x') \otimes f'(x', \mu^2)$$

## □ Input for QCD Global analysis/fitting:

### ✧ World data with “Q” > 2 GeV

### ✧ PDFs at an input scale:

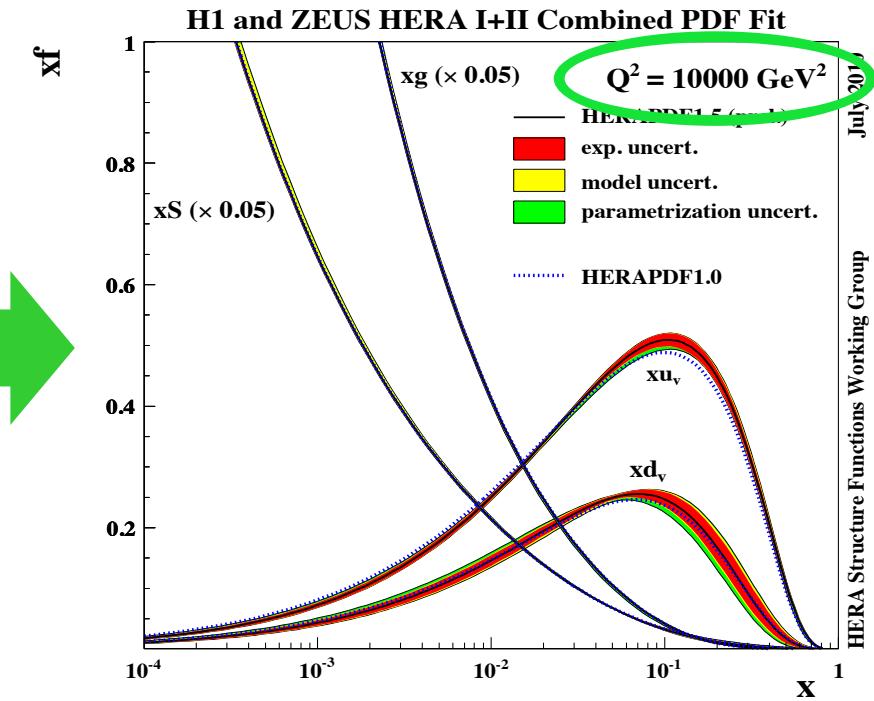
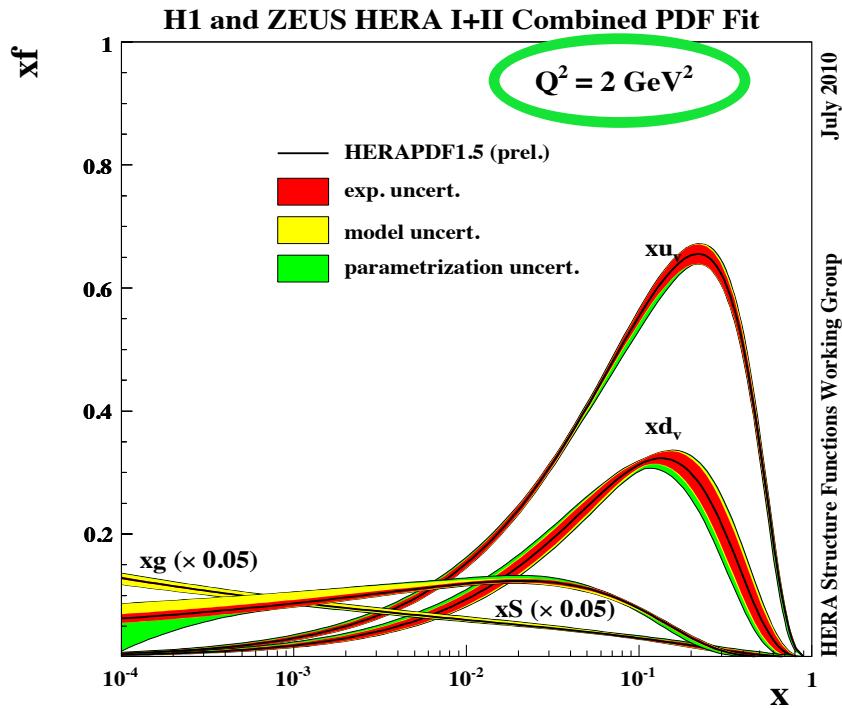
$$\phi_{f/h}(x, \mu_0^2, \{\alpha_j\})$$

Input scale ~ GeV

Fitting parameters

# PDFs from DIS

- Q<sup>2</sup>-dependence is a prediction of pQCD calculation:



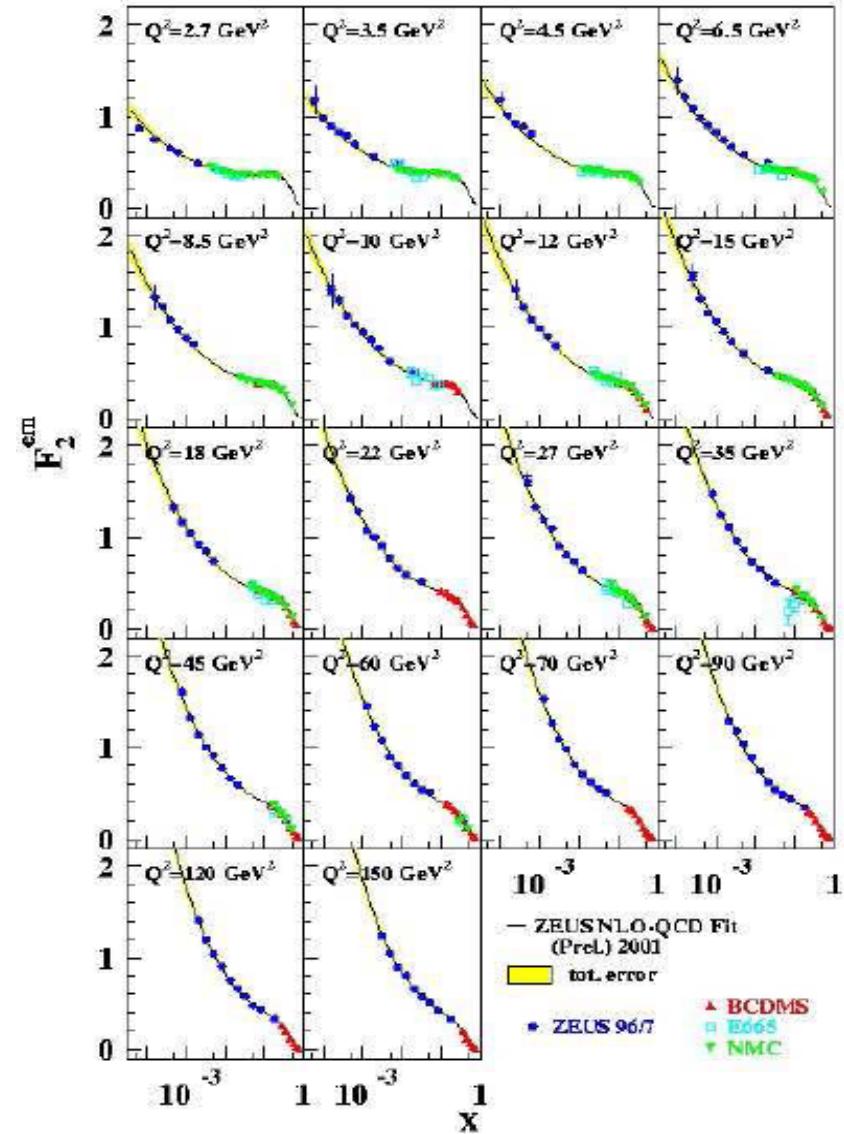
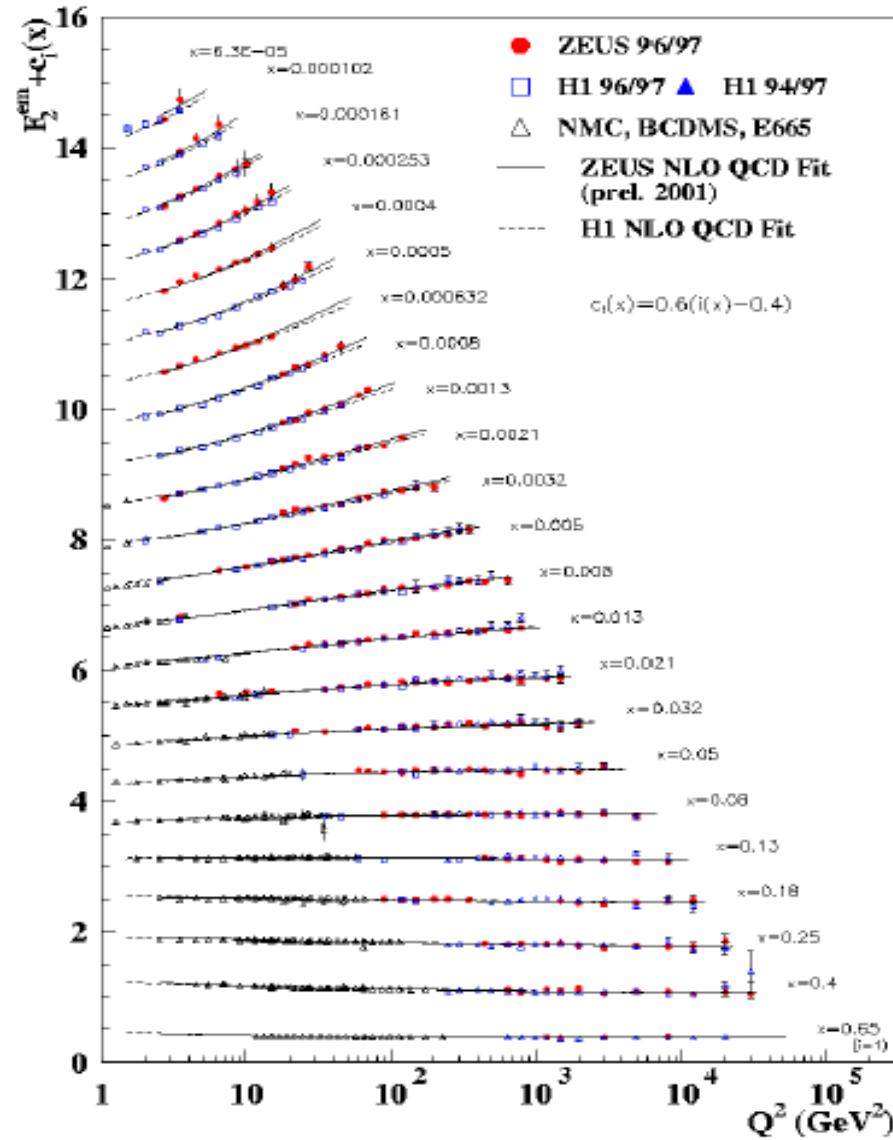
- Physics interpretation of PDFs:

$f(x, Q^2)$  : Probability density to find a parton of flavor “f” carrying momentum fraction “x”, probed at a scale of “Q<sup>2</sup>”

◇ Number of partons:  $\int_0^1 dx u_v(x, Q^2) = 2, \int_0^1 dx d_v(x, Q^2) = 1$

◇ Momentum fraction:  $\langle x(Q^2) \rangle_f = \int_0^1 dx x f(x, Q^2) \rightarrow \sum_f \langle x(Q^2) \rangle = 1$

# Scaling and scaling violation



*$Q^2$ -dependence is a prediction of pQCD calculation*

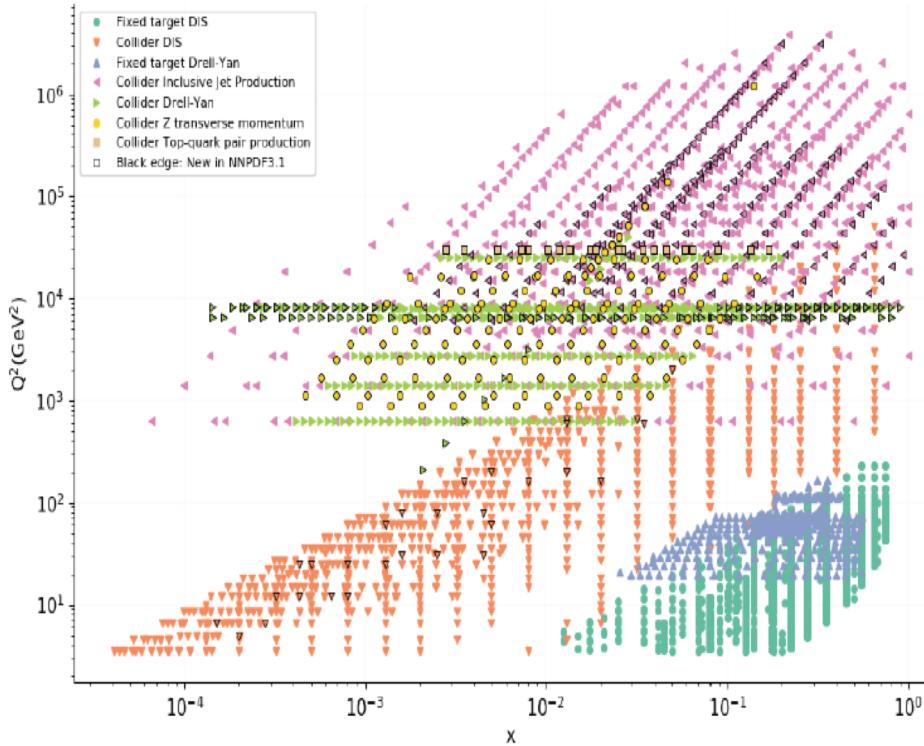
# QCD factorization works to the precision

## □ Data sets for Global Fits:

Process	Subprocess	Partons	$x$ range
$\ell^\pm [p, n] \rightarrow \ell^\pm + X$	$\gamma^* q \rightarrow q$	$q, \bar{q}, g$	$x \gtrsim 0.01$
$\ell^\pm n/p \rightarrow \ell^\pm + X$	$\gamma^* d/u \rightarrow d/u$	$d/u$	$x \gtrsim 0.01$
Fixed Target	$pp \rightarrow \mu^+ \mu^- + X$	$u\bar{u}, d\bar{d} \rightarrow \gamma^*$	$\bar{q}$
	$p\bar{n}/p \rightarrow \mu^+ \mu^- + X$	$(u\bar{d})/(u\bar{d}) \rightarrow \gamma^*$	$\bar{d}/u$
	$\nu(\bar{\nu}) N \rightarrow \mu^-(\mu^+) + X$	$W^* q \rightarrow q'$	$q, \bar{q}$
	$\nu N \rightarrow \mu^- \mu^+ + X$	$W^* s \rightarrow c$	$s$
	$\bar{\nu} N \rightarrow \mu^+ \mu^- + X$	$W^* \bar{s} \rightarrow \bar{c}$	$\bar{s}$
Collider DIS	$e^\pm p \rightarrow e^\pm + X$	$\gamma^* q \rightarrow q$	$0.0001 \lesssim x \lesssim 0.1$
	$e^+ p \rightarrow \bar{\nu} + X$	$W^* [d, s] \rightarrow [u, c]$	$d, s$
	$e^\pm p \rightarrow e^\pm c\bar{c} + X$	$\gamma^* c \rightarrow c, \gamma^* g \rightarrow c\bar{c}$	$c, g$
	$e^\pm p \rightarrow e^\pm b\bar{b} + X$	$\gamma^* b \rightarrow b, \gamma^* g \rightarrow b\bar{b}$	$b, g$
	$e^\pm p \rightarrow \text{jet} + X$	$\gamma^* g \rightarrow q\bar{q}$	$0.01 \lesssim x \lesssim 0.1$
Tevatron	$pp \rightarrow \text{jet} + X$	$gg, q\bar{q}, qq \rightarrow 2j$	$0.01 \lesssim x \lesssim 0.5$
	$pp \rightarrow (W^\pm \rightarrow \ell^\pm \nu) + X$	$ud \rightarrow W^+, \bar{u}\bar{d} \rightarrow W^-$	$u, d, \bar{u}, \bar{d}$
	$pp \rightarrow (Z \rightarrow \ell^+\ell^-) + X$	$uu, dd \rightarrow Z$	$u, d$
	$pp \rightarrow t\bar{t} + X$	$qq \rightarrow t\bar{t}$	$q$
LHC	$pp \rightarrow \text{jet} + X$	$gg, q\bar{q}, qq \rightarrow 2j$	$0.001 \lesssim x \lesssim 0.5$
	$pp \rightarrow (W^\pm \rightarrow \ell^\pm \nu) + X$	$ud \rightarrow W^+, d\bar{u} \rightarrow W^-$	$u, d, \bar{u}, \bar{d}, g$
	$pp \rightarrow (Z \rightarrow \ell^+\ell^-) + X$	$q\bar{q} \rightarrow Z$	$q, \bar{q}, g$
	$pp \rightarrow (Z \rightarrow \ell^+\ell^-) + X, p_\perp$	$gq(q) \rightarrow Zq(\bar{q})$	$g, q, \bar{q}$
	$pp \rightarrow (\gamma^* \rightarrow \ell^+\ell^-) + X, \text{Low mass}$	$q\bar{q} \rightarrow \gamma^*$	$q, \bar{q}, g$
	$pp \rightarrow (\gamma^* \rightarrow \ell^+\ell^-) + X, \text{High mass}$	$q\bar{q} \rightarrow \gamma^*$	$\bar{q}$
	$pp \rightarrow W^+ c, W^- \bar{c}$	$sg \rightarrow W^+ c, \bar{s}g \rightarrow W^- \bar{c}$	$s, \bar{s}$
	$pp \rightarrow t\bar{t} + X$	$gg \rightarrow t\bar{t}$	$g$

## □ Kinematic Coverage:

NNPDF3.1



## □ Fit Quality:

$$\chi^2/\text{dof} \sim 1 \Rightarrow \text{Non-trivial}$$

check of QCD

All data sets	3706 / 2763	3267 / 2996	2717 / 2663
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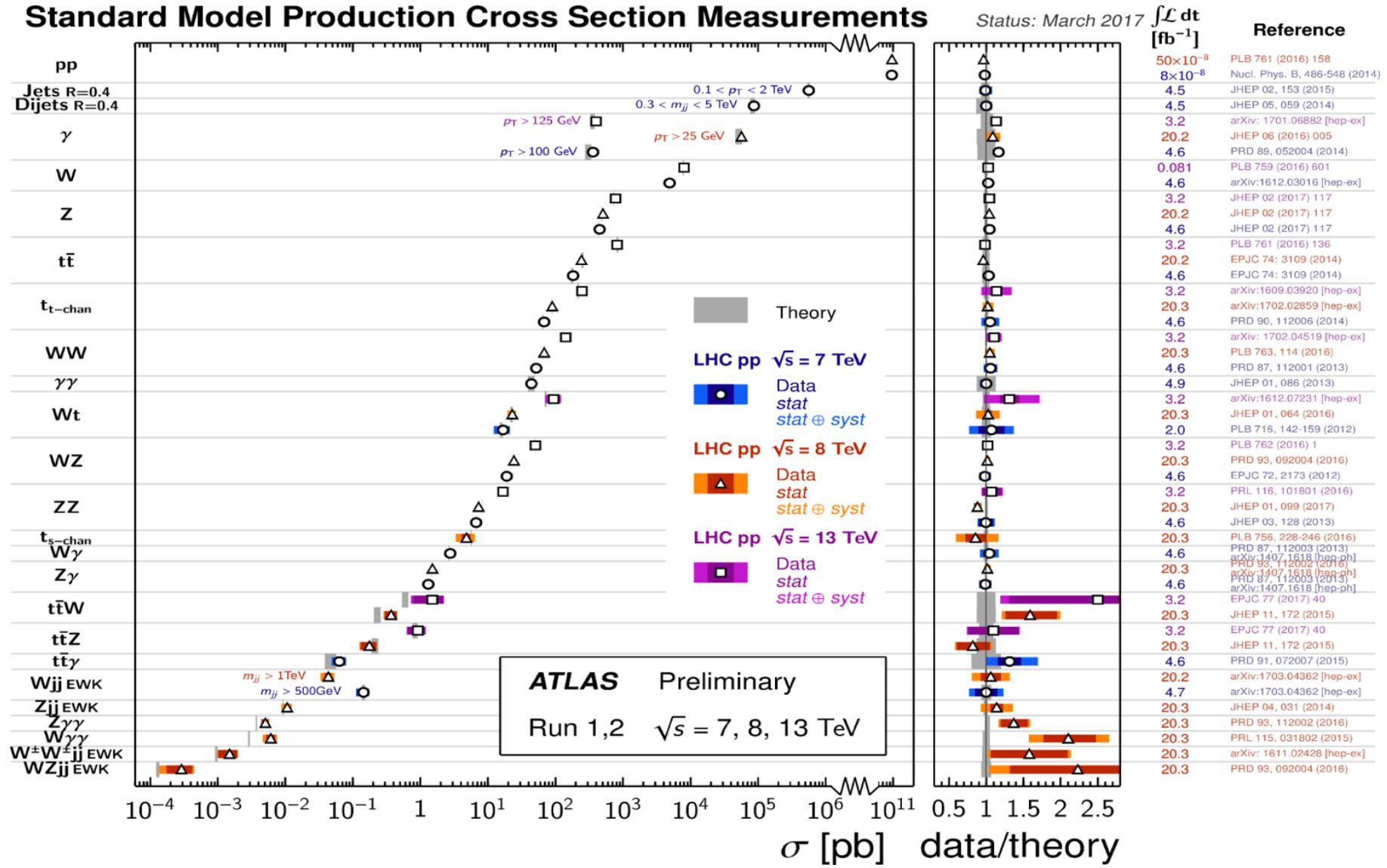
LO

NLO

NNLO

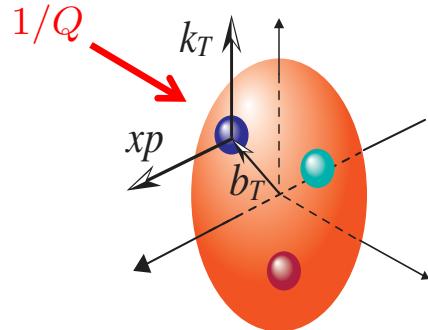
# Unprecedented Success of QCD and Standard Model

## Standard Model Production Cross Section Measurements



# Probes for 3D hadron structure

## □ Single scale hard probe is too “localized”:



- It pins down the particle nature of quarks and gluons
- But, not very sensitive to the detailed structure of hadron  $\sim \text{fm}$
- Transverse confined motion:  $k_T \sim 1/\text{fm} \ll Q$
- Transverse spatial position:  $b_T \sim \text{fm} \gg 1/Q$

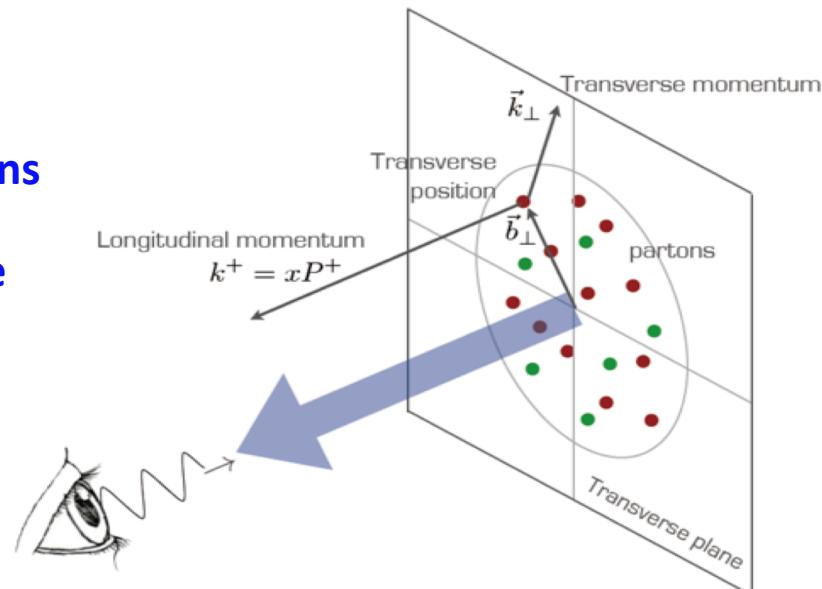
## □ Need new type of “Hard Probes” – Physical observables with TWO Scales:

$$Q_1 \gg Q_2 \sim 1/R \sim \Lambda_{\text{QCD}}$$

Hard scale:  $Q_1$  To localize the probe  
particle nature of quarks/gluons

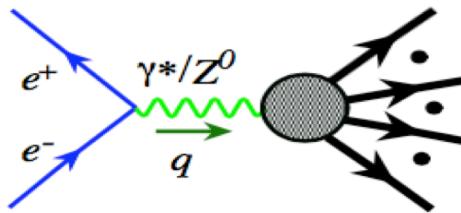
“Soft” scale:  $Q_2$  could be more sensitive to the  
hadron structure  $\sim 1/\text{fm}$

Hit the hadron “very hard” without breaking it,  
clean information on the structure!



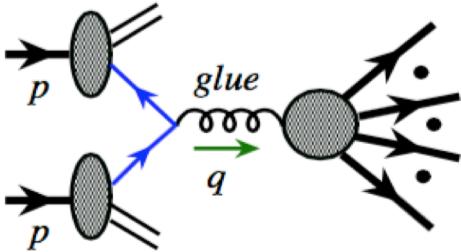
# QCD & hadron structure needs lepton-hadron facility

- Hadrons are produced from the energy in e+e- collisions:



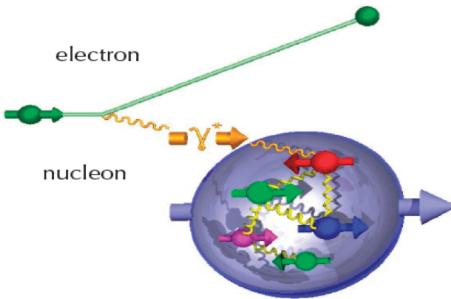
- No hadron to start with
- Emergence of hadrons

- Hadrons are produced in hadron-hadron collisions:



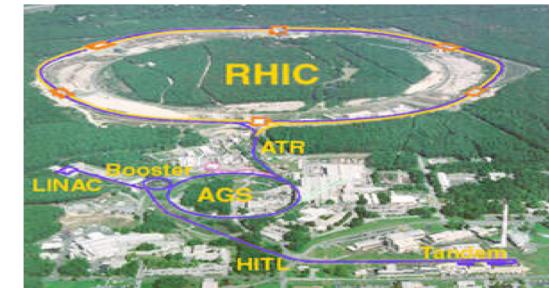
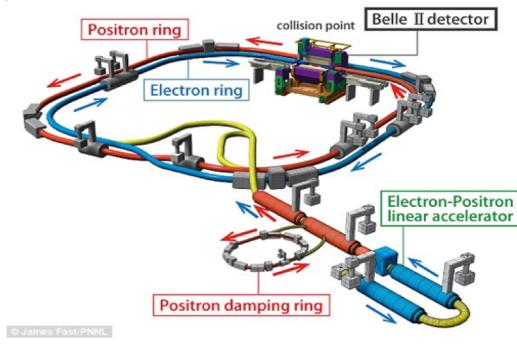
- Partonic structure
- Emergence of hadrons
- Heavy ion target or beam(s)

- Hadrons are produced in lepton-hadron collisions:

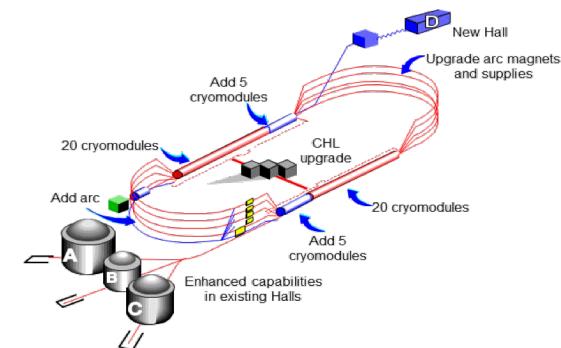


One facility covers all!

- Colliding hadron can be broken or stay intact!
- Imaging partonic structure
- Emergence of hadrons
- Heavy ion target or beam



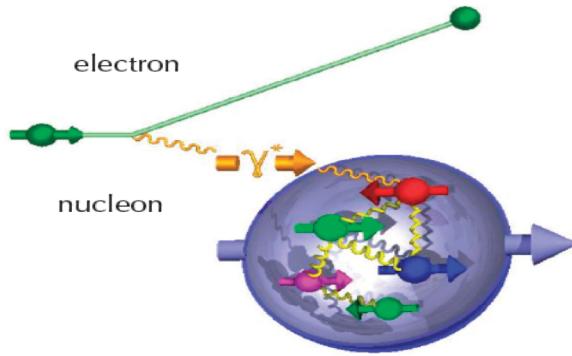
Also at the LHC



Also at COMPASS & future EIC

# Why a lepton-hadron facility is special?

## □ The new generation of “Rutherford” experiment:

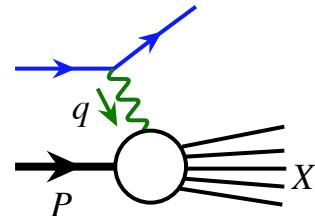


- ❖ A controlled “probe” – virtual photon
- ❖ Can either break or not break the hadron

*One facility covers all!*  
(JLab, COMPASS, EIC, ...)

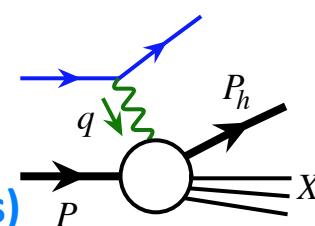
### ❖ Inclusive events: $e+p/A \rightarrow e'+X$

Detect only the scattered lepton in the detector  
(Modern Rutherford experiment!)



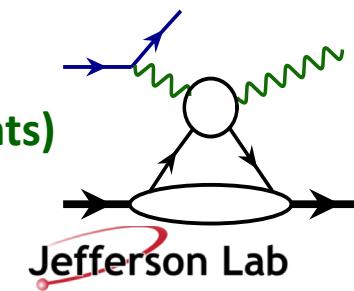
### ❖ Semi-Inclusive events: $e+p/A \rightarrow e'+h(p,K,p,jet)+X$

Detect the scattered lepton in coincidence with identified hadrons/jets  
(Initial hadron is broken – confined motion! – cleaner than h-h collisions)



### ❖ Exclusive events: $e+p/A \rightarrow e'+ p'/A' + h(p,K,p,jet)$

Detect every things including scattered proton/nucleus (or its fragments)  
(Initial hadron is NOT broken – tomography!  
– almost impossible for h-h collisions)

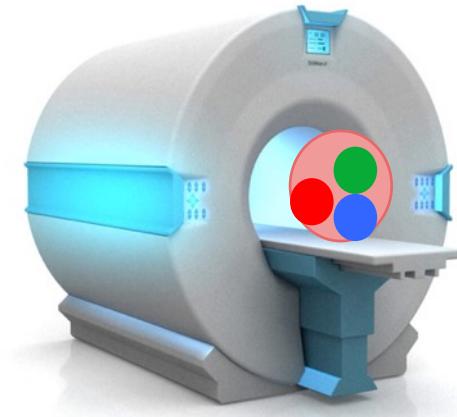


# The Electron-Ion Collider (EIC) – the Future!

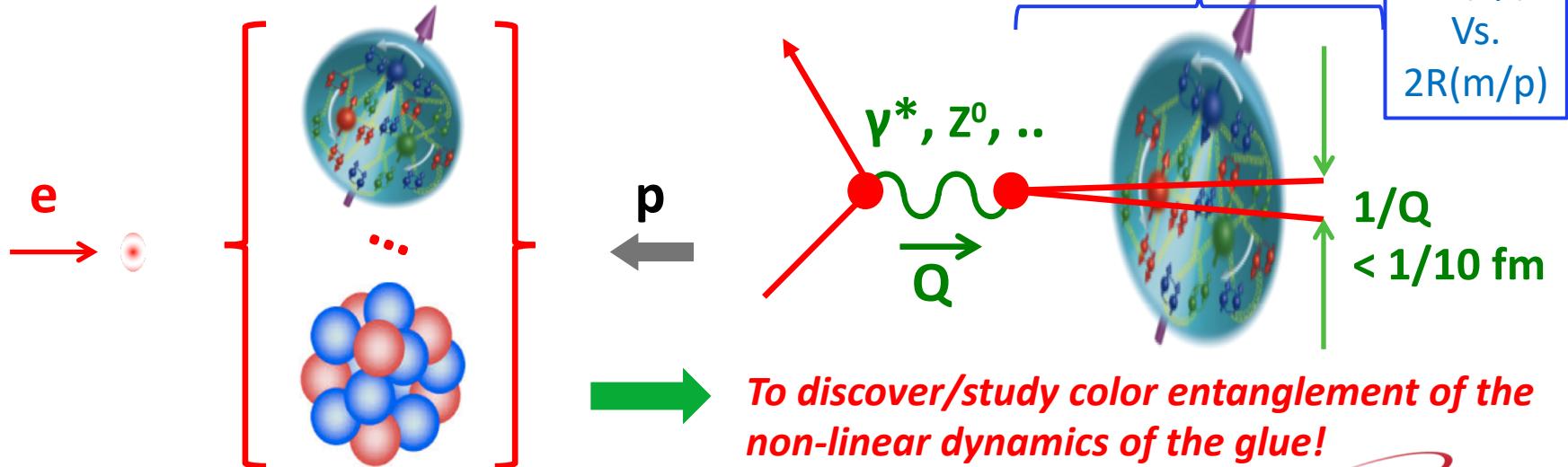
- A sharpest “CT” – “imagine” quark/gluon structure without breaking the hadron

- “cat-scan” the nucleon and nuclei with a better than  $1/10$  fm resolution
- “see” proton “radius” of quark/gluon density comparing with the radius of EM charge density

→ *To discover color confining radius, hints on confining mechanism!*



- A giant “Microscope” – “see” quarks and gluons by breaking the hadron



## Drell-Yan Factorization

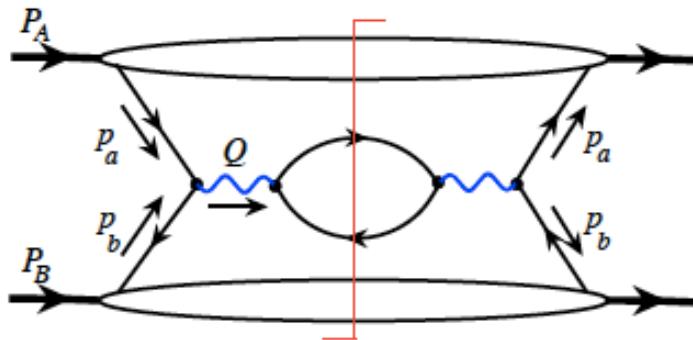
# Drell-Yan process in QCD – factorization

## □ Factorization – approximation:

Collins, Soper, Sterman, 1988

- ❖ **Suppression of quantum interference between short-distance ( $1/Q$ ) and long-distance ( $\text{fm} \sim 1/\Lambda_{\text{QCD}}$ ) physics**

➡ Need “long-lived” active parton states linking the two



$$\int d^4 p_a \frac{1}{p_a^2 + i\varepsilon} \frac{1}{p_a^2 - i\varepsilon} \rightarrow \infty$$

Perturbatively pinched at  $p_a^2 = 0$

➡ Active parton is effectively on-shell for the hard collision

- ❖ **Maintain the universality of PDFs:**

Long-range soft gluon interaction has to be power suppressed

- ❖ **Infrared safe of partonic parts:**

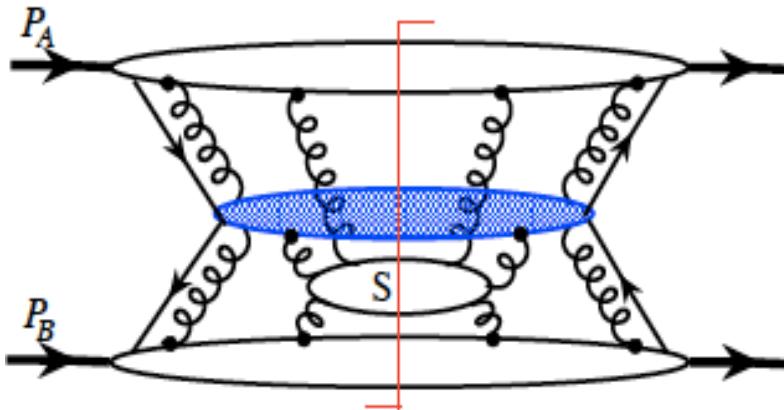
Cancelation of IR behavior

Absorb all CO divergences into PDFs

on-shell:  $p_a^2, p_b^2 \ll Q^2$ ;  
collinear:  $p_{aT}^2, p_{bT}^2 \ll Q^2$ ;  
higher-power:  $p_a^- \ll q^-$ ; and  
 $p_b^+ \ll q^+$

# Drell-Yan process in QCD – factorization

## □ Leading singular integration regions (pinch surface):



**Hard:** all lines off-shell by  $Q$

**Collinear:**

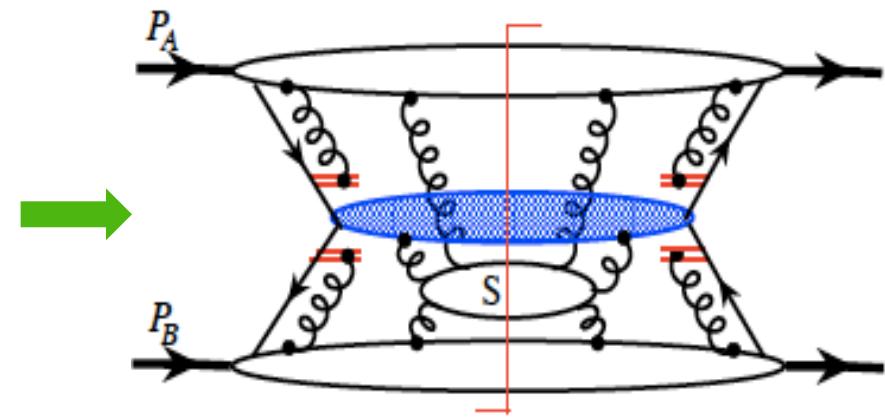
- ✧ lines collinear to A and B
- ✧ One “physical parton” per hadron

**Soft:** all components are soft

## □ Collinear gluons:

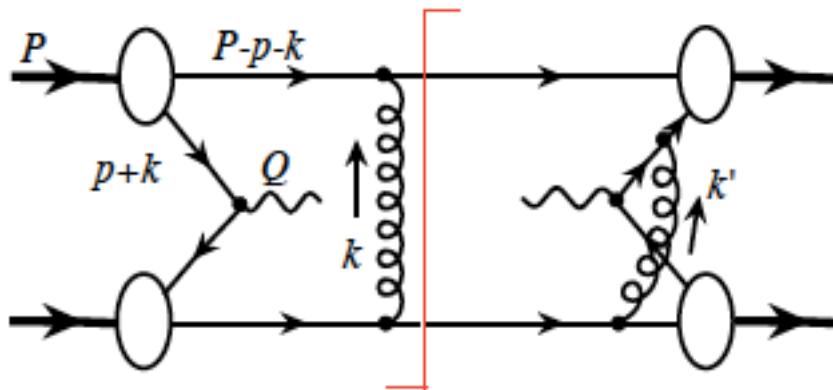
- ✧ Collinear gluons have the polarization vector:  $\epsilon^\mu \sim k^\mu$
- ✧ The sum of the effect can be represented by the eikonal lines,

*which are needed to make the PDFs gauge invariant!*



# Drell-Yan process in QCD – factorization

## □ Trouble with soft gluons:

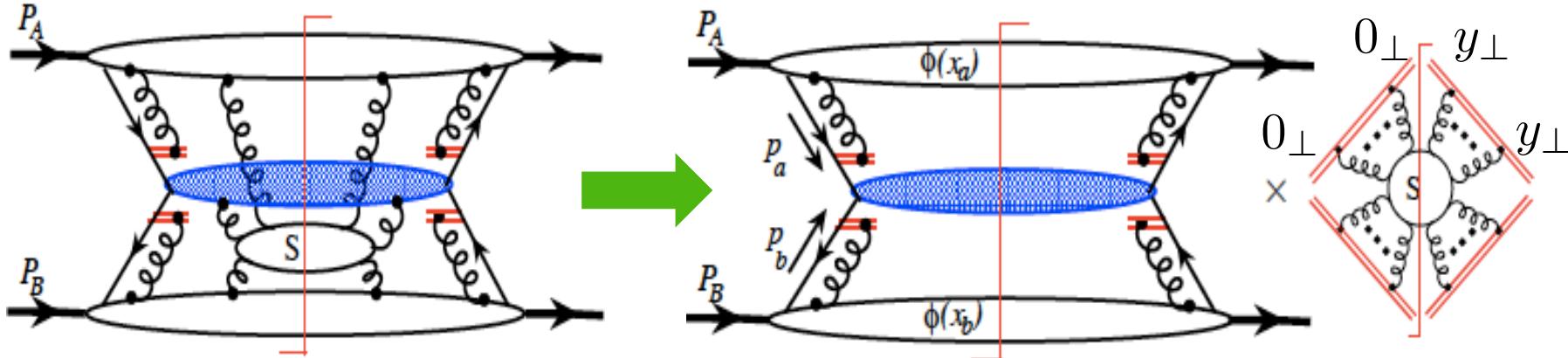


$$(xp + k)^2 + i\epsilon \propto k^- + i\epsilon$$
$$((1-x)p - k)^2 + i\epsilon \propto k^- - i\epsilon$$

- ✧ Soft gluon exchanged between a spectator quark of hadron B and the active quark of hadron A could rotate the quark's color and keep it from annihilating with the antiquark of hadron B
- ✧ The soft gluon approximations (with the eikonal lines) need  $k^\pm$  not too small. But,  $k^\pm$  could be trapped in “too small” region due to the pinch from spectator interaction:  $k^\pm \sim M^2/Q \ll k_\perp \sim M$   
*Need to show that soft-gluon interactions are power suppressed*

# Drell-Yan process in QCD – factorization

## □ Most difficult part of factorization:



- ❖ Sum over all final states to remove all poles in one-half plane
  - no more pinch poles
- ❖ Deform the  $k^\pm$  integration out of the trapped soft region
- ❖ Eikonal approximation → soft gluons to eikonal lines
  - gauge links
- ❖ Collinear factorization: Unitarity → soft factor = 1

*All identified leading integration regions are factorizable!*