Lattice QCD Calculation of TMDs with Large-Momentum Effective Theory

TMD Studies: from JLab to EIC
JLab, May 6—7, 2021

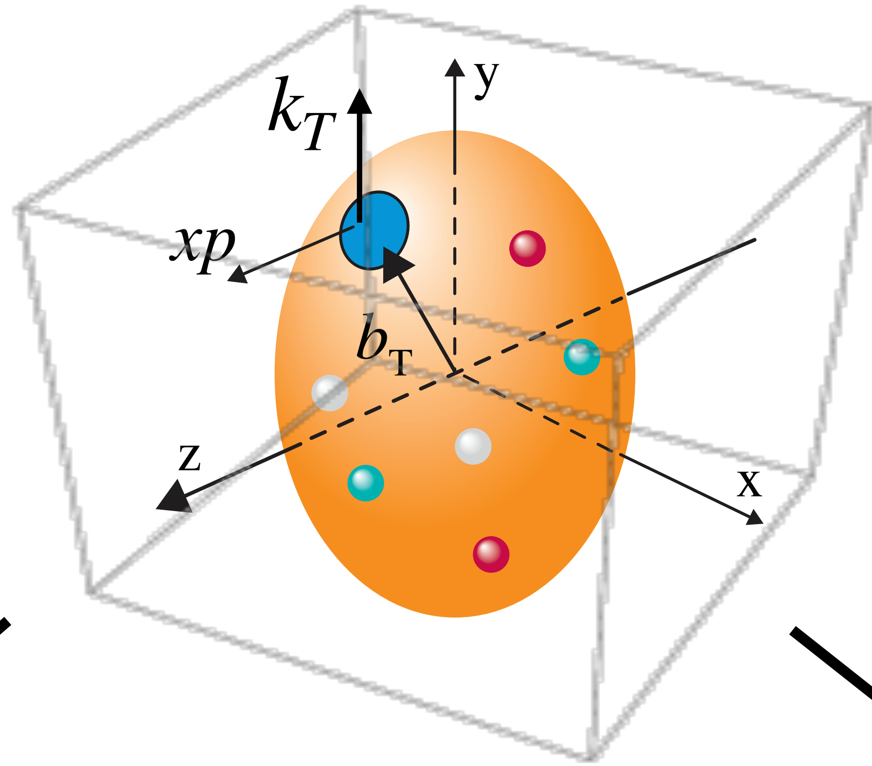
YONG ZHAO

MAY 6, 2021

Argonne NATIONAL LABORATORY

3D Tomography of the proton

Wigner distributions
(or GTMDs):
$W(x, \vec{k}_T, \vec{b}_T)$



Momentum space:
confined motion

$\int d^2 b_T$

Transverse momentum dependent (TMD) PDFs
$f(x, \vec{k}_T)$

Coordinate space:
spatial distribution

$\int d^2 k_T$

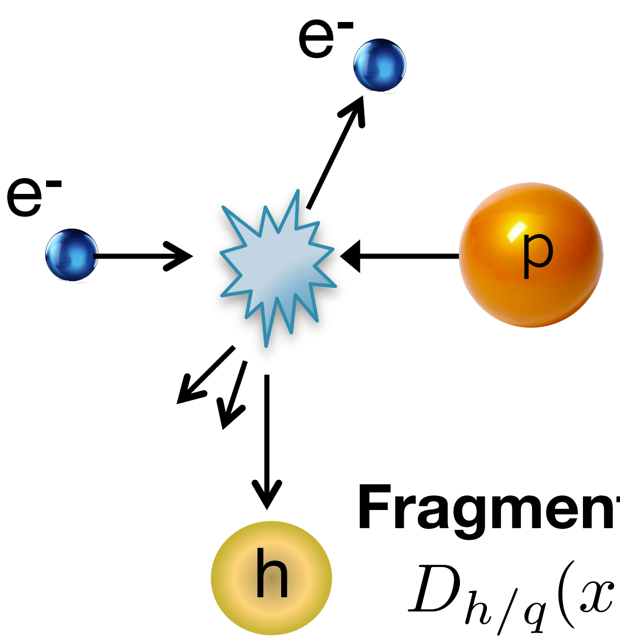
Generalized Parton Distributions (GPDs)
$f(x, \vec{b}_T)$

TMDPDFs from experiments

- TMD processes:

Semi-Inclusive DIS

$$\sigma \sim f_{q/P}(x,k_T)D_{h/q}(x,k_T)$$

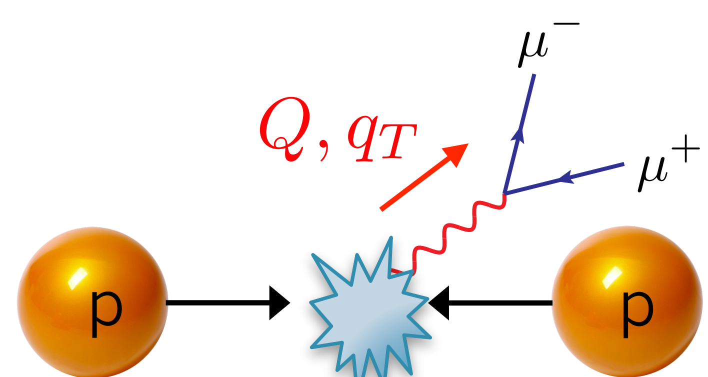


Fragmentation

$$D_{h/q}(x,k_T)$$

Drell-Yan

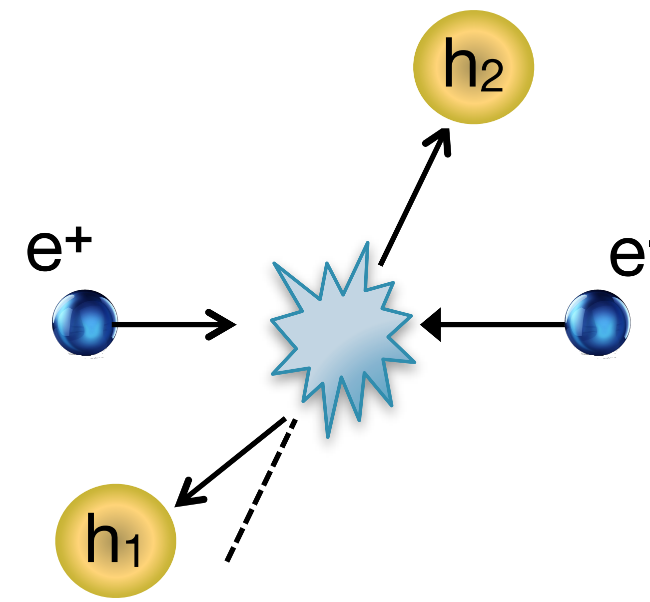
$$\sigma \sim f_{q/P}(x,k_T)f_{q/P}(x,k_T)$$



$$q_T \ll Q$$

Dihadron in e^+e^-

$$\sigma \sim D_{h_1/q}(x,k_T)D_{h_2/q}(x,k_T)$$



Many different schemes for TMD factorization in literature:

- Collins, Soper and Sterman, NPB250 (1985); Collins, 2011;
- Ji, Ma and Yuan, PRD71 (2005) 034005;
- Becher and Neubert, EPJC71 (2011);
- Echevarria, Idilbi and Scimemi, JHEP07 (2012), PLB726 (2013);
- Chiu, Jain, Neil and Rothstein, JHEP05 (2012), PRL108 (2012);
- Li, Neil and Zhu, arXiv: 1604.00392.

Definition of TMDPDF:

Collins-Soper scale: $\zeta = (2xP^+e^{-y_n})^2$

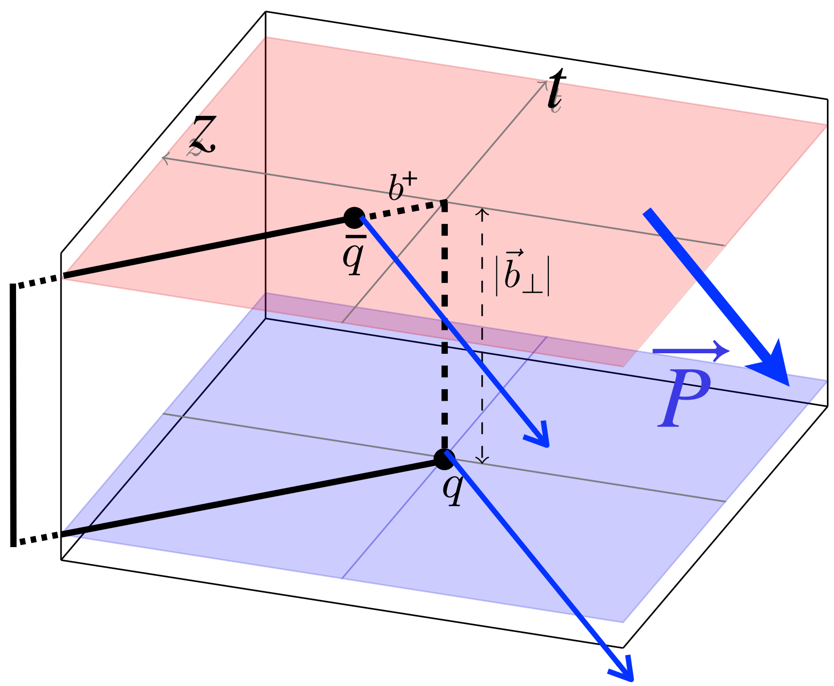
Rapidity divergence regulator

$$f_i^{\rm TMD}(x, \vec{b}_T, \mu, \zeta) = \lim_{\epsilon\to 0,\tau\to 0} Z_{\rm UV}(\epsilon, \mu, xP^+) B_i(x, \vec{b}_T, \epsilon, \tau, xP^+)\sqrt{S^i(b_T, \epsilon, \tau)}$$

Rapidity-regulator-independent

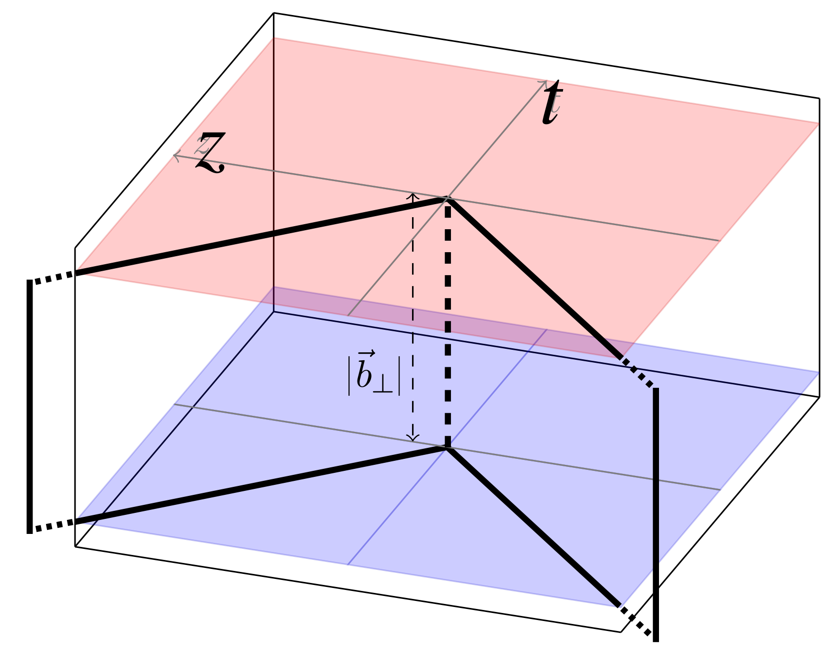
UV divergence regulator

- **Beam function :**



$$B^q(x, \vec{b}_T, \epsilon, \tau) = \int \frac{db^-}{2\pi} e^{-i(xP^+)b^-} \langle P | \bar{q}(b^\mu) W(b^\mu) \frac{\gamma^+}{2} \times W_T(-\infty\bar{n}; \vec{b}_T, \vec{0}_T) W^\dagger(0) q(0) \Big|_\tau |P\rangle$$

- **Soft function :**



$$S_q(b_T, \epsilon, \tau) = \frac{1}{N_c} \langle 0 | \mathrm{Tr}\Big[S_n^\dagger(\vec{b}_T) S_{\bar{n}}(\vec{b}_T) S_T \times S_{\bar{n}}^\dagger(\vec{0}_T) S_n(\vec{0}_T) S_T^\dagger\Big] \Big|_\tau |0\rangle$$

Definition of TMDPDF:

Collins-Soper scale: $\zeta = (2xP^+ e^{-y_n})^2$

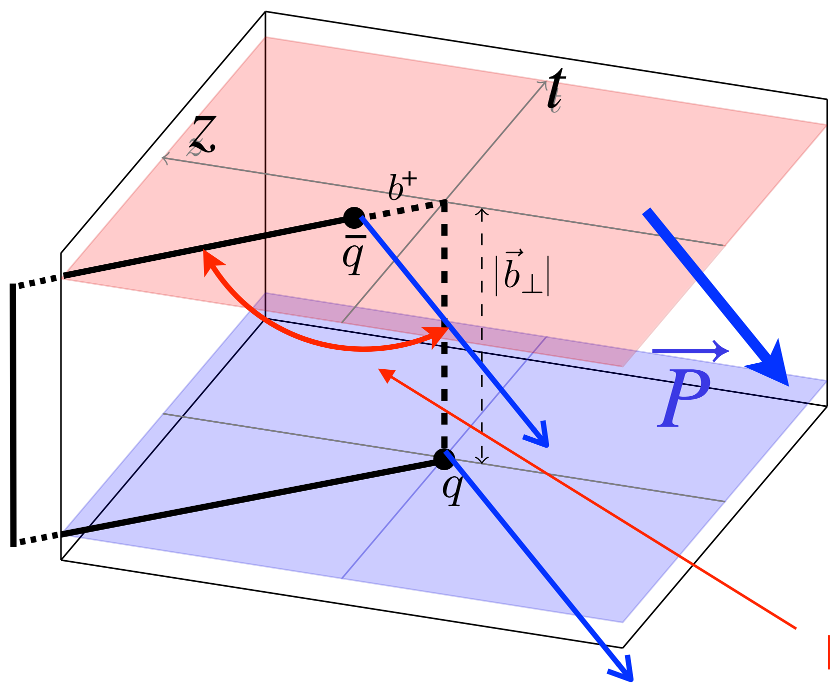
Rapidity divergence regulator

$$f_i^{\rm TMD}(x, \vec{b}_T, \mu, \zeta) = \lim_{\epsilon\to 0, \tau\to 0} Z_{\rm UV}(\epsilon, \mu, xP^+) B_i(x, \vec{b}_T, \epsilon, \tau, xP^+) \sqrt{S^i(b_T, \epsilon, \tau)}$$

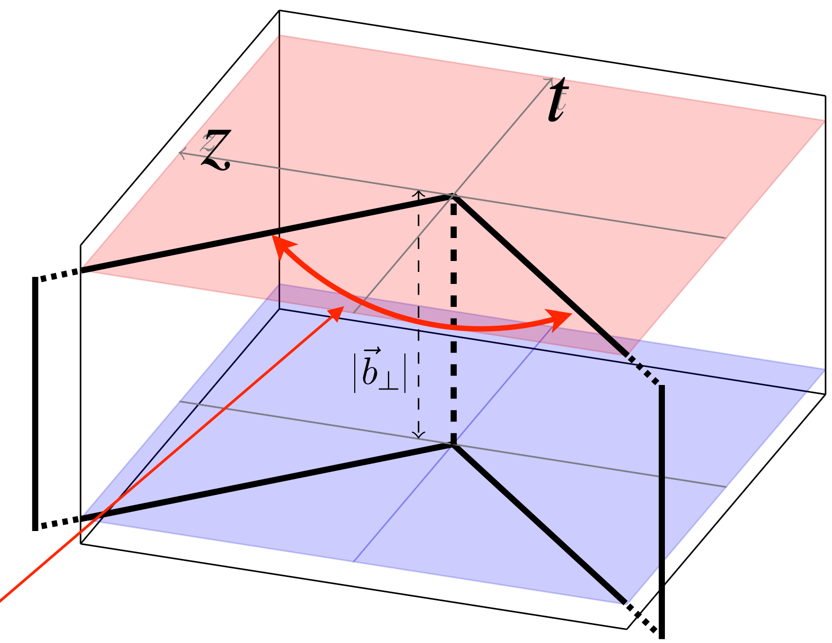
Rapidity-regulator-independent

UV divergence regulator

- Beam function :



- Soft function :



Rapidity divergences

$$B^q(x, \vec{b}_T, \epsilon, \tau) = \int \frac{db^-}{2\pi} e^{-i(xP^+)b^-} \langle P | \bar{q}(b^\mu) W(b^\mu) \frac{\gamma^+}{2} \times W_T(-\infty \bar{n}; \vec{b}_T, \vec{0}_T) W^\dagger(0) q(0) \Big|_\tau |P\rangle$$

$$S_q(b_T, \epsilon, \tau) = \frac{1}{N_c} \langle 0 | \mathrm{Tr}\Big[S_n^\dagger(\vec{b}_T) S_{\bar{n}}(\vec{b}_T) S_T \times S_{\bar{n}}^\dagger(\vec{0}_T) S_n(\vec{0}_T) S_T^\dagger\Big] \Big|_\tau |0\rangle$$

TMDPDF Evolution

$$\mu \frac{d \ln f_i^{\mathrm{TMD}}}{d\mu} = \gamma_\mu^i(\mu, \zeta)$$ Anomalous dimension for μ evolution, perturbatively calculable;

$$\frac{1}{2}\zeta \frac{d \ln f_i^{\mathrm{TMD}}}{d\zeta} = \gamma_\zeta^i(\mu, b_T)$$ Collins-Soper kernel. Nonperturbative when $b_T \sim 1/\Lambda_{\mathrm{QCD}}$.

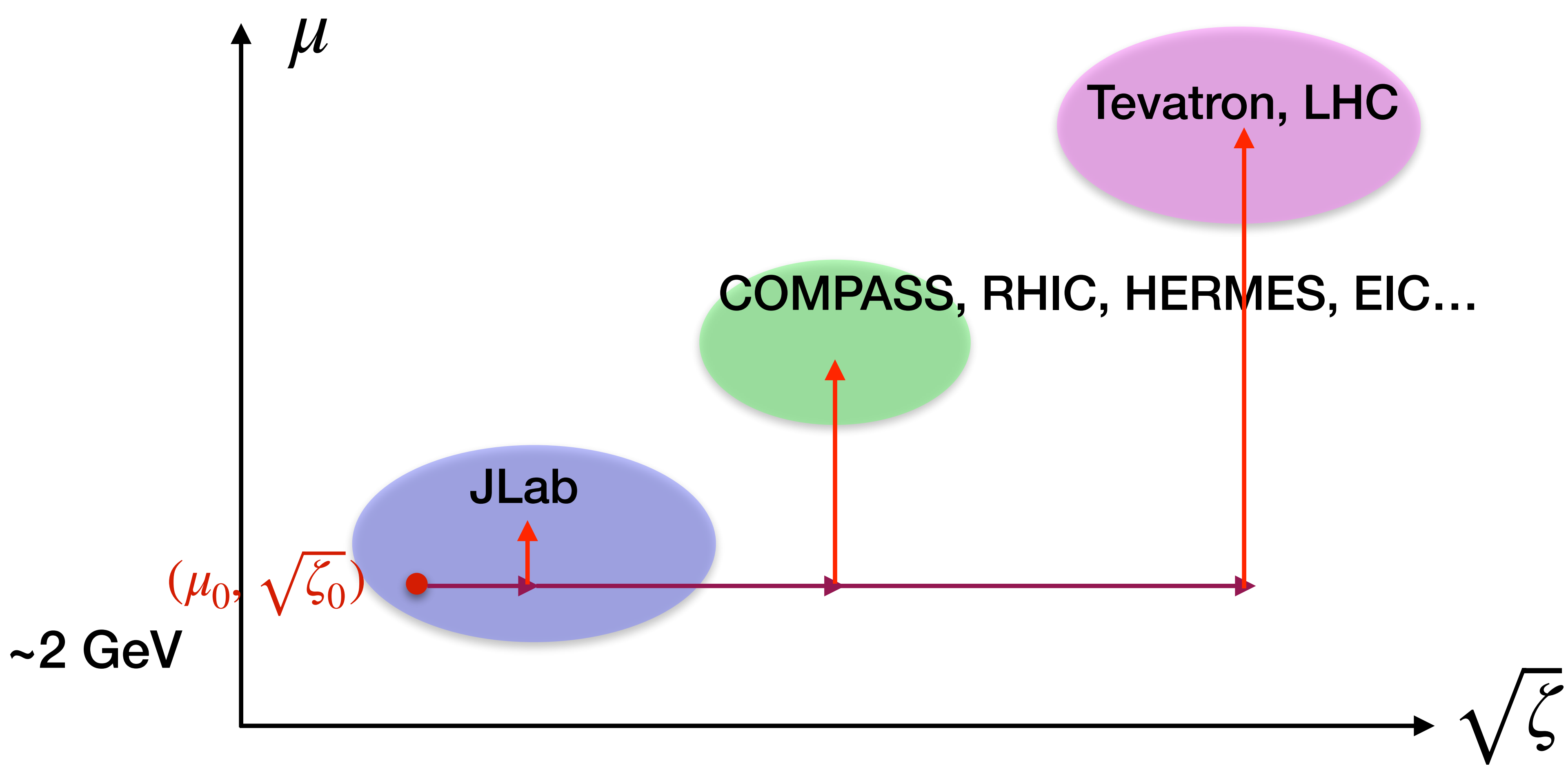
$$\frac{d\gamma_\zeta^i(\mu, b_T)}{d \ln \mu} = 2\frac{d\gamma_\mu^i(\mu, \zeta)}{d \ln \zeta} = -2\Gamma_{\mathrm{cusp}}^i[\alpha_s(\mu)]$$ **Analytical in the $\mu - \zeta$ plane.**

Global fitting of TMDPDF

$$\frac{d\sigma_{\rm DY}}{dQdYd^2q_T} = \sum_{i,j} H_{ij}(Q,\mu)\int d^2b_T\, e^{i\vec{b}_T\cdot\vec{q}_T} f_i^{\rm TMD}(x_a, \vec{b}_T, \mu, \zeta_a) f_j^{\rm TMD}(x_b, \vec{b}_T, \mu, \zeta_b)\left[1+\mathcal{O}\left(\frac{q_T^2}{Q^2}, \frac{\Lambda_{\rm QCD}^2}{Q^2}\right)\right]$$

$$f_i^{\rm TMD}(x, \vec{b}_T, \mu, \zeta) = f_i^{\rm TMD}(x, \vec{b}_T, \mu_0, \zeta_0)\ \exp\left[\int_{\mu_0}^{\mu}\frac{d\mu'}{\mu'}\gamma_\mu^i(\mu', \zeta_0)\right]\ \exp\left[\frac{1}{2}\gamma_\zeta^i(\mu, b_T)\ln\frac{\zeta}{\zeta_0}\right]$$

$\mu \sim Q,\ \zeta \sim Q^2$ Modelling Modelling

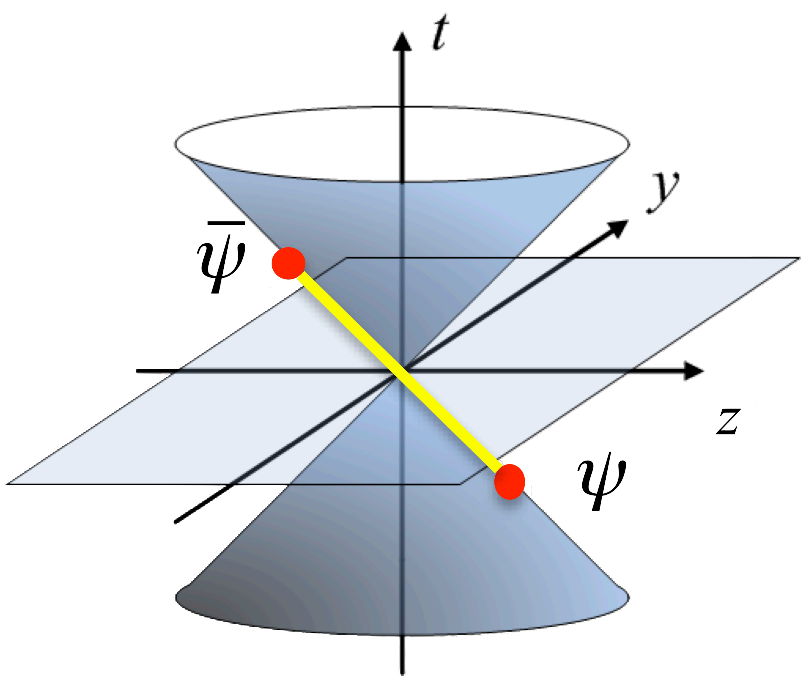


Lattice QCD Calculations in LaMET

- Large-Momentum Effective Theory (LaMET)
- Soft function
- Collins-Soper kernel
- Lattice QCD calculation of the full TMDPDF

Large-Momentum Effective Theory (LaMET)

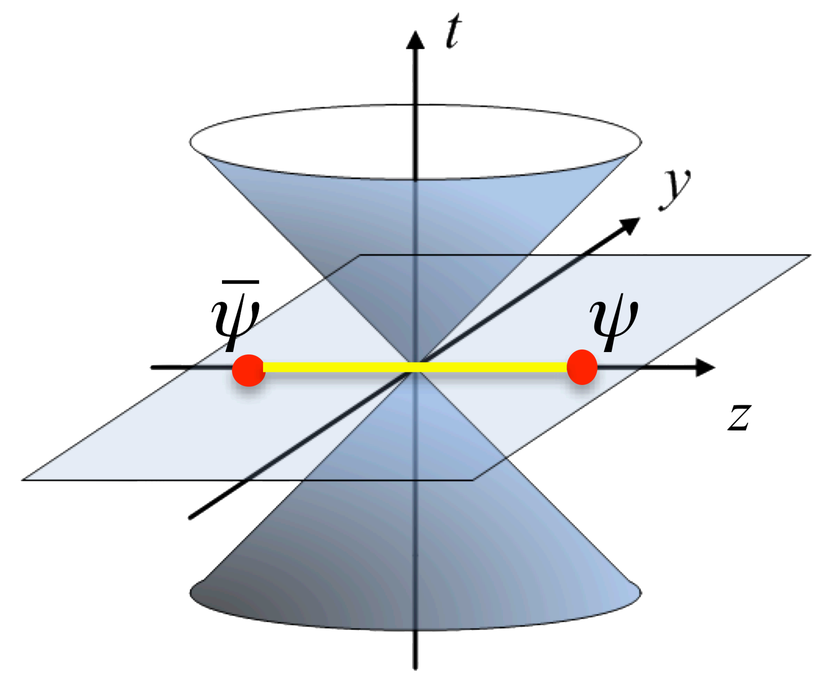
$z + ct = 0, \quad z - ct \neq 0$



PDF $f(x)$:
Cannot be calculated on the lattice

$$f(x) = \int \frac{db^-}{2\pi} e^{-ib^-(xP^+)} \langle P | \bar{\psi}(b^-) \times \frac{\gamma^+}{2} W[b^-,0] \psi(0) | P \rangle$$

$t = 0, \quad z \neq 0$



Quasi-PDF $\tilde{f}(x, P^z)$:
Directly calculable on the lattice

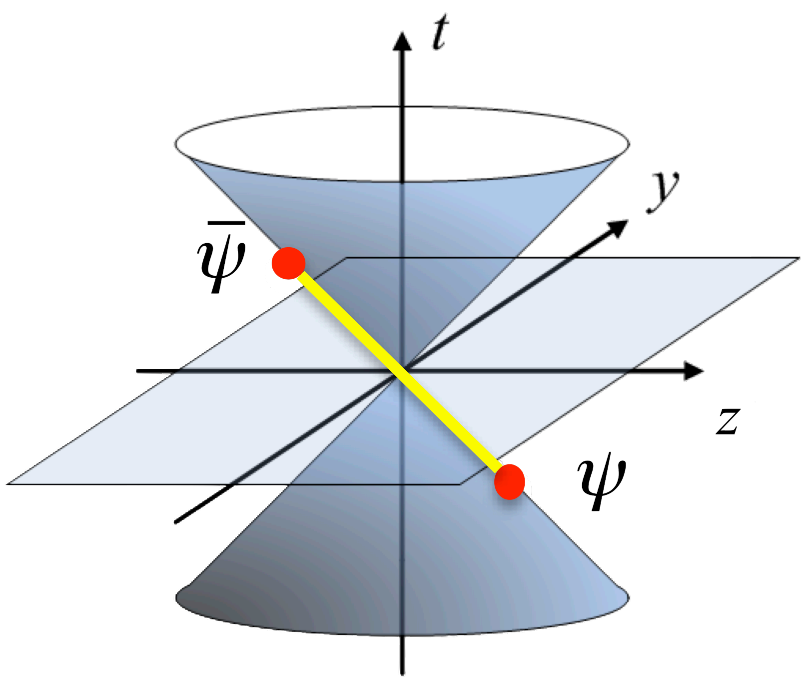
$$\tilde{f}(x, P^z) = \int \frac{dz}{2\pi} e^{ib^z(xP^z)} \langle P | \bar{\psi}(b^z) \times \frac{\gamma^z}{2} W[b^z,0] \psi(0) | P \rangle$$

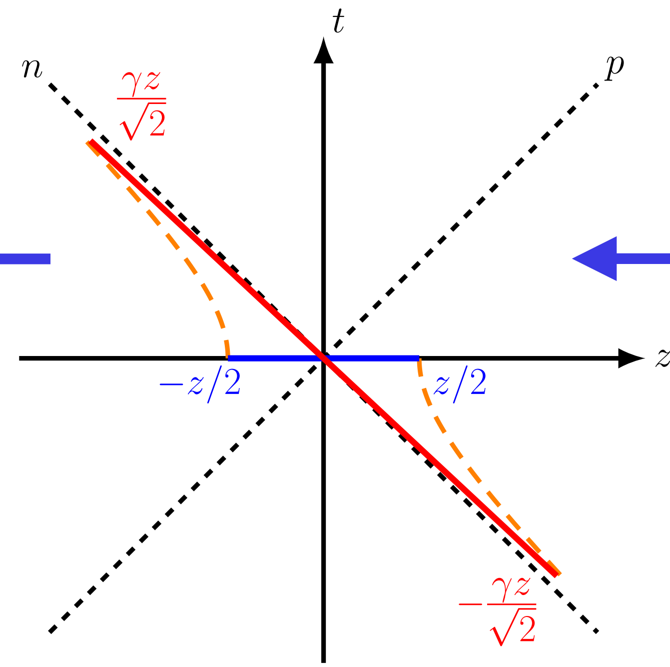
Large-Momentum Effective Theory (LaMET)

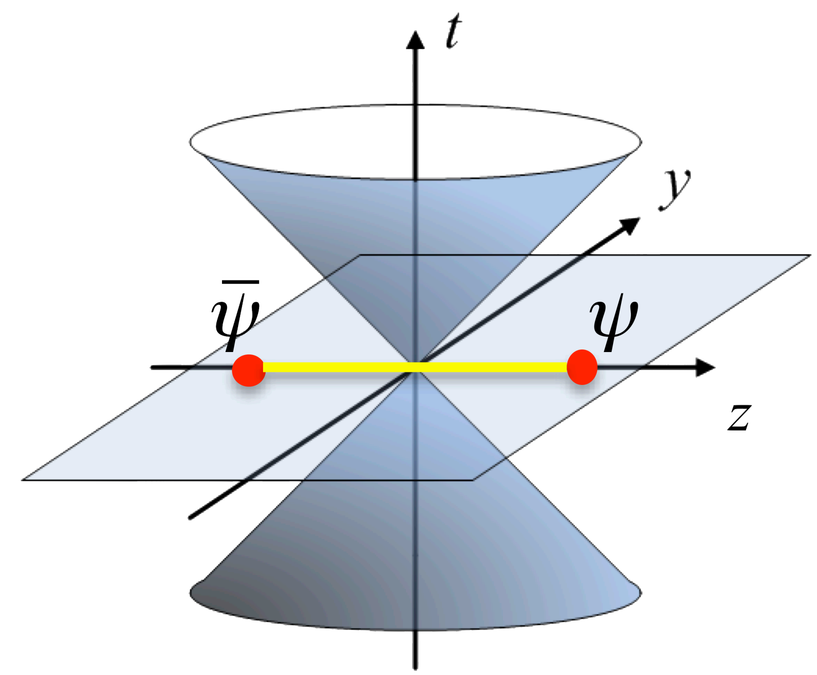
$z + ct = 0, \;\; z - ct \neq 0$

Related by Lorentz boost

$t = 0, \;\; z \neq 0$







PDF $f(x)$:
Cannot be calculated
on the lattice

$$f(x) = \int \frac{db^-}{2\pi} e^{-ib^-(xP^+)} \langle P | \bar{\psi}(b^-) \times \frac{\gamma^+}{2} W[b^-,0] \psi(0) | P \rangle$$

Quasi-PDF $\tilde{f}(x, P^z)$:
Directly calculable on
the lattice

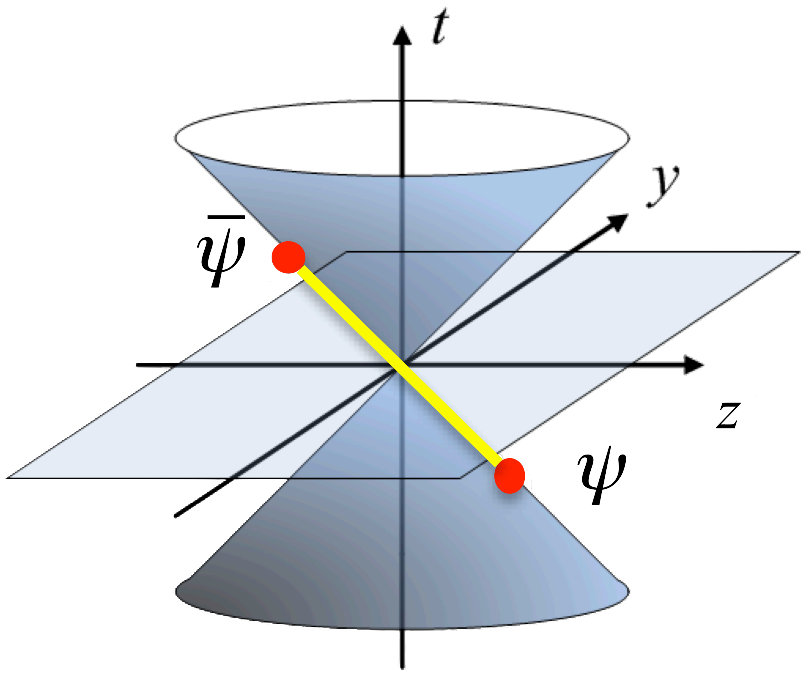
$$\tilde{f}(x, P^z) = \int \frac{dz}{2\pi} e^{ib^z(xP^z)} \langle P | \bar{\psi}(b^z) \times \frac{\gamma^z}{2} W[b^z,0] \psi(0) | P \rangle$$

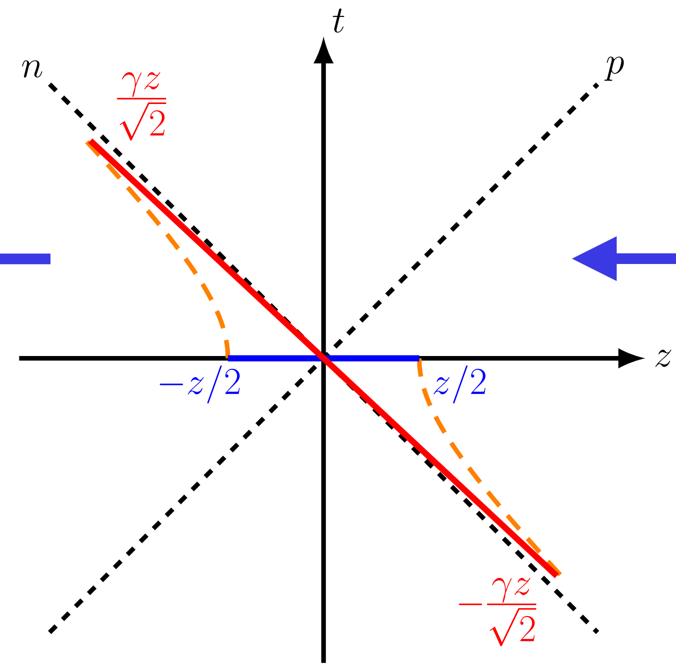
Large-Momentum Effective Theory (LaMET)

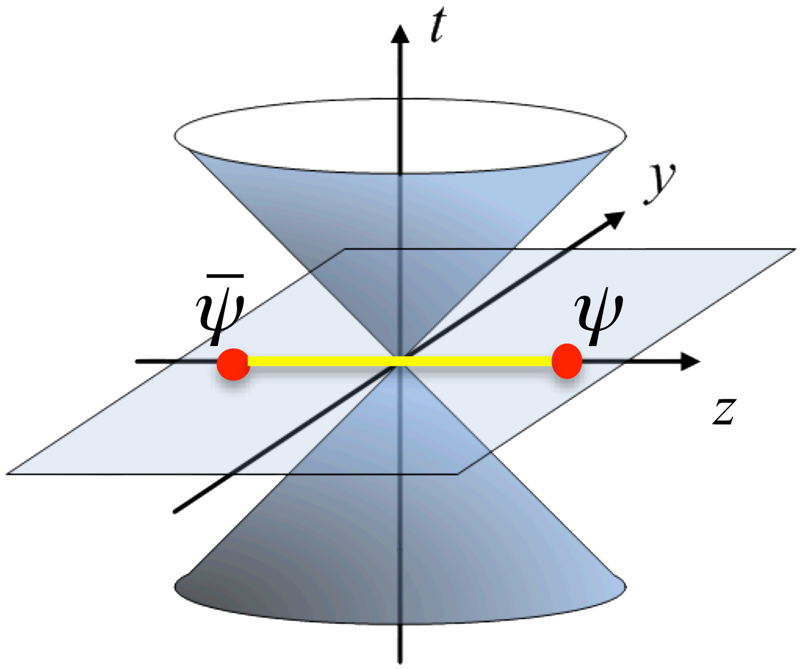
$z + ct = 0, \quad z - ct \neq 0$

Related by Lorentz boost

$t = 0, \quad z \neq 0$







PDF $f(x)$:
Cannot be calculated on the lattice

$$f(x) = \int \frac{db^-}{2\pi} e^{-ib^-(xP^+)} \langle P | \bar{\psi}(b^-) \times \frac{\gamma^+}{2} W[b^-, 0] \psi(0) | P \rangle$$

$$\lim_{P^z \to \infty} \tilde{f}(x, P^z) \stackrel{?}{=} f(x)$$

Quasi-PDF $\tilde{f}(x, P^z)$:
Directly calculable on the lattice

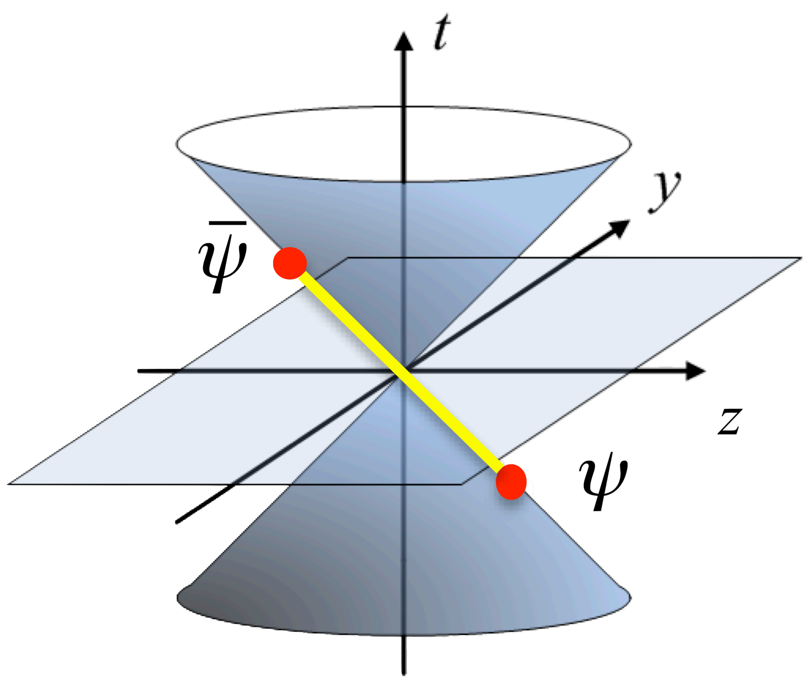
$$\tilde{f}(x, P^z) = \int \frac{dz}{2\pi} e^{ib^z(xP^z)} \langle P | \bar{\psi}(b^z) \times \frac{\gamma^z}{2} W[b^z, 0] \psi(0) | P \rangle$$

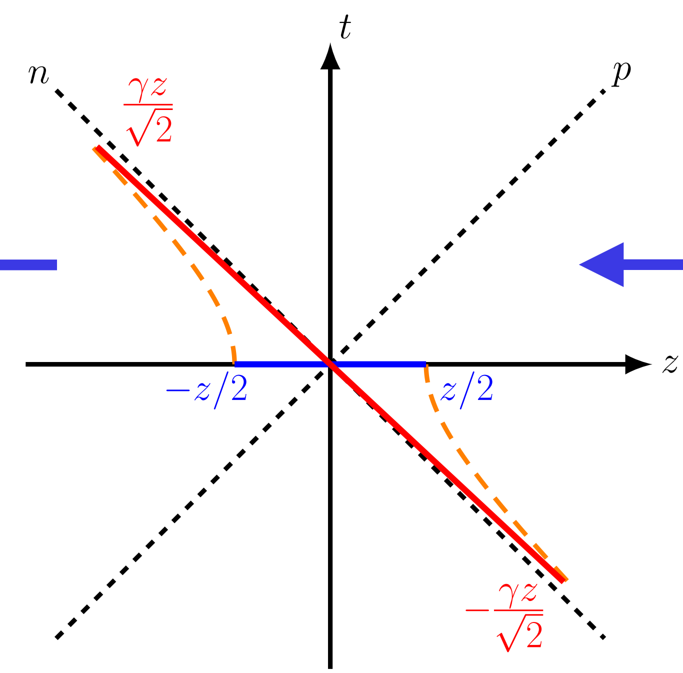
Large-Momentum Effective Theory (LaMET)

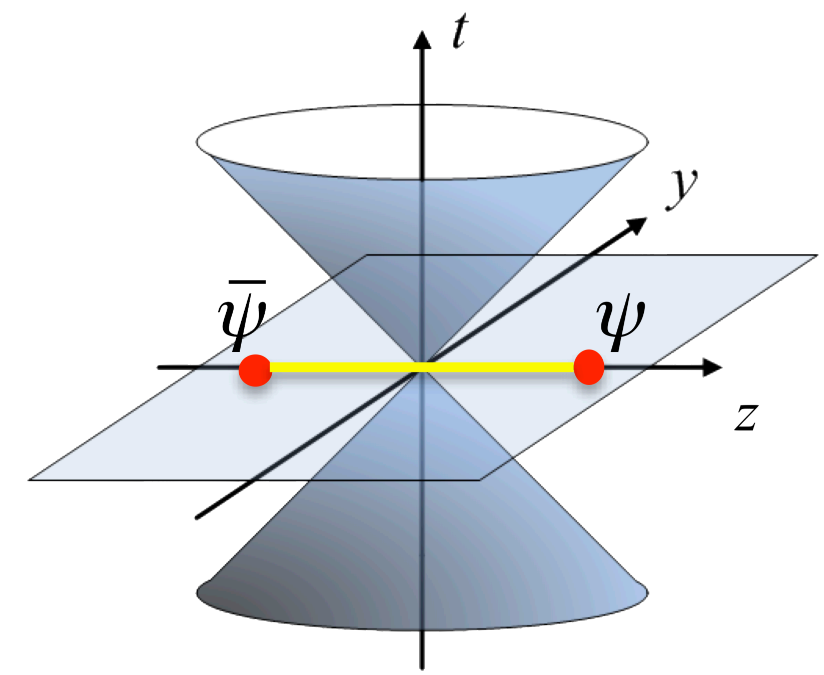
$z + ct = 0, \;\; z - ct \neq 0$

Related by Lorentz boost

$t = 0, \;\; z \neq 0$







PDF $f(x)$:
Cannot be calculated on the lattice

$$f(x) = \int \frac{db^-}{2\pi} e^{-ib^-(xP^+)} \langle P | \bar{\psi}(b^-) \times \frac{\gamma^+}{2} W[b^-, 0] \psi(0) | P \rangle$$

$$\lim_{P^z \to \infty} \tilde{f}(x, P^z) \stackrel{?}{=} f(x)$$

✘

Quasi-PDF $\tilde{f}(x, P^z)$:
Directly calculable on the lattice

$$\tilde{f}(x, P^z) = \int \frac{dz}{2\pi} e^{ib^z(xP^z)} \langle P | \bar{\psi}(b^z) \times \frac{\gamma^z}{2} W[b^z, 0] \psi(0) | P \rangle$$

Large-Momentum Effective Theory (LaMET)

- Quasi-PDF: $P^z \ll \Lambda$;

Λ: the ultraviolet lattice cutoff, $\sim 1/a$

- PDF: $P^z = \infty$, including $P^z \gg \Lambda$.
    - The limits $P^z \ll \Lambda$ and $P^z \gg \Lambda$ are not exchangeable;
    - For $P^z \gg \Lambda_{\rm QCD}$, their infrared (nonperturbative) physics are the same.

Large-momentum expansion:

$$f(x,\mu) = C\left(x, P^z/\mu\right) \otimes \tilde{f}(x,P^z) + O\left(\frac{\Lambda_{\rm QCD}^2}{x^2P_z^2}, \frac{\Lambda_{\rm QCD}^2}{(1-x)^2P_z^2}\right)$$

Perturbative matching

Power corrections

- It is the large-momentum state, instead of the operator, that filters out collinear modes in the field operators;
- Contribution from the collinear modes is identical to the PDF.

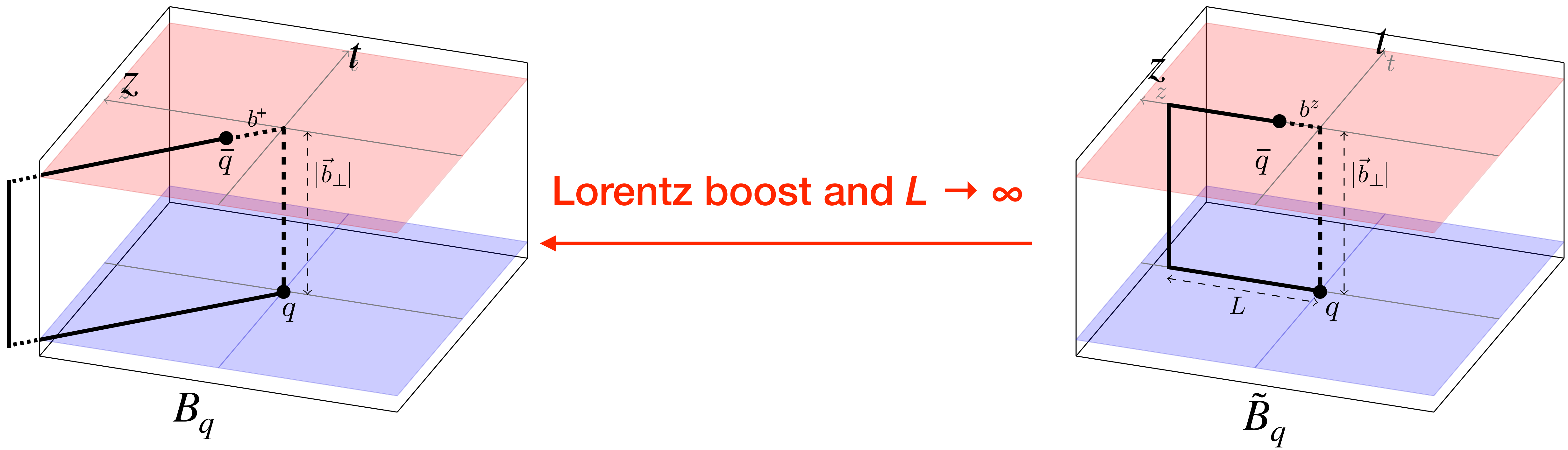
- X. Ji, PRL 110 (2013); SCPMA57 (2014).
- X. Xiong, X. Ji, J.-H. Zhang and YZ, PRD 90 (2014);
- X. Ji, Y.-S. Liu, Y. Liu, J.-H. Zhang and YZ, arXiv: 2004.03543.

Construction of Quasi-TMDPDF

- Quasi-beam function on lattice:

$$\tilde{B}^q_\Gamma(x, \vec{b}_T, a, L, P^z) = \int \frac{db^z}{2\pi} e^{ib^z(xP^z)} \tilde{B}_q(b^z, \vec{b}_T, a, L, P^z)$$

$$= \int \frac{db^z}{2\pi} e^{ib^z(xP^z)} \langle P | \bar{q}(b^\mu) W_{\hat{z}}(b^\mu; L - b^z) \frac{\Gamma}{2} W_T(L\hat{z}; \vec{b}_T, \vec{0}_T) W^\dagger_{\hat{z}}(0) q(0) | P \rangle$$

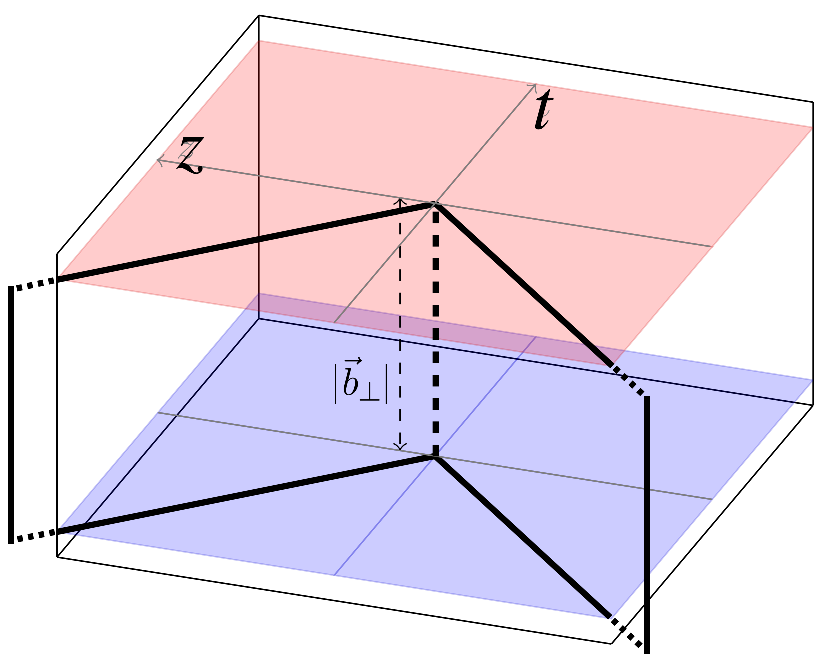


- Ji, Sun, Xiong and Yuan, PRD91 (2015);
- Ji, Jin, Yuan, Zhang and Y.Z., PRD99 (2019);
- M. Ebert, I. Stewart, Y.Z., PRD99 (2019), JHEP09 (2019) 037.
- Ji, Liu and Liu, Nucl.Phys.B 955 (2020), Phys.Lett.B 811 (2020).

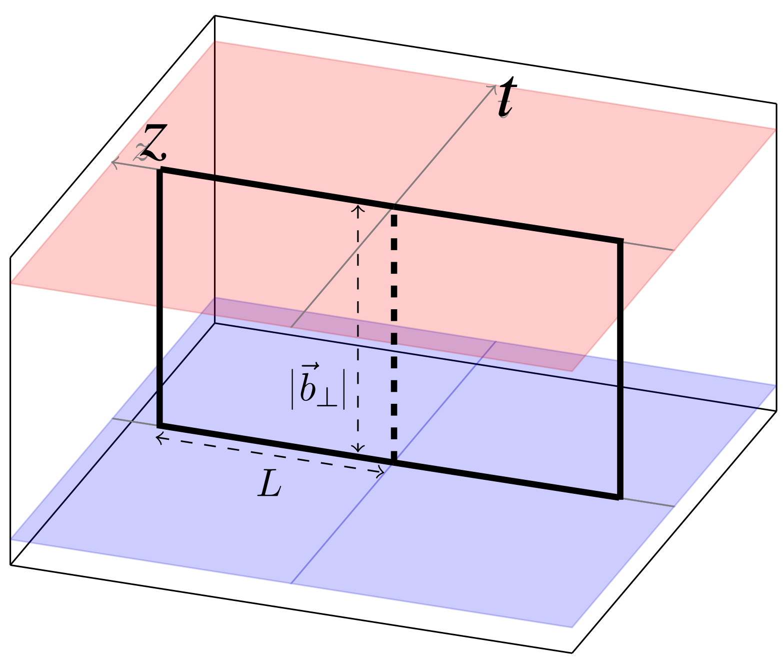
Construction of Quasi-TMDPDF

- Quasi-soft function on lattice (naive definition):

$$\tilde{S}_q(b_T, a, L) = \frac{1}{N_c}\langle 0 | \mathrm{Tr}\left[S^{\dagger}_{\hat{z}}(\vec{b}_T; L) S_{-\hat{z}}(\vec{b}_T; L) S_T(L\hat{z}; \vec{b}_T, \vec{0}_T) S^{\dagger}_{-\hat{z}}(\vec{0}_T; L) S_n(\vec{0}_T; L) S^{\dagger}_T(-L\hat{z}; \vec{b}_T, \vec{0}_T)\right] | 0\rangle$$



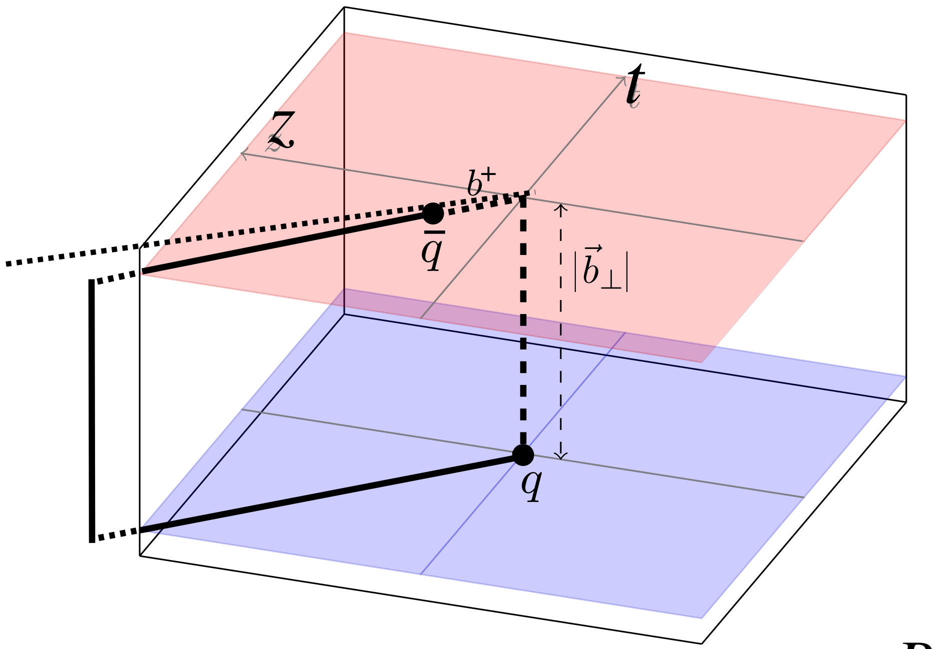
Cannot be related by Lorentz boost



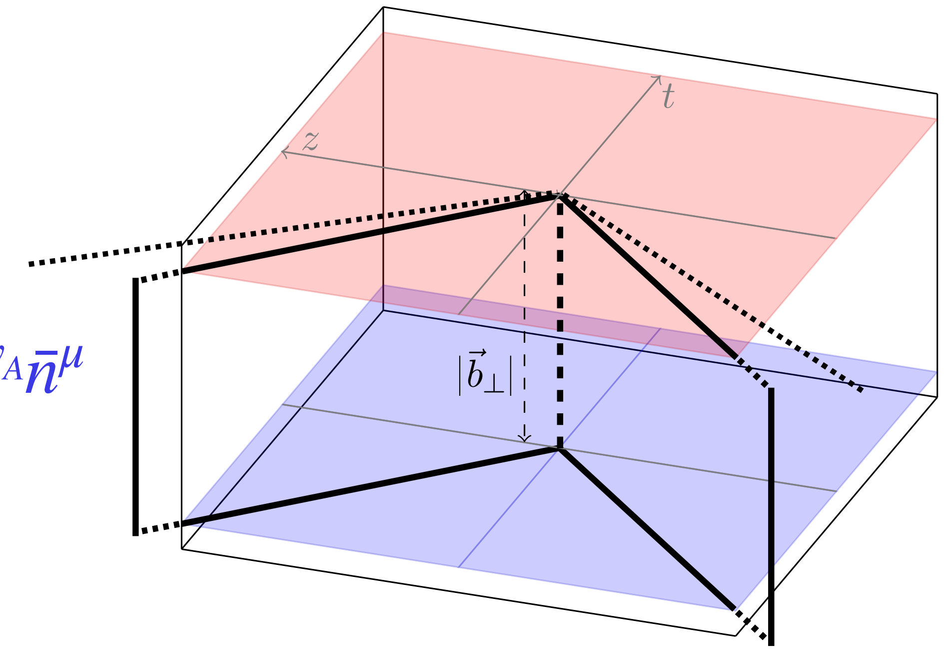
- Ji, Sun, Xiong and Yuan, PRD91 (2015);
- Ji, Jin, Yuan, Zhang and Y.Z., PRD99 (2019);
- M. Ebert, I. Stewart, Y.Z., PRD99 (2019), JHEP09 (2019) 037.
- Ji, Liu and Liu, Nucl.Phys.B 955 (2020), Phys.Lett.B 811 (2020).

Comparison to Collins-Soper-Sterman Scheme

- Wilson lines off the light-cone: Collins, Soper and Sterman, NPB250 (1985); Collins, 2011



$$n_A^\mu \equiv n^\mu - e^{-2y_A}\bar{n}^\mu$$



$$n_B^\mu \equiv \bar{n}^\mu - e^{2y_B}n^\mu$$

$$f_q^{\rm TMD}(x, \overrightarrow{b}_T, \mu, \zeta) = \lim_{y_B\to-\infty} Z_{UV}\frac{B_q(x, \overrightarrow{b}_T, \epsilon, y_P - y_B)}{\sqrt{S^q(b_T, \epsilon, 2(y_n - y_B))}}$$

$$\lim_{y_P-y_B\to-\infty} B_q(x, \overrightarrow{b}_T, \epsilon, y_P - y_B) \propto e^{(y_P-y_B)\gamma_\zeta(b_T,\mu)}$$

$$\lim_{y_n-y_B\to-\infty} S^q(b_T, \mu,\ 2(y_n - y_B)) = e^{2(y_n-y_B)\gamma_\zeta(b_T,\mu)+\mathscr{D}(b_T,\mu)}$$

$e^{\mathscr{D}(b_T,\mu)}$ is what is missing in the quasi soft functions, which is intrinsically Minkowskian.

A. Vladimirov, JHEP 04 (2018).

Ji, Liu and Liu, Nucl.Phys.B 955 (2020), Phys.Lett.B 811 (2020).

Factorization and the reduced soft function

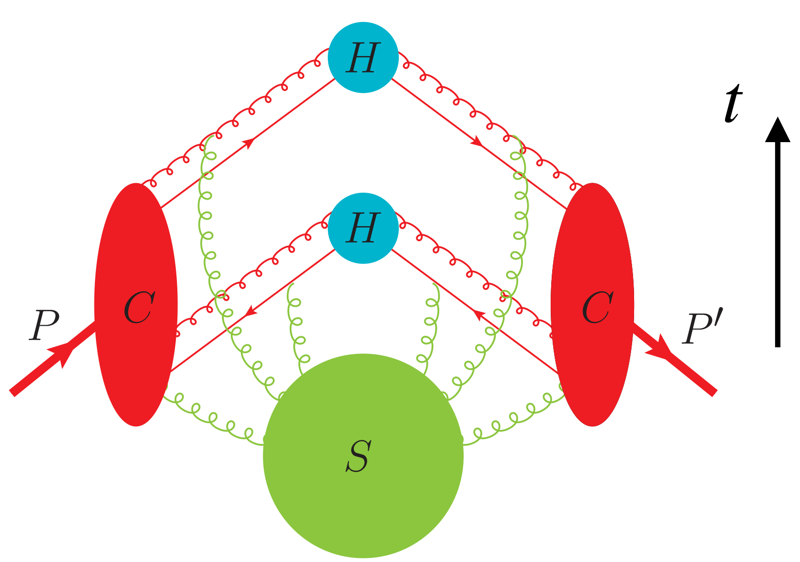
$$\frac{\tilde{f}_{\rm ns}^{\rm TMD}(x, \vec{b}_T, \mu, P^z)}{\sqrt{S_r^q(b_T, \mu)}} = C_{\rm ns}^{\rm TMD}(\mu, xP^z) \ \exp\left[\frac{1}{2}\gamma_\zeta^q(\mu, b_T)\ln\frac{(2xP^z)^2}{\zeta}\right]$$

$$\times f_{\rm ns}^{\rm TMD}(x, \vec{b}_T, \mu, \zeta) + \mathcal{O}\left(\frac{b_T}{L}, \frac{1}{b_T P^z}, \frac{1}{P^z L}\right)$$

$$S_r^q(b_T, \mu) = e^{\mathscr{D}(b_T, \mu)}$$

- M. Ebert, I. Stewart, Y.Z., PRD99 (2019), JHEP09 (2019) 037;
- Ji, Liu and Liu, Nucl.Phys.B 955 (2020), Phys.Lett.B 811 (2020);

- $S_r^q(b_T, \mu)$ from a light-meson form factor:



Ji, Liu and Liu, Nucl.Phys.B 955 (2020).

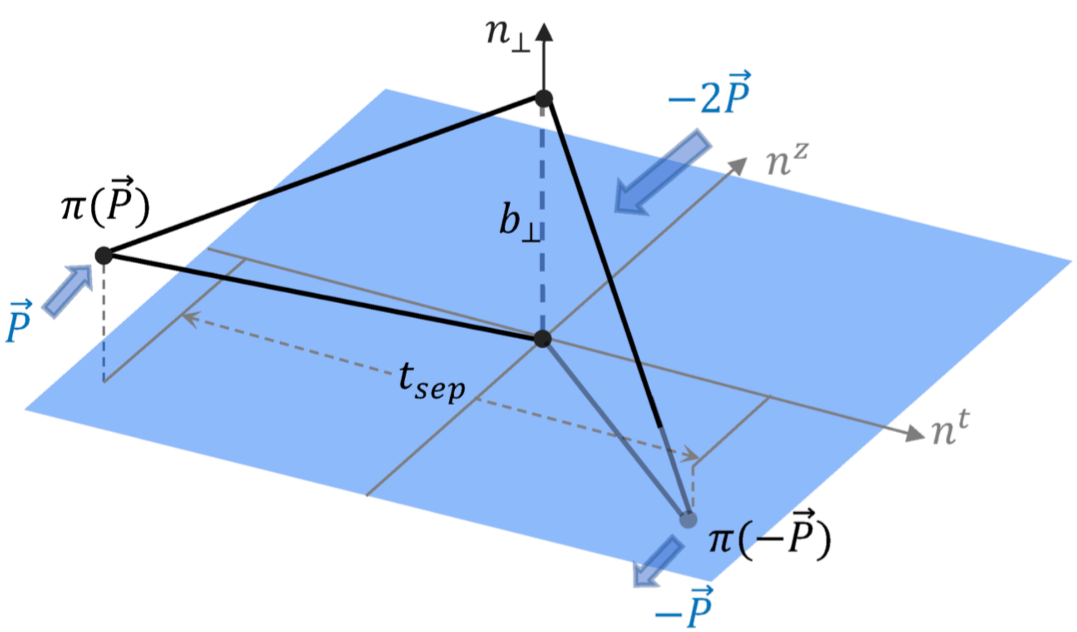
$$F(b_T, P^z)$$
$$= \langle \pi(-P) | j_1(b_T) j_2(0) | \pi(P) \rangle$$
$$= S_q^r(b_T, \mu)\ H(x, \mu) \otimes \Phi^\dagger(x, b_T, -P^z) \otimes \Phi(x, b_T, P^z)$$

Φ: Quasi-TMD distribution amplitude

$$\Phi(x, b_T, P^z) \equiv \int \frac{db^z}{2\pi} e^{ib^z(xP^z)} \langle 0 | \bar{q}(b^\mu) W_{\hat{z}} \frac{\Gamma}{2} W_T W_{\hat{z}}^\dagger q(0) | \pi(P) \rangle$$

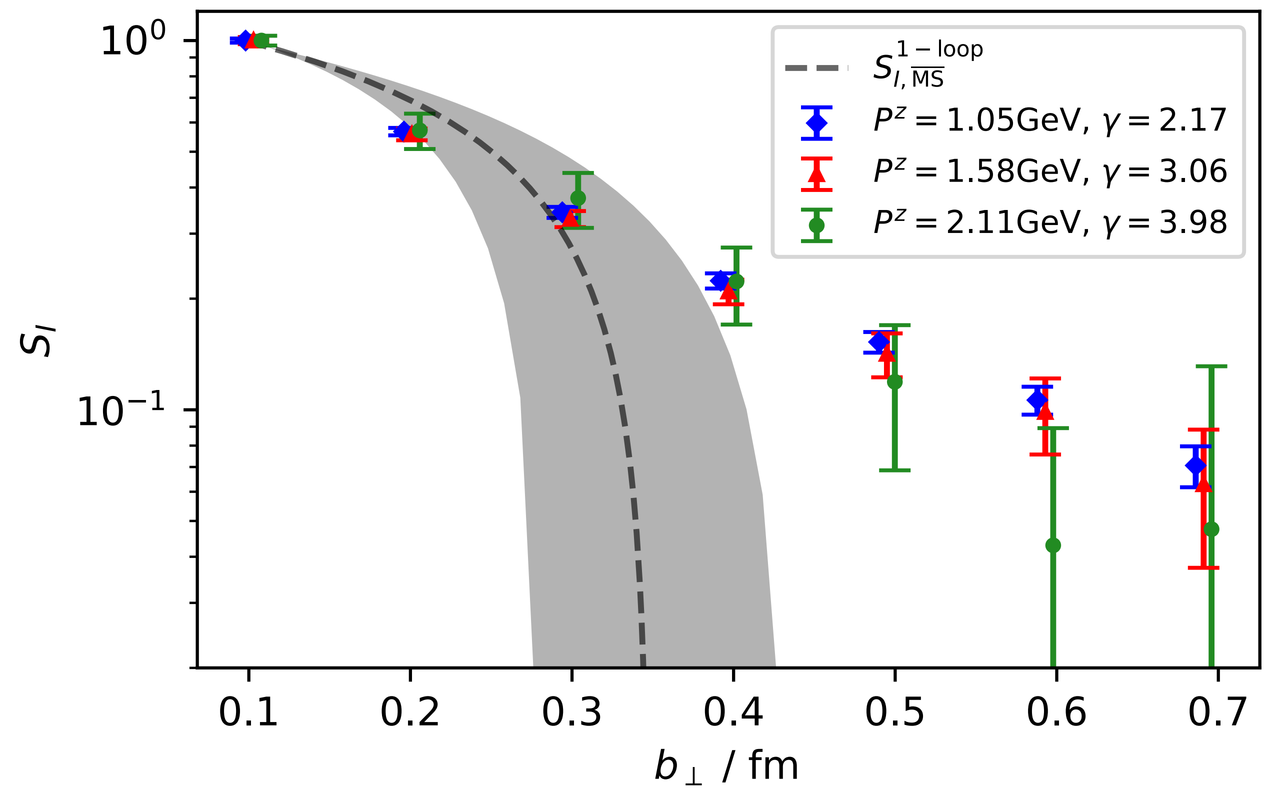
Reduced soft function from lattice QCD

First lattice calculation:



$F(b_T, P^z)$

$S_q^r(b_T, \mu)$



Q.-A. Zhang, et al. (LP Collaboration), Phys.Rev.Lett. 125 (2020).

Collins-Soper kernel from lattice QCD

Collins-Soper kernel from momentum evolution of quasi-TMDs:

$$\gamma_\zeta^q(\mu, b_T) = \frac{1}{\ln(P_1^z/P_2^z)} \ln \frac{C_{\rm ns}^{\rm TMD}(\mu, xP_2^z)\, \tilde{f}_{\rm ns}^{\rm TMD}(x, \vec{b}_T, \mu, P_1^z)}{C_{\rm ns}^{\rm TMD}(\mu, xP_1^z)\, \tilde{f}_{\rm ns}^{\rm TMD}(x, \vec{b}_T, \mu, P_2^z)}$$

$$= \frac{1}{\ln(P_1^z/P_2^z)} \ln \frac{C_{\rm ns}^{\rm TMD}(\mu, xP_2^z)\, \tilde{B}_{\rm ns}^{\rm TMD}(x, \vec{b}_T, \mu, P_1^z)}{C_{\rm ns}^{\rm TMD}(\mu, xP_1^z)\, \tilde{B}_{\rm ns}^{\rm TMD}(x, \vec{b}_T, \mu, P_2^z)}$$

Study of CS kernel through quasi-TMDs suggested in

- Ji, Sun, Xiong and Yuan, PRD91 (2015);

The concrete formalism first derived in

- Ebert, Stewart and YZ, Phys.Rev.D 99 (2019).

- Does not depend on the external hadron state;
- One can also calculate ratios of TMDPDFs with different spin structures.

- Ebert, Schindler, Stewart and YZ, JHEP 09 (2020).

Collins-Soper kernel from lattice QCD

Collins-Soper kernel from momentum evolution of quasi-TMDs:

$$\gamma_\zeta^q(\mu, b_T) = \frac{1}{\ln(P_1^z/P_2^z)} \ln \frac{C_{\rm ns}^{\rm TMD}(\mu, xP_2^z)\, \tilde{f}_{\rm ns}^{\rm TMD}(x, \vec{b}_T, \mu, P_1^z)}{C_{\rm ns}^{\rm TMD}(\mu, xP_1^z)\, \tilde{f}_{\rm ns}^{\rm TMD}(x, \vec{b}_T, \mu, P_2^z)}$$

$$= \frac{1}{\ln(P_1^z/P_2^z)} \ln \frac{C_{\rm ns}^{\rm TMD}(\mu, xP_2^z)\, \tilde{B}_{\rm ns}^{\rm TMD}(x, \vec{b}_T, \mu, P_1^z)}{C_{\rm ns}^{\rm TMD}(\mu, xP_1^z)\, \tilde{B}_{\rm ns}^{\rm TMD}(x, \vec{b}_T, \mu, P_2^z)}$$

Study of CS kernel through quasi-TMDs suggested in

- Ji, Sun, Xiong and Yuan, PRD91 (2015);

The concrete formalism first derived in

- Ebert, Stewart and YZ, Phys.Rev.D 99 (2019).

- Does not depend on the external hadron state;
- One can also calculate ratios of TMDPDFs with different spin structures.

- Ebert, Schindler, Stewart and YZ, JHEP 09 (2020).

The idea of forming ratios has been used in the calculation of ratios of x-moments of TMDPDFs:

Hagler, Musch, Engelhardt, Yoon, et al., EPL88 (2009), PRD83 (2011), PRD85 (2012), PRD93 (2016), arXiv:1601.05717, PRD96 (2017)

Collins-Soper kernel from lattice QCD

Collins-Soper kernel from momentum evolution of quasi-TMDs:

$$\gamma_\zeta^q(\mu, b_T) = \frac{1}{\ln(P_1^z/P_2^z)} \ln \frac{C_{\rm ns}^{\rm TMD}(\mu, xP_2^z)\, \tilde{f}_{\rm ns}^{\rm TMD}(x, \vec{b}_T, \mu, P_1^z)}{C_{\rm ns}^{\rm TMD}(\mu, xP_1^z)\, \tilde{f}_{\rm ns}^{\rm TMD}(x, \vec{b}_T, \mu, P_2^z)}$$

$$= \frac{1}{\ln(P_1^z/P_2^z)} \ln \frac{C_{\rm ns}^{\rm TMD}(\mu, xP_2^z)\, \tilde{B}_{\rm ns}^{\rm TMD}(x, \vec{b}_T, \mu, P_1^z)}{C_{\rm ns}^{\rm TMD}(\mu, xP_1^z)\, \tilde{B}_{\rm ns}^{\rm TMD}(x, \vec{b}_T, \mu, P_2^z)}$$

Study of CS kernel through quasi-TMDs suggested in

- Ji, Sun, Xiong and Yuan, PRD91 (2015);

The concrete formalism first derived in

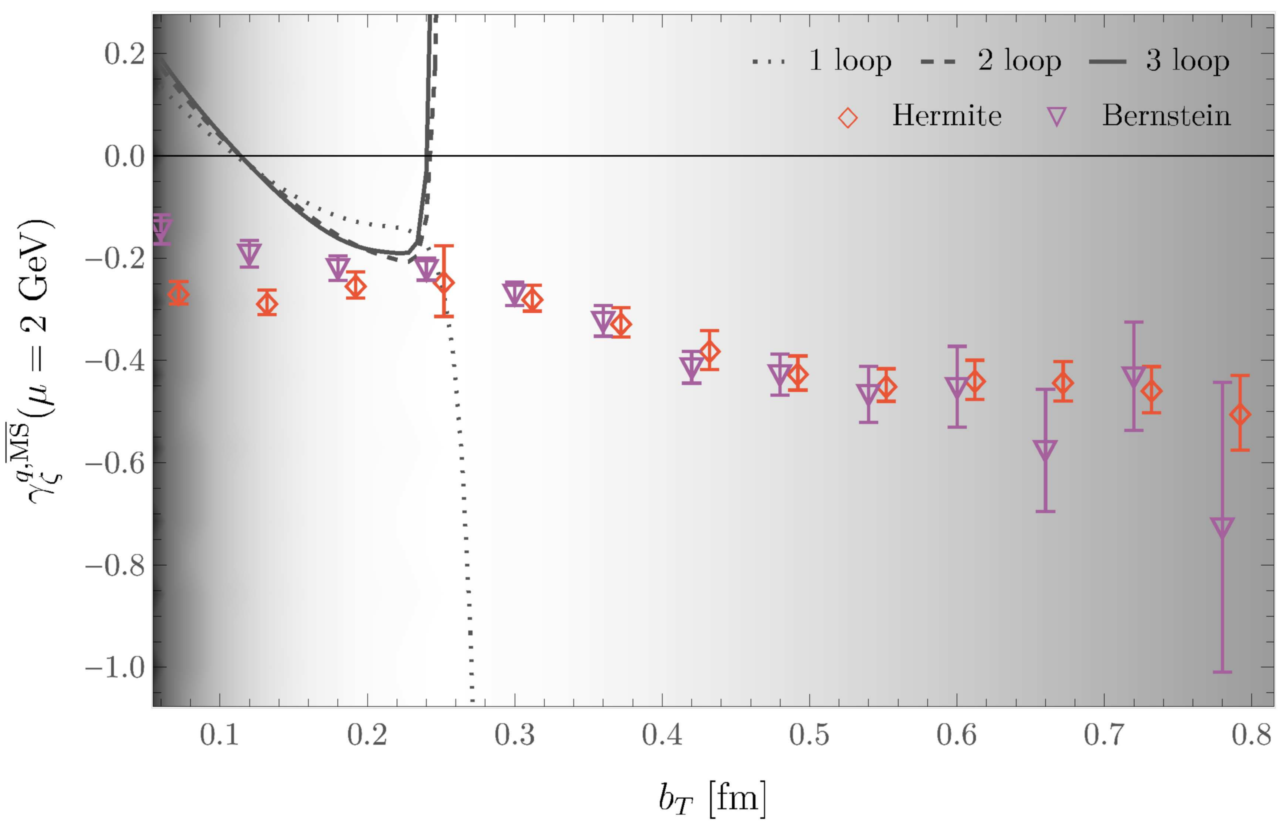
- Ebert, Stewart and YZ, Phys.Rev.D 99 (2019).

- Does not depend on the external hadron state;
- One can also calculate ratios of TMDPDFs with different spin structures.

- Ebert, Schindler, Stewart and YZ, JHEP 09 (2020).

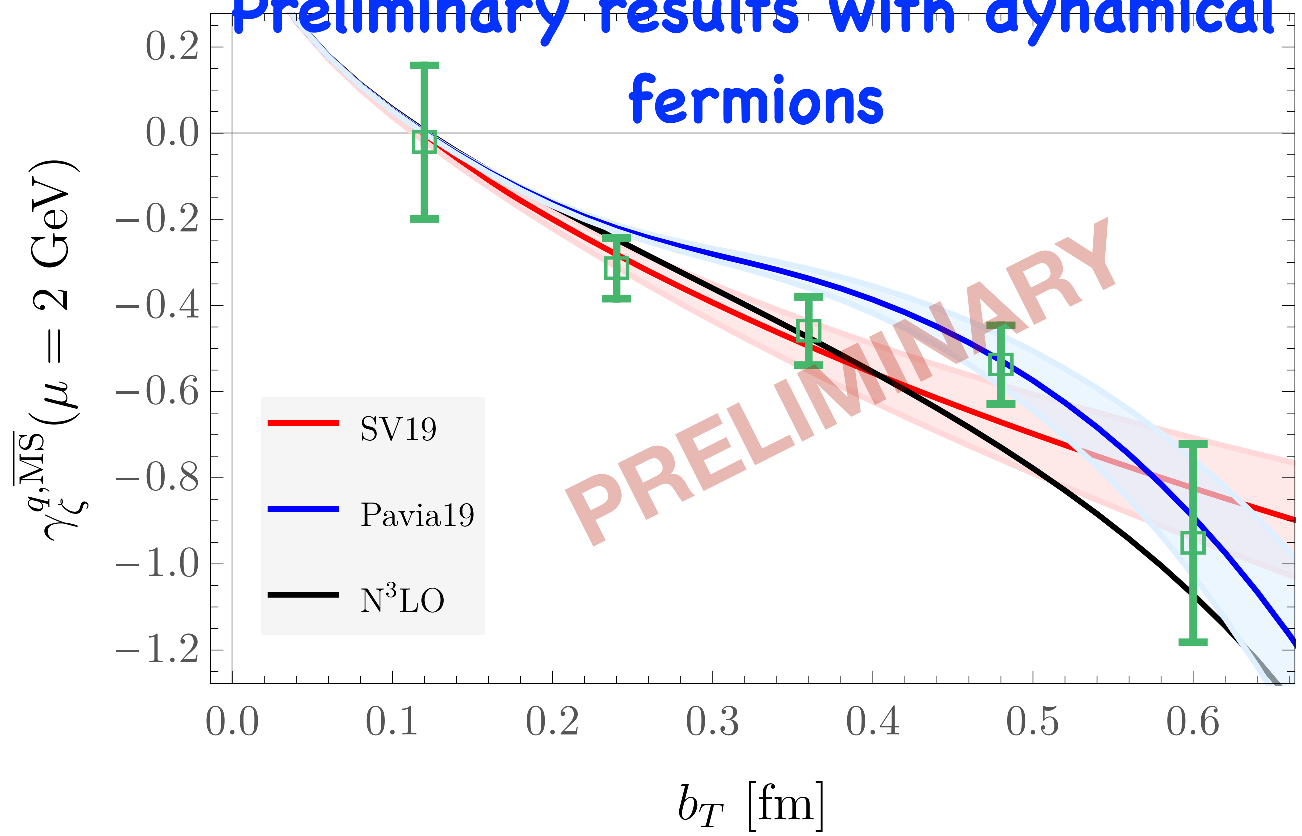
Collins-Soper kernel from lattice QCD

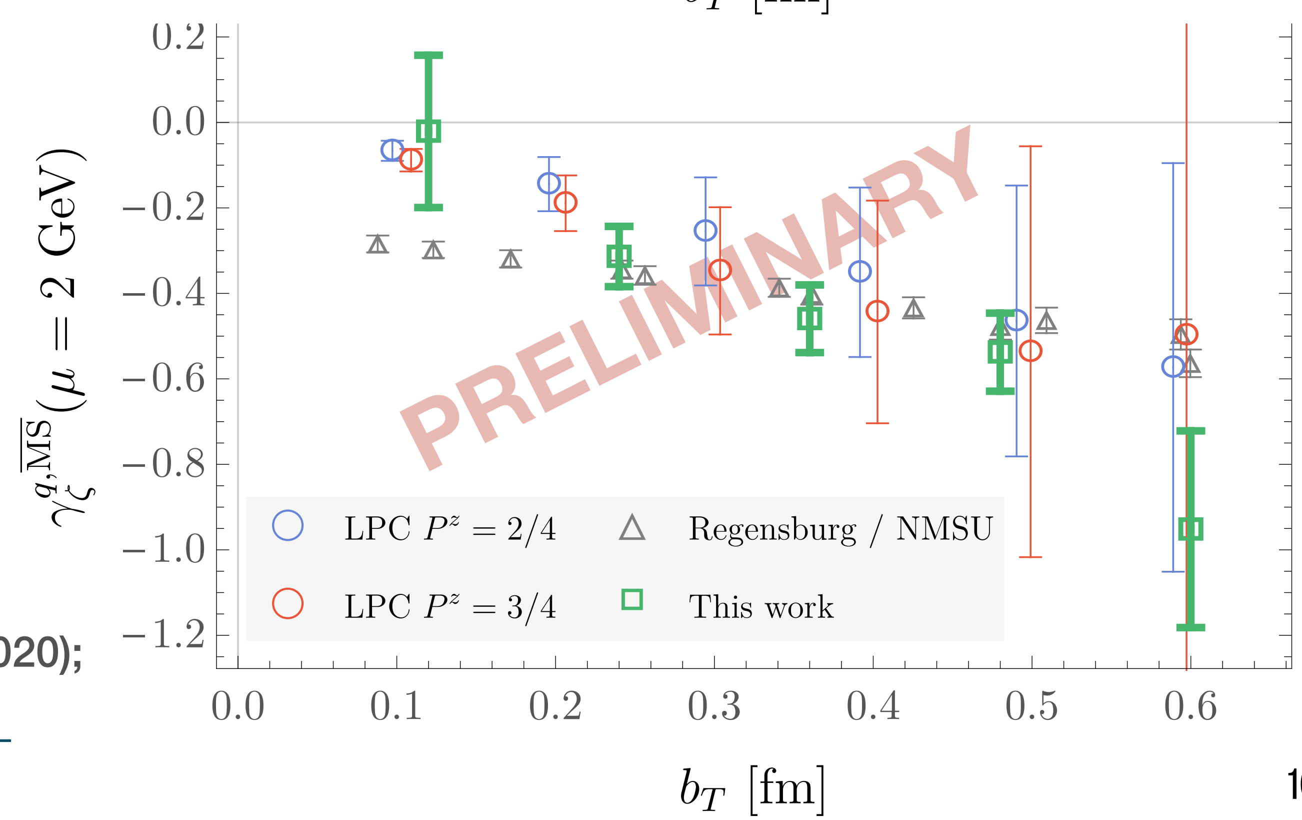
First exploratory calculation on a quenched lattice ensemble



Shanahan, Wagman, Y.Z., Phys.Rev.D 102 (2020).

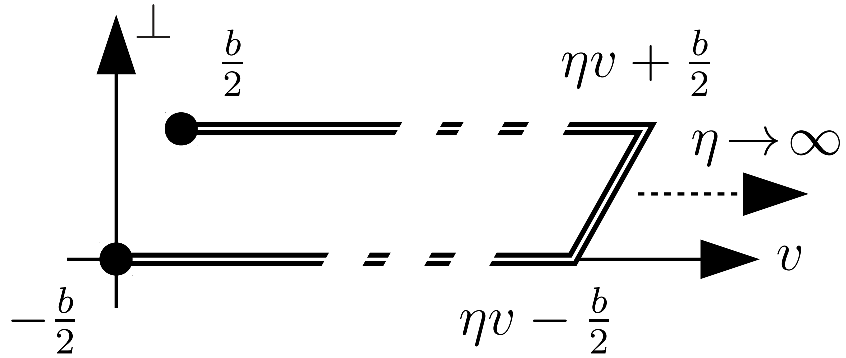
Preliminary results with dynamical fermions





Shanahan, Wagman, Y.Z., in preparation;
Q.-A. Zhang, et al. (LP Collaboration), Phys.Rev.Lett. 125 (2020);
M. Schlemmer, et al. (Regensburg), 2103.16991.

Comparison with the Lorentz-invariant approach



Hagler, Musch, Engelhardt, Yoon, et al., EPL88 (2009), PRD83 (2011), PRD85 (2012), PRD93 (2016), arXiv:1601.05717, PRD96 (2017).

Lorentz Invariant	Modern CS (y_B)	Euclidean Lattice	Quasi-beam
$P \cdot b$	$P^+ b^-$	$-P^z b^z$	$-P^z b^z$
b^2	$-\mathbf{b}_T^2$	$-b_z^2 - \mathbf{b}_T^2$	$-b_z^2 - b_T^2$
$\hat{\zeta} = \frac{v \cdot P}{m_p \sqrt{-v^2}}$	$\sinh(y_P - y_B)$	$\sinh(y_P)$	$\sinh(y_P)$
$\frac{v \cdot b}{\sqrt{-v^2}}$	$\frac{-e^{y_B} b^-}{\sqrt{2}}$	$\frac{-v^z b^z - \mathbf{v}_T \cdot \mathbf{b}_T}{\sqrt{v_z^2 + v_T^2}}$	$-b^z$
$\eta^2 v^2$	$-\infty$	$-\eta^2 (v_z^2 + v_T^2)$	$-\eta^2$

TMD Handbook by the TMD collaboration.

At fixed P, Fourier transform to obtain x-dependence:

$$\int \frac{d(P \cdot b)}{2\pi} e^{-ixP\cdot b} = -P^z \int \frac{db^z}{2\pi} e^{ixP^z b^z}$$

Need $xP^z \gg 1/b_T$, so that $b^z \ll b_T$, and $b^2 \approx -b_T^2$.

In this limit, the Lorentz-invariant approach leads to the same quasi-beam function.

Therefore, one should still need the reduced soft factor and perturbative matching $C^{\rm TMD}\left(\mu, xP^z\right)$ to extract the TMDPDF.

Lattice QCD calculation of full TMDPDF

$$\frac{\tilde{f}_{\rm ns}^{\rm TMD}(x, \vec{b}_T, \mu, P^z)}{\sqrt{S_r^q(b_T, \mu)}} = C_{\rm ns}^{\rm TMD}\left(\mu, xP^z\right) \exp\left[\frac{1}{2}\gamma_\zeta^q(\mu, b_T)\ln\frac{(2xP^z)^2}{\zeta}\right] \times f_{\rm ns}^{\rm TMD}(x, \vec{b}_T, \mu, \zeta) + \mathcal{O}\left(\frac{b_T}{L}, \frac{1}{b_T P^z}, \frac{1}{P^z L}\right)$$

- Calculation of the quasi-beam function, renormalization and matching to the MSbar scheme;
  - M. Ebert, I. Stewart, Y.Z., JHEP 03 (2020);
  - Shanahan, Wagman, Y.Z., Phys.Rev.D 101 (2020).
- Calculation of the reduced soft function;
- Calculation of the Collins-Soper kernel (to evolve to arbitrary Collins-Soper scale).

Conclusion

- LaMET uses large-momentum hadron states to filter out collinear mode contributions, thus allowing for the extraction of parton physics from lattice QCD;
- The TMD soft function and Collins-Soper evolution kernel can both be calculated from lattice, and first results show promising signs;
- Outlook: Prediction of the full TMDPDF at initial scales to provide inputs/constraints for global analysis.