

Entanglement,  
partial set of measurements, and diagonality  
of the density matrix in the parton model

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[arXiv.2001.01726](https://arxiv.org/abs/2001.01726), with Alex Kovner and Vladimir V. Skokov

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# Motivation

## ■ Proton:

- $\tau \sim \infty \Rightarrow H |\text{Proton}\rangle = M |\text{Proton}\rangle$ , (pure) energy eigenstate.
- Parton model treats proton as collection of **nearly** free particles
- Suggested resolution of this apparent paradox: quantum entanglement (arXiv.1702.03489, Kharzeev & Levin)
- Postulation: reduced density matrix for observed parton is diagonal in particle number basis

## ■ Color Glass Condensate:

- Hamiltonian is non-perturbative and unknown, so is the wavefunction
- A model for proton wavefunction

$$|\text{proton}\rangle = \sum_{\rho_a} |v; \rho_a\rangle \otimes |s; \rho_a, A_b\rangle$$

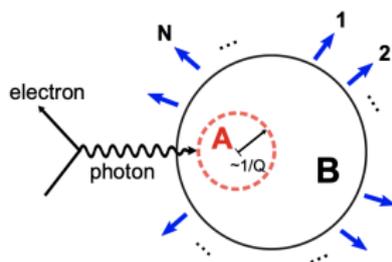
- Is CGC reduced density matrix diagonal in gluon number basis?

## Reduced density matrix

Density matrix  $\hat{\rho}(A, B) \rightarrow$  reduced density matrix

$$\hat{\rho}_A = \text{Tr}_B \hat{\rho}(A, B)$$

- Here  $A$  can be probed in DIS partons of the parton model;  $B$  is the unobserved part of the parton wavefunction.
- The property of interest: if  $\hat{\rho}(A, B)$  is pure, non-pure reduced density matrix  $\hat{\rho}_A \rightarrow$  entanglement



from arXiv.1904.11974

# Quantum entropies

Common entropies in quantum information theory:

- Renyi entropy  $S_R^N = \frac{1}{1-N} \ln \text{Tr}\{\hat{\rho}^N\}$
- von Neumann entropy  $S_V = \lim_{N \rightarrow 1} S_R^N = -\text{Tr}\{\hat{\rho} \ln \hat{\rho}\}$

**Entropy of entanglement:** entropy of the reduced density matrix

## Entropy of ignorance

- Any experimental measurement is limited: one can study only part of the full  $\hat{\rho}$ .
- Parton model: most (if not all) observables probe diagonal components of the density matrix in the number of parton representation.
- Ignorance density matrix: replace the off-diagonal elements of the density matrix with zeros. The Ignorance density matrix is positive-definite and is **definitely** not pure.
- **Entropy of ignorance**: entropy of the ignorance density matrix

## Example

Given a pure state  $|\phi_{AB}\rangle = \frac{\sqrt{2}}{2} |0_A\rangle \otimes |0_B\rangle + \frac{1}{2} |0_A\rangle \otimes (|0_B\rangle + |1_B\rangle)$

$$\hat{\rho}_A = \frac{1}{4} \begin{bmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 2 \end{bmatrix} \quad \hat{\rho}_A^I = \frac{1}{4} \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

The ignorance density matrix  $\rho_A^I$  is defined in particle number basis.

$$S_V(\hat{\rho}_A) \sim 0.426 \quad S_V(\hat{\rho}_A^I) \sim 0.693$$

# Proton wave function

CGC model for Proton wavefunction,

$$|\text{proton}\rangle = \sum_{\rho_a} |v; \rho_a\rangle \otimes |s; \rho_a, A_b\rangle$$

where

- $|v\rangle$  describes the valance dof
- $|s\rangle$  stands for soft gluons
- $\rho_a(x)$  is the color charge density of the valance modes
- $A_b$  is the gluon field generated from the source  $\rho_a$

# Reduced density matrix for soft gluons in MV model

Our goal is the reduced density matrix for soft gluons

$$\hat{\rho}_s = \text{Tr}_v(|v\rangle \langle v| \otimes |s\rangle \langle s|)$$

In MV model

$$\langle v|v\rangle = \exp\left\{-\int_k \frac{\rho_a(k)\rho_a^*(k)}{2\mu^2}\right\}$$

$$\text{Tr}_v \Rightarrow \int D[\rho_a]$$

$$|s\rangle = \mathcal{C} |0\rangle; \quad \mathcal{C} = \exp\left\{i \int_k b_a^i(k) \phi_a^{*i}\right\}$$

$$b_a^i(k) = \frac{igk^i}{k^2} \rho_a(k) + \mathcal{O}(\rho_a^2)$$

$$\phi_a^{*i}(k) = a_a^{i\dagger}(k) + a_a^i(-k)$$

## Entropy of entanglement

$$\hat{\rho}(\phi, \Phi) = \mathcal{N} \int D[\rho_a] e^{-\int_k \frac{\rho_a(k) \rho_a^*(k)}{2\mu^2}} \langle \phi | \mathcal{C} | 0 \rangle \langle 0 | \mathcal{C}^\dagger | \Phi \rangle$$

To compute the entanglement entropy, one recall

$$- \text{Tr}\{\hat{\rho} \ln(\hat{\rho})\} = \lim_{N \rightarrow 1} \frac{1}{1-N} \ln(\text{Tr}\{\hat{\rho}^N\})$$

and in terms of functional integrals

$$\text{Tr}\{\hat{\rho}^N\} = \int D[\phi_1, \phi_2, \dots, \phi_N] \rho(\phi_1, \phi_2) \rho(\phi_2, \phi_3) \dots \rho(\phi_N, \phi_1)$$

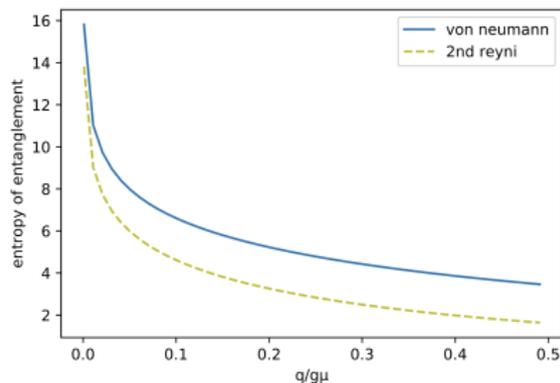
# Analytic results for entropy of entanglement (Leading order)

$$S_R^2 = \frac{1}{2}(N_c^2 - 1)S_\perp \int \frac{d^2 q}{(2\pi)^2} \ln\left(1 + 4\frac{g^2\mu^2}{q^2}\right).$$

$$S_V = \frac{1}{2}(N_c^2 - 1)S_\perp \int \frac{d^2 q}{(2\pi)^2} \left[ \ln\left(\frac{g^2\mu^2}{q^2}\right) + \sqrt{1 + 4\frac{g^2\mu^2}{q^2}} \ln\left(1 + \frac{q^2}{2g^2\mu^2} + \frac{q^2}{2g^2\mu^2} \sqrt{1 + 4\frac{g^2\mu^2}{q^2}}\right) \right].$$

- Extensive in terms of transverse area  $S_\perp$
- $\lim_{q \rightarrow \infty} S(q) = 0$

*ArXiv.1506.05394 by Alex Kovner,  
Michael Lublinsky*



# Soft gluon in particle number basis

Recall the definition of soft gluon state

$$|s\rangle = e^{i \int_k b_a(k)(a_a^\dagger(k) + a_a(-k))} |0\rangle$$

- Discretize the momentum  $\int \frac{dk^2}{(2\pi)^2} \rightarrow \sum \frac{\Delta^2}{(2\pi)^2}$
- The coherent operator can be rewritten as

$$\mathcal{C} |0\rangle = e^{i \int_k b_a(k) a_a^\dagger(k) + b_a^*(k) a_a(k)} |0\rangle = e^{i \int_k b_a(\mathbf{k}) a_a^\dagger(\mathbf{k})} e^{-\frac{1}{2} \int_k \frac{g^2}{k^2} |\rho_a|^2} |0\rangle$$

- Expanding  $e^{i \frac{\Delta^2}{(2\pi)^2} b_a(\mathbf{k}) a_a^\dagger(\mathbf{k})}$  will allow us to do calculation in particle number basis.

## Matrix elements in particle number basis

For a single momentum mode  $q$ , including normalization

$$\begin{aligned} & \langle n_c(q), m_c(-q) | \hat{\rho}_s(q) | \alpha_c(q), \beta_c(-q) \rangle \\ &= (1 - R) \frac{(n + \beta)!}{\sqrt{n! m! \alpha! \beta!}} \left( \frac{R}{2} \right)^{n + \beta} \delta_{(n + \beta), (m + \alpha)}; \quad R = \left( 1 + \frac{q^2}{2g^2 \mu^2} \right)^{-1} \end{aligned}$$

**Nonzero off-diagonal elements**, eg,  $\langle 0, 0 | \hat{\rho}_s(q) | 1, 1 \rangle = \frac{(1-R)R}{2}$

The delta function is from the gaussian integral of  $\rho_a(q)$ , the diagonal matrix elements are

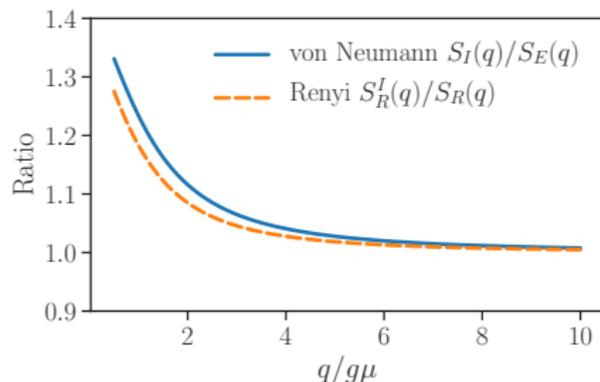
$$\begin{aligned} & \langle n_c(q), m_c(-q) | \hat{\rho}_s(q) | n_c(q), m_c(-q) \rangle \\ &= (1 - R) \frac{(n_c + m_c)!}{n_c! m_c!} \left( \frac{R}{2} \right)^{n_c + m_c} \end{aligned}$$

# Ratio between entropies of entanglement and ignorance

- $\rho_{nm} \propto (1 - R)R^{m_c+n_c}$  at momentum  $q$

$$R = \left(1 + \frac{q^2}{2g^2\mu^2}\right)^{-1}$$

- At large  $q$ ,  $S_I \simeq S_E$ 
  - $R \simeq 0$
  - vacuum contribution dominates
- At small  $q$ ,  $S_I > S_E$ 
  - $R \sim \mathcal{O}(1)$
  - higher states and **interference terms** are also important



# Experimental observation and the entropy of ignorance

- Observation was done in particle number basis in experiment

$$S_{\text{entanglement}} \Rightarrow S_{\text{hadron}} = - \sum P(N_h) \ln(P(N_h))$$

- Kharzeev & Levin

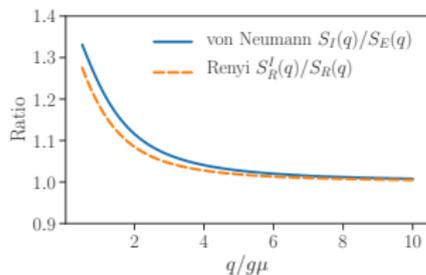
- The reduced density matrix  $\hat{\rho}_r = \sum_{N_p} P_{N_p} |N_p\rangle\langle N_p|$
- $S = - \sum P_{N_p} \ln(P_{N_p})$
- At small  $x$ , entropy of gluon  $\Rightarrow$  entropy of hadron

$$S \simeq \ln(xG(x, Q^2))$$

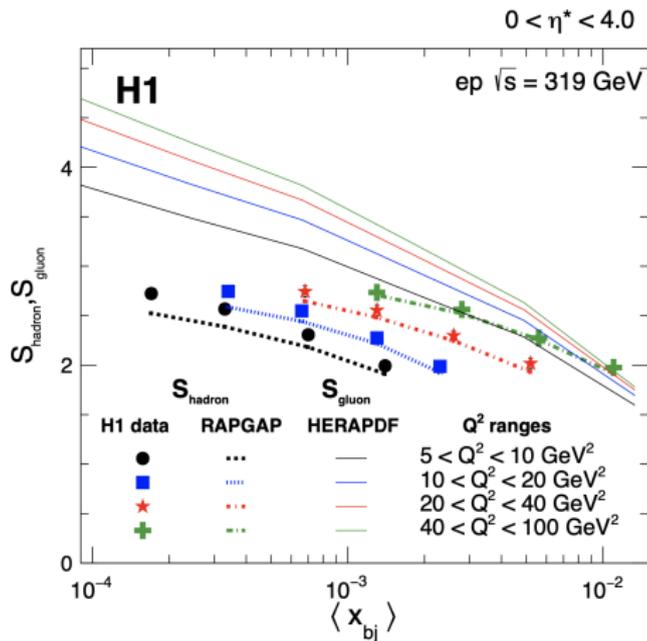
- $\ln(xG(x, Q^2))$  corresponds to entropy of ignorance in our calculation

# DIS data from HERA

- $\ln(xG(x, Q^2))$  overestimates hadron entropy
- Difference becomes small at large  $Q^2$  which is consistent with our analysis



arXiv 2011.01812, H1 Collaboration



In what basis  $S_I = S_E$ ?

## Thermal density matrix

A different perspective, consider the following **reduced** density matrix

$$\hat{\rho}_r = (1 - e^{-\beta\omega_0}) \sum_{n=0} e^{-n\beta\omega_0} |n\rangle\langle n|$$

where  $n$  is the energy level, and define  $f = \frac{1}{e^{\beta\omega_0} - 1}$ . The corresponding von Neumann entropy is

$$S_V = (1 + f) \ln(1 + f) - f \ln(f)$$

## A further examination of CGC $S_V$

$$S_V = \frac{1}{2}(N_c^2 - 1)S_\perp \int \frac{d^2q}{(2\pi)^2} \left[ \ln\left(\frac{g^2\mu^2}{q^2}\right) + \sqrt{1 + 4\frac{g^2\mu^2}{q^2}} \ln\left(1 + \frac{q^2}{2g^2\mu^2} + \frac{q^2}{2g^2\mu^2} \sqrt{1 + 4\frac{g^2\mu^2}{q^2}}\right) \right]$$

If set  $\beta\omega_0 = 2 \ln\left(\frac{q}{2g\mu} + \sqrt{1 + \frac{q^2}{4g^2\mu^2}}\right)$ , we recover the same structure

$$S_V = \frac{1}{2}(N_c^2 - 1)S_\perp \int \frac{d^2q}{(2\pi)^2} [(1 + f) \ln(1 + f) - f \ln(f)]$$

Which indicates the leading order CGC density matrix describe a thermal system of *quasi-particles*

$$c(q) = \frac{1}{2}(\sqrt{\alpha} + \frac{1}{\sqrt{\alpha}}) a(q) + \frac{1}{2}(\sqrt{\alpha} - \frac{1}{\sqrt{\alpha}}) a^\dagger(-q)$$

$$\text{where } \alpha = \sqrt{1 + \frac{4g^2\mu^2}{q^2}}$$

## Conclusion

- CGC provides a calculable model for proton wave function
- In CGC, density matrix for soft gluons is not diagonal in particle number basis; this contradicts to Kharzeev and Levin's assumption
- Entropy of ignorance overestimates entropy of hadrons
- CGC reduced density matrix can be diagonalized into thermal form

## Through diagonalization of $\hat{\rho}_r$

In field basis

$$\hat{\rho}_r = \int D[\phi, \Phi] \rho_r(\phi, \Phi) |\phi\rangle \langle \Phi|$$

To diagonalize it, we construct a wave functional

$$|\Psi\rangle = \int D[\psi] f(\psi) |\psi\rangle$$

and we then have the eigen-equation

$$\hat{\rho}_r |\Psi\rangle = \lambda |\Psi\rangle$$

## Through diagonalization of $\hat{\rho}_r$

In terms of field explicitly

$$\int D[\Phi] \rho_r(\phi, \Phi) f(\Phi) = \lambda f(\phi)$$

Our assumption is based on quantum harmonic oscillator such that the ground state is given by

$$f(\phi) = \exp\{-\alpha\phi\phi^*\}$$

One can build higher excited states use ladder operators.

## Thermal eigenvalues

It turns out, the reduced density matrix can be exactly diagonalized in the "quantum harmonic oscillator" basis, with Boltzmann weight eigenvalues.

$$\lambda_n = \exp\left\{-\left(\frac{1}{2} + n\right)\omega\beta\right\}$$

where  $n=0, 1, 2, \dots$

$$\beta\omega = 2 \ln\left(\frac{q}{2g\mu} + \sqrt{1 + \frac{q^2}{4g^2\mu^2}}\right)$$

## Connection to $xG^{(1)}(x, Q)$ and $xh^{(1)}(x, Q)$

Leading order:

$$\langle b_a^i(k) b_b^j(k) \rangle = g^2 \mu^2 \delta^{ab} \frac{k^i k^j}{k^4} \quad (1)$$

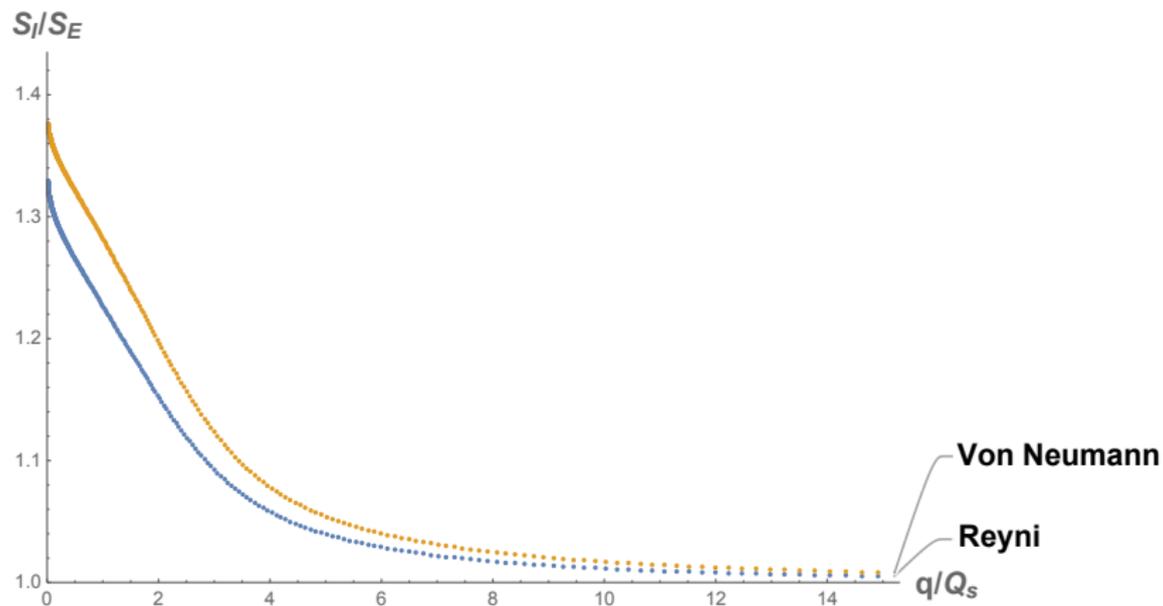
Full result:

$$\langle b_i^a(k) b_j^b(q) \rangle = \frac{(2\pi)^6}{4\pi S_{\perp}^2 (N_c^2 - 1)} \delta_{ab} \delta^{(2)}(k + q) xG_{WW}^{ij}(x, k) \quad (2)$$

where

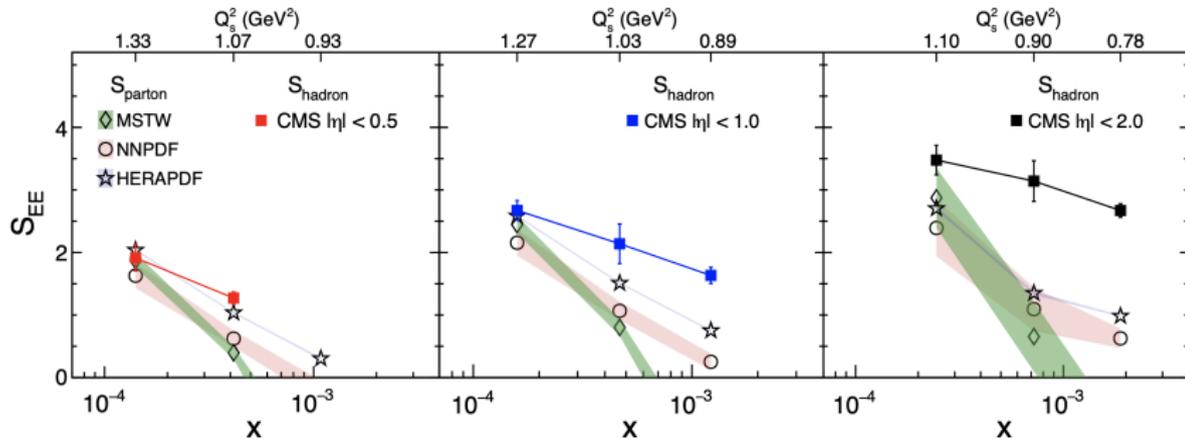
$$xG_{WW}^{ij}(x, k) = \frac{1}{2} xG^{(1)}(x, k) \delta^{ij} - \frac{1}{2} \left( \delta^{ij} - \frac{2k^i k^j}{k^2} \right) xh^{(1)}(x, k) \quad (3)$$

## Ratio of entropies for full result



# PP collision data from LHC

ArXiv.1904.11974, by Zhoudunming Tu, Dmitri E. Kharzeev,  
Thomas Ullrich



$$S_{Parton} = \ln(xG(x, Q^2))$$

$$S_{hadron} = - \sum P(N) \ln P(N)$$