Entanglement, partial set of measurements, and diagonality of the density matrix in the parton model

Haowu Duan

North Carolina State University
arXiv.2001.01726, with Alex Kovner and Vladimir V. Skokov

EIC Early Career Workshop 2021

NC STATE UNIVERSITY

Motivation

- Proton:

  $\tau \gg 1 \Rightarrow H|\text{Proton}\rangle = M|\text{Proton}\rangle$, (pure) energy eigenstate.
  Parton model treats proton as collection of **nearly** free particles
  Suggested resolution of this apparent paradox: quantum entanglement (arXiv.1702.03489,Kharzeev & Levin)
  Postulation: reduced density matrix for observed parton is diagonal in particle number basis

- Color Glass Condensate:

  Hamiltonian is non-perturbative and unknown, so is the wavefunction
  A model for proton wavefunction

$$|\text{proton}\rangle = \sum_{\rho_a} |v; \rho_a\rangle \otimes |s; \rho_a, A_b\rangle$$

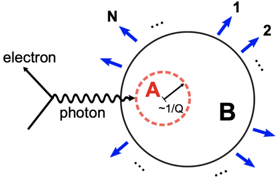
- Is CGC reduced density matrix diagonal in gluon number basis?

Reduced density matrix

Density matrix $\hat{\rho}(A, B) \to$ reduced density matrix

$$\hat{\rho}_A = \mathrm{Tr}_B\, \hat{\rho}(A, B)$$

- Here A can be probed in DIS partons of the parton model; B is the unobserved part of the parton wavefunction.
- The property of interest: if $\hat{\rho}(A, B)$ is pure, non-pure reduced density matrix $\hat{\rho}_A \to$ entanglement



from arXiv.1904.11974

NC STATE UNIVERSITY

Quantum entropies

Common entropies in quantum information theory:

- Renyi entropy $S_R^N = \frac{1}{1-N} \ln \mathrm{Tr}\{\hat{\rho}^N\}$
- von Neumann entropy $S_V = \lim_{N \to 1} S_R^N = -\mathrm{Tr}\{\hat{\rho} \ln \hat{\rho}\}$

Entropy of entanglement: entropy of the reduced density matrix

NC STATE
UNIVERSITY

Entropy of ignorance

- Any experimental measurement is limited: one can study only part of the full $\hat{\rho}$.
- Parton model: most (if not all) observables probe diagonal components of the density matrix in the number of parton representation.
- Ignorance density matrix: replace the off-diagonal elements of the density matrix with zeros. The Ignorance density matrix is positive-definite and is **definitely** not pure.
- **Entropy of ignorance**: entropy of the ignorance density matrix

NC STATE
UNIVERSITY

Example

Given a pure state $|\phi_{AB}\rangle = \frac{\sqrt{2}}{2}\,|0_A\rangle \otimes |0_B\rangle + \frac{1}{2}\,|0_A\rangle \otimes (|0_B\rangle + |1_B\rangle)$

$$\hat{\rho}_A = \frac{1}{4}\begin{bmatrix} 2 & \sqrt{2} \\ \sqrt{2} & 2 \end{bmatrix} \qquad \hat{\rho}^I_A = \frac{1}{4}\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}$$

The ignorance density matrix ρ^I_A is defined in particle number basis.

$$S_V(\hat{\rho}_A) \sim 0.426 \quad S_V(\hat{\rho}^I_A) \sim 0.693$$

NC STATE
UNIVERSITY

Proton wave function

CGC model for Proton wavefunction,

$$|\text{proton}\rangle = \sum_{\rho_a} |v; \rho_a\rangle \otimes |s; \rho_a, A_b\rangle$$

where

- $|v\rangle$ describes the valance dof
- $|s\rangle$ stands for soft gluons
- $\rho_a(x)$ is the color charge density of the valance modes
- A_b is the gluon field generated from the source ρ_a

NC STATE
UNIVERSITY

Reduced density matrix for soft gluons in MV model

Our goal is the reduced density matrix for soft gluons

$$\hat{\rho}_s = \mathrm{Tr}_v(|v\rangle\langle v| \otimes |s\rangle\langle s|)$$

In MV model

$$\langle v|v\rangle = \exp\left\{-\int_k \frac{\rho_a(k)\rho_a(k)}{2\mu^2}\right\}$$

$$\mathrm{Tr}_v \Rightarrow \int D[\rho_a]$$

$$|s\rangle = \mathcal{C}\,|0\rangle\,; \quad \mathcal{C} = \exp\left\{i\int_k b_a^i(k){\phi_a}^i\right\}$$

$$b_a^i(k) = \frac{igk^i}{k^2}\rho_a(k) + \mathcal{O}\big(\rho_a^2\big)$$

$$ {\phi_a}^i(k) = a_a^{i\dagger}(k) + a_a^i(-k)$$

NC STATE UNIVERSITY

Entropy of entanglement

$$\hat{\rho}(\phi, \Phi) = \mathcal{N} \int D[\rho_a] e^{\quad \int_k \frac{\rho_a(k)\rho_a^*(k)}{2\mu^2}} \langle \phi | \mathcal{C} | 0 \rangle \langle 0 | \mathcal{C}^y | \Phi \rangle$$

To compute the entanglement entropy, one recall

$$-\mathrm{Tr}\{\hat{\rho} \ln(\hat{\rho})\} = \lim_{N \,!\, 1} \frac{1}{1 - N} \ln\big(\mathrm{Tr}\big\{\hat{\rho}^N\big\}\big)$$

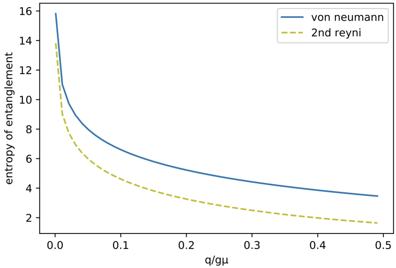
and in terms of functional integrals

$$\mathrm{Tr}\big\{\hat{\rho}^N\big\} = \int D[\phi_1, \phi_2, ..\phi_N] \rho(\phi_1, \phi_2) \rho(\phi_2, \phi_3) ... \rho(\phi_N, \phi_1)$$

Analytic results for entropy of entanglement (Leading order)

$$S_R^2 = \frac{1}{2}(N_c^2 - 1)S_\perp \int \frac{d^2q}{(2\pi)^2} \ln\left(1 + 4\frac{g^2\mu^2}{q^2}\right).$$

$$S_V = \frac{1}{2}(N_c^2 - 1)S_\perp \int \frac{d^2q}{(2\pi)^2} \left[\ln\left(\frac{g^2\mu^2}{q^2}\right) + \sqrt{1 + 4\frac{g^2\mu^2}{q^2}} \ln\left(1 + \frac{q^2}{2g^2\mu^2} + \frac{q^2}{2g^2\mu^2}\sqrt{1 + 4\frac{g^2\mu^2}{q^2}}\right)\right].$$



- Extensive in terms of transverse area $S_?$
- $\lim_{q\,!\,1} S(q) = 0$

ArXiv.1506.05394 by Alex Kovner, Michael Lublinsky

NC STATE UNIVERSITY

Soft gluon in particle number basis

Recall the definition of soft gluon state

$$|s\rangle = e^{i\int_k b_a(k)(a_a^\dagger(k)+a_a(\ \ k))}\,|0\rangle$$

- Discretize the momentum $\int \frac{dk^2}{(2\pi)^2} \to \sum \frac{\Delta^2}{(2\pi)^2}$
- The coherent operator can be rewritten as

$$\mathcal{C}\,|0\rangle = e^{i\int_k b_a(k)a_a^\dagger(k)+b_a^*(k)a_a(k)}\,|0\rangle = \boldsymbol{e^{i\int_k b_a(k)a_a^\dagger(k)}}\,e^{\ \ \frac{1}{2}\int_k \frac{g^2}{k^2}|\rho_a|^2}\,|0\rangle$$

- Expanding $\boldsymbol{e^{i\frac{\Delta^2}{(2\pi)^2}b_a(k)a_a^\dagger(k)}}$ will allow us to do calculation in particle number basis.

NC STATE UNIVERSITY

Matrix elements in particle number basis

For a single momemtum mode q, including normalization

$$\langle n_c(q), m_c(-q)|\hat{\rho}_s(q)|\alpha_c(q), \beta_c(-q)\rangle$$
$$= (1-R)\frac{(n+\beta)!}{\sqrt{n!m!\alpha!\beta!}}\left(\frac{R}{2}\right)^{n+\beta}\delta_{(n+\beta),(m+\alpha)}\,; \quad R = (1+\frac{q^2}{2g^2\mu^2})^{\ 1}$$

Nonzero off-diagonal elements, eg, $\langle 0,0|\hat{\rho}_s(q)|1,1\rangle = \frac{(1\ \ R)R}{2}$
The delta function is from the gaussian integral of $\rho_a(q)$, the diagonal matrix elements are

$$\langle n_c(q), m_c(-q)|\hat{\rho}_s(q)|n_c(q), m_c(-q)\rangle$$
$$= (1-R)\frac{(n_c+m_c)!}{n_c!m_c!}\left(\frac{R}{2}\right)^{n_c+m_c}$$

NC STATE
UNIVERSITY

Ratio between entropies of entanglement and ignorance

- $\rho_{nm} \propto (1-R)R^{m_c+n_c}$ at momentum q

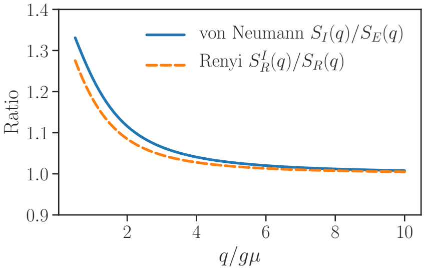
$$R = (1 + \frac{q^2}{2g^2\mu^2})^{-1}$$

- At large q, $S_I \simeq S_E$

  R ' 0
  vacuum contribution dominates

- At small q, $S_I > S_E$

  R $O(1)$
  higher states and **interference terms** are also important



NC STATE
UNIVERSITY

Experimental observation and the entropy of ignorance

- Observation was done in particle number basis in experiment
  $S_{\text{entanglement}} \Rightarrow S_{\text{hadron}} = -\sum P(N_h) \ln(P(N_h))$
- Kharzeev & Levin

  The reduced density matrix $\hat{\rho}_r = \sum_{N_p} P_{N_p} |N_p\rangle\langle N_p|$

  $S = -\sum P_{N_p} \ln(P_{N_p})$

  At small x, entropy of gluon $\Rightarrow$ entropy of hadron

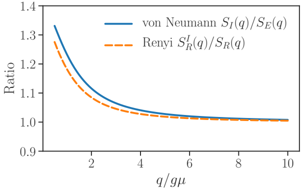
$$S \simeq \ln\big(xG(x, Q^2)\big)$$

- $\ln\big(xG(x, Q^2)\big)$ corresponds to entropy of ignorance in our calculation

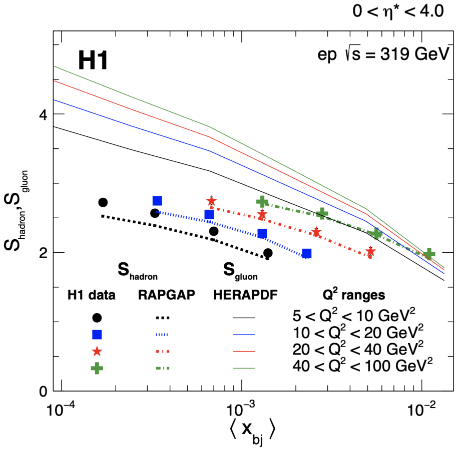
NC STATE UNIVERSITY

DIS data from HERA

- $\ln\left(xG(x, Q^2)\right)$ overestimates hadron entropy
- Difference becomes small at large Q^2 which is consistent with our analysis



arXiv 2011.01812, H1 Collaboration



NC STATE UNIVERSITY

In what basis $S_I = S_E$?

NC STATE UNIVERSITY

Thermal density matrix

A different perspective, consider the following **reduced** density matrix

$$\hat{\rho}_r = (1 - e^{\quad \beta\omega_0}) \sum_{n=0} e^{\quad n\beta\omega_0} |n\rangle\langle n|$$

where n is the energy level, and define $f = \frac{1}{e^{\beta\omega_0} \quad 1}$. The corresponding von Neumann entropy is

$$S_V = (1 + f)\ln(1 + f) - f\ln(f)$$

A further examination of CGC S_V

$$S_V = \frac{1}{2}(N_c^2-1)S_\perp \int \frac{d^2q}{(2\pi)^2}\left[\ln\left(\frac{g^2\mu^2}{q^2}\right) + \sqrt{1+4\frac{g^2\mu^2}{q^2}}\ln\left(1+\frac{q^2}{2g^2\mu^2}+\frac{q^2}{2g^2\mu^2}\sqrt{1+4\frac{g^2\mu^2}{q^2}}\right)\right]$$

If set $\beta\omega_0 = 2\ln\left(\frac{q}{2g\mu}+\sqrt{1+\frac{q^2}{4g^2\mu^2}}\right)$, we recover the same structure

$$S_V = \frac{1}{2}(N_c^2-1)S_? \int \frac{d^2q}{(2\pi)^2}\left[(1+f)\ln(1+f) - f\ln(f)\right]$$

Which indicates the leading order CGC density matrix describe a thermal system of $quasi-particles$

$$c(q) = \frac{1}{2}(\sqrt{\alpha}+\frac{1}{\sqrt{\alpha}})\ a(q) + \frac{1}{2}(\sqrt{\alpha}-\frac{1}{\sqrt{\alpha}})\ a^y(-q)$$

where $\alpha = \sqrt{1+\frac{4g^2\mu^2}{q^2}}$

NC STATE UNIVERSITY

Conclusion

- CGC provides a calculable model for proton wave function
- In CGC, density matrix for soft gluons is not diagonal in particle number basis; this contradicts to Kharzeev and Levin's assumption
- Entropy of ignorance overestimates entropy of hadrons
- CGC reduced density matrix can be diagonalized into thermal form

NC STATE UNIVERSITY

Through diagonalization of $\hat{\rho}_r$

In field basis

$$\hat{\rho}_r = \int D[\phi, \Phi] \;\; \rho_r(\phi, \Phi)|\phi\rangle\langle\Phi|$$

To diagonalize it, we construct a wave functional

$$|\Psi\rangle = \int D[\psi] f(\psi)|\psi\rangle$$

and we then have the eigen-equation

$$\hat{\rho}_r|\Psi\rangle = \lambda|\Psi\rangle$$

NC STATE
UNIVERSITY

Through diagonalization of $\hat{\rho}_r$

In terms of field explicitly

$$\int D[\Phi]\ \ \rho_r(\phi, \Phi) f(\Phi) = \lambda f(\phi)$$

Our assumption is based on quantum harmonic oscillator such that the ground state is given by

$$f(\phi) = \exp\{-\alpha\phi\phi\ \}$$

One can build higher excited states use ladder operators.

NC STATE
UNIVERSITY

Thermal eigenvalues

It turns out, the reduced density matrix can be exactly diagonalized in the "quantum harmonic oscillator" basis, with Boltzmann weight eigenvalues.

$$\lambda_n = \exp\left\{-(\frac{1}{2}+n)\omega\beta\right\}$$

where n=0, 1, 2,.....

$$\beta\omega = 2\ln\left(\frac{q}{2g\mu}+\sqrt{1+\frac{q^2}{4g^2\mu^2}}\right)$$

NC STATE
UNIVERSITY

Connection to $xG^{(1)}(x,Q)$ and $xh^{(1)}(x,Q)$

Leading order:

$$\langle b_a^i(k) b_b^j(k)\rangle = g^2\mu^2\delta^{ab}\frac{k^i k^j}{k^4} \tag{1}$$

Full result:

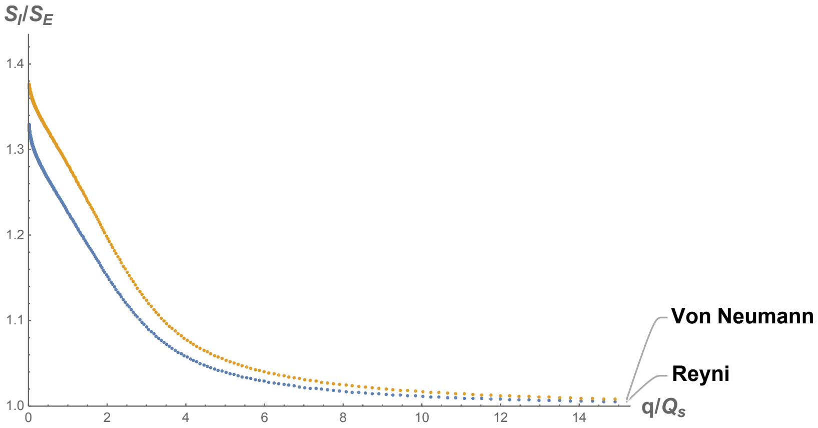
$$\langle b_i^a(k) b_j^b(q)\rangle = \frac{(2\pi)^6}{4\pi S_?^2(N_c^2-1)}\delta_{ab}\delta^{(2)}(k+q)xG_{WW}^{ij}(x,k) \tag{2}$$

where

$$xG_{WW}^{ij}(x,k) = \frac{1}{2}xG^{(1)}(x,k)\delta^{ij} - \frac{1}{2}(\delta^{ij} - \frac{2k^i k^j}{k^2})xh^{(1)}(x,k) \tag{3}$$

NC STATE UNIVERSITY

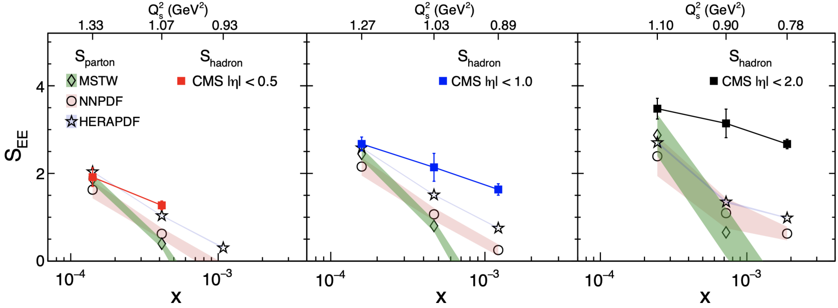
Ratio of entropies for full result



NC STATE
UNIVERSITY

PP collision data from LHC

ArXiv.1904.11974,by Zhoudunming Tu, Dmitri E. Kharzeev, Thomas Ullrich



$$S_{Parton} = \ln\left(xG(x, Q^2)\right)$$
$$S_{hadron} = -\sum P(N) \ln P(N)$$

NC STATE UNIVERSITY