New digitization strategies for relativistic quantum field theories

Niklas Mueller
University of Maryland

based on Phys. Rev. A 103, 042410

9th Workshop of the APS Topical Group on Hadronic Physics

April 15, 2021
Motivation

Computing quantum many-body systems is exponentially hard (most of the time)

Static problems
Thermal or ground state properties, bound states etc.

Dynamical problems
Scattering at the Electron-Ion Collider

Ultra-relativistic heavy ion collisions
Motivation

\[ H = \sum_n \left[ \frac{\pi_n^2}{2} + \frac{1}{2} (\nabla \phi_n)^2 + \frac{m^2}{2} \phi_n^2 + \frac{\lambda}{4!} \phi_n^4 \right] \]

(Real scalar field theory \( d+1 \) dim.)

\[ \phi_n | \phi \rangle_n = \bar{\phi}_n | \phi \rangle_n \quad \text{or} \quad \pi_n | \pi \rangle_n = \bar{\pi}_n | \pi \rangle_n \]

Available Hilbert spaces (“hardware”)

**Digital** (sc-qubit, ions …)

**Analog** (atoms, molecules, …)

**Digital-Analog** (cQED, optical…)

For bosonic theories spectrum continuous, infinite dimensional local HS

\[ \phi_x | \phi \rangle_x = \bar{\phi}_x | \phi \rangle_x \quad \text{or} \quad \pi_x | \pi \rangle_x = \bar{\pi}_x | \pi \rangle_x \]

Here will focus mostly on digital implementations
**Digitization Strategies**

- Challenge in NISQ era: how to do best with limited resources?

("Strategy 1") - Field Based Digitization

\[
\phi_x |\phi\rangle_x = \tilde{\phi}_x |\phi\rangle_x \quad \tilde{\phi} \in [-\phi^{\text{max}}, \phi^{\text{max}}]
\]

\[
\tilde{\phi} = -\phi^{\text{max}} + \beta\phi\delta\phi \quad \beta \in 0, 1, \ldots \phi^{\text{max}}/\delta\phi \quad \text{(integer)}
\]

\[
|\phi\rangle \rightarrow |\beta\phi\rangle = |0\rangle, |1\rangle, |2\rangle, \ldots
\]

\[
\pi_x |\pi\rangle_x = \tilde{\pi}_x |\pi\rangle_x \quad \tilde{\pi} \in [-\pi^{\text{max}}, \pi^{\text{max}}]
\]

- Binary representation

("Strategy 2") - Harmonic Oscillator Based Digitization

\[
\phi_x = \frac{1}{V^{1/2}} \sum_q \frac{1}{(2\omega_q)^{1/2}} [a_q + a_q^+] e^{2\pi i q/N}
\]

\[
\pi_x = -\frac{i}{V^{1/2}} \sum_q \frac{\omega_q}{2} [a_q - a_q^+] e^{2\pi i q/N}
\]

- Position space vs. momentum space

\[
a_q^\dagger = \frac{\phi_x - i\pi_x}{2} \quad a_q = \frac{\phi_x + i\pi_x}{2}
\]

- Truncation \(n_q, n_x < n^{\text{max}}\), binary representation

\[
|n = 0\rangle = |\ldots 000\rangle, |n = 1\rangle = |\ldots 001\rangle, |n = 2\rangle = |\ldots 010\rangle, \ldots
\]

\[O(V)\]

Digitization Strategies

(“Strategy 3”): Particle basis

- Non-relativistic system: $|p\rangle = |0\rangle, |1\rangle, |2\rangle, \ldots$

$$|x\rangle = \frac{1}{\sqrt{V}} \sum_p e^{ipx} |p\rangle$$

- Binary encoding: $|n\rangle = |0\rangle, |1\rangle, \ldots |L-1\rangle = |0\ldots01\rangle, |0\ldots10\rangle, \ldots, |1\ldots11\rangle$ using $d \log(L) = \log(V)$ qubits

In relativistic QFT more complicated!

**Particle register**

$\mathcal{H} \equiv \text{span}(|p\rangle, |\Omega\rangle)$

- $|p\rangle = |1011\ldots; 1\rangle$
- $|\Omega\rangle = |0000\ldots; 0\rangle$

Relation to many-body states and quantum field theory

- $|\text{vac}\rangle = \bigotimes_{i=0}^{M-1} |\Omega\rangle$ is the Fock vacuum of the theory
- $|p, p'\rangle = \mathcal{N} \{ |p\rangle |p'\rangle |\Omega\rangle \ldots + |p'\rangle |p\rangle |\Omega\rangle + \ldots\} \equiv \{ n_p = 1, n_{p'} = 1 \}$ are the Fock states

... actually a hard core boson representation

$$a_p^\dagger = \frac{1}{\sqrt{M}} \sum_{i=0}^{M-1} a_p^{(i)}^\dagger \quad a_p^\dagger |\text{vac}\rangle = |p\rangle \quad [a_p, a_p^\dagger] = \delta_{p,p'} + O\left(\frac{n}{M}\right)$$

If $n/V < 1$, compact $\log(V)$ versus $V$
If $n/V \sim 1$, similar $V \log(V)$
If $n/V > 1$, worse, still polynomial

note dimensionless operators/units
+ different normalization as “textbook” single particle states $(2a_p)^{1/2} a_p^\dagger |\text{vac}\rangle = |p\rangle$
Algorithms: Scattering problem

- Interesting because reveals **dynamical information** of quantum system
  (the only “tomography tool” for high energy experimentalists)

- **S-matrix**

\[
S = U(\infty, -\infty) \quad S_{\beta\alpha} = \langle \psi^{\text{free}}_{\beta} | U_{\infty,-\infty} | \psi^{\text{free}}_{\alpha} \rangle \quad U_{t,t'} = \exp\{-iH(t - t')\}
\]

- **Computational perspectives**

  (a) Heisenberg picture, quantum variational algorithms
  Yeter-Aydeniz, Sipsis, Pooser 2008.08763

\[
|\psi^{\text{in/out}}_{\alpha}\rangle = (V - VG_0 V)^{-1} V |\phi_{\alpha}\rangle \quad H = H_0 + V \quad G_0 \equiv (E_{\alpha} - H_0 \pm i\epsilon)^{-1}
\]

  (b) Finite volume energies vs. scattering amplitudes (from euclidean and minkowski correlation functions)

  (c) Schroedinger picture, real time evolution
**Algorithms: Scattering problem**

### Schroedinger picture

(a) Prepare single particle states of non-interacting theory

\[ |\psi(-\infty)\rangle = |\phi(-\infty)\rangle \]

(b) Evolve into single particle states of interacting theory

\[ U_{t,-\infty} |\psi(-\infty)\rangle \]

\[ |\psi(t)\rangle = \int d\alpha g(\alpha) e^{-iE_\alpha t} |\psi\rangle \]

(wave packet)

\[ H(t) = H_0 + (1 - t/t_0)H_I \]

(c) Wave packets overlap / interact

\[ U_{t_0} |\psi(t_0)\rangle \]

(d) Measure outgoing particle states or energy density

\[ |\psi_{out}\rangle \]

(e) Renormalization

Relate to continuum scattering cross sections

- bosonic truncation
- continuum limit
- RG
Time Evolution Algorithm - Single Particle basis

- Trotter scheme
  \[ U = \prod_t U(\delta t) \quad U(\delta t) = e^{-iH_0\delta t} e^{-iH_I\delta t} + O(\delta t^2) \]

- Split up Hamiltonian
  \[ H = \sum_x \left[ \frac{\pi_x^2}{2} + \frac{1}{2} (\nabla \phi_x)^2 + \frac{m^2}{2} \phi_x^2 + \frac{\lambda}{4!} \phi_x^4 \right] \]
  \[ H_I = \sum_{q_1 q_2 q_3 q_4} \frac{\lambda}{4!} \phi_{q_1} \phi_{q_2} \phi_{q_3} \phi_{q_4} \delta(q_1 + q_2 + q_3 + q_4) \]

- Diagonal in momentum space
- Very complicated in momentum space
- Complicated in position space
- Local in position space

\[ U(\delta t) = e^{-iH_0 t} S \quad \text{Squeezing} \quad \text{qFT} \quad e^{-iH_I t} S \quad \text{Squeezing} \quad \text{qFT} \]

reminder:
\[ a_x^\dagger = \frac{\phi_x - i\pi_x}{2} \quad a_x = \frac{\phi_x + i\pi_x}{2} \]
are Fock operators in position space

\[ \omega_p^2 = a_p^\dagger a_p \]
Time Evolution Algorithm - Single Particle basis

Squeezing transformation?

- Fock operators in position space
  \[ a_x \equiv \frac{1}{\sqrt{2}}(\phi_x + i\pi_x) \quad a_x^\dagger \equiv \frac{1}{\sqrt{2}}(\phi_x - i\pi_x) \]

- Fourier conjugates
  \[ A_q \equiv \frac{1}{\sqrt{V}} \sum_q A_q e^{2\pi i q x / N} \]

- but \( A_q \neq a_q \) instead
  \[ A_q \equiv \frac{1}{2} \left[ \frac{1}{\sqrt{\omega_q}} + \sqrt{\omega_q} \right] a_q + \frac{1}{2} \left[ \frac{1}{\sqrt{\omega_q}} - \sqrt{\omega_q} \right] a_q^\dagger = S_q a_q S_q^\dagger \]

- Particle/Fock states in position vs momentum space
  \[ |x\rangle \neq \int \frac{d^d p}{(2\pi)^d} e^{ipx} | p \rangle \quad \text{but actually} \quad |x, x^0\rangle = \int \frac{d^4 p}{((2\pi)^4)^d} \delta(p^2 - m^2) \theta(p^0) e^{ipx} | p, p^0 \rangle \]

- Circuits

<table>
<thead>
<tr>
<th></th>
<th>Elementary gate operations</th>
<th>Ancilla qubits</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Time Evolution</strong></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Free part ( U_0 )</td>
<td>( O(M \text{poly log}(V)t) )</td>
<td>( O(\log(V)/d) )</td>
</tr>
<tr>
<td>Squeezing transform ( S )</td>
<td>( O(M^2V \text{poly log}(V)t) )</td>
<td>( 0 )</td>
</tr>
<tr>
<td>quantum Fourier transform</td>
<td>( O(M \text{poly log}(V)t) )</td>
<td>( 0 )</td>
</tr>
<tr>
<td>Interaction part ( U_I )</td>
<td>( O(M^4V \text{poly log}(V)t) )</td>
<td>( O(\log(V)/d) )</td>
</tr>
<tr>
<td><strong>Total</strong></td>
<td>( O(M^4V \text{poly log}(V)t) )</td>
<td>( O(\log(V)/d) )</td>
</tr>
</tbody>
</table>

State Preparation and measurement

- Vacuum and single particle states are computational basis states

$$\mathcal{H} \equiv \text{span}(|p\rangle, |\Omega\rangle) \quad |p\rangle = |1011...; 1\rangle \quad |\Omega\rangle = |0000...; 0\rangle$$

- **Initial state preparation:** Wave packet superposition and Bose symmetrization

- Measurement: computational basis = particle number basis

$$|\psi(t)\rangle = \sum_\epsilon \alpha_\epsilon |\psi_\epsilon\rangle = \sum_{\text{basis states}} \left[ \frac{\alpha_{q,q',...}}{N_{q,q'}}^{1/2} (t) |q,q',\Omega,.. + \text{sym} \right]$$

will collapse onto particle number state upon measurement

- If particle (parton) number not meaningful (such as in QCD):
  
  **Phase Estimation Algorithm (*)** circuits are the same as for $e^{-iH_0\delta t}$, $e^{-iH_I\delta t}$
Non-perturbative Renormalization

Renormalization scheme should match (the precision of) the actual computation. Thus in principle the same computational complexity.

- Good understanding for static quantities, but not so much for dynamics
- Hamiltonian operator renormalization

\[
H = \begin{pmatrix} H_{\ell\ell} & H_{\ell h} \\ H_{h\ell} & H_{hh} \end{pmatrix} \quad \quad H^{\text{eff}} = P_\ell (THT^\dagger)P_\ell \quad T = e^{i\eta}
\]

- Perturbation theory: Schrieffer Wolf transformation

Non-perturbative: Similarity RG


In practice (oversimplified)

1. Quantum compute dimensionless static low energy quantity, e.g. ratio of low lying excitations for some \(a_s, M\).
   Adjust \(M, \lambda\) to reproduce physical value.
2. Repeat at different \(a_s, M\) along \(a_s \to 0, M \to \infty\), adjust \(M, \lambda\) (line of constant physics)
3. Once \(\lambda^{\text{ren}}, m^{\text{ren}}\) known for many \(a_s, M\), use as input for scattering experiment + operator renormalization.
4. Continuum extrapolation of scattering results.
Summary

• Many approaches to digitization of scalar field theory and scattering algorithms

• Few are systematically explored

• Single particle digitization may be very efficient for dilute systems, not for all systems

• Initial State preparation and measurement of scattering cross sections very simple
<table>
<thead>
<tr>
<th></th>
<th>Field Based (JLP)</th>
<th>Particle Based</th>
<th>Harmonic Oscillator based (*)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>position space</td>
</tr>
<tr>
<td><strong>State rep.</strong></td>
<td>(O(V \log(V)))</td>
<td>(O(\log(V)))\footnote{**} best case</td>
<td>(O(V \log(V)))</td>
</tr>
<tr>
<td></td>
<td>(&gt; O(V \log(V))) worst case (still polynomial)</td>
<td>(O(V \log(V)))</td>
<td>(O(V \log(V)))</td>
</tr>
<tr>
<td><strong>initial state prep</strong></td>
<td>additional Trotter complexity (O(V \times \text{const}))</td>
<td>easy</td>
<td>?? (<strong>\footnote{</strong>) easy</td>
</tr>
<tr>
<td><strong>time evolution</strong></td>
<td>(O(V)) polynomial</td>
<td>(O(V)) best case</td>
<td>it depends, see Klco, Savage PRA 98, 052335 (2019)</td>
</tr>
<tr>
<td></td>
<td>(+ \text{continuum dispersion}) worst case</td>
<td>(O(V)) worst case</td>
<td>(O(V^4)) + continuum dispersion</td>
</tr>
<tr>
<td><strong>measurement</strong></td>
<td>additional Trotter complexity (O(V^2 \times \text{const}))</td>
<td>easy, integrated CS for free</td>
<td>?? (<strong>\footnote{</strong>) easy</td>
</tr>
<tr>
<td></td>
<td>lattice perturbation theory or kinetic theory (at weak coupling)</td>
<td>lattice perturbation theory or kinetic theory (at weak coupling)</td>
<td>?? (<strong>\footnote{</strong>) lattice perturbation theory or kinetic theory (at weak coupling)</td>
</tr>
<tr>
<td><strong>verification</strong></td>
<td>classical-statistical (at weak coupling) (large (\phi^{\text{max}}))</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

\footnote{(*) Note: no strict separation. Connected by unitary transformations. (**) Reflects my ignorance, not necessarily that of others.}