Helicity at Small x: The Single-Logarithmic Contribution

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Background

Proton helicity can be decomposed into spin and orbital angular momentum (OAM) of quarks and gluons [Jaffe and Manohar, 1990]

$$\frac{1}{2} = S_q + S_G + L_q + L_G \tag{1}$$

where

$$S_q(Q^2) = \frac{1}{2}\int_0^1 dx\, \Delta\Sigma(x, Q^2) = \frac{1}{2}\int_0^1 dx \sum_f \left[\Delta q_f(x, Q^2) + \Delta\overline{q}_f(x, Q^2)\right]. \tag{2}$$

Experiments have measured S_q but can only include $0 < x_{\text{min}} \leq x \leq 1$.

Objective: Find the contribution to S_q coming from $\Delta\Sigma$ as $x \to 0$.

Background

At small Bjorken-x, quark helicity distribution satisfies

$$\Delta\Sigma(x, Q^2) = \frac{N_c N_f}{2\pi^3} \int_{\Lambda^2/s}^{1} \frac{dz}{z} \int_{1/zs}^{1/zQ^2} \frac{dx_{10}^2}{x_{10}^2} \int d^2\underline{b}\; Q(\underline{x}_1, \underline{x}_0, z), \quad (3)$$

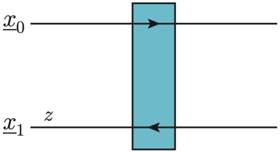
where $\underline{b} = \frac{\underline{x}_0 + \underline{x}_1}{2}$ and $\underline{x}_{10} = \underline{x}_1 - \underline{x}_0$.

Here, $Q(\underline{x}_1, \underline{x}_0, z) \equiv Q_{10}(z)$ is the quark (longitudinally) polarized dipole amplitude.

Quark Unpolarized Dipole Amplitude

$S_{10}(z)$ corresponds to a minus-moving quark dipole interacting with a plus-moving target proton, represented by the blue rectangle.

$S_{10}(z) =$



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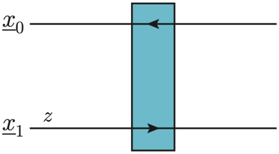


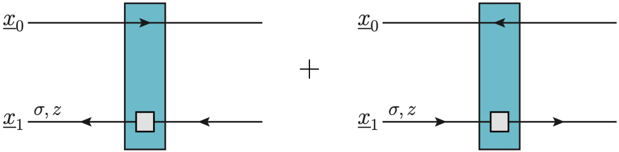
$$= \frac{1}{2N_c} \,\mathrm{Re}\left[\left\langle \mathcal{T}\ \mathrm{tr}\left[V_{\underline{0}} V_{\underline{1}}^{\dagger}\right]\right\rangle(z) + \left\langle \mathcal{T}\ \mathrm{tr}\left[V_{\underline{1}} V_{\underline{0}}^{\dagger}\right]\right\rangle(z)\right]. \tag{4}$$

- $V_{\underline{i}}$ is the fundamental Wilson's line at $\underline{x}_i$.
- The angle brackets average over target proton's wave function.

Quark Polarized Dipole Amplitude

Diagrammatically, $Q_{10}(z)$ corresponds to a quark dipole, one of which has helicity σ, interacting with a polarized target proton (blue rectangle).

$Q_{10}(z) =$



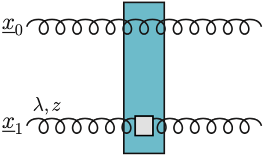
$$= \frac{zs}{2N_c} \, \mathrm{Re} \left[\left\langle \mathcal{T} \, \mathrm{tr} \left[V_{\underline{0}} V_{\underline{1}}^{\mathrm{pol} \, \dagger} \right] \right\rangle (z) + \left\langle \mathcal{T} \, \mathrm{tr} \left[V_{\underline{1}}^{\mathrm{pol}} V_{\underline{0}}^{\dagger} \right] \right\rangle (z) \right] . \quad (5)$$

- $V_{\underline{0}}$ is the fundamental unpolarized Wilson's line at $\underline{x}_0$.
- $V_{\underline{1}}^{\mathrm{pol}}$ is the fundamental *polarized Wilson's line* at $\underline{x}_1$.
- The angle brackets average over target proton's wave function.
- At Born level, $Q_{10}(z) \sim 1$ because the helicity-dependent tree-level cross-section $\sim \frac{1}{zs}$.

Gluon Polarized Dipole Amplitude

Similar to quark, consider a gluon dipole, one of which has helicity λ, interacting with a polarized target proton (blue rectangle).

$$G_{10}(z) =$$

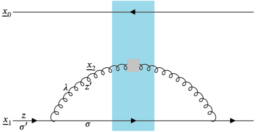


$$= \frac{zs}{2(N_c^2-1)} \,\mathrm{Re}\left[\left\langle \mathcal{T}\ \mathrm{tr}\left[U_{\underline{0}} U_{\underline{1}}^{\mathrm{pol}\ \dagger}\right]\right\rangle(z) + \left\langle \mathcal{T}\ \mathrm{tr}\left[U_{\underline{1}}^{\mathrm{pol}} U_{\underline{0}}^{\dagger}\right]\right\rangle(z)\right]. \quad (6)$$

- $U_{\underline{0}}$ is the adjoint unpolarized Wilson's line at $\underline{x}_0$.
- $U_{\underline{1}}^{\mathrm{pol}}$ is the adjoint *polarized Wilson's line* at $\underline{x}_1$.
- The angle brackets average over target proton's wave function.

Evolution

- The polarized dipole amplitudes obey integral equations resulting from quark/gluon splitting outside the target shockwave, e.g.



- To the first order in α_s, both dipole amplitudes evolve as

$$\delta\langle\ldots\rangle \sim \alpha_s \Bigg[\underbrace{\int \frac{dz'}{z'} \int \frac{dx_{21}^2}{x_{21}^2}}_{\text{Double-logarithmic}} + \underbrace{\int dz' \int \frac{dx_{32}^2}{x_{32}^2} + \int \frac{dz'}{z'} \int dx_{21}^2}_{\text{Single-logarithmic}} + \ldots \Bigg]$$

$$\times \,(\text{dipole amplitudes})\,.$$

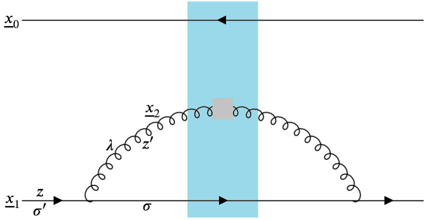
- The single-logarithmic (SLA) terms (resumming $\alpha_s \ln \frac{1}{x}$) are subleading to the double-logarithmic (DLA) term (resumming $\alpha_s \ln^2 \frac{1}{x}$).

Longitudinally Soft Parton Emission

- In a splitting in the limit $z' \ll z$, the longitudinal z'-integral is logarithmic, giving the evolution terms

$$\alpha_s \int \frac{dz'}{z'} \Bigg[\underbrace{A \int \frac{dx_{21}^2}{x_{21}^2}}_{\text{DLA}} + \underbrace{B \int dx_{21}^2}_{\text{SLA}_\text{L}} \Bigg] \text{(dipole amps)} \tag{7}$$

- SLA_L: single-logarithmic term with logarithmic longitudinal integral.



DLA Evolution Equations

With only DLA terms included, the evolution equations are [Kovchegov et al, 2016] [Kovchegov and Sievert, 2019]

$$
\begin{aligned}
&\frac{1}{N_c}\left\langle\!\!\left\langle \mathrm{tr}\left[V_{\underline{0}}^{unp}\,V_{\underline{1}}^{pol\,\dagger}\right]\right\rangle\!\!\right\rangle(z) = \frac{1}{N_c}\left\langle\!\!\left\langle \mathrm{tr}\left[V_{\underline{0}}^{unp}\,V_{\underline{1}}^{pol\,\dagger}\right]\right\rangle\!\!\right\rangle_0(z) + \frac{\alpha_s}{2\pi^2}\int\limits_{z_i}^{z}\frac{dz'}{z'}\int\limits_{\rho'^2}\frac{d^2x_2}{x_{21}^2} \\
&\times\left\{\theta(x_{10}-x_{21})\,\frac{2}{N_c}\left\langle\!\!\left\langle \mathrm{tr}\left[t^b\,V_{\underline{0}}^{unp}\,t^a\,V_{\underline{1}}^{unp\,\dagger}\right]\,U_{\underline{2}}^{pol\,ba}\right\rangle\!\!\right\rangle(z') + \theta(x_{10}^2 z - x_{21}^2 z')\,\frac{1}{N_c}\left\langle\!\!\left\langle \mathrm{tr}\left[t^b\,V_{\underline{0}}^{unp}\,t^a\,V_{\underline{2}}^{pol\,\dagger}\right]\,U_{\underline{1}}^{unp\,ba}\right\rangle\!\!\right\rangle(z')\right. \\
&\left.+\,\theta(x_{10}-x_{21})\,\frac{1}{N_c}\left[\left\langle\!\!\left\langle \mathrm{tr}\left[V_{\underline{0}}^{unp}\,V_{\underline{2}}^{unp\,\dagger}\right]\,\mathrm{tr}\left[V_{\underline{2}}^{unp}\,V_{\underline{1}}^{pol\,\dagger}\right]\right\rangle\!\!\right\rangle(z') - N_c\left\langle\!\!\left\langle \mathrm{tr}\left[V_{\underline{0}}^{unp}\,V_{\underline{1}}^{pol\,\dagger}\right]\right\rangle\!\!\right\rangle(z')\right]\right\}.
\end{aligned}
$$

$$
\begin{aligned}
&\frac{1}{N_c^2-1}\left\langle\!\!\left\langle \mathrm{Tr}\left[U_{\underline{0}}^{unp}\,U_{\underline{1}}^{pol\,\dagger}\right]\right\rangle\!\!\right\rangle(z) = \frac{1}{N_c^2-1}\left\langle\!\!\left\langle \mathrm{Tr}\left[U_{\underline{0}}^{unp}\,U_{\underline{1}}^{pol\,\dagger}\right]\right\rangle\!\!\right\rangle_0(z) + \frac{\alpha_s}{2\pi^2}\int\limits_{z_i}^{z}\frac{dz'}{z'}\int\limits_{\rho'^2}\frac{d^2x_2}{x_{21}^2} \\
&\times\left\{\theta(x_{10}-x_{21})\,\frac{4}{N_c^2-1}\left\langle\!\!\left\langle \mathrm{Tr}\left[T^b\,U_{\underline{0}}^{unp}\,T^a\,U_{\underline{1}}^{unp\,\dagger}\right]\,U_{\underline{2}}^{pol\,ba}\right\rangle\!\!\right\rangle(z')\right. \\
&-\,\theta(x_{10}^2 z - x_{21}^2 z')\,\frac{N_f}{N_c^2-1}\left\langle\!\!\left\langle \mathrm{tr}\left[t^b\,V_{\underline{1}}^{unp}\,t^a\,V_{\underline{2}}^{pol\,\dagger}\right]\,U_{\underline{0}}^{unp\,ba} + \mathrm{tr}\left[t^b\,V_{\underline{2}}^{pol}\,t^a\,V_{\underline{1}}^{unp\,\dagger}\right]\,U_{\underline{0}}^{unp\,ba}\right\rangle\!\!\right\rangle(z') \\
&\left.+\theta(x_{10}-x_{21})\frac{2}{N_c^2-1}\left[\left\langle\!\!\left\langle \mathrm{Tr}\left[T^b\,U_{\underline{0}}^{unp}\,T^a\,U_{\underline{1}}^{pol\,\dagger}\right]\,U_{\underline{2}}^{unp\,ba}\right\rangle\!\!\right\rangle(z') - N_c\left\langle\!\!\left\langle \mathrm{Tr}\left[U_{\underline{0}}^{unp}\,U_{\underline{1}}^{pol\,\dagger}\right]\right\rangle\!\!\right\rangle(z')\right]\right\}
\end{aligned}
$$

where $\left\langle\!\left\langle \cdots \right\rangle\!\right\rangle = zs\left\langle \cdots \right\rangle$.

DLA Evolution Equations

- At large N_c and large $N_c\&N_f$, the equations become closed and linear [Kovchegov et al, 2016] [Kovchegov and Sievert, 2019].
- For example, at large N_c, the equations are:

$$G_{10}(z) = G_{10}^{(0)}(z) + \frac{\alpha_s N_c}{2\pi} \int\limits_{\frac{1}{s\, x_{10}^2}}^{z} \frac{dz'}{z'} \int\limits_{\frac{1}{z' s}}^{x_{10}^2} \frac{dx_{21}^2}{x_{21}^2} \left[\Gamma_{10,21}(z') + 3\, G_{21}(z')\right]$$

$$\Gamma_{10,21}(z') = \Gamma_{10,21}^{(0)}(z') + \frac{\alpha_s N_c}{2\pi} \int\limits_{\min\{\Lambda^2, \frac{1}{x_{10}^2}\}/s}^{z'} \frac{dz''}{z''} \int\limits_{\frac{1}{z'' s}}^{\min\{x_{10}^2, x_{21}^2 z'/z''\}} \frac{dx_{32}^2}{x_{32}^2} \left[\Gamma_{10,32}(z'') + 3\, G_{32}(z'')\right]$$

where Γ is an auxiliary function.

DLA Evolution Equations

These DLA equations have been analytically solved at large N_c [Kovchegov et al, 2017] and numerically solved at large $N_c \& N_f$ [Kovchegov and Tawabutr, 2020].

- At large N_c, the quark helicity PDF has the asymptotic form

$$\Delta\Sigma(x, Q^2) \sim (1/x)^{\alpha_h^q} . \tag{8}$$

- At large $N_c \& N_f$, the asymptotic form displays oscillation pattern

$$\Delta\Sigma(x, Q^2) \sim (1/x)^{\alpha_h^q} \cos\left[\omega_q \ln(1/x) + \varphi_q\right] . \tag{9}$$

In both cases, $\alpha_h^q \approx \frac{4}{\sqrt{3}}\sqrt{\frac{\alpha_s N_c}{2\pi}}$, and ω_q is small and increases with N_f.

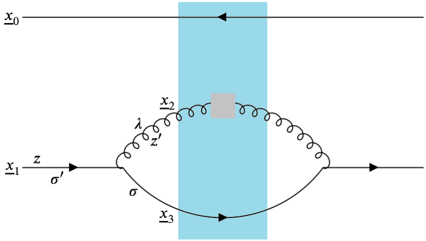
- Phenomenological implications were studied in [Adamiak et al, 2021].

Longitudinally Hard Parton Emission

- In the limit $z' \sim z$, a parton splitting yields the SLA_T terms, i.e. single-logarithmic terms with logarithmic transverse integral

$$\alpha_s \int dz' \, \Delta P(z'/z) \int \frac{dx_{32}^2}{x_{32}^2} \, (\text{dipole amps}) \tag{10}$$

- Here, $\Delta P(z'/z)$ is the polarized DGLAP splitting function.
- Since $z' \sim (z - z') \sim z$, neither $\underline{x}_3$ nor $\underline{x}_2$ is close to $\underline{x}_1$, but we still have $x_{32} \ll x_{10}$.



SLA Evolution Equations

Including both double-log (DLA) and single-log (SLA) terms, we have

$$\begin{aligned}
&\frac{1}{N_c}\left\langle\!\!\left\langle \mathrm{tr}\left[V_{\underline{0}}\,V_{\underline{1}}^{\mathrm{pol}\,\dagger}\right]\right\rangle\!\!\right\rangle(z_{\min},z_{\mathrm{pol}}) = \frac{1}{N_c}\left\langle\!\!\left\langle \mathrm{tr}\left[V_{\underline{0}}\,V_{\underline{1}}^{\mathrm{pol}\,\dagger}\right]\right\rangle\!\!\right\rangle_0(z_{\mathrm{pol}}) + \frac{1}{2\pi^2}\int\limits_{\Lambda^2/s}^{z_{\min}}\frac{dz'}{z'}\int\limits_{1/(z's)} d^2x_2 \\
&\quad\times\left[\left(\frac{\alpha_s(1/x_{21}^2)}{x_{21}^2}\,\theta(x_{10}^2 z_{\min}-x_{21}^2 z') - \alpha_s(\min\{1/x_{21}^2,1/x_{20}^2\})\,\frac{\underline{x}_{21}\cdot\underline{x}_{20}}{x_{21}^2\,x_{20}^2}\,\theta(x_{10}^2z_{\min}-\max\{x_{21}^2,x_{20}^2\}z')\right)\right.\\
&\qquad\times\frac{2}{N_c}\left\langle\!\!\left\langle \mathrm{tr}\left[t^b\,V_{\underline 0}\,t^a\,V_{\underline 1}^\dagger\right]\,U_{\underline 2}^{\mathrm{pol}\,ba}\right\rangle\!\!\right\rangle(z',z')\\
&\qquad\left.+\frac{\alpha_s(1/x_{21}^2)}{x_{21}^2}\,\theta(x_{10}^2z_{\min}-x_{21}^2z')\,\frac{1}{N_c}\left\langle\!\!\left\langle \mathrm{tr}\left[t^b\,V_{\underline0}\,t^a\,V_{\underline2}^{\mathrm{pol}\,\dagger}\right]\,U_{\underline1}^{ba}\right\rangle\!\!\right\rangle(z',z')\right]\\
&+\frac{1}{2\pi^2}\int\limits_{\Lambda^2/s}^{z_{\min}}\frac{dz'}{z'}\int\limits_{1/(z's)} d^2x_2\,K_{\mathrm{rcBK}}(\underline x_0,\underline x_1;\underline x_2)\,\theta(x_{10}^2z_{\min}-x_{21}^2z')\\
&\quad\times\frac{1}{N_c}\left[\left\langle\!\!\left\langle \mathrm{tr}\left[V_{\underline0}\,V_{\underline2}^\dagger\right]\,\mathrm{tr}\left[V_{\underline2}\,V_{\underline1}^{\mathrm{pol}\,\dagger}\right]\right\rangle\!\!\right\rangle(z',z_{\mathrm{pol}}) - N_c\left\langle\!\!\left\langle\mathrm{tr}\left[V_{\underline0}\,V_{\underline1}^{\mathrm{pol}\,\dagger}\right]\right\rangle\!\!\right\rangle(z',z_{\mathrm{pol}})\right]\\
&\color{blue}{-\frac{1}{2\pi^2}\int\limits_0^{z_{\mathrm{pol}}}\frac{dz'}{z_{\mathrm{pol}}}\int\limits_{\frac{z_{\mathrm{pol}}}{z'(z_{\mathrm{pol}}-z')s}}\frac{d^2x_{32}}{x_{32}^2}\,\theta(x_{10}^2z_{\min}z_{\mathrm{pol}}-x_{32}^2z'(z_{\mathrm{pol}}-z'))\;\alpha_s\left(\frac{1}{x_{32}^2}\right)}\\
&\quad\color{blue}{\times\left[\frac{1}{N_c}\left\langle\!\!\left\langle\mathrm{tr}\left[t^b\,V_{\underline0}\,t^a\,V^\dagger_{\underline x_1-\frac{z'}{z_{\mathrm{pol}}}\underline x_{32}}\right]\,U^{\mathrm{pol}\,ba}_{\underline x_1+\left(1-\frac{z'}{z_{\mathrm{pol}}}\right)\underline x_{32}}\right\rangle\!\!\right\rangle(\min\{z_{\min},z',z_{\mathrm{pol}}-z'\},z')\right.}\\
&\qquad\color{blue}{\left.+\frac{1}{N_c}\left\langle\!\!\left\langle\mathrm{tr}\left[t^b\,V_{\underline0}\,t^a\,V^{\mathrm{pol}\,\dagger}_{\underline x_1+\left(1-\frac{z'}{z_{\mathrm{pol}}}\right)\underline x_{32}}\right]\,U^{ba}_{\underline x_1-\frac{z'}{z_{\mathrm{pol}}}\underline x_{32}}\right\rangle\!\!\right\rangle(\min\{z_{\min},z',z_{\mathrm{pol}}-z'\},z')\right]}\\
&\color{blue}{+\frac{C_F}{2\pi^2}\int\limits_0^{z_{\mathrm{pol}}}\frac{dz'}{z_{\mathrm{pol}}}\left(1+\frac{z'}{z_{\mathrm{pol}}}\right)\int\limits_{\frac{z_{\mathrm{pol}}}{z'(z_{\mathrm{pol}}-z')s}}\frac{d^2x_{32}}{x_{32}^2}\,\theta(x_{10}^2z_{\min}z_{\mathrm{pol}}-x_{32}^2z'(z_{\mathrm{pol}}-z'))\;\alpha_s\left(\frac{1}{x_{32}^2}\right)}\\
&\quad\color{blue}{\times\frac{1}{N_c}\left\langle\!\!\left\langle\mathrm{tr}\left[V_{\underline0}\,V_{\underline1}^{\mathrm{pol}\,\dagger}\right]\right\rangle\!\!\right\rangle(\min\{z_{\min},z',z_{\mathrm{pol}}-z'\},z_{\mathrm{pol}}).}
\end{aligned}$$

$$\begin{aligned}
&\frac{1}{N_c^2-1}\left\langle\!\!\left\langle \mathrm{Tr}\left[U_{\underline{0}}\,U_{\underline{1}}^{\mathrm{pol}\,\dagger}\right]\right\rangle\!\!\right\rangle(z_{\min},z_{\mathrm{pol}}) = \frac{1}{N_c^2-1}\left\langle\!\!\left\langle \mathrm{Tr}\left[U_{\underline{0}}\,U_{\underline{1}}^{\mathrm{pol}\,\dagger}\right]\right\rangle\!\!\right\rangle_0(z_{\mathrm{pol}}) + \frac{1}{2\pi^2}\int\limits_{\Lambda^2/s}^{z_{\min}}\frac{dz'}{z'}\int\limits_{1/(z's)} d^2x_2 \\
&\quad\times\left[\left(\frac{\alpha_s(1/x_{21}^2)}{x_{21}^2}\,\theta(x_{10}^2 z_{\min}-x_{21}^2 z') - \alpha_s(\min\{1/x_{21}^2,1/x_{20}^2\})\,\frac{\underline{x}_{21}\cdot\underline{x}_{20}}{x_{21}^2\,x_{20}^2}\,\theta(x_{10}^2z_{\min}-\max\{x_{21}^2,x_{20}^2\}z')\right)\right.\\
&\qquad\times\frac{4}{N_c^2-1}\left\langle\!\!\left\langle \mathrm{Tr}\left[T^b\,U_{\underline 0}\,T^a\,U_{\underline 1}^\dagger\right]\,U_{\underline 2}^{\mathrm{pol}\,ba}\right\rangle\!\!\right\rangle(z',z')\\
&\qquad\left.-\frac{\alpha_s(1/x_{21}^2)}{x_{21}^2}\,\theta(x_{10}^2z_{\min}-x_{21}^2z')\,\frac{N_f}{N_c^2-1}\left\langle\!\!\left\langle \mathrm{tr}\left[t^b\,V_{\underline1}\,t^a\,V_{\underline2}^{\mathrm{pol}\,\dagger}\right]\,U_{\underline0}^{ba}+\mathrm{tr}\left[t^b\,V_{\underline2}^{\mathrm{pol}}\,t^a\,V_{\underline1}^{\dagger}\right]\,U_{\underline0}^{ba}\right\rangle\!\!\right\rangle(z',z')\right]\\
&+\frac{1}{\pi^2}\int\limits_{\Lambda^2/s}^{z_{\min}}\frac{dz'}{z'}\int\limits_{1/(z's)} d^2x_2\,K_{\mathrm{rcBK}}(\underline x_0,\underline x_1;\underline x_2)\,\theta(x_{10}^2z_{\min}-x_{21}^2z')\\
&\quad\times\frac{1}{N_c^2-1}\left[\left\langle\!\!\left\langle \mathrm{Tr}\left[T^b\,U_{\underline0}\,T^a\,U_{\underline1}^{\mathrm{pol}\,\dagger}\right]\,U_{\underline2}^{ba}\right\rangle\!\!\right\rangle(z',z_{\mathrm{pol}}) - N_c\left\langle\!\!\left\langle\mathrm{Tr}\left[U_{\underline0}\,U_{\underline1}^{\mathrm{pol}\,\dagger}\right]\right\rangle\!\!\right\rangle(z',z_{\mathrm{pol}})\right]\\
&\color{blue}{-\frac{1}{2\pi^2}\int\limits_0^{z_{\mathrm{pol}}}\frac{dz'}{z_{\mathrm{pol}}}\int\limits_{\frac{z_{\mathrm{pol}}}{z'(z_{\mathrm{pol}}-z')s}}\frac{d^2x_{32}}{x_{32}^2}\,\theta(x_{10}^2z_{\min}z_{\mathrm{pol}}-x_{32}^2z'(z_{\mathrm{pol}}-z'))\;\alpha_s\left(\frac{1}{x_{32}^2}\right)}\\
&\quad\color{blue}{\times\frac{4}{N_c^2-1}\left\langle\!\!\left\langle\mathrm{Tr}\left[T^b\,U_{\underline0}\,T^a\,U^\dagger_{\underline x_1-\frac{z'}{z_{\mathrm{pol}}}\underline x_{32}}\right]\,U^{\mathrm{pol}\,ba}_{\underline x_1+\left(1-\frac{z'}{z_{\mathrm{pol}}}\right)\underline x_{32}}\right\rangle\!\!\right\rangle(\min\{z_{\min},z',z_{\mathrm{pol}}-z'\},z')}\\
&\color{blue}{+\frac{1}{\pi^2}\int\limits_0^{z_{\mathrm{pol}}}\frac{dz'}{z_{\mathrm{pol}}}\int\limits_{\frac{z_{\mathrm{pol}}}{z'(z_{\mathrm{pol}}-z')s}}\frac{d^2x_{32}}{x_{32}^2}\,\theta(x_{10}^2z_{\min}z_{\mathrm{pol}}-x_{32}^2z'(z_{\mathrm{pol}}-z'))\;\alpha_s\left(\frac{1}{x_{32}^2}\right)\frac{N_f}{N_c^2-1}}\\
&\quad\color{blue}{\times\left\langle\!\!\left\langle\mathrm{tr}\left[t^b\,V_{\underline x_1-\frac{z'}{z_{\mathrm{pol}}}\underline x_{32}}\,t^a\,V^{\mathrm{pol}\,\dagger}_{\underline x_1+\left(1-\frac{z'}{z_{\mathrm{pol}}}\right)\underline x_{32}}\right]\,U_{\underline0}^{ba}\right.}\\
&\qquad\color{blue}{\left.+\mathrm{tr}\left[t^b\,V^{\mathrm{pol}}_{\underline x_1+\left(1-\frac{z'}{z_{\mathrm{pol}}}\right)\underline x_{32}}\,t^a\,V^\dagger_{\underline x_1-\frac{z'}{z_{\mathrm{pol}}}\underline x_{32}}\right]\,U_{\underline0}^{ba}\right\rangle\!\!\right\rangle(\min\{z_{\min},z',z_{\mathrm{pol}}-z'\},z')}\\
&\color{blue}{+\frac{1}{2\pi^2}\int\limits_0^{z_{\mathrm{pol}}}\frac{dz'}{z_{\mathrm{pol}}}\int\limits_{\frac{z_{\mathrm{pol}}}{z'(z_{\mathrm{pol}}-z')s}}\frac{d^2x_{32}}{x_{32}^2}\left[N_c\left(2-\frac{z'}{z_{\mathrm{pol}}}+\frac{z'^2}{z_{\mathrm{pol}}^2}\right)-\frac{N_f}{2}\left(\frac{z'^2}{z_{\mathrm{pol}}^2}+\left(1-\frac{z'}{z_{\mathrm{pol}}}\right)^2\right)\right]}\\
&\quad\color{blue}{\times\theta\left(x_{10}^2z_{\min}z_{\mathrm{pol}}-x_{32}^2z'(z_{\mathrm{pol}}-z')\right)\alpha_s\left(\frac{1}{x_{32}^2}\right)\frac{1}{N_c^2-1}\left\langle\!\!\left\langle\mathrm{Tr}\left[U_{\underline0}\,U_{\underline1}^{\mathrm{pol}\,\dagger}\right]\right\rangle\!\!\right\rangle(\min\{z_{\min},z',z_{\mathrm{pol}}-z'\},z_{\mathrm{pol}}).}
\end{aligned}$$

- SLA_T terms are written in blue.
- The running coupling correction is SLA and also has to be included.

SLA Evolution Equations

Including both double-log (DLA) and single-log (SLA) terms, we have

$$
\begin{aligned}
&\frac{1}{N_c}\left\langle\!\!\left\langle \mathrm{tr}\left[V_{\underline{0}}\,V_{\underline{1}}^{\mathrm{pol}\,\dagger}\right]\right\rangle\!\!\right\rangle(z_{\min},z_{\mathrm{pol}}) = \frac{1}{N_c}\left\langle\!\!\left\langle \mathrm{tr}\left[V_{\underline{0}}\,V_{\underline{1}}^{\mathrm{pol}\,\dagger}\right]\right\rangle\!\!\right\rangle_0(z_{\mathrm{pol}}) + \frac{1}{2\pi^2}\int\limits_{\Lambda^2/s}^{z_{\min}}\frac{dz'}{z'}\int\limits_{1/(z's)} d^2x_2 \\
&\quad\times\left[\left(\frac{\alpha_s(1/x_{21}^2)}{x_{21}^2}\,\theta(x_{10}^2 z_{\min}-x_{21}^2 z') - \alpha_s(\min\{1/x_{21}^2,1/x_{20}^2\})\,\frac{\underline{x}_{21}\cdot\underline{x}_{20}}{x_{21}^2\,x_{20}^2}\,\theta(x_{10}^2 z_{\min}-\max\{x_{21}^2,x_{20}^2\}z')\right)\right. \\
&\qquad\times\frac{2}{N_c}\left\langle\!\!\left\langle \mathrm{tr}\left[t^b\,V_{\underline{0}}\,t^a\,V_{\underline{1}}^{\dagger}\right]\,U_{\underline{2}}^{\mathrm{pol}\,ba}\right\rangle\!\!\right\rangle(z',z') \\
&\qquad\left.+\frac{\alpha_s(1/x_{21}^2)}{x_{21}^2}\,\theta(x_{10}^2 z_{\min}-x_{21}^2 z')\,\frac{1}{N_c}\left\langle\!\!\left\langle \mathrm{tr}\left[t^b\,V_{\underline{0}}\,t^a\,V_{\underline{2}}^{\mathrm{pol}\,\dagger}\right]\,U_{\underline{1}}^{ba}\right\rangle\!\!\right\rangle(z',z')\right] \\
&+\frac{1}{2\pi^2}\int\limits_{\Lambda^2/s}^{z_{\min}}\frac{dz'}{z'}\int\limits_{1/(z's)} d^2x_2\,K_{\mathrm{rcBK}}(\underline{x}_0,\underline{x}_1;\underline{x}_2)\,\theta(x_{10}^2 z_{\min}-x_{21}^2 z') \\
&\quad\times\frac{1}{N_c}\left[\left\langle\!\!\left\langle \mathrm{tr}\left[V_{\underline{0}}\,V_{\underline{2}}^{\dagger}\right]\,\mathrm{tr}\left[V_{\underline{2}}\,V_{\underline{1}}^{\mathrm{pol}\,\dagger}\right]\right\rangle\!\!\right\rangle(z',z_{\mathrm{pol}}) - N_c\left\langle\!\!\left\langle \mathrm{tr}\left[V_{\underline{0}}\,V_{\underline{1}}^{\mathrm{pol}\,\dagger}\right]\right\rangle\!\!\right\rangle(z',z_{\mathrm{pol}})\right] \\
&-\frac{1}{2\pi^2}\int\limits_0^{z_{\mathrm{pol}}}\frac{dz'}{z_{\mathrm{pol}}}\int\limits_{\frac{z_{\mathrm{pol}}}{z'(z_{\mathrm{pol}}-z')s}}\frac{d^2x_{32}}{x_{32}^2}\,\theta(x_{10}^2 z_{\min}z_{\mathrm{pol}}-x_{32}^2 z'(z_{\mathrm{pol}}-z'))\;\alpha_s\left(\frac{1}{x_{32}^2}\right) \\
&\quad\times\left[\frac{1}{N_c}\left\langle\!\!\left\langle \mathrm{tr}\left[t^b\,V_{\underline{0}}\,t^a\,V^{\dagger}_{\underline{x}_1-\frac{z'}{z_{\mathrm{pol}}}\underline{x}_{32}}\right]\,U^{\mathrm{pol}\,ba}_{\underline{x}_1+\left(1-\frac{z'}{z_{\mathrm{pol}}}\right)\underline{x}_{32}}\right\rangle\!\!\right\rangle(\min\{z_{\min},z',z_{\mathrm{pol}}-z'\},z')\right. \\
&\qquad\left.+\frac{1}{N_c}\left\langle\!\!\left\langle \mathrm{tr}\left[t^b\,V_{\underline{0}}\,t^a\,V^{\mathrm{pol}\,\dagger}_{\underline{x}_1+\left(1-\frac{z'}{z_{\mathrm{pol}}}\right)\underline{x}_{32}}\right]\,U^{ba}_{\underline{x}_1-\frac{z'}{z_{\mathrm{pol}}}\underline{x}_{32}}\right\rangle\!\!\right\rangle(\min\{z_{\min},z',z_{\mathrm{pol}}-z'\},z')\right] \\
&+\frac{C_F}{2\pi^2}\int\limits_0^{z_{\mathrm{pol}}}\frac{dz'}{z_{\mathrm{pol}}}\left(1+\frac{z'}{z_{\mathrm{pol}}}\right)\int\limits_{\frac{z_{\mathrm{pol}}}{z'(z_{\mathrm{pol}}-z')s}}\frac{d^2x_{32}}{x_{32}^2}\,\theta(x_{10}^2 z_{\min}z_{\mathrm{pol}}-x_{32}^2 z'(z_{\mathrm{pol}}-z'))\;\alpha_s\left(\frac{1}{x_{32}^2}\right) \\
&\quad\times\frac{1}{N_c}\left\langle\!\!\left\langle \mathrm{tr}\left[V_{\underline{0}}\,V_{\underline{1}}^{\mathrm{pol}\,\dagger}\right]\right\rangle\!\!\right\rangle(\min\{z_{\min},z',z_{\mathrm{pol}}-z'\},z_{\mathrm{pol}}).
\end{aligned}
$$

- SLA$_T$ terms are written in blue.
- The running coupling correction is SLA and also has to be included.

SLA Evolution Equations

- At large N_c and large $N_c \& N_f$, the equations become closed but *non-linear* because the evolution of unpolarized dipoles are also SLA.
- For example, at large N_c, (half of) the equations are:

$$
\begin{aligned}
G_{10}(z_{\min}, z_{\rm pol}) &= G_{10}^{(0)}(z_{\rm pol}) + \frac{N_c}{\pi^2} \int\limits_{\Lambda^2/s}^{z_{\min}} \frac{dz'}{z'} \int\limits_{1/(z's)} d^2 x_2 \\
&\times \left(\frac{\alpha_s(1/x_{21}^2)}{x_{21}^2}\, \theta\left(x_{10}^2 z_{\min} - x_{21}^2 z'\right) - \alpha_s(\min\{1/x_{21}^2, 1/x_{20}^2\})\, \frac{\underline{x}_{21} \cdot \underline{x}_{20}}{x_{21}^2 x_{20}^2}\, \theta\left(x_{10}^2 z_{\min} - \max\left\{x_{21}^2, x_{20}^2\right\} z'\right) \right) \\
&\times \left[G_{21}(z', z')\, S_{20}(z') + \Gamma_{20,21}^{gen}(z', z')\, S_{21}(z') \right] \\
&+ \frac{N_c}{2\pi^2} \int\limits_{\Lambda^2/s}^{z_{\min}} \frac{dz'}{z'} \int\limits_{1/z's} d^2 x_2\, K_{\rm rcBK}(\underline{x}_0, \underline{x}_1; \underline{x}_2)\, \theta\left(x_{10}^2 z_{\min} - x_{21}^2 z'\right) \left[G_{21}(z', z_{\rm pol})\, S_{20}(z') - \Gamma_{10,21}^{gen}(z', z_{\rm pol}) \right] \\
&- \frac{N_c}{\pi^2} \int\limits_0^{z_{\rm pol}} \frac{dz'}{z_{\rm pol}} \int\limits_{\frac{z_{\rm pol}}{z'(z_{\rm pol} - z')s}} \frac{d^2 x_{32}}{x_{32}^2}\, \alpha_s\left(\frac{1}{x_{32}^2}\right) \theta(x_{10}^2 z_{\min} z_{\rm pol} - x_{32}^2 z'(z_{\rm pol} - z')) \\
&\qquad \times \left[G_{\underline{x}_1 + \left(1 - \frac{z'}{z_{\rm pol}}\right)\underline{x}_{32},\, \underline{x}_1 - \frac{z'}{z_{\rm pol}}\underline{x}_{32}}(z_{\min}, z')\, S_{10}(z_{\min}) + \Gamma_{10,32}(z_{\min}, z') \right] \\
&+ \frac{N_c}{2\pi^2} \int\limits_0^{z_{\rm pol}} \frac{dz'}{z_{\rm pol}} \left(2 - \frac{z'}{z_{\rm pol}} + \frac{z'^2}{z_{\rm pol}^2}\right) \int\limits_{\frac{z_{\rm pol}}{z'(z_{\rm pol} - z')s}} \frac{d^2 x_{32}}{x_{32}^2}\, \alpha_s\left(\frac{1}{x_{32}^2}\right) \theta(x_{10}^2 z_{\min} z_{\rm pol} - x_{32}^2 z'(z_{\rm pol} - z')) \\
&\times \Gamma_{10,32}(z_{\min}, z_{\rm pol}).
\end{aligned}
$$

Conclusion

- Quark's helicity contribution to proton spin follows evolution equations that contain leading DLA terms and subleading SLA terms.
- The SLA terms have been derived, with the effects of running coupling included, as the latter is also single-logarithmic.
- In the large-N_c and large-$N_c\&N_f$ limits, the equations are closed but non-linear, since the unpolarized evolution is single-logarithmic.
- Future work:
  - Connection to the polarized DGLAP evolution
  - Numerical solutions to the SLA evolution equations at large N_c and large $N_c\&N_f$
  - Phenomenological implications

The End

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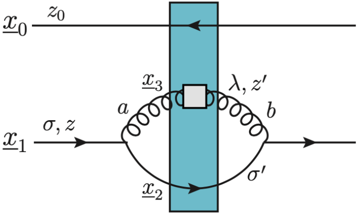
Running Coupling

- For soft unpolarized parton emission, the strong coupling constant runs the same way as the unpolarized BK evolution [Kovchegov and Weigert, 2007] [Balitsky, 2007]
- For other splitting terms, i.e. soft polarized or hard parton emission, the strong coupling constant runs with the transverse size of the daughter dipole, e.g.



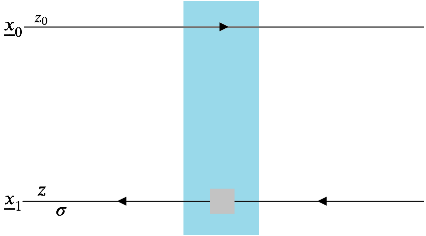
$$\sim \alpha_s(1/x_{32}^2)$$

With this prescription, there is no double counting with the SLA$_T$ terms.

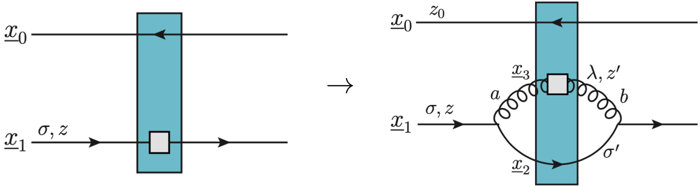
SLA Caveats

The polarized dipole amplitudes depend on two momentum fractions.

$$\left\langle \mathrm{tr}\left[V_{\underline{x}_0} V^{\dagger}_{\underline{x}_1}(\sigma)\right]\right\rangle(z) \quad \Rightarrow \quad \left\langle \mathrm{tr}\left[V_{\underline{x}_0} V^{\dagger}_{\underline{x}_1}(\sigma)\right]\right\rangle(\underbrace{\min\{z, z_0\}}_{\text{minimum}}, \underbrace{z}_{\text{polarized}})$$



SLA Evolution Equations



Include not only the terms coming from the splitting of the polarized line.

- Since the evolutions of both polarized and unpolarized lines are SLA, they have to be included.
- The running coupling correction is SLA and also has to be included.

At SLA, the evolution of the dipole amplitude depends on momentum fractions of two lines – the polarized line and the softest-parton line.