Precise determination of the $\kappa$ resonance from a dispersive $\pi K$ analysis

GHP 2021, April 13, 2021

Arkaitz Rodas
1 Introduction
   1.1 Motivation
   1.2 First principles

2 Results
   2.1 $\pi K$ dispersive analysis
   2.2 Spectroscopy and dispersion relations

3 Summary
Motivation: $\kappa$

- Debated for decades
- “We are beginning to think that $\kappa$ should be classified along with flying saucers, the Loch Ness Monster, and the Abominable Snowman”
  
  (Data on Particles and Resonant States, 1967)

- “Confirmed soon”
  
  Anonymous PDG member

- One of the broadest resonances

- Cannot be interpreted as pure $q\bar{q}$

- Vicinity of the $\pi K S^{1/2}$ threshold

---

Dispersive determination of the $\kappa$ resonance

A. Rodas
Motivation

- Most of its determinations → simple models
- Scalar nonet, and $\kappa \sim \sigma$

- Too broad to be determined using simple models
- Threshold behavior (ChPT), Adler Zero and LHC play a role
- Same problems in Lattice QCD at low $m_\pi$ mass
Motivation: $\pi K$

- $\pi K$ scattering $\rightarrow$ final state in hadronic strange processes
- Heavy decays, CP violation, $\tau$ decays JHEP 09 031, JHEP 09 042, PLB 804 135371
- $\pi \pi \rightarrow K\bar{K}$ $\Rightarrow$ new physics, $g-2$...

- $\pi, K$ pseudo-Goldstone Bosons $\rightarrow$ ChPT $\rightarrow$ Scattering Lengths
- UChPT $\rightarrow$ Good description, not suited for high precision
- Experimental groups need robust params $\rightarrow$ LHCb for CP
- New experiment $\rightarrow$ KLF
  see Justin Stevens’ talk
\( \pi K \) scattering lengths

- Tension between Lattice and ChPT calculations
- \( SU(3) \) ChPT does not seem to be converging well

For all these reasons \( \iff \) Dispersive determination of the \( \kappa \) resonance

Dispersive determination of the \( \kappa \) resonance

A. Rodas
Motivation

- Experiment cannot access $\pi K$ directly
- No precise data at threshold
- Big systematic uncertainties
Table of Contents

1 Introduction
   1.1 Motivation
   1.2 First principles

2 Results
   2.1 $\pi K$ dispersive analysis
   2.2 Spectroscopy and dispersion relations

3 Summary
S-matrix principles: Unitarity

- **UNITARITY** ⇔ probability $\sum |\langle f | S | i \rangle|^2 = 1$
- Both right and left branch cuts $SS^\dagger = I \Rightarrow F - F^\dagger = iFF^\dagger$.
- Elastic unitarity $\rightarrow S^{II}(z) = \frac{1}{S^I(z)}$
- Zero of $S^I(z) \rightarrow$ pole of $S^{II}(z)$

Dispersive determination of the $\kappa$ resonance

A. Rodas
S-matrix principles: Analyticity and Crossing

- **CAUSALITY ⇔ ANALITICITY**

- No poles in the first sheet

\[ F(s,t) = \frac{1}{\pi} \int_{s_{th}}^{\infty} ds' \frac{\text{Im} F(s',t)}{s' - s} + LHC \]

- Structures → unitarity, bound states, cusp

- Together with CROSSING → Mandelstam analyticity
Table of Contents

1 Introduction
   1.1 Motivation
   1.2 First principles

2 Results
   2.1 $\pi K$ dispersive analysis
   2.2 Spectroscopy and dispersion relations

3 Summary
Amplitudes

- Two independent amplitudes $I=1/2, 3/2$.
- $s$-channel $\pi K$ and $t$-channel $\pi\pi \to K\bar{K}$

$$F^+(s,t) = \frac{1}{3} F^{1/2}(s,t) + \frac{2}{3} F^{3/2}(s,t) = \frac{G^{I=0}_t(t,s)}{\sqrt{6}},$$

$$F^-(s,t) = \frac{1}{3} F^{1/2}(s,t) - \frac{1}{3} F^{3/2}(s,t) = \frac{G^{I=1}_t(t,s)}{2}.$$

- Symmetric and antisymmetric amplitudes under $s \leftrightarrow u$ exchange
- Customary decomposition in partial waves

$$F^I(s,t) = 16\pi \sum_\ell (2\ell + 1) f^I_\ell(s) P(z_s(t)),$$

$$G^I(t,s) = 16\pi \sqrt{2} \sum_\ell (2\ell + 1) (q_\pi q_K)\ell g^I_\ell(t) P(z_t(s)).$$
Combining the First Principles

Example, amplitude DR, $t = 0$

\[
\text{Re} F^I(s) = F^I(s_{th}) + \frac{(s - s_{th})}{\pi}
\]

\[
PV \int_{s_{th}}^{\infty} ds' \left[ \frac{\text{Im} F^I(s')}{(s' - s)(s' - s_{th})} + (-1)^I \frac{\text{Im} F^I(s')}{(s' + s - 2\Sigma_{\pi K})(s' + s_{th} - 2\Sigma_{\pi K})} \right],
\]

- If we project $F^I(s) \rightarrow f^I_\ell(s)$
  1. We need Input $\rightarrow F^I(s), f^I_\ell(s)$
  2. We get DR

- We recover $\text{Re} F^I(s)$
  1. Stringent constrains
  2. Perform stable analytic continuation
UFD Input: Elastic region

- Unitarity for partial waves

\[ f^I_l(s) = \frac{1}{\sigma(s) \cot \delta^I_l(s)} \]

- with \( \cot \delta^I_l(s) = \frac{\sqrt{s}}{2q^{2l+1}} \sum B_n \omega(s)^n \)
- Inelastic region \( \rightarrow \) pheno fits
- 8 \( \pi K \) PW \( \sim 1.8 \text{ GeV} \)
- 5 \( \pi\pi \rightarrow K\bar{K} \) PW \( \sim 2 \text{ GeV} \)
Forward Dispersion relations

- Amplitudes built using the whole tower of partial waves
- Two independent amplitudes $F^+$ and $F^-$
- We define a penalty function $\hat{d}^2 = \frac{1}{N} \sum_i^N \left( \frac{F_{\text{out}}^I(s_i) - F_{\text{fit}}^I(s_i)}{\Delta(F_{\text{out}}^I - F_{\text{fit}}^I)(s_i)} \right)^2$
- Above 1.8 GeV discrepancies too big

- Room for improvement $\rightarrow$ Constrained fits
Forward Dispersion relations

- Amplitudes built using the whole tower of partial waves
- Two independent amplitudes $F^+$ and $F^-$
- We define a penalty function $\hat{d}^2 = \frac{1}{N} \sum_{i}^{N} \left( \frac{F_{out}^I(s_i) - F_{fit}^I(s_i)}{\Delta(F_{out}^I-F_{fit}^I)(s_i)} \right)^2$
- Above 1.8 GeV discrepancies too big

- Very good agreement
DR for $\pi K$ and $\pi\pi \to K\bar{K}$  


1. We build $DR$

2. Define Penalty function

$$d_{DR}^2 = \frac{1}{N} \sum_i^N \left( \frac{f_{out}(s_i) - f_{fit}(s_i)}{(f_{out} - f_{fit})(s_i)} \right)^2$$

3. We minimize a global

$$\chi^2 = W_1 \chi^2_{data} + W_2 d_{DR}^2$$

4. Weights ($W_i$) $\sim$ d.o.f

Forward Dispersion Relations

Phys.Rev.D 93 074025

1. Very simple

2. Applicable to arbitrary high energies

PWDR for $\pi K$ and $\pi\pi \to K\bar{K}$


1. Fixed-$t$ DR for $\pi K$ only

2. Hyperbolic dispersion relations for both

3. Omnès-Muskhelishvili problem

4. Aplicable $\sim \mathcal{O}(1)$ GeV

Dispersive determination of the $\kappa$ resonance

A. Rodas
2 Results

2.1 $\pi K$ dispersive analysis

HDR/Fixed $- t$ both $\pi K$ and $\pi \pi \to K \bar{K}$  2010.11222, Invited to Phys.Rep.

- Fixed-$t$ only used for $\pi K \to \pi K$ inputs dominate

- HDR used for both $\pi K$ and $\pi \pi \to K \bar{K}$ channels

- $(s - a_i)(u - a_i) = b$ with $a_s, a_t$ used to maximize to applicability region

\[
f_0^+(s) = a_0^+ + \frac{1}{\pi} \sum_l \left( \int_{s_{th}}^\infty ds' K_{0l}^+(s, s') \text{Im} f_l^+(s') + \int_4^\infty dt' G_{02l}^+(s, t') \text{Im} g_{2l}^0(t') \right)
\]

\[
g_0^0(t) = \frac{\sqrt{3}}{2} m_+ a_0^+ + \frac{1}{\pi} \sum_l \left( \int_{m_+^2}^s ds' G_{0,l}^+(t, s') \text{Im} f_l^+(s') + t \int_4^\infty \frac{dt'}{t'} G_{0,2l}^0(t, t') \text{Im} g_{2l}^0(t') \right)
\]

- Both channels are coupled
Some of the dispersion relations are severely deviated
The scattering lengths are not compatible with the DR
2 Results

2.1 $\pi K$ dispersive analysis

- Remarkable agreement
- All DR now compatible from threshold on

Dispersive determination of the $\kappa$ resonance

A. Rodas
- Average $d^2/DR \simeq 5.5$ (UFD) $\rightarrow 0.6$ (CFD)
- **13** partial waves $\rightarrow \chi^2/dof \simeq 1$ (UFD) $\rightarrow 1.6$ (CFD)
### CFD result for scattering lengths:

<table>
<thead>
<tr>
<th></th>
<th>UFD</th>
<th>CFD</th>
<th>Paris group</th>
</tr>
</thead>
<tbody>
<tr>
<td>$a_0^{1/2}$</td>
<td>$0.241 \pm 0.013$</td>
<td>$0.224 \pm 0.011$</td>
<td>$0.224 \pm 0.022$</td>
</tr>
<tr>
<td>$a_0^{3/2}$</td>
<td>$-0.067 \pm 0.014$</td>
<td>$-0.048 \pm 0.006$</td>
<td>$-0.0448 \pm 0.0077$</td>
</tr>
</tbody>
</table>

![Graph showing CFD results for scattering lengths](image-url)
2 Results

2.2 Spectroscopy and dispersion relations

**CFD** $K_{0}^{*}(700)/\kappa$ pole

- Stable result **AFTER** constraining
- All uncertainties have been taken into account

\[ \sqrt{s_{p}} = (648 \pm 7) - i(560 \pm 32)/2 \text{ MeV} \quad \text{HDR} \]
\[ \sqrt{s_{p}} = (658 \pm 13) - i(557 \pm 24)/2 \text{ MeV} \quad \text{Descotes-Genon, Moussallam} \]
\[ \sqrt{s_{p}} = (680 \pm 50) - i(600 \pm 80)/2 \text{ MeV} \quad \text{PDG} \]
1 Introduction
   1.1 Motivation
   1.2 First principles

2 Results
   2.1 $\pi K$ dispersive analysis
   2.2 Spectroscopy and dispersion relations

3 Summary
\(\pi K\) dispersive analysis: Summary

- DR analysis on almost all \(\pi K\) available data
  1. Pruning on the data
  2. Result → simple params compatible with both Data and DR
  3. Model independent determination of the scattering lengths

- DR applied to spectroscopy
  1. Extraction of the \(\kappa/K_0^*(700)\) with 2 DR → exists
  2. Extraction of the \(K^*(892)\) using 3 DR
Spare slides!
\[ F^I(s_{th}, 0) = 8\pi m_+ a_0^I, \]

where \( m_+ = m_\pi + m_K \)

- **At LO**
  \[ a_0^- \propto \frac{1}{f_\pi^2} \quad a_0^+ = \mathcal{O}(m_{+4}). \]

- **NLO \rightarrow LECS \ L_{1-8}**
  \[ a_0^- \propto \frac{L_5}{f_\pi^4} \quad a_0^+ \rightarrow 7L_i. \]

- **NNLO \rightarrow 32C_i, a_0^- \rightarrow 10C_i, a_0^+ \rightarrow 23C_i.**
Forward dispersion relations

- Simple set of DR, \( t = 0 \)
  \[
  \text{Re} \, F^I(s) = F^I(s_{th}) + \frac{(s - s_{th})}{\pi} \\
  \text{PV}\int_{s_{th}}^{\infty} ds' \left[ \frac{\text{Im} \, F^I(s')}{(s' - s)(s' - s_{th})} + (-1)^I \frac{\text{Im} \, F^I(s')}{(s' + s - 2\Sigma_{\pi K})(s' + s_{th} - 2\Sigma_{\pi K})} \right],
  \]

- For the symmetric \( s \leftrightarrow u \) amplitude one subtraction is needed
  \[
  \text{Re} \, F^+(s) = F^+(s_{th}) + \frac{(s - s_{th})}{\pi} \\
  P\int_{s_{th}}^{\infty} ds' \left[ \frac{\text{Im} \, F^+(s')}{(s' - s)(s' - s_{th})} - \frac{\text{Im} \, F^+(s')}{(s' + s - 2\Sigma_{\pi K})(s' + s_{th} - 2\Sigma_{\pi K})} \right],
  \]
  where \( \Sigma_{\pi K} = m_{\pi}^2 + m_{K}^2 \).

- For the antisymmetric amplitude no subtraction is needed
  \[
  \text{Re} \, F^-(s) = \frac{(2s - 2\Sigma_{\pi K})}{\pi} P\int_{s_{th}}^{\infty} ds' \frac{\text{Im} \, F^-(s')}{(s' - s)(s' + s - 2\Sigma_{\pi K})}.
  \]
In the inelastic region $f^I_l = \frac{\eta^I_l(s)e^{2i\delta^I_l(s)} - 1}{2i} = |f^I_l|e^{i\phi^I_l}$.

- We use complex rational functions that near each resonance look like BW.
- Focusing on simple parameterizations, no EFT included here.
- We impose matching conditions on the inelastic $\eta K$ threshold.
- We use up to $G^{1/2} \rightarrow 8$ partial waves.
- Although we use for our analysis the $P^{3/2}, D^{3/2}, F^{1/2}$ and $G^{1/2}$ their contribution is small. Not shown here.
\( \pi K \) scattering lengths

- Sum rule from FDR \( \rightarrow a_0^- = 0.292 \pm 0.01 \)
- However, sum rule coming from \( G^1 \) channel suggests:
  \[ a_0^- = 0.253 \pm 0.015 \]
- New sum rule closer to Lattice works.

---

Dispersive determination of the \( \kappa \) resonance

A. Rodas
HDR/Fixed—$t$ both $\pi K$ and $\pi \pi \to K\bar{K}$

- HDR used for both $\pi K$ and $\pi \pi \to K\bar{K}$ channels
- $(s - a_c)(u - a_c) = b$ with $a_s, a_t$ used to maximize to applicability region

\[
f_0^\pm(s) = a_0^\pm + \frac{1}{\pi} \sum_l \int_{s_{th}}^\infty ds' K_0^{\pm}(s, s') \text{Im} f_l^\pm(s')
\]
\[
+ \frac{1}{\pi} \sum_l \int_{4m_{\pi}^2}^\infty dt' G_0^{\pm(2l-2, 2l-1)}(s, t') \text{Im} g_{0,1}^{(2l-2, 2l-1)}(t')
\]
\[
g_0^0(t) = \frac{\sqrt{3}}{2} m + a_0^+ + \frac{t}{\pi} \int_{4m_{\pi}^2}^\infty \frac{\text{Im} g_0^0(t')}{t'(t' - t)} dt'
\]
\[
+ \frac{t}{\pi} \sum_l \int_{4m_{\pi}^2}^\infty \frac{dt'}{t'} G_0^{0,2l-2}(t, t') \text{Im} f_{0,2l-2}^0(t') + \sum_l \int_{m_{\pi}^2}^\infty ds' G_{0,l}^+(t, s') \text{Im} f_{l}^+(s').
\]

- Fixed-$t$ only used for $\pi K \to \pi K$ inputs dominate
- Tension between FDR, HDR and Lattice.
- Scarcity of $\pi K$ data $\rightarrow$ SL poorly determined.
- $K_0^*(700)$ pole out of FDR/fixed-t range of validity $\rightarrow$ only HDR here.
Crossed channel HDR partial wave with one substraction

\[ g_0^0(t) = \frac{\sqrt{3}}{2} m + a_0^+ + \frac{t}{\pi} \int_{4m_\pi^2}^{\infty} \frac{\text{Im} g_0^0(t')}{t'(t' - t)} dt' \]

\[ + \frac{t}{\pi} \sum_l \int_{4m_\pi^2}^{\infty} \frac{dt'}{t'} G_{0,2l-2}^0(t, t') \text{Im} g_{2l-2}^0(t') + \sum_l \int_{m_+^2}^{\infty} ds' G_{0,l}^+(t, s') \text{Im} f_l^+(s') \]

\[ = \frac{t}{\pi} \int_{4m_\pi^2}^{\infty} \frac{\text{Im} g_0^0(t')}{t'(t' - t)} dt' + \Delta_0^0(t) \]

- \( \Delta_0^0(t) \) contains the left cut.
- Unknown value of \( |g_l^I(t)| \) below \( K\bar{K} \) threshold
- Phase shift below \( K\bar{K} \to \text{Watson Theorem} \)
- Define \( \hat{g}_l^I(t) = \frac{g_l^I(t) - \Delta_l^I(t)}{\Omega_l^I(t)} \) with \( \Omega_l^I(t) = e^{\frac{t}{\pi} \int_{4m_\pi^2}^{t_m^2} \phi_l^I(t')/t'(t' - t) dt'} \).
- We develop a DR for the new function \( \hat{g}_l^I(t) \)
The set of final Omnès-Muskhelishvili DR:

\[
\begin{align*}
    g_0^0(t) &= \Delta_0^0(t) + \frac{t \Omega_0^0(t)}{t_m - t} \left[ \alpha + \frac{t}{\pi} \int_{4m^2_{\pi}}^{t_m} dt' \frac{(t_m - t') \Delta_0^0(t') \sin \phi_0^0(t')}{\Omega_{0,R}^0(t') t'^2 (t' - t)} \right] \\
    &+ \frac{t}{\pi} \int_{t_m}^{\infty} dt' \frac{(t_m - t') |g_0^0(t')| \sin \phi_0^0(t')}{\Omega_{0,R}^0(t') t'^2 (t' - t)} \\
    g_1^1(t) &= \Delta_1^1(t) + \Omega_1^1(t) \left[ \frac{1}{\pi} \int_{4m^2_{\pi}}^{t_m} dt' \frac{\Delta_1^1(t') \sin \phi_1^1(t')}{\Omega_{1,R}^1(t') (t' - t)} \right] \\
    &+ \frac{1}{\pi} \int_{t_m}^{\infty} dt' \frac{|g_1^1(t')| \sin \phi_1^1(t')}{\Omega_{1,R}^1(t') (t' - t)}
\end{align*}
\]

When \( s \) real we obtain \( |g_1^I(t)|_{out} \).
Some of the dispersion relations are severely deviated

The scattering lengths are not compatible with the DR

Dispersive determination of the $\kappa$ resonance

A. Rodas
**Remarkable agreement**

**All DR now compatible from threshold on**
Again, much better agreement after the constrains
16 dispersion relations $\rightarrow$ 2 FDR, 4 OM, 4 fixed-t, 6 HDR.

HDR with less subtractions $\rightarrow$ worst discrepancies.

UFD deviations of more than 3 sigmas.

Up to 8 low energy parameters can be obtained with high precision.

Up to 13 partial waves included in this analysis $\rightarrow$ 7 constrained
Preliminary: $\pi K$ CFD

- The $\chi^2/dof$ worsen by a 30% on average.

- Most regions for most partial waves $\rightarrow$ nice data description
Conformal map

- Simple, yet powerful in the elastic region

\[ \cot \delta_l(s) = \frac{\sqrt{s}}{2q^{2l+1}} F(s) \sum_n B_n \omega(s)^n, \text{ where } F(s) \text{ can have zeroes or poles.} \]

- Can mimic the LHC $\rightarrow$ fit/poles should be more stable
Preliminary: $\pi\pi \rightarrow K\bar{K}$ CFD
Mandelstamm Analyticity in Relativistic scattering

If one combines analyticity and crossing \(\rightarrow\) Mandelstamm Hypothesis

- Only one analytic function which

\[
T(s,t,u) = \begin{cases} 
T_{12\rightarrow34}(s,t,u), & s \geq (m_1 + m_2)^2, \quad t \leq 0, \quad u \leq 0, \\
T_{13\rightarrow24}(t,s,u), & t \geq (m_1 + m_3)^2, \quad s \leq 0, \quad u \leq 0, \\
T_{14\rightarrow32}(u,t,s), & u \geq (m_1 + m_4)^2, \quad s \leq 0, \quad t \leq 0.
\end{cases}
\]

- No more non-analytic structures

- Cauchy theorem: Let \(D\) be a domain of the complex plane where the function \(f(z)\) is analytic and let \(C\) be the closed curve defined by its boundary. Then, for any \(z \in D\)

\[
f(z) = \oint_C \frac{f(z')}{z'-z} \, dz'
\]
Analyticity in Relativistic scattering: $\pi\pi$

- Fixed-$t$ right and left hand cuts starting at $s = 4m^2_\pi$ and $s = -t$

- If $T(s,t) \to 1/s$ when $s \to \infty$ then

$$T(s,t) = \frac{1}{\pi} \int_{4m^2_\pi}^{\infty} ds' \frac{\text{Im}
T(s',t)}{(s' - s)}$$

$$+ \frac{1}{\pi} \int_{-\infty}^{-t} ds' \frac{\text{Im}
T(s',t)}{(s' - s)}$$

- If not $\to$ subtractions

$$T(s,t) = T(s_0,t) + \frac{(s - s_0)}{\pi} \int_{4m^2_\pi}^{\infty} ds' \frac{\text{Im}
T(s',t)}{(s' - s)(s' - s_0)}$$

$$+ \frac{(s - s_0)}{\pi} \int_{-\infty}^{-t} ds' \frac{\text{Im}
T(s',t)}{(s' - s)(s' - s_0)}$$
Analyticity in Relativistic scattering: $\pi\pi$

- If we make the change of variables $s' \rightarrow u' = 4m_{\pi}^2 - t - s'$

$$T(s, t) = \frac{1}{\pi} \int_{4m_{\pi}^2}^{\infty} ds' \left( \frac{\text{Im}T(s', t)}{(s' - s)} - \frac{\text{Im}T(4m_{\pi}^2 - s' - t, t)}{(u' - u)} \right)$$

- $u'$ is a dummy variable
- The LHC can be always rewritten as RHC terms
- Due to crossing $T^{Is}(s, t, u) = \sum_{It} C_{su} T^{Iu}(u, t, s)$ and

$$T(s, t) = \frac{1}{\pi} \int_{4m_{\pi}^2}^{\infty} ds' \left( \frac{\text{Im}T(s', t)}{(s' - s)} - \sum C_{su}^{II'} \text{Im}T^{II'}(s', t) \frac{\text{Im}T^{II'}(s', t)}{(s' - u)} \right)$$

- Here we have our closed dispersion relation

Dispersive determination of the $\kappa$ resonance

A. Rodas
Analyticity in Relativistic scattering: $\pi\pi$

- However this is a “toy DR”, we actually need more elaborated stuff.
- Sometimes we will not fix $t$, but move it as a function of the other two $(s, u)$ variables.
- In particular, by using $T(s, t) = 32\pi \sum \ell (2\ell + 1) P_\ell(z_s) t_\ell(s)$ we can project

$$t_\ell(s) = \frac{1}{32\pi} \int_0^1 dz_s T(s, t) P_{\ell'}(z_s),$$

- $P_{\ell'}(z_s)$ are the so called Legendre Polynomials (project the amplitude into defined angular momentums).
Analyticity in Relativistic scattering: $\pi\pi$

- The most sound dispersion relations for meson-meson scattering $\rightarrow$ Roy-Steiner eqs.

$$\vec{T}(s,t,u) = \text{S.T.} + \int_{4m_\pi^2}^{\infty} ds' g_2(s,t;s') \text{Im} \vec{T}(s',0,u')$$

$$+ \int_{4m_\pi^2}^{\infty} ds' g_3(s,t;s') \text{Im} \vec{T}(s',t,u')$$

$$\text{Re} \vec{t}_J(s) = \frac{1}{32\pi} \int_0^1 dx P_J(x) \vec{T}(s,t(x)) = \frac{1}{32\pi} \int_0^1 dx P_J(x) \text{S.T.} +$$

$$\sum_{J'} (2J' + 1) \int_{4m_\pi^2}^{\infty} ds' \int_0^1 dx P_{J'}(x) \left[ g_2(s,t(x);s') + P_{J'}(x) g_3(s,t(x);s') \right] \text{Im} \vec{t}_{J'}(s')$$

- $g_2, g_3$ are matrices of polynomials in the Mandelstamm variables
**ππ and σ: Fixed-t**

- Commonly known as Roy Eqs. (2-sub Bern group)  

- **Approach:**
  1. Matching conditions $\rightarrow$ unique solution  
  2. Numerical matching $\rightarrow$ Analyticity, Crossing and Unitarity  
  3. ChPT+ROY $\rightarrow$ very precise pwmtwililightictction below 850 MeV

---

Dispersive determination of the $\kappa$ resonance

A. Rodas
\(\pi\pi\) and \(\sigma\): Fixed-t

- Or GKPY Eqs. (1-sub Madrid group).

\[
\text{Re } F^{(I)}(s,t) = \sum_{l'} C^{l'l'}_{st} F^{(l')}(4M_{\pi}^2,0) + \frac{s}{\pi} \text{P.P.} \int_{4M_{\pi}^2}^{\infty} ds' \left[ \frac{\text{Im } F^{(I)}(s',t)}{s'(s'-s)} - \frac{\sum_{l''} C^{l''l'l'}_{st} \text{Im } F^{(l')}(s',t)}{(s'+t-4M_{\pi}^2)(s'+s+t-4M_{\pi}^2)} \right] \\
+ \frac{t-4M_{\pi}^2}{\pi} \text{P.P.} \int_{4M_{\pi}^2}^{\infty} ds' \sum_{l''} C^{l'l''}_{st} \left[ \frac{\text{Im } F^{(l'')}(s',0)}{(s'-t)(s'-4M_{\pi}^2)} - \frac{\sum_{l'''} C^{l''''l''l'''}_{st} \text{Im } F^{(l'')}(s',0)}{s'(s'+t-4M_{\pi}^2)} \right]
\]

\[
t^{(I)}_c(s) = \overline{S} T^I_c(s) + \sum_{l=0}^{\ell_{\max}} \sum_{\ell'=0}^{2 \ell_{\max}} \int_{4M_{\pi}^2}^{s_{\max}} ds' \bar{K}_{l \ell'}^{H'}(s, s') \text{Im } t^{(I)}_{c}(s') + \overline{D} T^I_c(s),
\]

- Approach:
  1. Use data as constrain
  2. Numerical minimization of the distances
  3. Very precise determination of LEP
Omnès-Muskheilishvili equations

- Omnès-Muskheilishvili DR with as less subtractions as possible
- S-channel and T-channel coupled in a complicated non-linear way

\[
g_0^0(t) = \Delta_0^0(t) + \frac{t \Omega_0^0(t)}{t_m - t} \left[ \alpha + \frac{t}{\pi} \int_{t_m}^{t_m} dt' \frac{(t_m - t') \Delta_0^0(t') \sin \phi_0^0(t')}{\Omega_{0,R}^0(t') t'^2 (t' - t)} \right. \\
+ \left. \frac{t}{\pi} \int_{t_m}^{\infty} dt' \frac{(t_m - t') |g_0^0(t')| \sin \phi_0^0(t')}{\Omega_{0,R}^0(t') t'^2 (t' - t)} \right],
\]

\[
g_1^1(t) = \Delta_1^1(t) + \Omega_1^1(t) \left[ \frac{1}{\pi} \int_{4m_\pi^2}^{t_m} dt' \frac{\Delta_1^1(t') \sin \phi_1^1(t')}{\Omega_{1,R}^1(t')(t' - t)} \right. \\
+ \left. \frac{1}{\pi} \int_{t_m}^{\infty} dt' \frac{|g_1^1(t')| \sin \phi_1^1(t')}{\Omega_{1,R}^1(t')(t' - t)} \right].
\]

- If more subtractions \(\Rightarrow\) scalar and vector partial waves coupled in a non-linear way.
Omnès-Muskhelishvili matching condition

- $\Omega^I_\ell(t) = \exp \left( \frac{t}{\pi} \int_{4m^2}^{t_m} \frac{\phi^I_\ell(t')dt'}{t'(t'-t)} \right)$
- Unique/Perfect solution $\rightarrow$ not $t_m$ dependence

Dispersive determination of the $\kappa$ resonance

A. Rodas
There are 2 possible $g_0(t)$ even after imposing the DR.
\( \pi \pi \rightarrow K\bar{K} \)


- Different \( f_0(980) \) behaviors yet almost same \( \pi K \) and \( \kappa/K_0^*(700) \) results
- Both \( g_1^1(t) \) fully compatible in the pseudo-threshold region

Dispersive determination of the \( \kappa \) resonance

A. Rodas
The $\kappa$ resonance

- Several different models and methods used to determine its parameters.
- Clear convergence with the use of analytic techniques.
- Model dependent determinations not suitable for this scenario.
- Model independent: $\rightarrow$ Padé (before), HDR (next)

$$S^{II}(s) = \frac{1}{S^{I}(s)}.$$
- Dispersion relations obeying \((s - a)(u - a) = b\). Most previous works \(\rightarrow a = 0\).
- This work: \(a\) used to maximize applicability region.
Regge physics constrains

Dispersive determination of the $\kappa$ resonance

A. Rodas
alternative \( P \)-wave


- Compatible with \( D^+ \rightarrow K^- \pi^+ \mu^+ \nu \) by the FOCUS collab.
- Compatible with previous dispersive approaches to \( \tau \) decays and form factors
- Compatible with \( K_{\ell 3} \) decays.

Dispersive determination of the \( \kappa \) resonance

A. Rodas
Scattering lengths

<table>
<thead>
<tr>
<th></th>
<th>SL</th>
<th>UFD</th>
<th>CFD</th>
<th>Roy-Steiner result</th>
</tr>
</thead>
<tbody>
<tr>
<td>$m_\pi a_0^{1/2}$</td>
<td>0.222±0.014</td>
<td>0.218±0.014</td>
<td>0.224± 0.022</td>
<td></td>
</tr>
<tr>
<td>$m_\pi a_0^{3/2}$</td>
<td>-0.101±0.03</td>
<td>-0.054±0.014</td>
<td>-0.0448± 0.0077</td>
<td></td>
</tr>
<tr>
<td>$m_\pi a_1^{1/2}$</td>
<td>0.031±0.008</td>
<td>0.024±0.005</td>
<td>0.019± 0.001</td>
<td></td>
</tr>
</tbody>
</table>

- Dirac collaboration measured the difference between the scalar scattering lengths.

\[
\frac{1}{3} \left( a_0^{1/2} - a_0^{3/2} \right) = 0.11^{+0.09}_{-0.04} m_\pi^{-1}, \quad \text{(DIRAC)}
\]

- Our results are compatible with Roy-Steiner equations, although there is tension with $\pi\pi \rightarrow K\bar{K}$ Sum Rule

\[
\frac{1}{3} \left( a_0^{1/2} - a_0^{3/2} \right) = 0.091^{+0.006}_{-0.005} m_\pi^{-1}. \quad \text{(CFD)}
\]

\[
\frac{1}{3} \left( a_0^{1/2} - a_0^{3/2} \right) = 0.075 \pm 0.006 m_\pi^{-1}. \quad \text{(Sum rule)}
\]
## Scattering lengths

<table>
<thead>
<tr>
<th></th>
<th>This work sum rules with CFD input</th>
<th>This work direct</th>
<th>Sum rules</th>
<th>NNLO ChPT</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Fixed-(t)</td>
<td>HDR</td>
<td>HDR(_{sub})</td>
<td>UFD</td>
</tr>
<tr>
<td>(m_\pi a_0^{1/2}) &amp; 0.222±0.009 &amp; 0.222±0.013 &amp; 0.224±0.011 &amp; 0.241±0.012 &amp; 0.224±0.011 &amp; 0.224±0.022 &amp; 0.224*</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(m_\pi b_0^{1/2} \times 10) &amp; 1.04±0.06 &amp; 1.07±0.08 &amp; 1.15±0.06 &amp; 0.90±0.04 &amp; 0.95±0.04 &amp; 0.85±0.04 &amp; 1.278</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(m_\pi a_0^{3/2} \times 10) &amp; -0.471±0.053 &amp; -0.469±0.067 &amp; -0.481±0.062 &amp; -0.67±0.12 &amp; -0.48±0.06 &amp; -0.448±0.077 &amp; -0.471*</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(m_\pi b_0^{3/2} \times 10) &amp; -0.42±0.02 &amp; -0.42±0.03 &amp; -0.45±0.02 &amp; -0.44±0.04 &amp; -0.36±0.04 &amp; -0.37±0.03 &amp; -0.326</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(m_\pi a_1^{1/2} \times 10) &amp; 0.227±0.012 &amp; 0.221±0.008 &amp; 0.223±0.007 &amp; 0.18±0.04 &amp; 0.21±0.05 &amp; 0.19±0.01 &amp; 0.152</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(m_\pi b_1^{1/2} \times 10^2) &amp; 0.87±0.05 &amp; 0.87±0.03 &amp; 0.89±0.03 &amp; 0.8±0.1 &amp; 0.5±0.3 &amp; 0.18±0.02 &amp; 0.032</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(m_\pi a_1^{3/2} \times 10^2) &amp; 0.17±0.07 &amp; 0.19±0.06 &amp; 0.18±0.05 &amp; 0.05±0.09 &amp; 0.15±0.13 &amp; 0.065±0.044 &amp; 0.293</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(m_\pi b_1 \times 10^3) &amp; -0.73±0.12 &amp; -0.77±0.11 &amp; -0.82±0.08 &amp; -0.57±0.9 &amp; -1.08±1.2 &amp; -0.92±0.17 &amp; 0.544</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(m_\pi a_2^{1/2} \times 10^3) &amp; 0.59±0.11 &amp; 0.55±0.04 &amp; 0.56±0.04 &amp; 0.41±0.04 &amp; 0.53±0.05 &amp; 0.47±0.03 &amp; 0.142</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(m_\pi b_2^{1/2} \times 10^4) &amp; 0.57±0.29 &amp; 0.42±0.09 &amp; 0.46±0.08 &amp; 0.16±0.01 &amp; 0.20±0.02 &amp; -1.4±0.3 &amp; -1.98</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(m_\pi a_2^{3/2} \times 10^4) &amp; -0.47±0.44 &amp; -0.09±0.16 &amp; -0.15±0.15 &amp; -0.14±0.06 &amp; -0.08±0.03 &amp; -0.11±0.27 &amp; -0.45</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>(m_\pi b_2 \times 10^4) &amp; -1.19±0.16 &amp; -1.14±0.08 &amp; -1.17±0.07 &amp; -0.06±0.03 &amp; -0.03±0.01 &amp; -0.96±0.26 &amp; 0.61</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

---

**Dispersive determination of the \(\kappa\) resonance**

A. Rodas
More parameters

<table>
<thead>
<tr>
<th></th>
<th>This work sum rules with CFD input</th>
<th>Sum rules</th>
<th>NNLO ChPT</th>
<th>Sum rules</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Fixed-( t )</td>
<td>HDR</td>
<td>HDR(_{sub})</td>
<td>Büttiker et al.</td>
</tr>
<tr>
<td>( C_{00}^+ )</td>
<td>1.5±0.5</td>
<td>1.5±0.5</td>
<td></td>
<td>2.01±1.10</td>
</tr>
<tr>
<td>( C_{10}^+ )</td>
<td>0.97±0.11</td>
<td>1.05±0.12</td>
<td></td>
<td>0.87±0.08</td>
</tr>
<tr>
<td>( C_{01}^+ )</td>
<td>2.34±0.06</td>
<td>2.34±0.06</td>
<td></td>
<td>2.07±0.10</td>
</tr>
<tr>
<td>( C_{11}^+ )</td>
<td>-0.046±0.006</td>
<td>-0.049±0.006</td>
<td></td>
<td>-0.066±0.010</td>
</tr>
<tr>
<td>( C_{00}^- )</td>
<td>9.0±0.3</td>
<td>9.6±0.4</td>
<td>9.1±0.4</td>
<td>8.92±0.38</td>
</tr>
<tr>
<td>( C_{10}^- )</td>
<td>0.45±0.04</td>
<td>0.39±0.02</td>
<td>0.40±0.01</td>
<td>0.31±0.01</td>
</tr>
<tr>
<td>( C_{01}^- )</td>
<td>0.68±0.02</td>
<td>0.67±0.02</td>
<td>0.68±0.02</td>
<td>0.62±0.06</td>
</tr>
<tr>
<td>( F_{CD}^+ )</td>
<td>3.6±0.6</td>
<td>3.7±0.6</td>
<td></td>
<td>3.90±1.50</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th></th>
<th>UFD ( I = 1/2 )</th>
<th>CFD ( I = 1/2 )</th>
<th>UFD ( I = 3/2 )</th>
<th>CFD ( I = 3/2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \sqrt{s_A, fixed-t} )</td>
<td>0.479(^{+0.006}_{−0.012} )</td>
<td>0.466(^{+0.006}_{−0.005} )</td>
<td>0.530(^{+0.014}_{−0.011} )</td>
<td>0.550(^{+0.009}_{−0.009} )</td>
</tr>
<tr>
<td>( \sqrt{s_A, HDR} )</td>
<td>0.472(^{+0.011}_{−0.009} )</td>
<td>0.466(^{+0.005}_{−0.005} )</td>
<td>0.538(^{+0.016}_{−0.019} )</td>
<td>0.550(^{+0.009}_{−0.009} )</td>
</tr>
<tr>
<td>( \sqrt{s_A, HDR_{sub}} )</td>
<td>0.481(^{+0.009}_{−0.008} )</td>
<td>0.470(^{+0.006}_{−0.005} )</td>
<td>0.531(^{+0.014}_{−0.016} )</td>
<td>0.552(^{+0.009}_{−0.010} )</td>
</tr>
</tbody>
</table>
Spectroscopy for strange states

- Precise determination using model independent techniques.
- We can study more than 6 resonances appearing in $\pi K$.
- Another 4 appearing in $\pi\pi \rightarrow K\bar{K}$ scattering.
- Used to determine the $f_0(500)/\sigma$, the $K_0^*(700)/\kappa$, etc...
- Resonances \( \rightarrow \) poles in unphysical sheets
- Analytic continuation is usually model dependent \( \rightarrow \) precise and model independent determination using S-matrix principles.
- High $L$ or broad resonance parameters not stable when using simple models. Customary $(q(s)/q(s_r))^L$ and $B_L(q,q_r) \Rightarrow$ deviations.
- Rigorous dispersive techniques cannot get to the poles at higher energies.
- Partial wave is described by a Padé approximant

$$t_l(s) \approx P^N_1(s,s_0) = \sum_{k=0}^{N-1} a_K(s-s_0)^k + \frac{a_N(s-s_0)^N}{1 - \frac{a_{N+1}}{a_N}(s-s_0)}.$$
- We stop at a $N (N + 1$ derivatives) where the systematic uncertainty is smaller than the statistical one (usually $N = 4$ is enough).
- $s_0$ fixed $\rightarrow$ gives the minimum difference between $N$ and $N + 1$.
- Run a Montecarlo for every fit to calculate the parameters and errors of each resonance.
- Different fitting functions included as systematics.
Meson Spectroscopy

- $K_0^*(700)$ Padé → trigewmtwilght the change of name from $K_0^*(800)$.

$$\sqrt{s_p} = (670 \pm 18) - i(295 \pm 28) \text{ MeV}$$

$$\sqrt{s_p} = (682 \pm 29) - i(274 \pm 12) \text{ MeV (PDG)}$$

Dispersive determination of the $\kappa$ resonance

- $K^*(1430)$, $K^*(1410)$, $K^*(1430)$ and $K^*(1780)$ vs PDG list.

A. Rodas
Very preliminary: $f_0(1370)$
- Original $\pi\pi$ CFD $\rightarrow$ a pole exists $\rightarrow$ too unstable

  1. Padè extraction $\sqrt{s_p} \simeq (1.23 \pm 0.02) - i(0.21 \pm 0.02)$ GeV
  2. Continuous fractions $\sqrt{s_p} \simeq (1.24 \pm 0.02) - i(0.22 \pm 0.02)$ GeV

Phys.Lett.B 774 411-416
- However the systematics are large $\rightarrow$ deviations from this particular param.
- Could the pole even disappear?
Very preliminary: $f_0(1370)$

- We extend $\pi\pi$ DR beyond original region $\sqrt{s_{\text{max}}} = 1.15 \rightarrow 1.3$ GeV

- Original and new CFD $\rightarrow \sqrt{s_p} \simeq (1.31 \pm 0.04) - i(0.22 \pm 0.03)$ GeV

- Crossed channel $\pi\pi \rightarrow K\bar{K} \rightarrow$ another stable pole

- CFD $\rightarrow \sqrt{s_p} \simeq (1.35 \pm 0.05) - i(0.24 \pm 0.04)$ GeV

Dispersive determination of the $\kappa$ resonance

A. Rodas
The $\kappa$ resonance

- $K_0^*(700)$ Padé → triggered the change of name from $K_0^*(800)$.

\[ \sqrt{s_p} = (670 \pm 18) - i(295 \pm 28) \text{MeV} \]

\[ \sqrt{s_p} = (682 \pm 29) - i(274 \pm 12) \text{MeV} \text{(PDG)} \]
For ordinary resonances: All hadrons are classified in linear $(J, M^2)$ trajectories.

$\sigma$ and $\kappa$-mesons are not included in these plots.
The contribution of a single pole to a partial wave is

\[ t(J, s) = t_{\text{background}} + \frac{\beta(s)}{J - \alpha(s)} \approx \frac{\beta(s)}{J - \alpha(s)} \]

- \( \alpha(s) \) is the position of the pole, whereas \( \beta(s) \) is the residue.
- Unitarity condition on the real axis implies

\[ \text{Im} \alpha(s) = \rho(s) \beta(s) \]

- The analytical properties of \( \beta(s) \) implies

\[ \beta(s) = \frac{\hat{s} \alpha(s)}{\Gamma(\alpha(s) + 3/2) \gamma(s)} \]
Following coupled integral eqs.

\[ \text{Re} \alpha(s) = \alpha_0 + \alpha' s + \frac{s}{\pi} \text{PV} \int_{m^2_+}^{\infty} ds' \frac{\text{Im} \alpha(s')}{s'(s' - s)} \]

\[ \text{Im} \alpha(s) = \frac{\rho(s) b_0 \hat{s}^{\alpha_0 + \alpha'} s}{|\Gamma(\alpha(s) + \frac{3}{2})|} \exp \left( -\alpha' s \left[ 1 - \log(\alpha' s_0) \right] \right) \]

\[ + \frac{s}{\pi} \text{PV} \int_{m^2_+}^{\infty} ds' \frac{\text{Im} \alpha(s') \log \frac{\hat{s}'}{\hat{s}} + \text{arg} \ \Gamma(\alpha(s') + \frac{3}{2})}{s'(s' - s)} \]

\[ \beta(s) = \frac{b_0 \hat{s}^{\alpha_0 + \alpha'} s}{\Gamma(\alpha(s) + \frac{3}{2})} \exp \left( -\alpha' s \left[ 1 - \log(\alpha' s_0) \right] \right) \]

\[ + \frac{s}{\pi} \int_{m^2_+}^{\infty} ds' \frac{\text{Im} \alpha(s') \log \frac{\hat{s}'}{\hat{s}} + \text{arg} \ \Gamma(\alpha(s') + \frac{3}{2})}{s'(s' - s)} \],

3 Constants fixed \( \leftrightarrow \) fitting pole position and residue
**κ resonance**

- **Slope→almost 10 times smaller**

![Graph showing slope comparison]

- **Striking similarity with Yukawa potentials at low energy:**
  \( V(r) = Ga \times \exp(r/a)/r \).

- **Similar order of magnitude for range:** \( a_{\pi\pi} = 0.5 \text{ GeV}^{-1} \) and \( a_{\pi K} = 0.32 \text{ GeV}^{-1} \).

- **We obtain that** \( a_{\pi\pi}/a_{\pi K} \approx \mu_{\pi K}/\mu_{\pi\pi} \).
The result obtained with our method is compatible near the pole.

It is almost linear.

Intercept $\alpha_0 = -1.15^{+0.23}_{-0.15}$, and Slope $\alpha' = 0.81 \pm 0.1 \text{GeV}^{-2}$. 
- Imposing a linear Regge trajectory $\rightarrow$ huge deviation from data.
- Trajectory very far from real, slope 6 times smaller than usual.
- Intercept $\alpha_0 = -0.28 \pm 0.02$, slope $\alpha' = 0.16 \pm 0.03 \text{GeV}^{-2}$. 

Dispersive determination of the $\kappa$ resonance

A. Rodas
Future project: New HDR

- It’s been shown that symmetric variables under $s, t, u$ exchanges offer the biggest convergence in the complex plane.
- Maximum energy in the real axis $\rightarrow$ 1.7 GeV.
- It offers two possibilities:
  1- Select between incompatible data sets above 1.4 GeV.
  2- Determine if the $f_0(1370), f_0(1500)$ appear in this process $\rightarrow$ glueball related.