# Three-body interactions from the finite-volume QCD spectrum

Ruairí Brett (GWU)

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# GWUQCD

GWU (Lattice):

- Andrei Alexandru
- Frank Lee
- Ruairí Brett

GWU (Pheno):

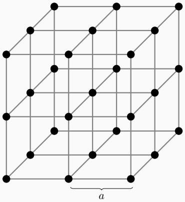
- Michael Döring
- Maxim Mai

Liverpool (Lattice):

- Chris Culver

Support:

THE GEORGE WASHINGTON UNIVERSITY
WASHINGTON, DC



# Light hadron spectrum

- two-body scattering reasonably well understood from LQCD and phenomenology
  - $\rightarrow$ David Wilson's plenary & multiple parallel talks

- two-body treatment of many states only sufficient for "special cases" $\rightarrow$ heavy $m_\pi$, etc.

- many unsolved three-body problems:

  **Roper:** $N(1440)\ 1/2^+$

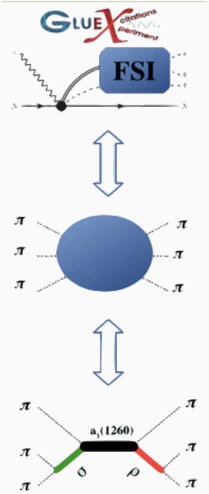
  mass pattern doesn't match quark model

  large BR to $N\pi\pi$

  $a_1(1260)$

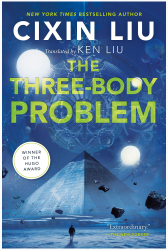
  decays to $\rho\pi, \sigma\pi \rightarrow \pi\pi\pi$

  model $\rho, \sigma$ as isobar



# Three-body amplitudes

- three-body amplitudes difficult even in infinite-volume
  - 8 kinematic variables
  - sub-channel dynamics




- three leading approaches:

RFT (Hansen, Sharpe, Blanton, Briceño, Romero-Lopez)
relativistic, diagrammatic approach

FVU (Mai, Döring)
relativistic, from unitarity

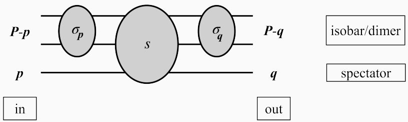
NREFT (Rusetsky, Peng et. al.)
non-relativistic EFT

Reviews: Hansen, Sharpe 1901.00483, Mai, Döring, Rusetsky 2103.00577

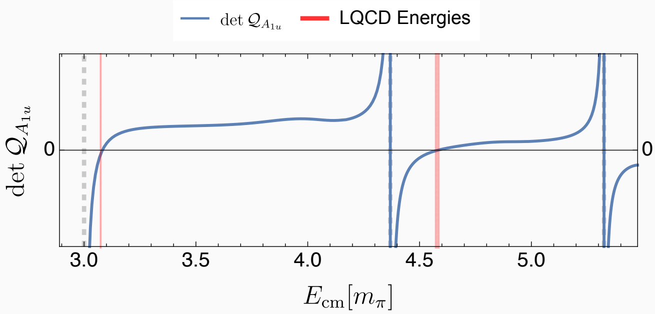
Comparison of RFT & FVU amplitudes: Jackura et. al. 1905.12007

# Three-body quantization condition (FVU)

finite-vol. energies $\sqrt{s}$ satisfy:




$$\det \mathcal{Q} \equiv \det \left[ B_0(\sqrt{s}) + C_0(\sqrt{s}) + E_{L\eta} \tau^{-1}_{L\eta P}(\sqrt{s}) \right] = 0.$$




Mai, Döring `1709.08222`, RB et. al. `2101.06144`

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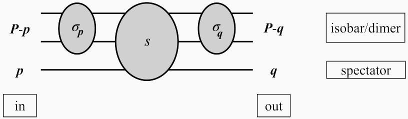


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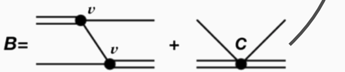


$B$: one particle exchange
$C$: isobar-spectator interaction

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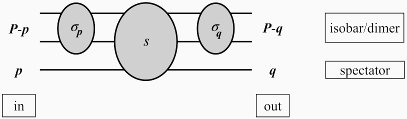


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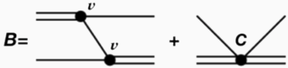


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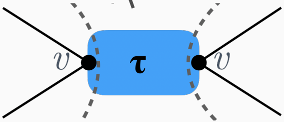



2-body input
($K$-matrix, effective range, etc.)

Mai, Döring `1709.08222`, RB et. al. `2101.06144`

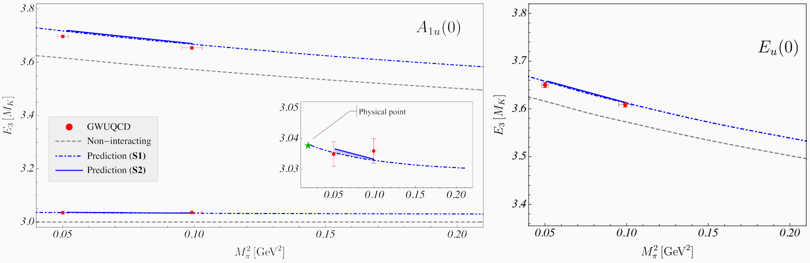
# Warming up: predicting $3K^-$

- NLO ChPT for 2-body interactions:
  **S1**: $f_\pi$ extrapolated using physical point data and NLO chiral expressions
  **S2**: meson decay constants determined on the lattice

- vanishing isobar-spectator interaction ($C = 0$)

- decent qualitative agreement*: rough $m_\pi$ dependence to $\approx 315$ MeV

- no fitting of 2- or 3-body parameters, haven't extracted any $\infty$-vol physics yet



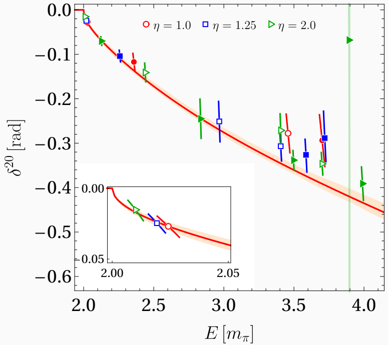

*quenched strange quark

Alexandru et. al. `2009.12358`

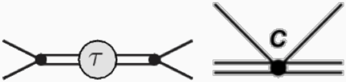
## $I = 3$, $3\pi^+$ SCATTERING

$\pi^+\pi^+\pi^+$ scattering:

- simplest three-body channel to test formalism(s)
- two-body sub-channel: $I = 2\ \pi^+\pi^+$ (repulsive, no resonances)
- inputs to $\infty$-volume amplitude:
  - $\rightarrow$ $I = 2\ \pi\pi$ amplitude
  - $\rightarrow$ three-body isobar-spectator interaction $C$




Mai et. al. 1908.01847



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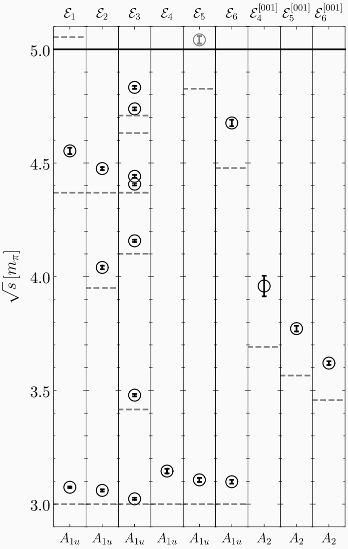



Culver et. al. 1911.09047, RB et. al. 2101.06144

# Lattice details

- cubic ($\eta = 1$) and lattices elongated in the $z$-direction ($L_z = \eta L_{x,y}$)

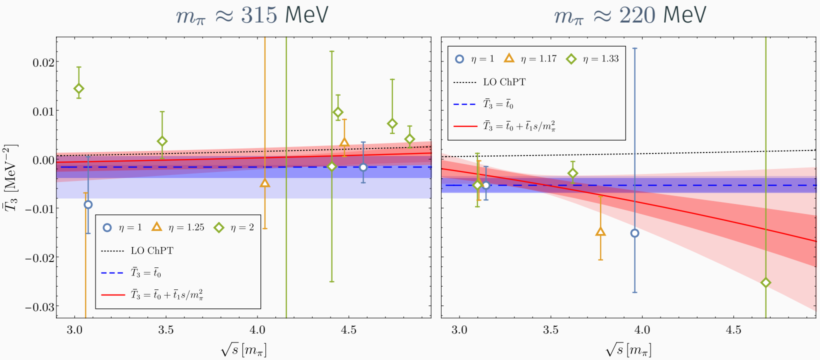
- $N_f = 2$ nHYP-smeared (clover) Wilson fermions at
  $m_\pi \approx 315$ MeV ($\eta = 1.0, 1.25, 2.0$) and
  $m_\pi \approx 220$ MeV ($\eta = 1.0, 1.17, 1.33$)

- two-flavor simulations offer insight into importance of strange quark
  - $\rightarrow$ e.g. strange quark content of $\rho$ resonance: `1605.03993`

- distillation for all quark propagation

- elongations help to sample scattering region (similar to moving frames), much cheaper than increasing cubic volume

# $3\pi^+$ CONTACT INTERACTION




real, three-body contact term:

$$\bar{T}_3 = \frac{3}{2}\left(\frac{K^{-1}}{32\pi}\right)^{-1}\frac{C_0}{\mathbb{1} - C_0 E_{L\eta}^{-1}\left(\frac{K^{-1}}{32\pi}\right)^{-1}}\left(\frac{K^{-1}}{32\pi}\right)^{-1}$$

RB et. al. `2101.06144`

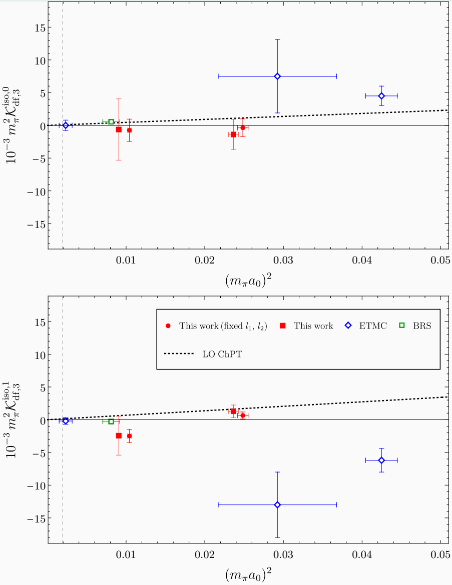
# Comparing to RFT

matching RFT and FVU at the level of the three-body amplitudes leads to

$$\mathcal{K}^{\text{iso},0}_{\text{df},3} \simeq 6(\bar{t}_0 + 9\bar{t}_1)$$
$$\mathcal{K}^{\text{iso},1}_{\text{df},3} \simeq 54\bar{t}_1$$

where the RFT 3-body term is

$$\mathcal{K}_{3,\text{df}} = \mathcal{K}^{\text{iso},0}_{\text{df},3} + \mathcal{K}^{\text{iso},1}_{\text{df},3}\left(\frac{s - 9m_\pi^2}{9m_\pi^2}\right)$$

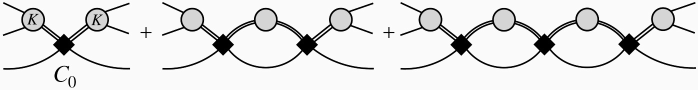


# Summary

- three-body (meson) spectra now realistically accessible from the lattice
- (unsurprisingly) more levels needed to constrain amplitudes than in 2-body scattering
- next challenge for the lattice is (coupled) resonant subchannels/isobars
  - $\rightarrow$ e.g. $a_1(1260) \rightarrow \pi\rho, \pi\sigma \rightarrow 3\pi$
- connecting to $\infty$-volume amplitudes/poles much more complex than for 2-body
  - $\rightarrow$ see previous two talks from Max Hansen & Andrew Jackura

door is open now for 3-body resonances to be studied over the next few years

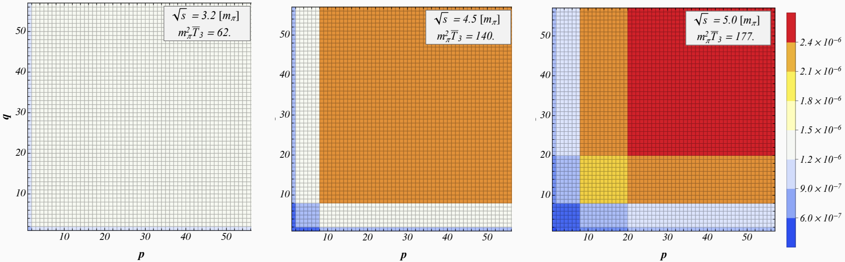
# Three-body contact interaction

$\bar{T}_3 =$  $+ \dots$



$$\bar{T}_3 = \frac{3}{2}\left(\frac{K^{-1}}{32\pi}\right)^{-1} \frac{C_0}{\mathbb{1} - C_0 E_{L\eta}^{-1}\left(\frac{K^{-1}}{32\pi}\right)^{-1}} \left(\frac{K^{-1}}{32\pi}\right)^{-1}$$

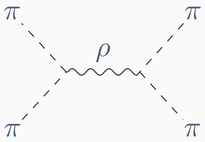
$C_0$ for fixed $\bar{T}_3$: (isotropic $\bar{T}_3 \leftrightarrow$ anisotropic $C$ and vice versa)



# Finite volume spectra

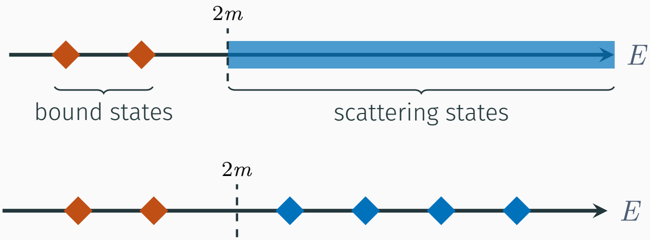
Scattering process: eg.

$I = 1 \quad \pi\pi \to \pi\pi$




$\infty$-volume

Finite volume

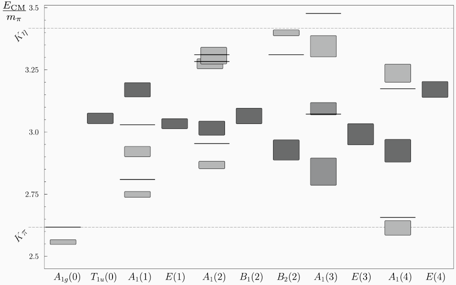


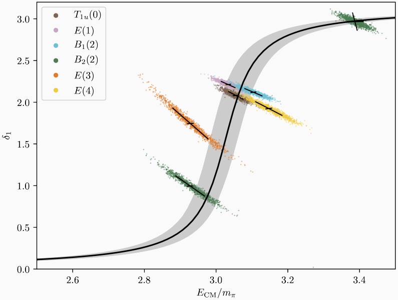

$\to$ how to access $\infty$-vol. physics?

# Two-body quantization condition

- quantisation condition (Lüscher formula):

$$\det[\, \widetilde{K}^{-1}(E_{\rm cm}) - B(E_{\rm cm}, L)\,] = 0$$




RB, PhD. Thesis

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$\widetilde{K}_\ell^{-1} \sim \cot \delta_l$ → $\widetilde{K}^{-1}$

$B$ ↑ known, mixes partial waves

- determinant over decay channel, partial waves (truncation!)

- implementation with group theory for cubic box (& boosted frames): `github.com/cjmorningstar10/TwoHadronsInBox`

Morningstar et. al. `1707.05817`

# Extracting the finite-vol. spectrum

- temporal correlation *matrix*:

$$C_{ij}(t) \equiv \langle 0|\mathcal{O}_i(t+t_0)\overline{\mathcal{O}}_j(t_0)|0\rangle$$
$$= \sum_n \langle 0|\mathcal{O}_i|n\rangle\langle n|\overline{\mathcal{O}}_j|0\rangle\, e^{-E_n t}$$

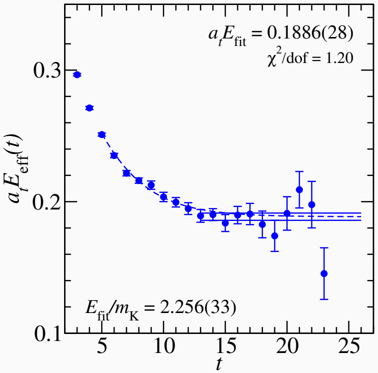
- solve generalized eigenvalue problem

$$C(\tau_0)^{-1/2}C(t)C(\tau_0)^{1/2}v_n(t,\tau_0)$$
$$= \lambda_n(t,\tau_0)v_n(t,\tau_0)$$

- eigenvalues tend to lowest $N$ energies

$$\lim_{t\to\infty}\lambda_n(t) = b_n e^{-E_n t}[1+\mathcal{O}(e^{\Delta_n t})]$$

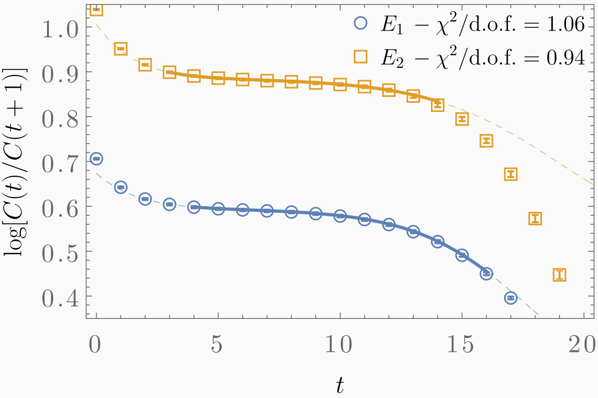
$$\Rightarrow E^n_{\text{eff}}(t) = \frac{1}{\Delta t}\ln\left(\frac{\lambda_n(t)}{\lambda_n(t+\Delta t)}\right)$$



# Lattice Details

| | $N_t \times N_{x,y}^2 \times N_z$ | $\eta$ | $a$[fm] | $N_{\mathsf{cfg}}$ | $am_\pi$ |
|---|---|---|---|---|---|
| $\mathcal{E}_1$ | $48 \times 24^2 \times 24$ | 1.00 | 0.1210(2)(24) | 300 | 0.1931(4) |
| $\mathcal{E}_2$ | $48 \times 24^2 \times 30$ | 1.25 | – | – | 0.1944(3) |
| $\mathcal{E}_3$ | $48 \times 24^2 \times 48$ | 2.00 | – | – | 0.1932(3) |
| $\mathcal{E}_4$ | $64 \times 24^2 \times 24$ | 1.00 | 0.1215(3)(24) | 400 | 0.1378(6) |
| $\mathcal{E}_5$ | $64 \times 24^2 \times 28$ | 1.17 | – | – | 0.1374(5) |
| $\mathcal{E}_6$ | $64 \times 24^2 \times 32$ | 1.33 | – | – | 0.1380(5) |

# Fitting thermal states




fit ansatz:

$$a_1 e^{-m_1 t} + a_2 e^{-m_2 t} + a_3 e^{-\Delta E t}$$

$aN_t = 5.80(1)(12)$ fm

| $n$ | $a_1$ | $m_1$ | $a_2$ | $m_2$ | $a_3$ | $\Delta E$ |
|---|---|---|---|---|---|---|
| 1 | 0.810(4) | 0.594(1) | 0.151(4) | 1.27(4) | 0.00029(3) | 0.1587(7) |
| 2 | 0.752(4) | 0.884(3) | 0.211(3) | 1.62(2) | 0.005(4) | 0.50(6) |