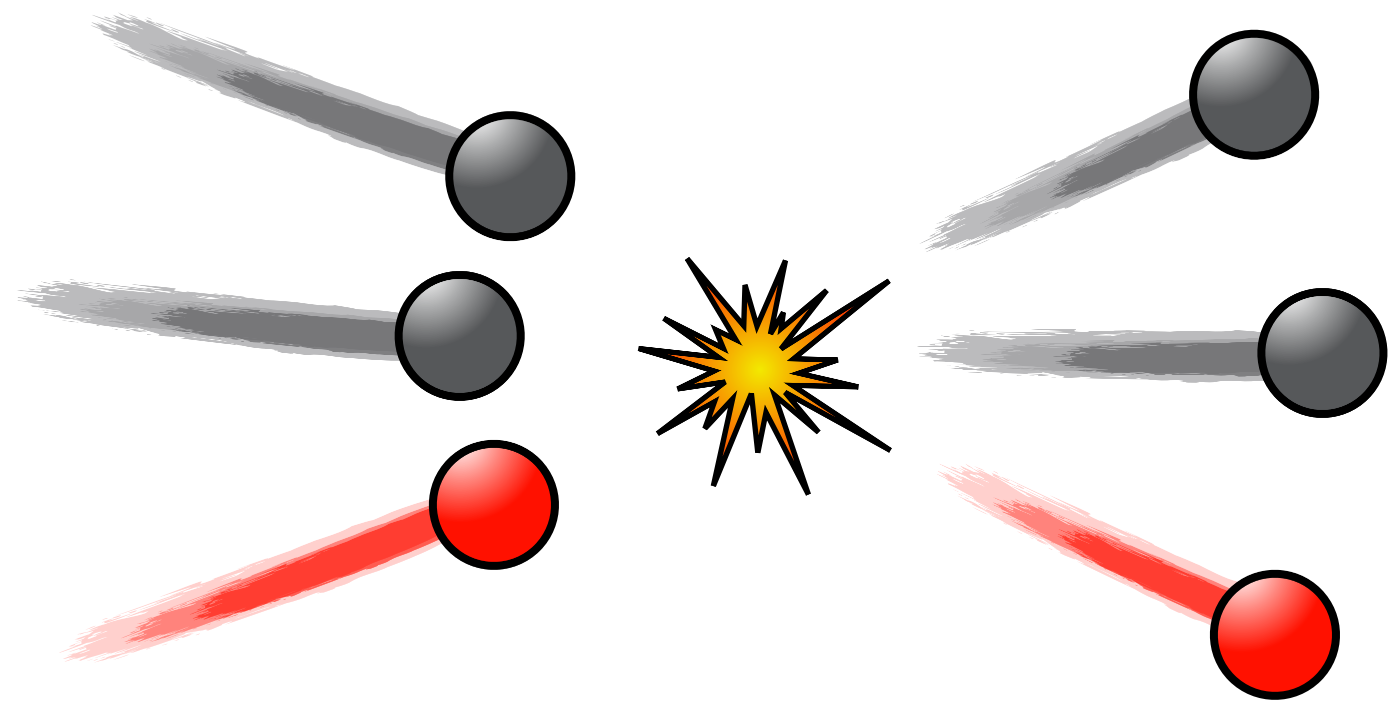
Integral equations for relativistic three-hadron scattering

Andrew W. Jackura

9th Workshop of the APS Topical Group on Hadronic Physics
Friday 16th, April 2021



Old Dominion University
Jefferson Lab
Thomas Jefferson National Accelerator Facility

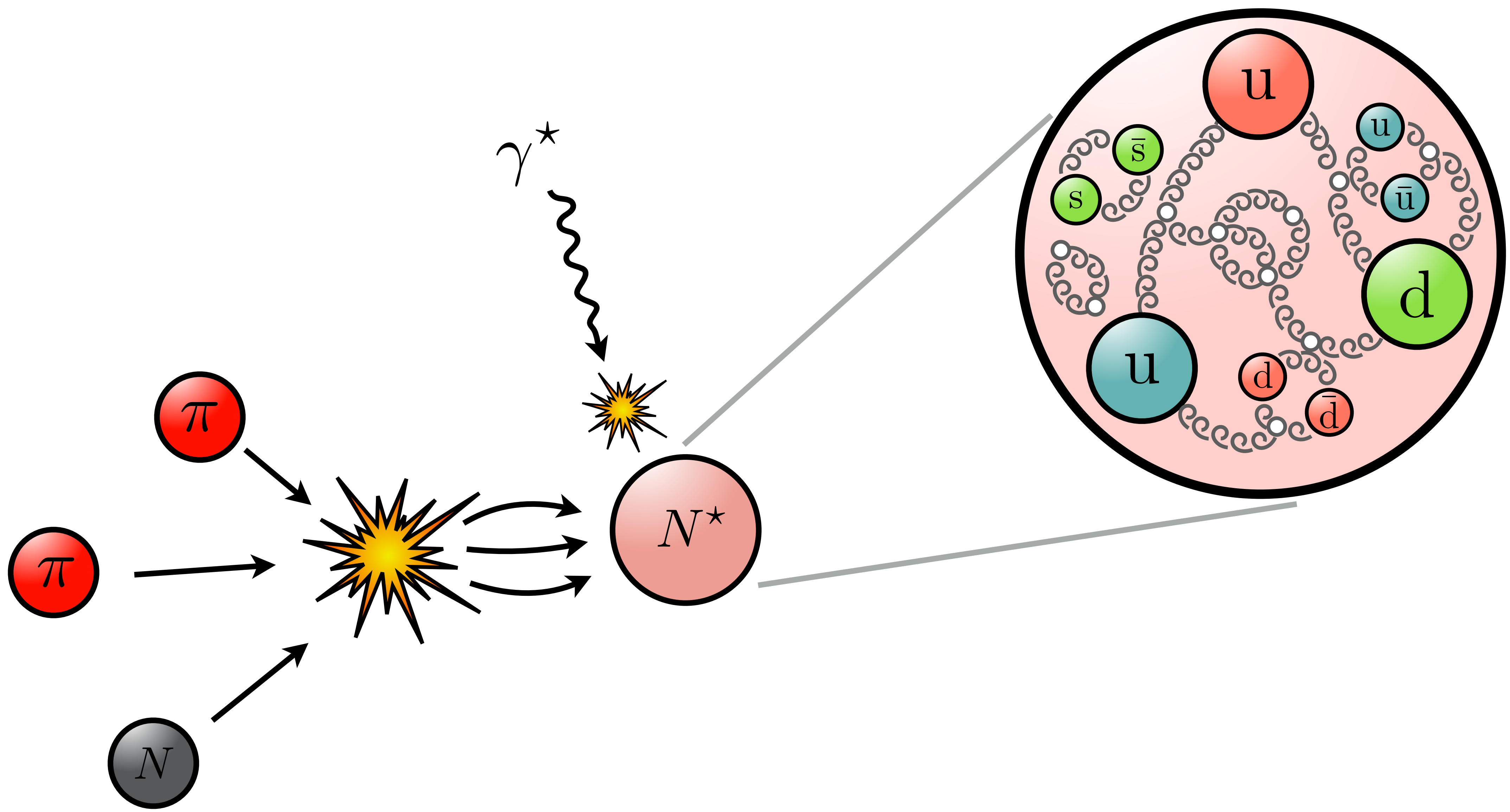
APS physics Topical Group on Hadronic Physics

Why three-body interactions?

Three-body interactions play a key role in many outstanding problems in hadron physics

- Most excited states strongly couple to three (or more) particle channels

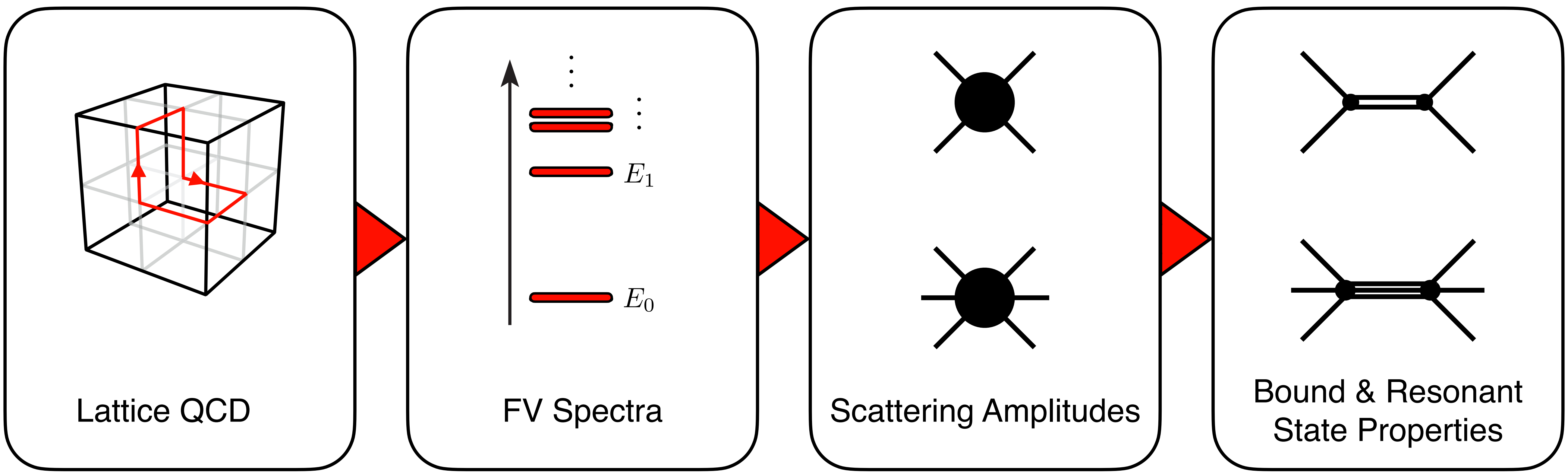
e.g. the Roper



Path to hadronic physics from QCD

Lattice QCD offers a systematic avenue to compute multi-hadron scattering amplitudes

- Connect spectra to scattering amplitudes via Lüscher methodology



R. Briceño, J. Dudek, and R. Young
Rev.Mod.Phys. **90**, 025001 (2018)

See Dave Wilson's talk for status of this program

Path to three-body physics from QCD

Past decade has seen tremendous progress extending these ideas to three particles

$$\{E_{\mathfrak{n}}\} \longrightarrow \mathcal{K}_3 \longrightarrow \mathcal{M}_3 \longrightarrow s_{\mathrm{res}}$$

Quantization Condition

Integral Equations

Analytic Continuation

Path to three-body physics from QCD

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$$\{E_{\mathfrak{n}}\} \longrightarrow \mathcal{K}_3 \longrightarrow \mathcal{M}_3 \longrightarrow s_{\mathrm{res}}$$

Quantization Condition | Integral Equations | Analytic Continuation

See Max Hansen's talk for applications

All-orders relativistic field theory (RFT)

M. Hansen and S. Sharpe
Phys. Rev. D **90**, 116003 (2014), Phys. Rev. D **95**, 034501 (2017)

R. Briceño, M. Hansen, and S. Sharpe
Phys. Rev. D **95**, 074510 (2017), Phys. Rev. D **98**, 014506 (2018), Phys. Rev. D **99**, 014516 (2019)

T. Blanton, F. Romero-Lopéz, and S. Sharpe
JHEP **03**, 106 (2019)

F. Romero-Lopéz, S. Sharpe, T. Blanton, R. Briceño, and M. Hansen
JHEP **10**, 007 (2019)

M. Hansen, F. Romero-Lopéz, and S. Sharpe
JHEP **07**, 047 (2020)

T. Blanton, and S. Sharpe
arXiv:2007.16188 (2020), arXiv:2007.16190 (2020)

Equivalence

Non-relativistic EFT (NREFT)

H. Hammer, J. Pang, and A. Rusetsky
JHEP **09**, 109 (2017), JHEP **10**, 115 (2017)

See Ruairí Brett's talk for applications

Finite-volume unitarity (FVU)

M. Mai and M. Döring
Eur. Phys. J. A **53**, 240 (2017), Phys. Rev. Lett. **122**, 062503 (2019)

M. Döring, H. Hammer, M. Mai, J. Pang, A. Rusetsky, and J. Wu
Phys. Rev. D **97**, 114508 (2018)

M. Mai, M. Döring, C. Culver, and A. Alexandru
Phys.Rev.D **101**, 054510 (2020)

Path to three-body physics from QCD

Past decade has seen tremendous progress extending these ideas to three particles

$$\{E_n\} \longrightarrow \mathcal{K}_3 \longrightarrow \mathcal{M}_3 \longrightarrow s_{\text{res}}$$

Quantization Condition

Integral Equations

Analytic Continuation

Focus of this talk

All-orders relativistic field theory (RFT)

M. Hansen and S. Sharpe
Phys. Rev. D **92**, 114509 (2015)

R. Briceño, M. Hansen, S. Sharpe, and A. Szczepaniak
Phys. Rev. D **100**, 054508 (2019)

R. Briceño, S. Dawid, M.H. Islam, AJ, and C. McCarty
arXiv:2010.09820 (2020)

S-matrix unitarity

M. Mai, B. Hu, M. Döring, A. Pilloni, and A. Szczepaniak
Eur. Phys. J. A **53**, 177 (2017)

AJ et al. [JPAC]
Eur. Phys. J. C **79**, no. 1, 56 (2019)

M. Mikhasenko, AJ et al. [JPAC]
Phys. Rev. D **98**, 096021 (2018)

M. Mikhasenko, AJ et al. [JPAC]
JHEP **08**, 080 (2019)

AJ et al.
Phys. Rev. D **100**, 034508 (2019)

Dawid and Szczepaniak
arXiv:2010.08084 (2020)

Equivalence

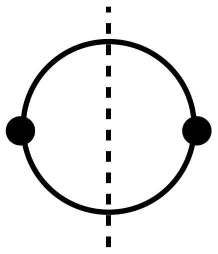
On-shell scattering equations

S-matrix unitarity fixes on-shell structure of scattering amplitudes

$2 \to 2$ scattering

$$\mathcal{M}_2 = \mathcal{K}_2 - \mathcal{K}_2 \mathcal{I} \mathcal{M}_2$$

On-shell two-particle rescattering

$\mathrm{Im}\, \mathcal{I} = -\rho \sim$ 

K-matrices

- *Unknown real function characterizing short-distance physics*
- *Parameterize with analytic function and determine from lattice QCD*
- *Scheme dependent (unphysical)*

On-shell scattering equations

S-matrix unitarity fixes on-shell structure of scattering amplitudes

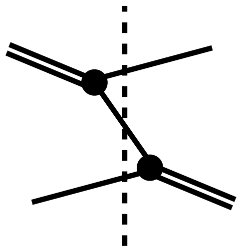
$\mathbf{2 \to 2}$ **scattering**

$$\mathcal{M}_2 = \mathcal{K}_2 - \mathcal{K}_2 \mathcal{I} \mathcal{M}_2$$

$\mathbf{3 \to 3}$ **scattering**

$$\mathcal{M}_3 = \mathcal{K}_3 - \mathcal{K}_3 \mathcal{I} \mathcal{M}_2 - \mathcal{K}_2 \mathcal{I} \mathcal{M}_3 - \int \mathcal{K}_3 \mathcal{I} \mathcal{M}_3$$
$$- \mathcal{K}_2 G \mathcal{M}_2 - \int \mathcal{K}_3 G \mathcal{M}_2 - \mathcal{K}_2 \int G \mathcal{M}_3 - \int\int \mathcal{K}_3 G \mathcal{M}_3$$

On-shell exchange

$$\mathrm{Im}\, G = -\Delta \;\sim$$ 

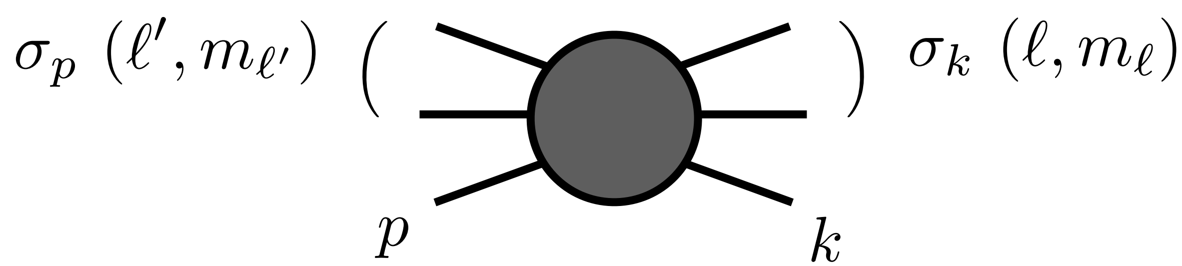
K-matrices

- *Unknown real function characterizing short-distance physics*
- *Parameterize with analytic function and determine from lattice QCD*
- *Scheme dependent (unphysical)*

Solving three-body scattering equations

Given $\mathcal{K}_2$, and $\mathcal{K}_3$, solve integral equation for $\mathcal{M}_3$

$$
\begin{aligned}
\mathcal{M}_3 = \; & \mathcal{K}_3 - \mathcal{K}_3 \mathcal{I} \mathcal{M}_2 - \mathcal{K}_2 \mathcal{I} \mathcal{M}_3 - \int \mathcal{K}_3 \mathcal{I} \mathcal{M}_3 \\
& - \mathcal{K}_2 G \mathcal{M}_2 - \int \mathcal{K}_3 G \mathcal{M}_2 - \mathcal{K}_2 \int G \mathcal{M}_3 - \iint \mathcal{K}_3 G \mathcal{M}_3
\end{aligned}
$$



- *Many equivalent forms (Hansen-Sharpe, Blanton-Sharpe, Döring-Mai, JPAC-AJ,Mikhasenko)*
- *Matrix equation in pair angular momenta*
- *Integral equation in spectator momenta*
- *Singular kernels*
- *Scheme dependent K-matrices (unphysical)*

Solving three-body scattering equations

Given $\mathcal{K}_2$, and $\mathcal{K}_3$, solve integral equation for $\mathcal{M}_3$

$$\mathcal{M}_3 = \mathcal{K}_3 - \mathcal{K}_3 \mathcal{I} \mathcal{M}_2 - \mathcal{K}_2 \mathcal{I} \mathcal{M}_3 - \int \mathcal{K}_3 \mathcal{I} \mathcal{M}_3$$
$$- \mathcal{K}_2 G \mathcal{M}_2 - \int \mathcal{K}_3 G \mathcal{M}_2 - \mathcal{K}_2 \int G \mathcal{M}_3 - \iint \mathcal{K}_3 G \mathcal{M}_3$$

Consider toy model: $3\varphi \to 3\varphi$ such that $2\varphi \to b$

AJ, R. Briceño, S. Dawid, M.H. Islam, and C. McCarty
arXiv:2010.09820

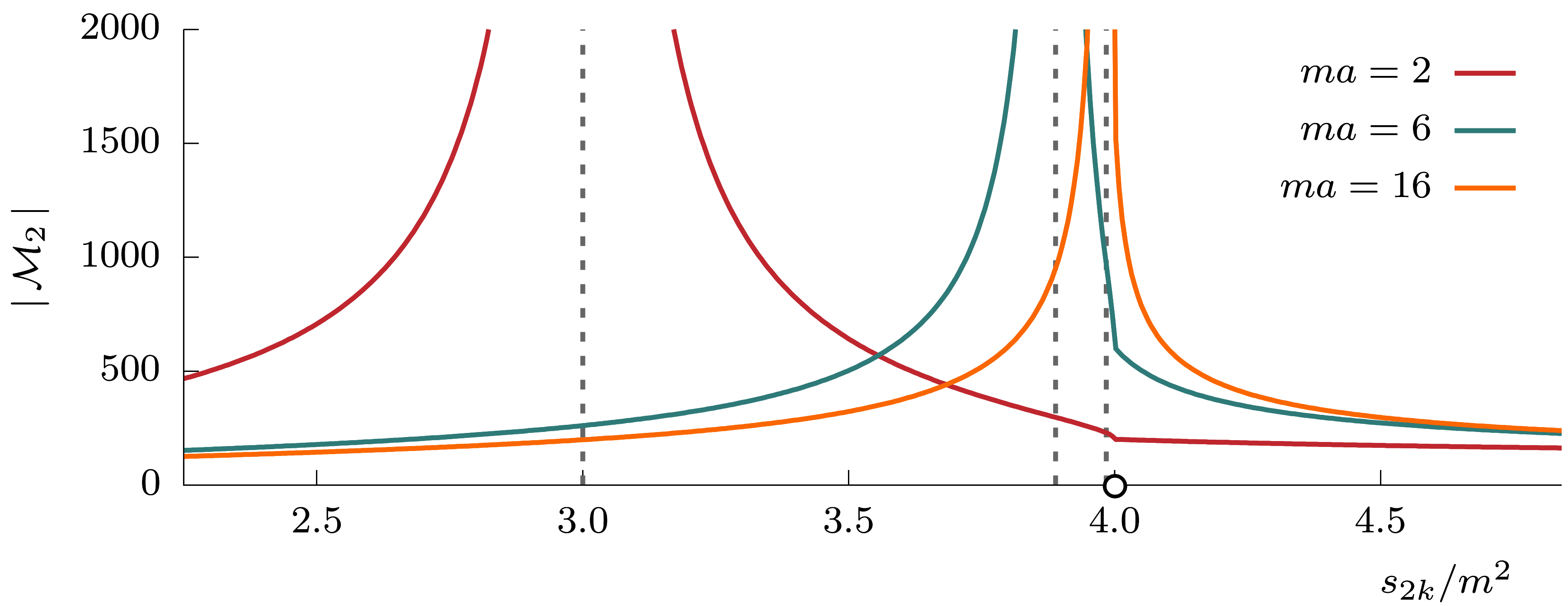
Solving three-body scattering equations

Given $\mathcal{K}_2$, and $\mathcal{K}_3$, solve integral equation for $\mathcal{M}_3$

$$\mathcal{M}_3 = \mathcal{K}_3 - \mathcal{K}_3\mathcal{I}\mathcal{M}_2 - \mathcal{K}_2\mathcal{I}\mathcal{M}_3 - \int \mathcal{K}_3\mathcal{I}\mathcal{M}_3$$
$$- \mathcal{K}_2 G \mathcal{M}_2 - \int \mathcal{K}_3 G \mathcal{M}_2 - \mathcal{K}_2 \int G \mathcal{M}_3 - \int\int \mathcal{K}_3 G \mathcal{M}_3$$

Consider toy model: $3\varphi \to 3\varphi$ such that $2\varphi \to b$

$$\mathcal{K}_2^{-1} \sim -\frac{1}{a}$$



Solving three-body scattering equations

Given $\mathcal{K}_2$, and $\mathcal{K}_3$, solve integral equation for $\mathcal{M}_3$

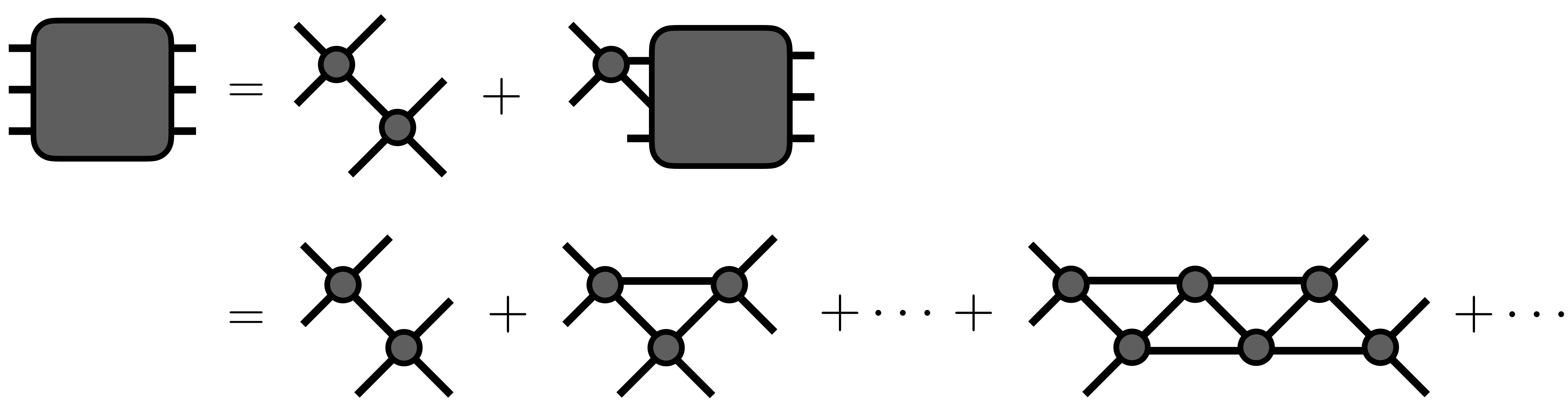
$$\mathcal{M}_3 = \cancel{\mathcal{K}_3} - \cancel{\mathcal{K}_3}\mathcal{I}\mathcal{M}_2 - \mathcal{K}_2\mathcal{I}\mathcal{M}_3 - \int \cancel{\mathcal{K}_3}\mathcal{I}\mathcal{M}_3$$
$$- \mathcal{K}_2 G \mathcal{M}_2 - \int \cancel{\mathcal{K}_3} G \mathcal{M}_2 - \mathcal{K}_2 \int G \mathcal{M}_3 - \int\int \cancel{\mathcal{K}_3} G \mathcal{M}_3$$

Consider toy model: $3\varphi \to 3\varphi$ such that $2\varphi \to b$

Assume $\mathcal{K}_3 = 0$

$$\mathcal{M}_3\big|_{\mathcal{K}_3=0} \equiv \mathcal{D} \qquad \Longrightarrow \qquad \mathcal{D} = -\mathcal{M}_2 G \mathcal{M}_2 - \mathcal{M}_2 \int G\mathcal{D}$$

$i\mathcal{D} =$

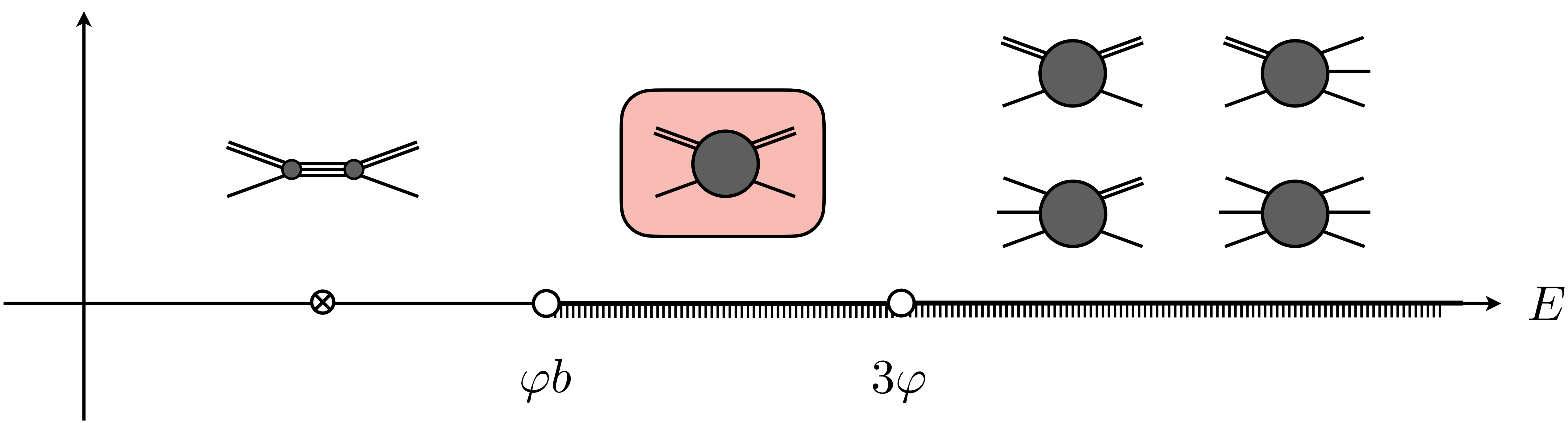


Solving three-body scattering equations

Focus on case where 2-body systems forms bound state

- Consider energies below three particle threshold

$$\lim_{\sigma_p,\sigma_k\to\sigma_b} i\mathcal{M}_3 = ig_b \frac{i}{\sigma_p-\sigma_b}\, i\mathcal{M}_{\varphi b}\, \frac{i}{\sigma_k-\sigma_b} ig_b$$



Solving three-body scattering equations

Convert integral equation to linear equation

- Introduce regulators N (matrix size) and ϵ (pole shift)
- Recover amplitude in $N \to \infty,\ \epsilon \to 0^+$ limit

$$\mathcal{M}_{\varphi b} = \lim_{N\to\infty} \lim_{\epsilon\to 0^+} \mathcal{M}_{\varphi b}(N, \epsilon)$$

S-matrix unitarity provides a way to check quality of solutions

- Deviation from unitarity guides quality of solution

$$\mathrm{Im}\, \mathcal{M}_{\varphi b}^{-1}(E) = -\rho_{\varphi b}(E)$$

$$\Delta\rho_{\varphi b}(E;N) \equiv \left| \frac{\mathrm{Im}\left[\mathcal{M}_{\varphi b}^{-1}(E;N)\right] + \rho_{\varphi b}(E)}{\rho_{\varphi b}(E)} \right| \times 100$$

Several methods

- *Brute-force*
- *Remove bound-state pole explicitly*
- *Splines — Glöckle, Hasberg, Neghabian Z. Phys.* ***A305*** *(1982) 217*

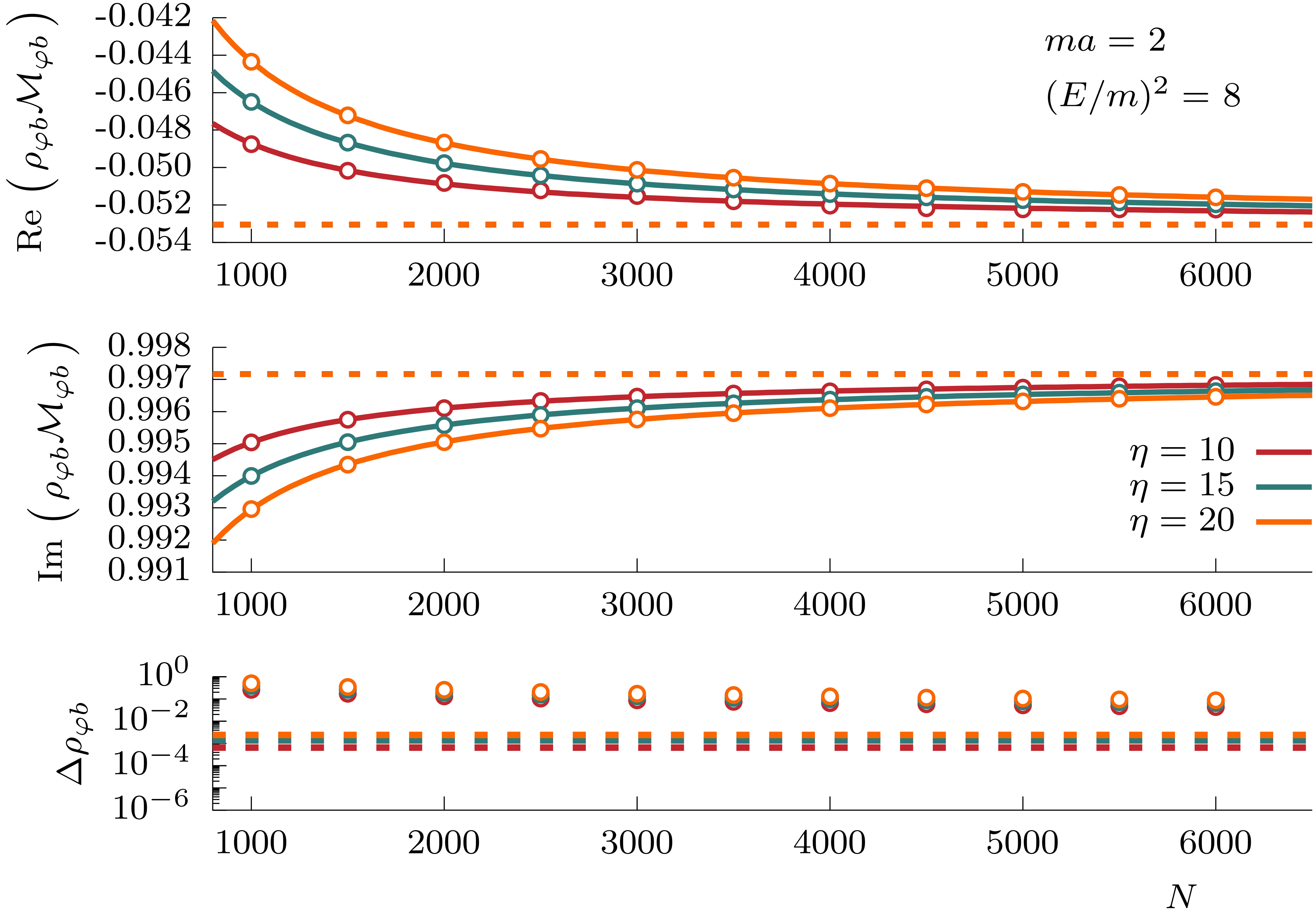
$N \to \infty$ Extrapolations

Compute multiple N solutions — extrapolate to $N \to \infty$ limit

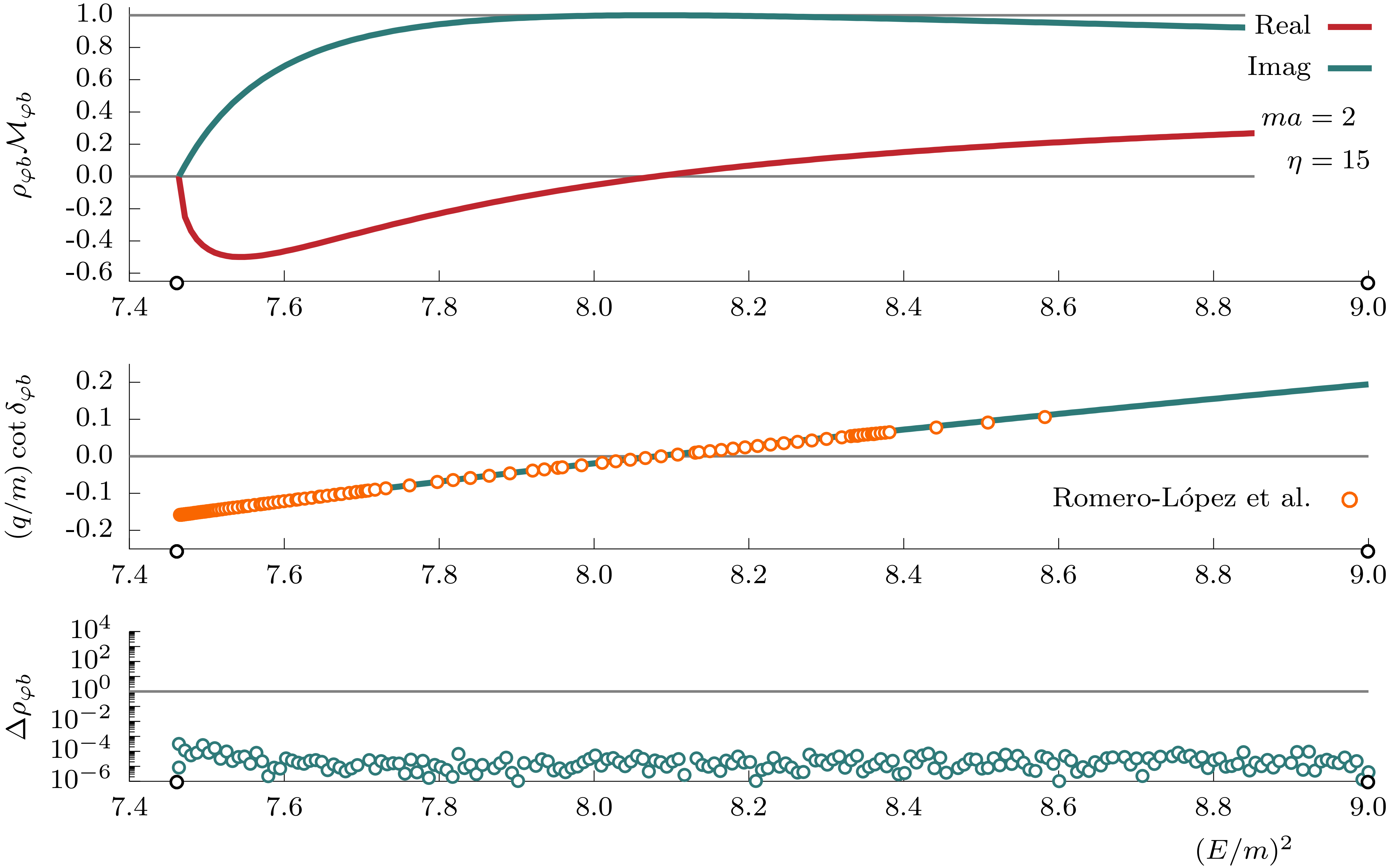
$$\mathcal{M}_{\varphi b}(E;N) \approx \mathcal{M}_{\varphi b}(E) + \frac{\alpha}{N}$$

- Unitarity deviation greatly improved

$$\epsilon \propto \frac{\eta}{N}$$



Example of solution



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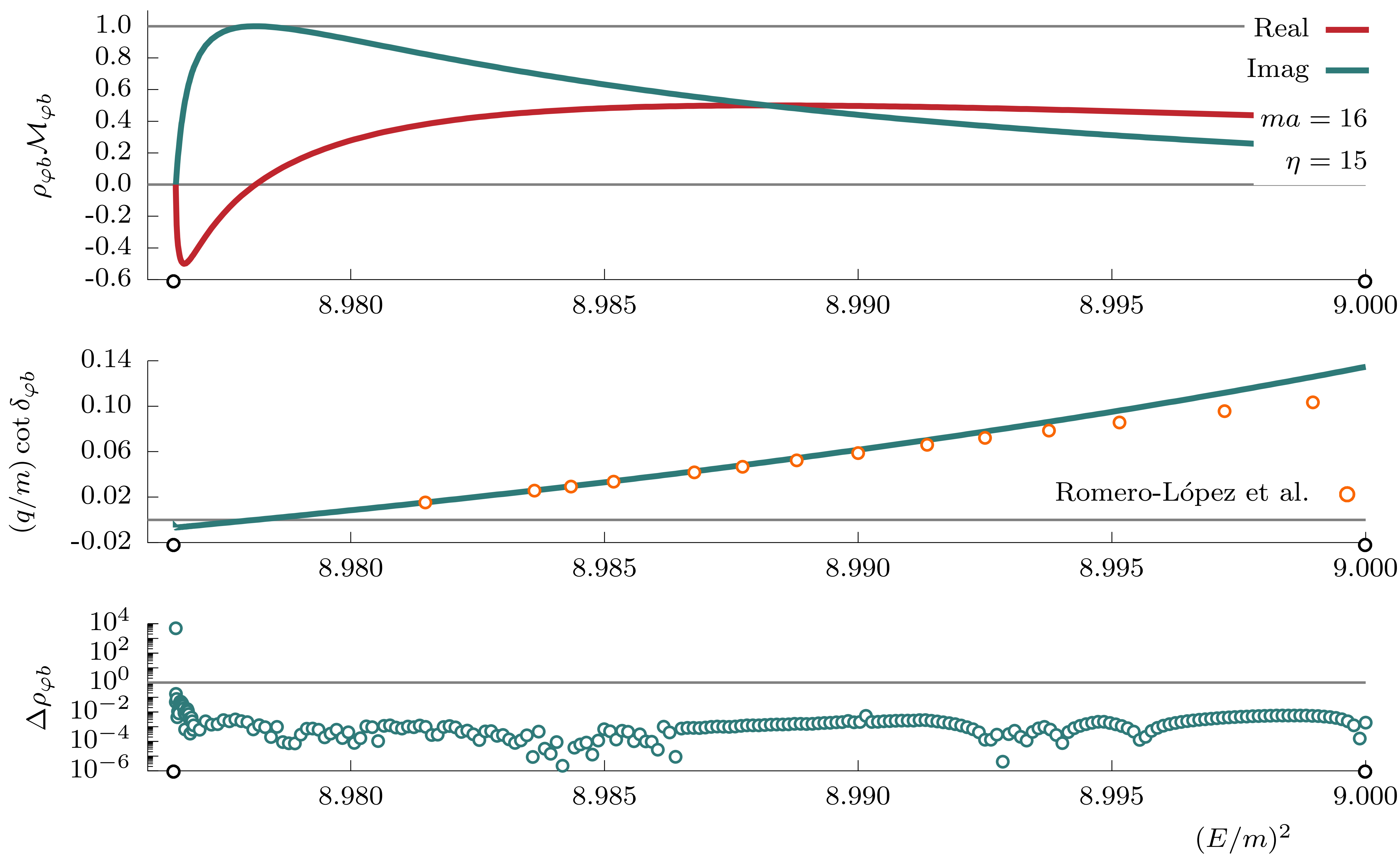
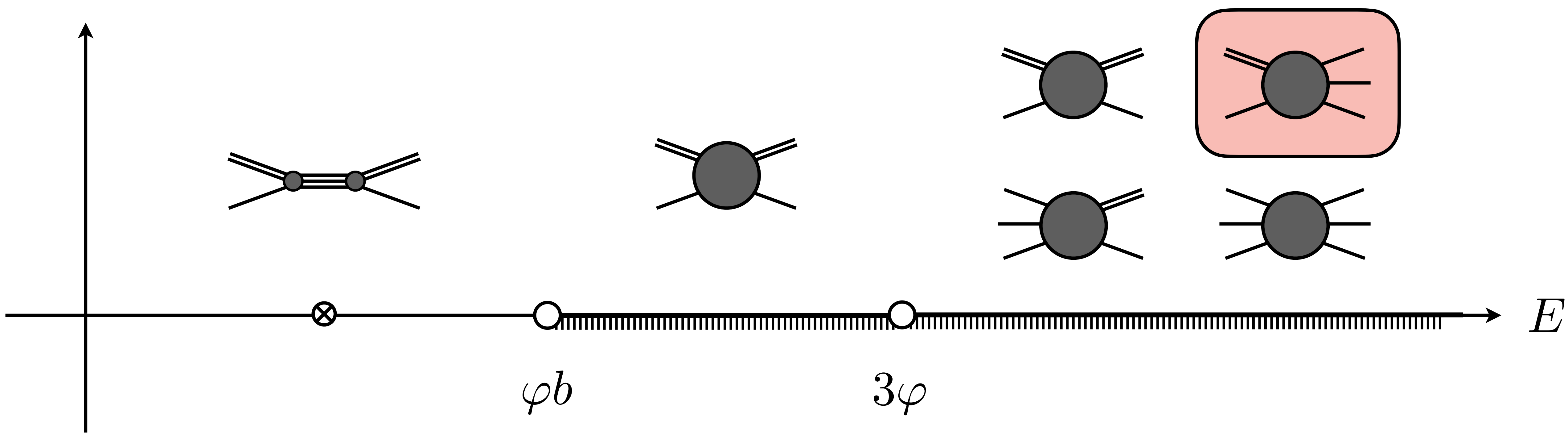


Above the three-body threshold

Methodology not limited to below 3-body threshold

- Allows for calculation of breakup / recombination amplitude

$$\mathcal{M}_{\varphi\to 3\varphi}(\sigma_p) = -\lim_{\sigma_k\to\sigma_b}\frac{\sigma_k-\sigma_b}{g}\,\mathcal{M}_3(\sigma_p,\sigma_k)$$



Above the three-body threshold

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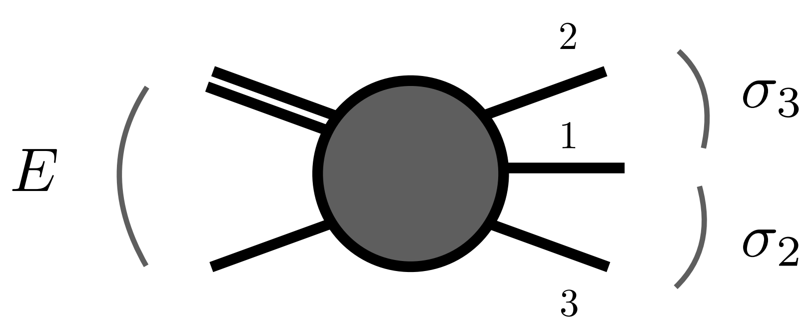
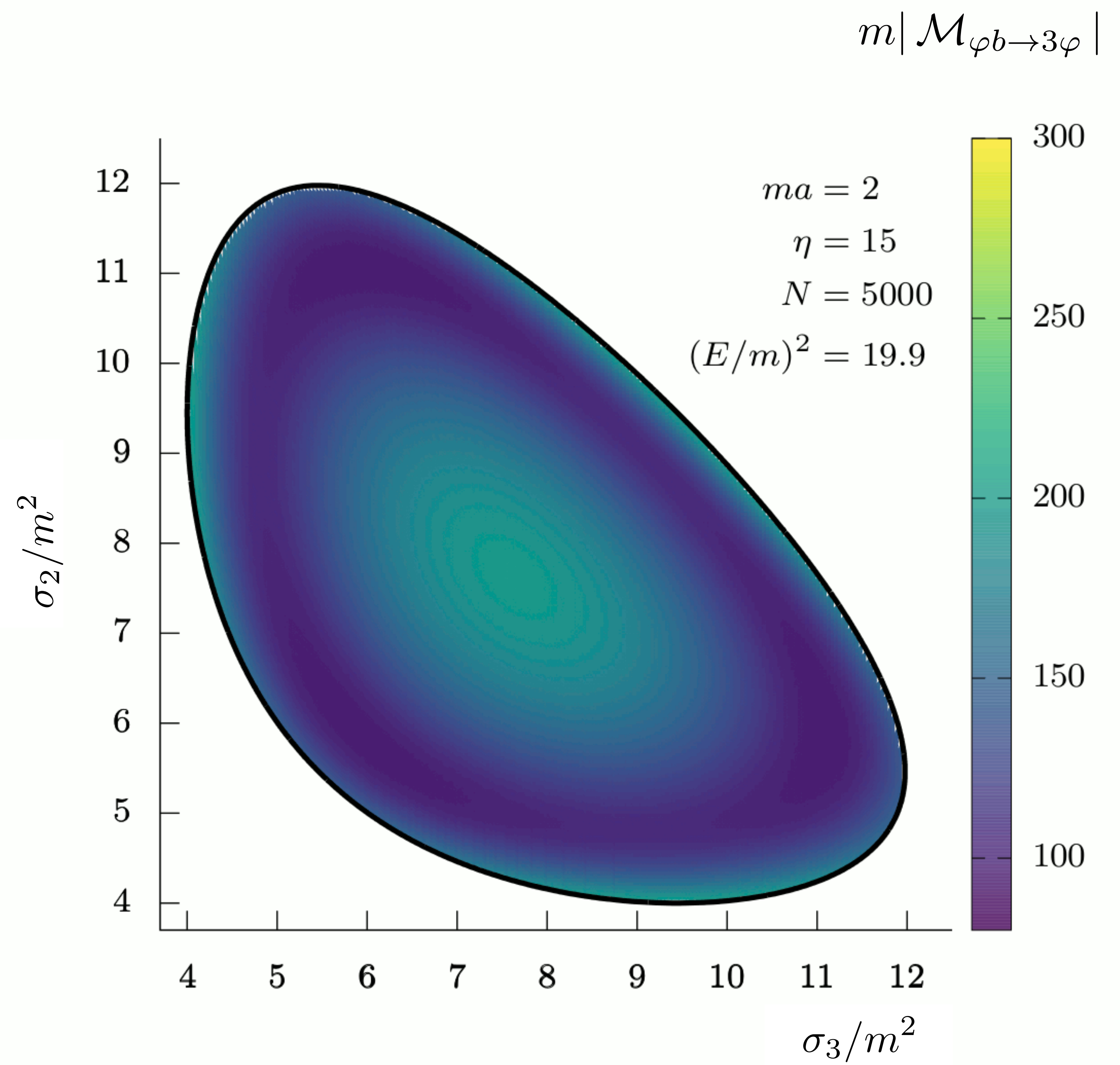




Unfortunately… pretty boring…

Path to three-body physics from QCD

Three-body resonances from QCD are within reach

$$\{E_{\mathfrak{n}}\} \longrightarrow \mathcal{K}_3 \longrightarrow \mathcal{M}_3 \longrightarrow s_{\mathrm{res}}$$

Quantization Condition

Integral Equations

Analytic Continuation

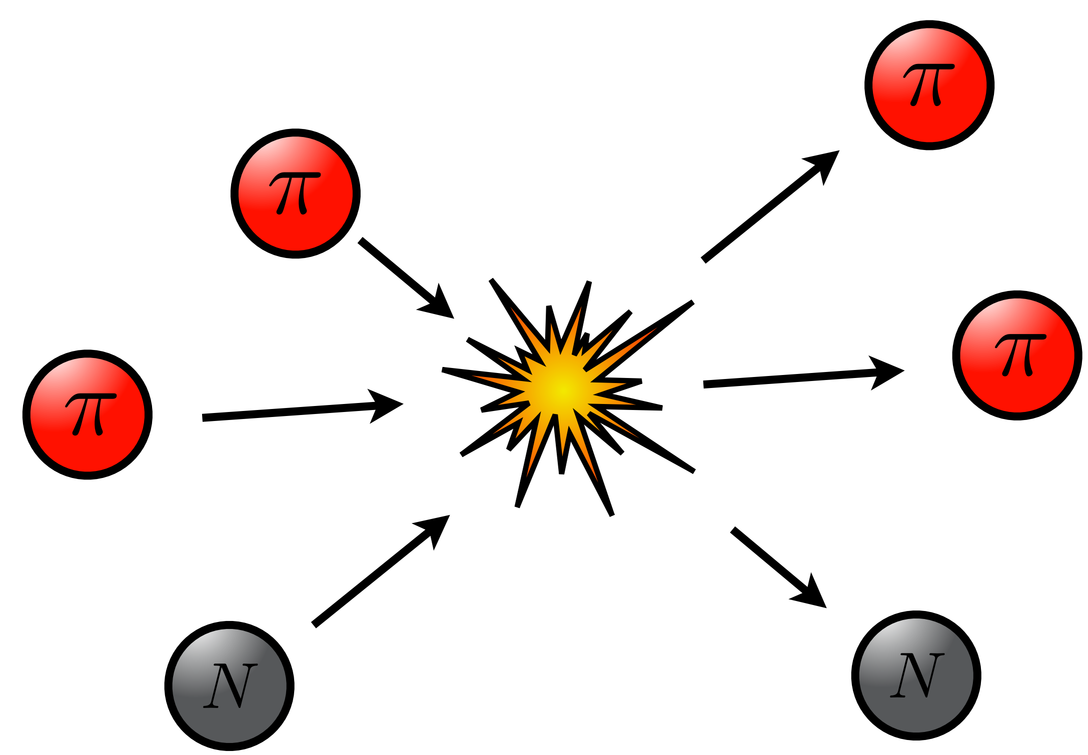
Plenty of work ahead of us

- Effects of scheme/cutoff dependencies?
- Consistent analytic continuation to complex energies?
- Efficient versions of quantization conditions/ integral equations?
- Generalizations to multiple channels, arbitrary spins?

Summary

Three-body resonances from QCD are within reach

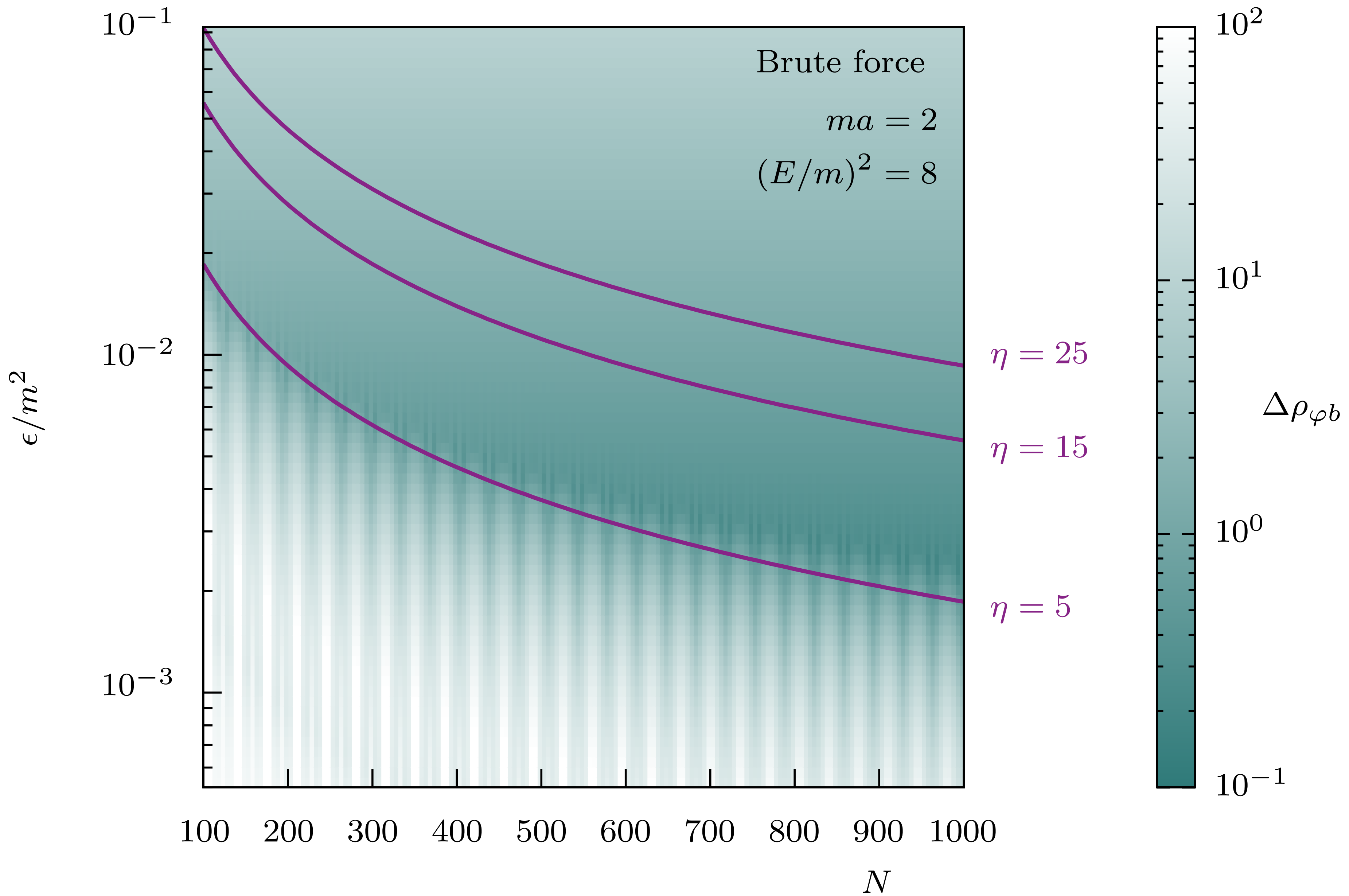
- First lattice calculations for $3\pi^+, 3K$, first $\mathbf{3} \to \mathbf{3}$ amplitude from QCD
- Integral equations for strongly interacting two-body (bound) states examined
- No restrictions above three-particle threshold
- ‘Easily’ generalized to resonating systems, coupled waves, …



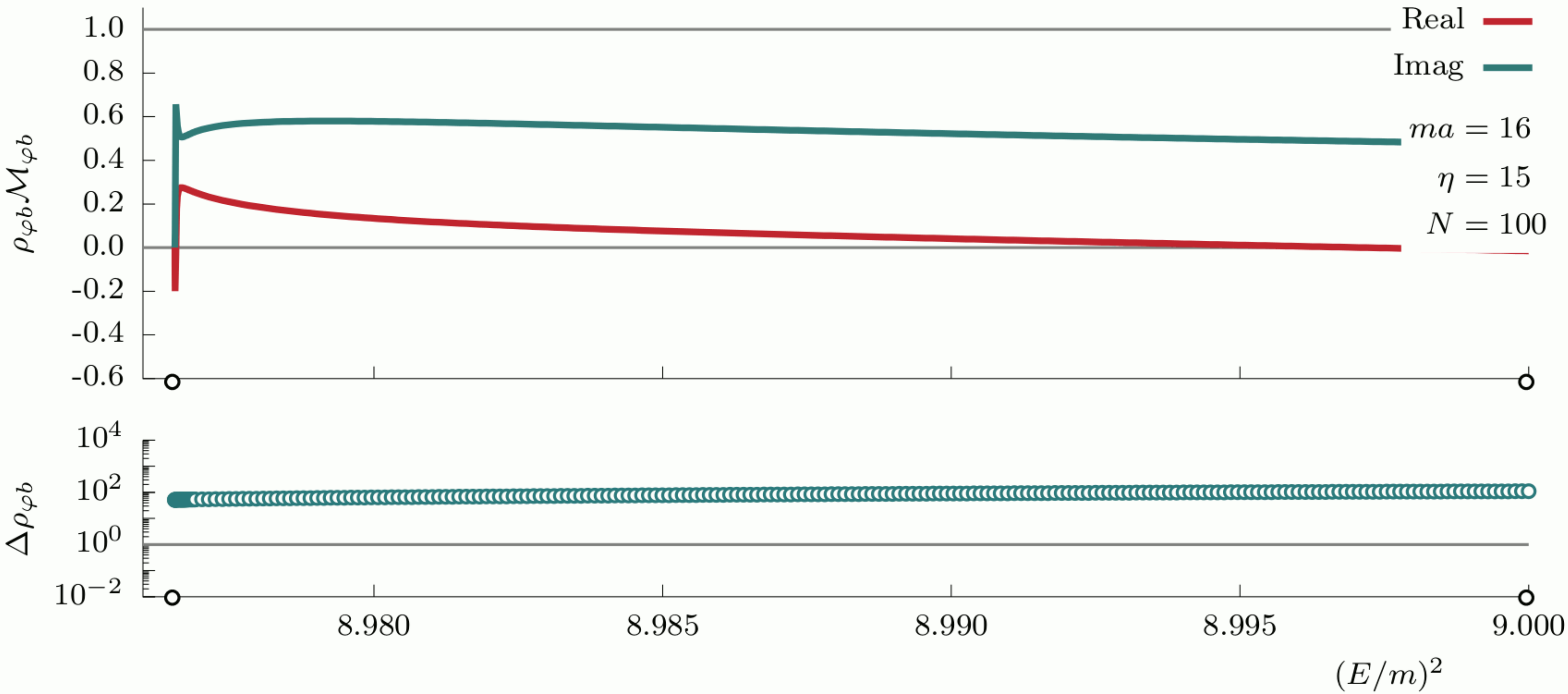
$\epsilon \to 0$ limit

Ensure $\epsilon \to 0$ through $N \to \infty$ limit

$$\left[\sum_x \Delta x - \int \mathrm{d}x\right] \frac{1}{x^2 - x_0^2 + i\epsilon} \sim e^{-2\pi\epsilon/\Delta x} \qquad \Longrightarrow \qquad \epsilon \propto \frac{\eta}{N}$$



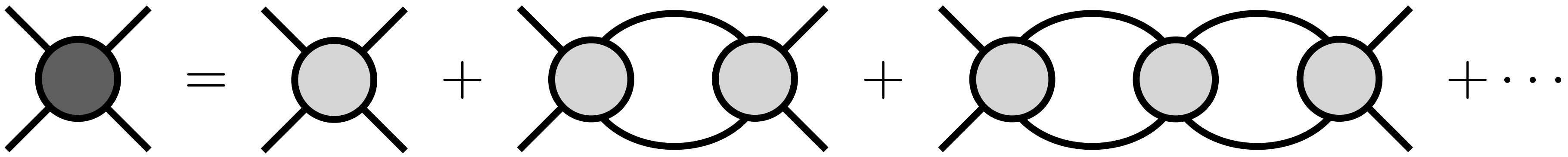
Evolution of solutions



On-shell scattering amplitudes from RFT

Sum to all orders in generic EFT all relevant cuts leading to singularities in physical region

- e.g. $\mathbf{2} \to \mathbf{2}$



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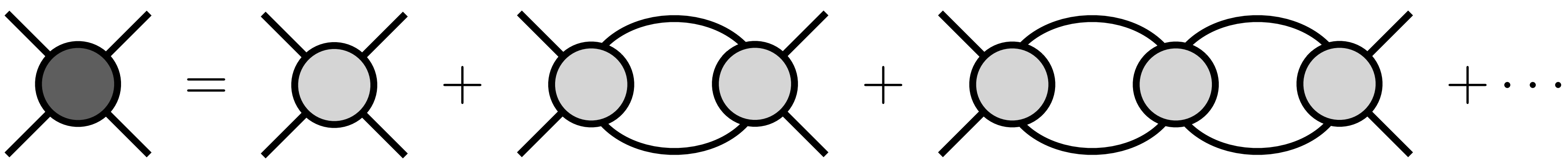

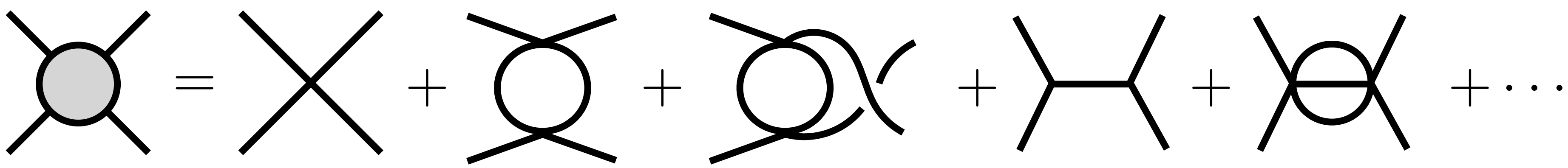

All 2PI diagrams - left hand cuts and higher multi-particle thresholds

On-shell scattering amplitudes from RFT

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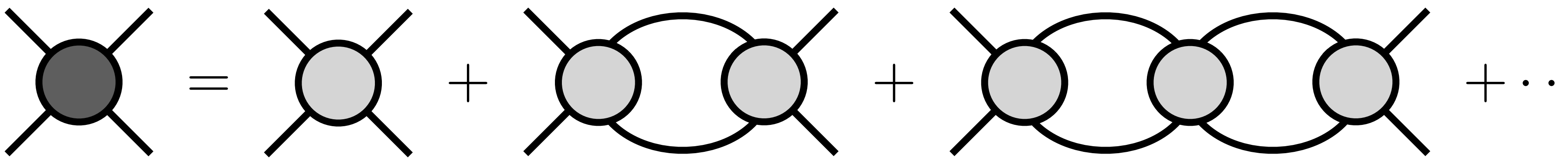
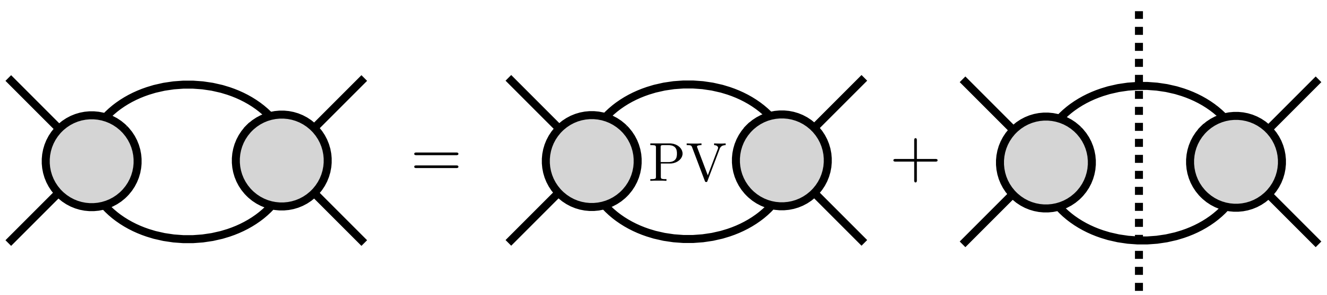


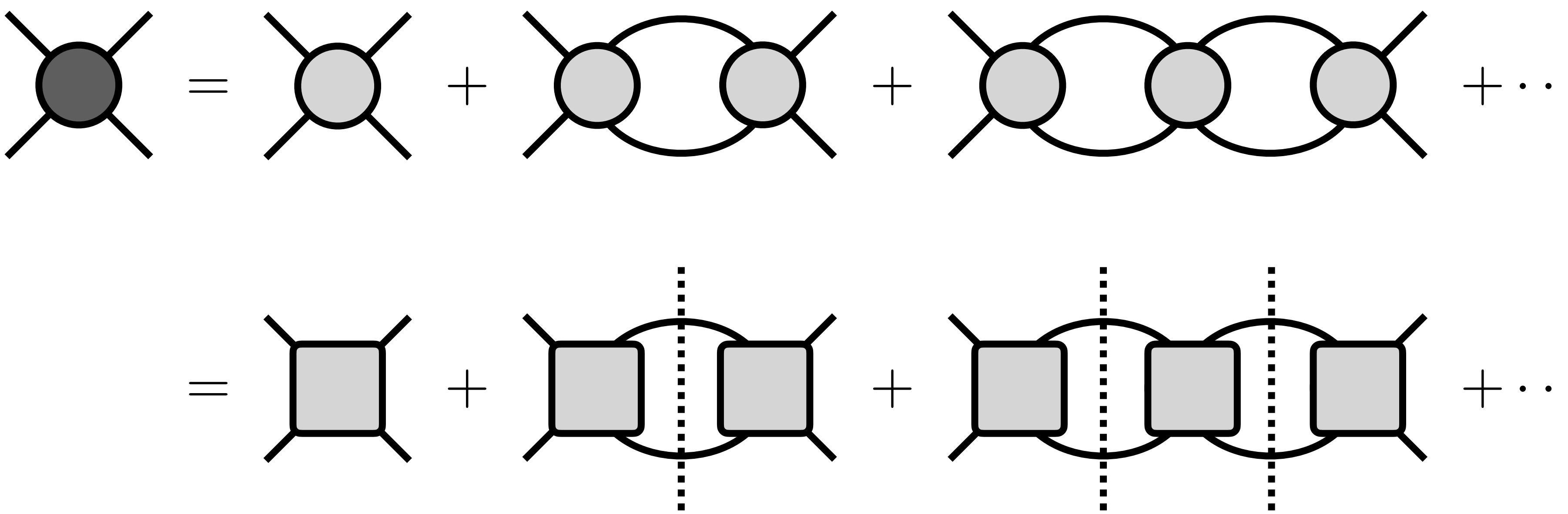


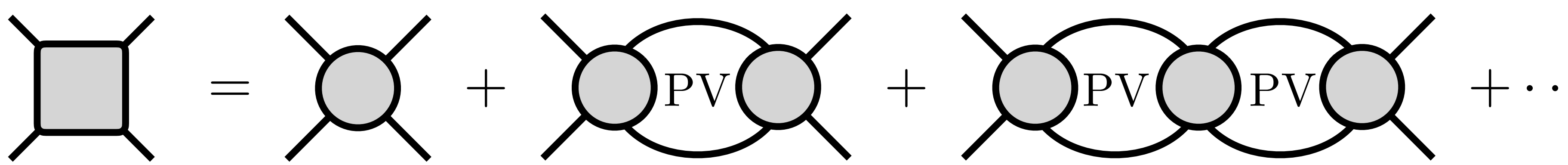
$$\rho = \frac{q}{8\pi E} \sim \sqrt{s - s_{\text{th}}}$$

On-shell scattering amplitudes from RFT

Sum to all orders in generic EFT all relevant cuts leading to singularities in physical region

- e.g. $\mathbf{2} \to \mathbf{2}$



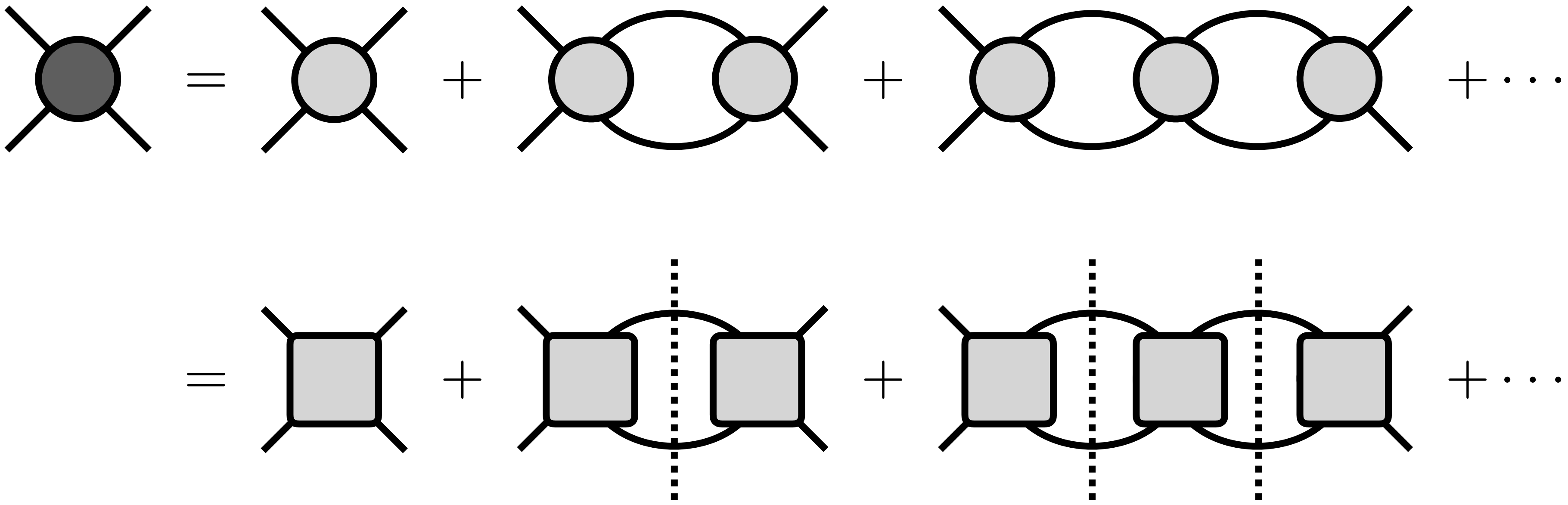


K matrix — unknown dynamical function unconstrained by unitarity

On-shell scattering amplitudes from RFT

Sum to all orders in generic EFT all relevant cuts leading to singularities in physical region

- e.g. $\mathbf{2} \to \mathbf{2}$



$$\mathcal{M}_2 = \mathcal{K}_2 + \mathcal{K}_2 i\rho\mathcal{M}_2$$

For given K matrix, obtain on-shell solution for amplitude

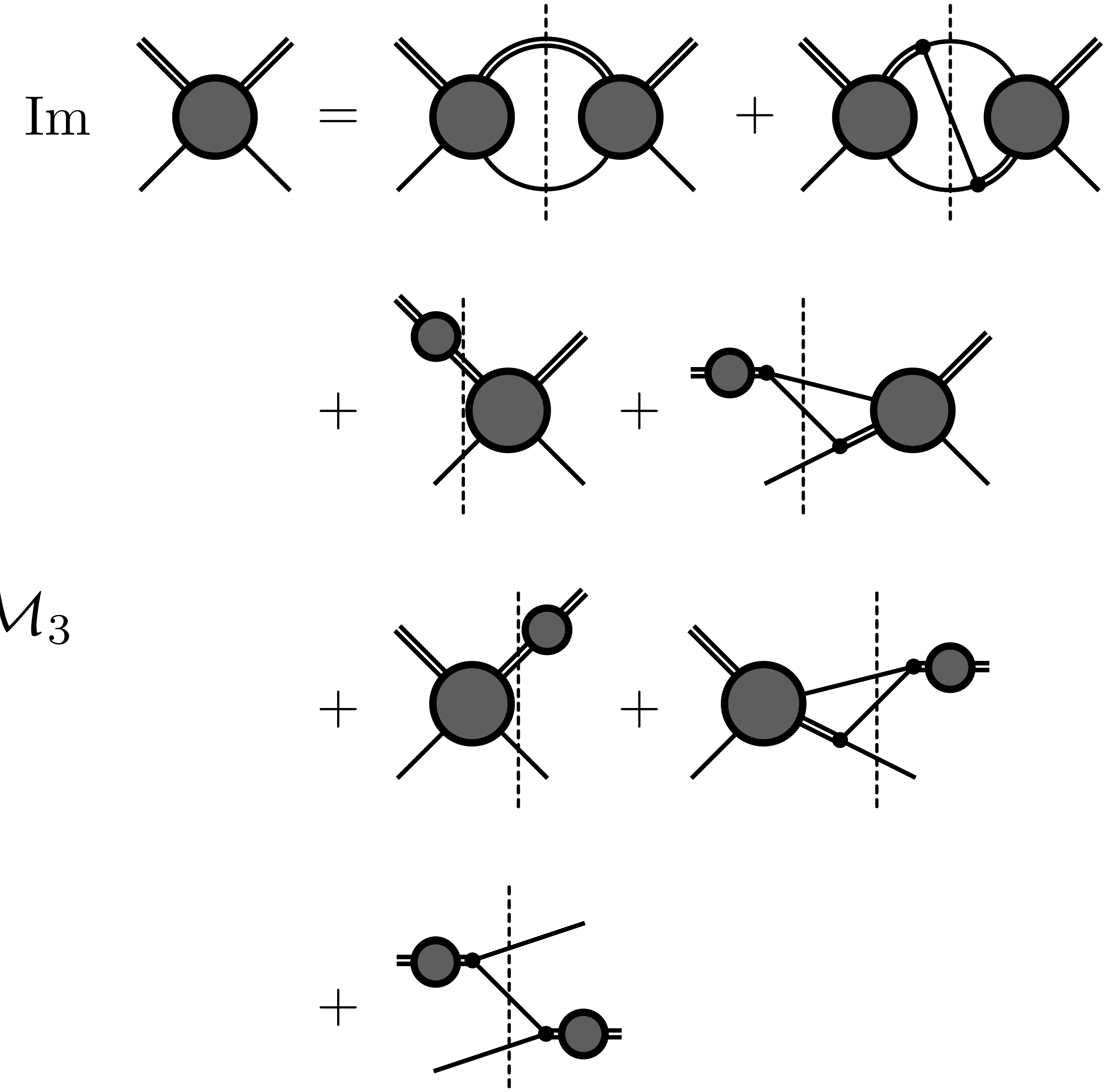
On-shell scattering amplitudes from unitarity

Start with S matrix unitarity

- Provides constrain for amplitude on real axis in physical region
- On-shell amplitude satisfies linear equation — check unitarity constraint

$$\mathcal{M}_3 = \mathcal{M}_2(\mathcal{R} - G)\mathcal{M}_2 + \int \mathcal{M}_2(\mathcal{R} - G)\mathcal{M}_3$$

R is a different short-distance function



M. Mai, B. Hu, M. Döring, A. Pilloni, and A. Szczepaniak
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M. Mikhasenko, AJ et al. [JPAC]
Phys. Rev. D **98**, 096021 (2018)

M. Mikhasenko, AJ et al. [JPAC]
JHEP **08**, 080 (2019)

Extension to FV

M. Mai and M. Döring
Eur. Phys. J. A **53**, 240 (2017)

Equivalence of relativistic methods

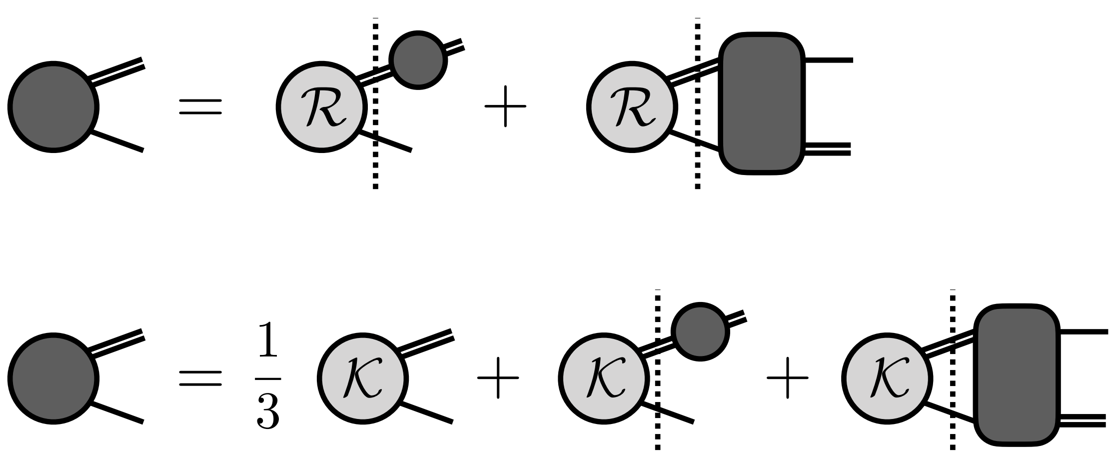
The RFT vs FVU methods can be summarized as “Bottom up” vs “Top down”

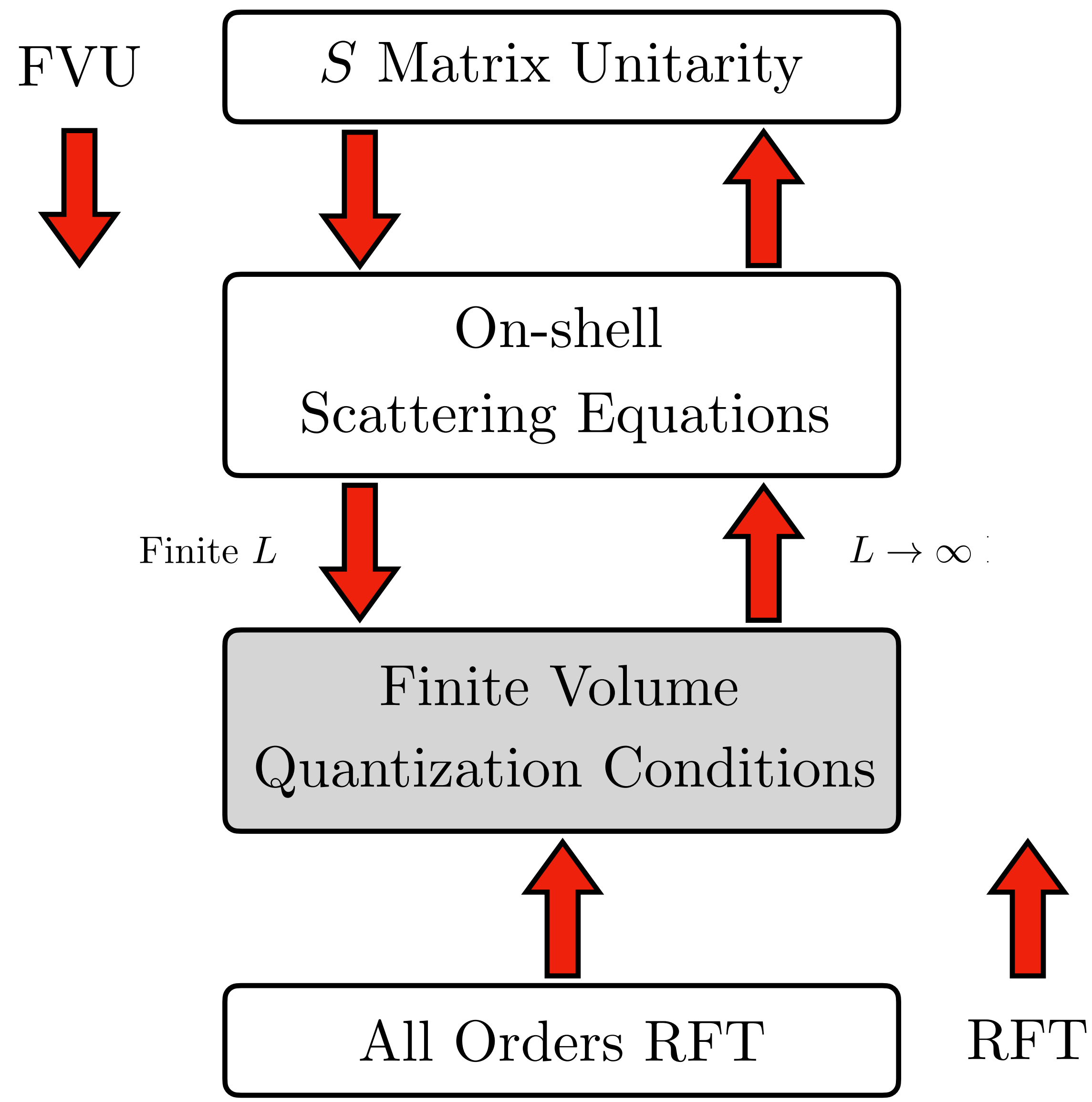
On-shell scattering equations are equivalent

AJ et al.
Phys. Rev. D **100**, 034508 (2019)

Quantization conditions are equivalent

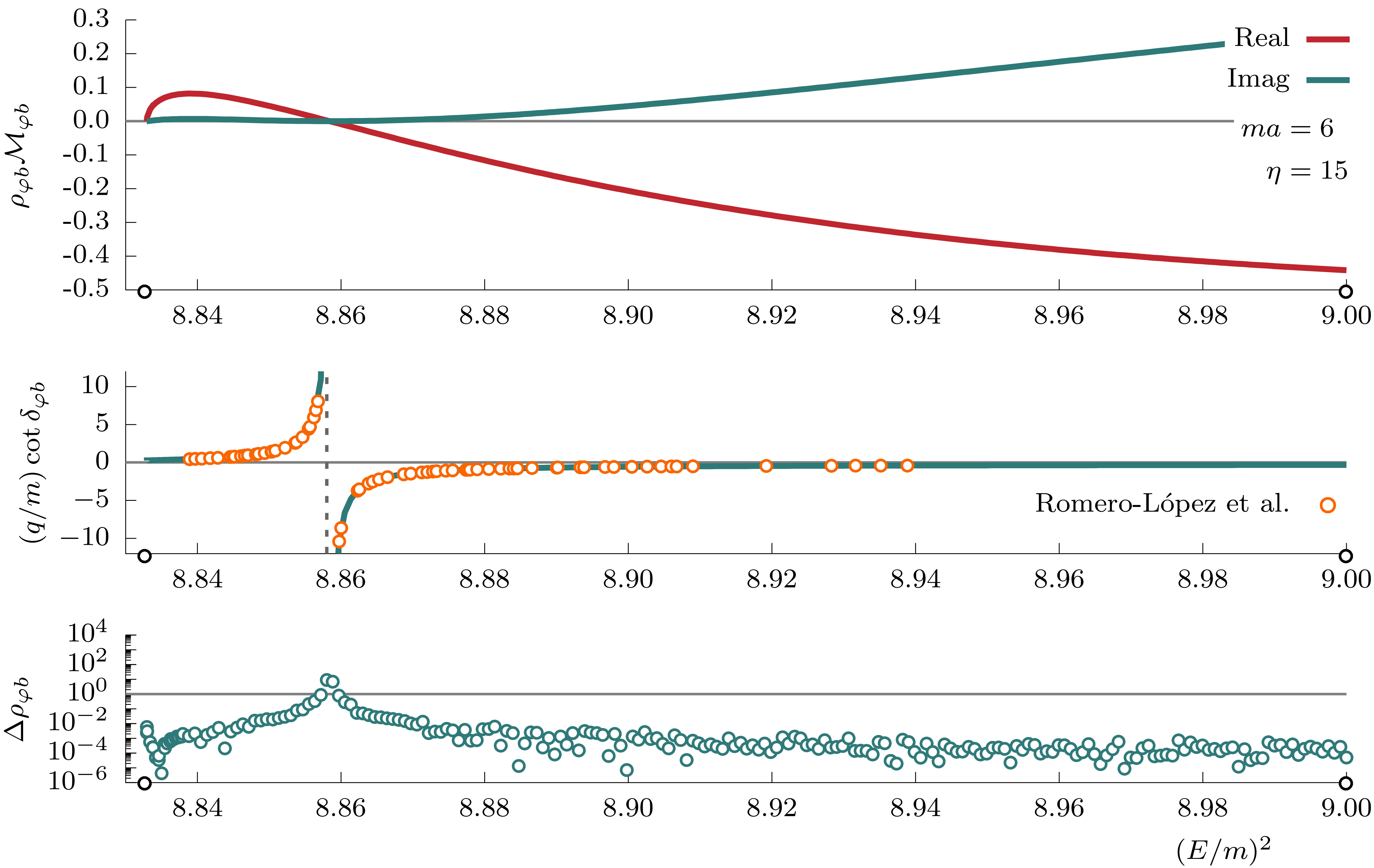
T. Blanton and S. Sharpe
arXiv:2007.16190 (2020)





Small details, e.g. aspects of symmetrization — see literature for information

Example of solution — Extrapolated result



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