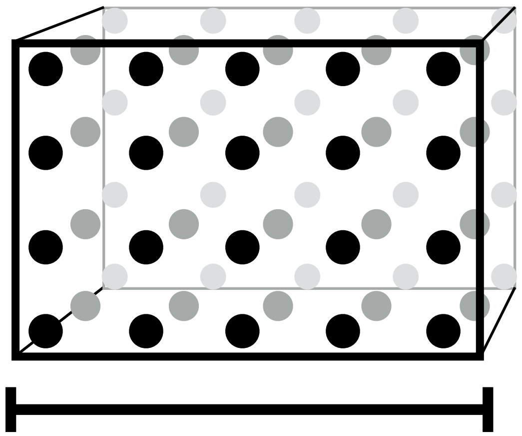
Three-pion scattering from lattice QCD

Maxwell T. Hansen

April 16th, 2021

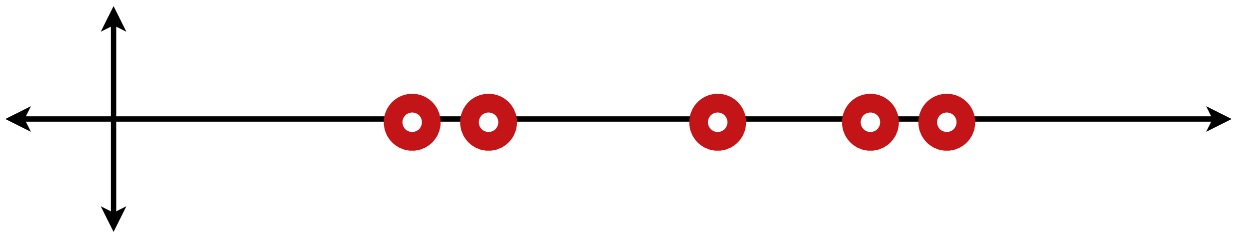
THE UNIVERSITY of EDINBURGH

Lattice QCD = QCD in a box

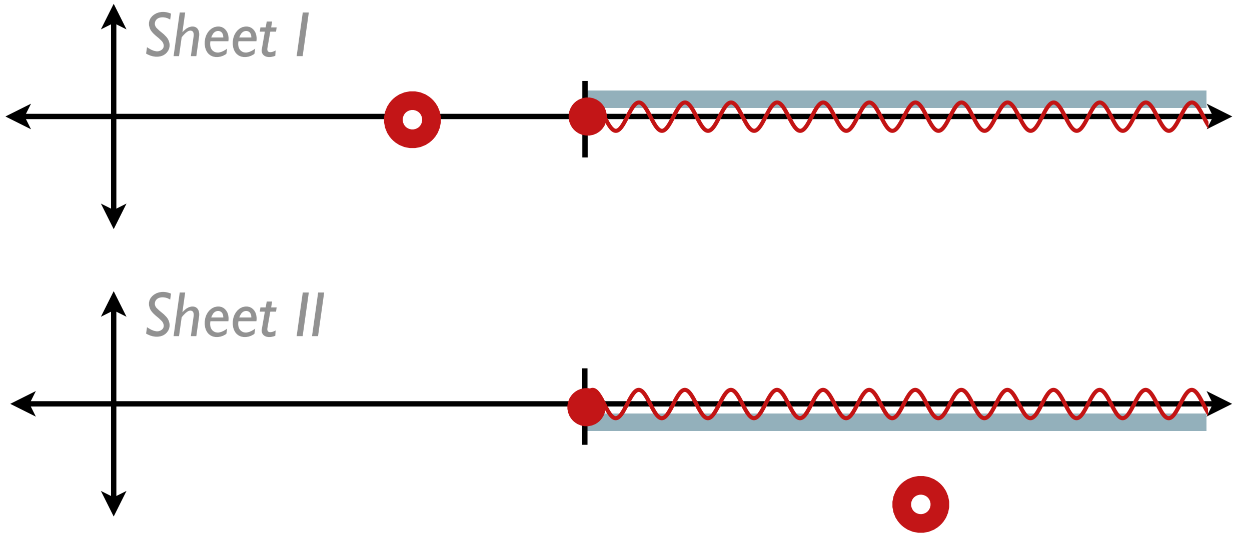


- The ***finite volume***…
  - ***Discretizes*** the spectrum
  - ***Eliminates*** the branch cuts and multiple sheets
  - ***Hides*** the resonance poles

Finite-volume analytic structure



Infinite-volume analytic structure



The volume as a tool

Our task is relate $E_n(L)$ and $\langle E_{m'}|\mathcal{J}(0)|E_m\rangle$ to *experimental observables*

Hadron resonances from lattice QCD *David Wilson*
17:00 - 17:30

On determining coupled-channel photoproduction amplitudes in finite-volume
Luka Leskovec

Elastic form-factors of resonances from lattice QCD
Felipe Ortega-Gama

Decays of an exotic 1^{-+} hybrid meson resonance in QCD
Jozef Dudek

$D K$ and $D \bar{K}$ scattering and the $D_{s0}(2317)$ from lattice QCD
Christopher Thomas

The lightest $D_0^\star$ resonance from QCD
Nicolas Lang

Two-nucleon interactions from lattice QCD with a variational method: where are the bound states
Andrew Hanlon

The excited and exotic bottomonium spectrum from latti QCD
Sinead Ryan

Radiative Transitions in Charmonium from Lattice QCD
James Delaney

Excited J^{--} meson resonances at the SU(3) flavor point from lattice QCD
Christopher Johnson

Integral equations for relativistic three-hadron scattering
Andrew Jackura

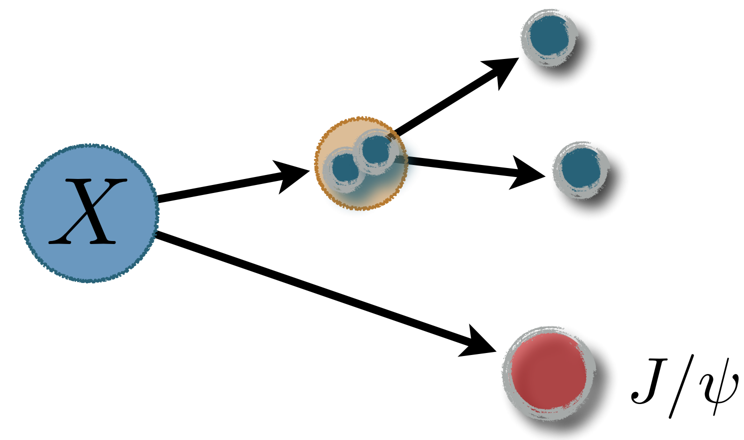
Three-body interactions from the finite-volume QCD spectrum
Ruairí Brett

Long-range processes in QCD *Raul Briceno*
19:50 - 20:10

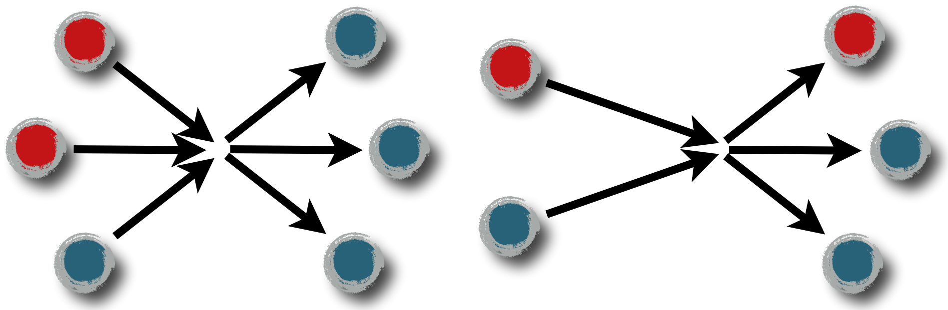
3-particle amplitudes

2-to-2 only samples J^P 0^+ 1^- $2^+ \cdots$

many interesting resonances have significant 3-body decays



Goal: ***finite-volume + unitarity formalism*** for generic two- and three-particle systems



Applications…

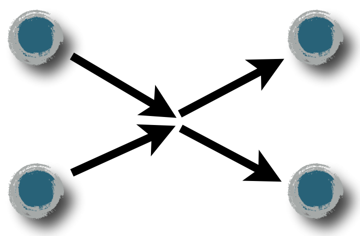
exotic resonance pole positions, couplings, quantum numbers

$\omega(782),\, a_1(1420) \to \pi\pi\pi$ $\quad X(3872) \to J/\psi\pi\pi$ $\quad N(1440) \to N\pi, N\pi\pi$

form factors and transitions

and much more!… *(3-body forces, weak transitions, gluons content)*

Complication: degrees of freedom

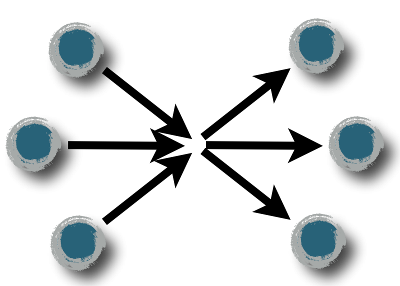


12 momentum components

-10 Poincaré generators

2 degrees of freedom

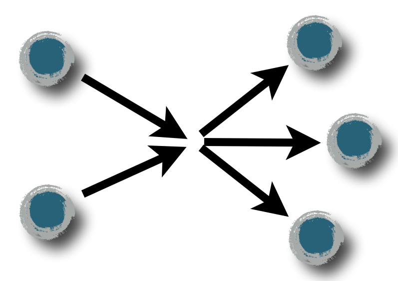
$$\vec{p}_1 + \vec{p}_2 \rightarrow \vec{p}_3 + \vec{p}_4 \longrightarrow \text{Mandelstam } s, t$$



18 momentum components

-10 Poincaré generators

8 degrees of freedom



15 momentum components

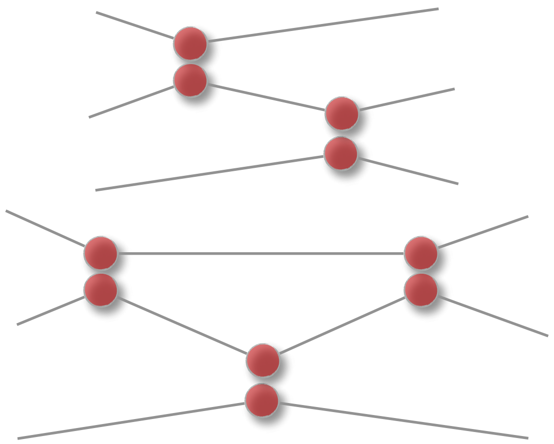
-10 Poincaré generators

5 degrees of freedom

Complication: on-shell states

- Classical pairwise scattering

for $m_1 = m_2 = m_3$ up to ***3*** binary collisions are possible



Dispersion Relations for Three-Particle Scattering Amplitudes. I*

$$b = \frac{(m_1+m_3)(m_2+m_3)}{m_1 m_2}$$

It follows that if

$$b^{n-2}(b-1) > 1, \qquad \text{(IV.18)}$$

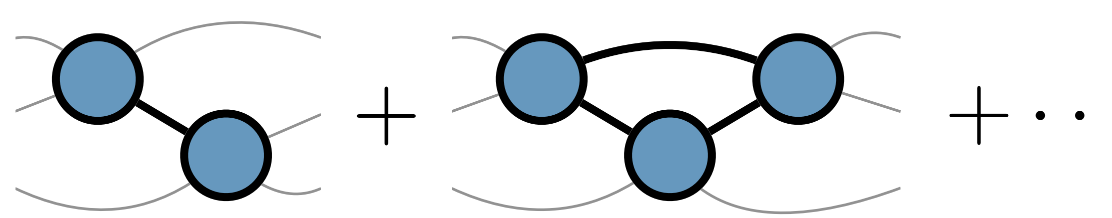
then $2n+1$ successive binary collisions are kinematically impossible.

$m_1 = m_2 = m_3 - \varepsilon$:
4 collisions possible
$\pi\pi K$

$b < 2$
5 collisions possible
$\pi K K$

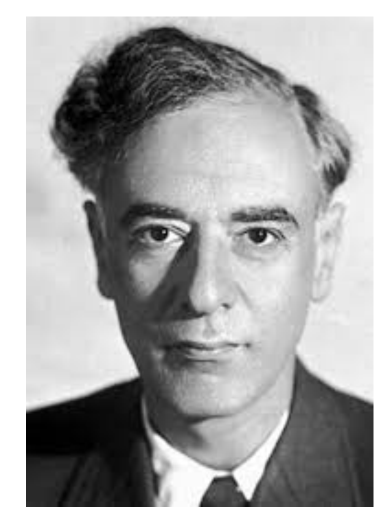
- Correspond to Landau singularities

$i\mathcal{M}_{3\to3} \equiv$ fully connected correlator $=$



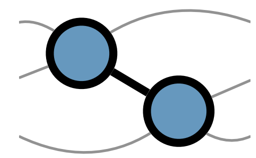
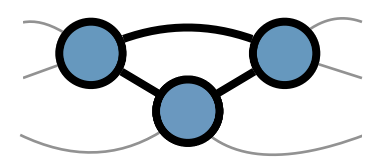
complicate analyticity & unitarity

difficult to disentangle kinematic singularities from resonance poles



Two key observations

- Intermediate $K_{\mathrm{df},3}$ removes singularities

$\mathcal{K}_{\mathrm{df},3} \equiv$ fully connected diagrams w/ ***PV pole prescription*** $-$  $+$  $+\cdots$

same degrees of freedom as *M3*

smooth real function

relation to *M3* = known

- $K_{\mathrm{df},3}$ has a systematic low-energy expansion

$$\mathcal{K}_{\mathrm{df},3}(p_3,p_2,p_1;k_3,k_2,k_1) = \mathcal{K}_{\mathrm{df},3}^{\mathrm{iso},0} + \mathcal{K}_{\mathrm{df},3}^{\mathrm{iso},1}\,\Delta + \cdots \qquad \Delta = \frac{s-(3m)^2}{(3m)^2}$$

analogous to effective range expansion

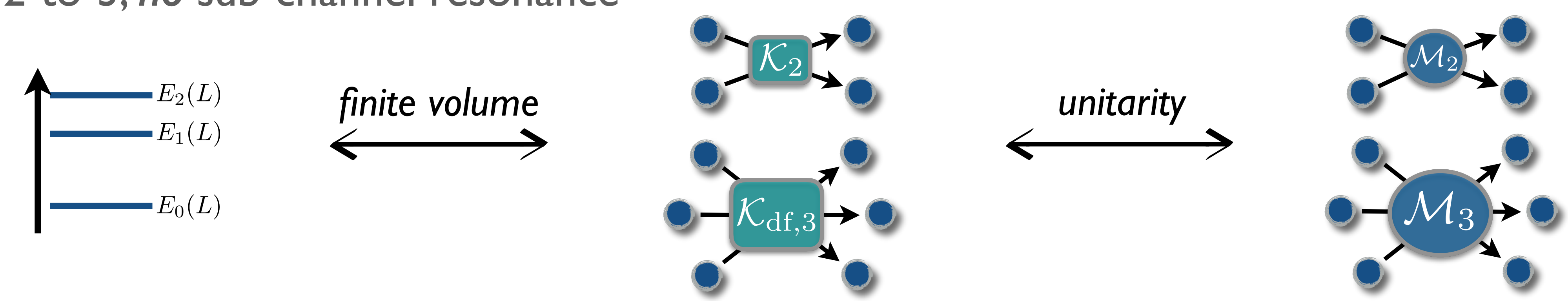
$$p\cot\delta = -\frac{1}{a} + \frac{1}{2}\,r\,p^2 + \mathcal{O}(p^4)$$

gives handle on many *degrees of freedom*
(DOFs enter order by order)

Status…

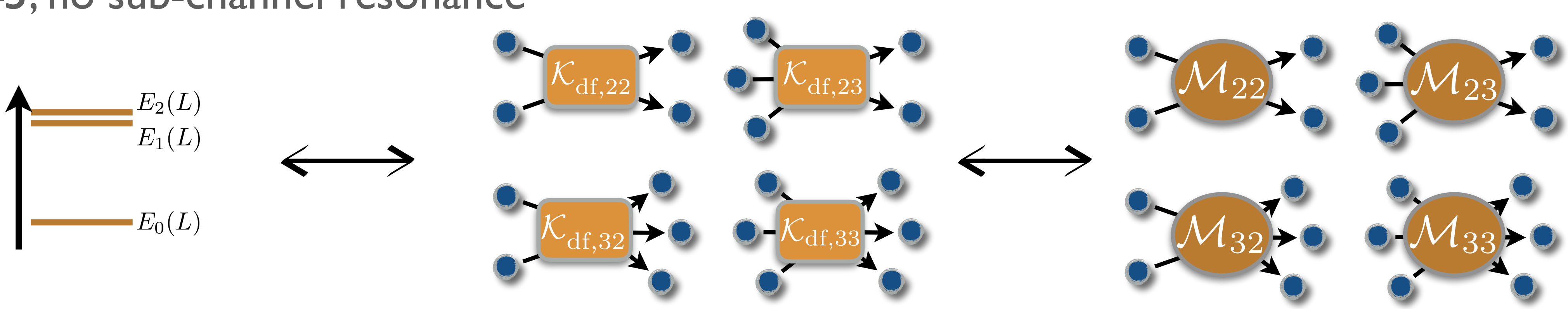
- General relation between *energies* and *two-and-three scalar scattering*

No **2-to-3,** ***no*** **sub-channel resonance**



• MTH, Sharpe (2014, 2015) •

2-to-3**, no sub-channel resonance**



• Briceño, MTH, Sharpe (2017) •

Including sub-channel resonances + *different isospins* + *non-degenerate*

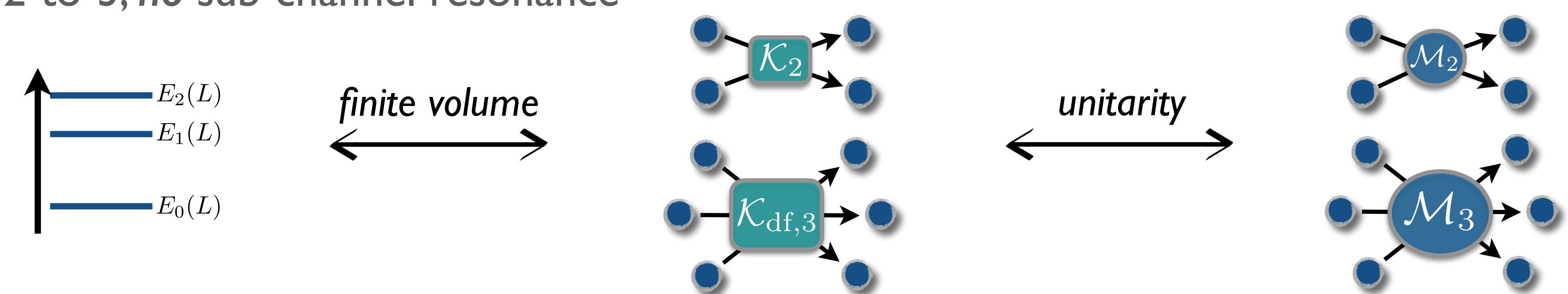
$$\pi\pi\pi \longrightarrow \rho\pi \longrightarrow \omega \longrightarrow \rho\pi \longrightarrow \pi\pi\pi$$

• Briceño, MTH, Sharpe (2018) • MTH, Romero-López, Sharpe (2020) • Blanton, Sharpe (2020)

Status…

- [ ] General relation between ***energies*** and ***two-and-three scalar scattering***

No 2-to-3, ***no*** sub-channel resonance



• MTH, Sharpe (2014, 2015) •

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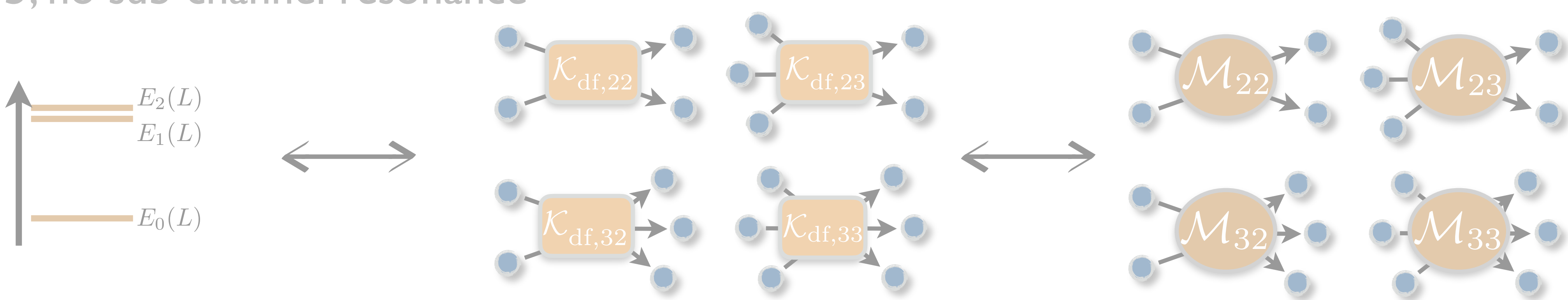


• Briceño, MTH, Sharpe (2017) •

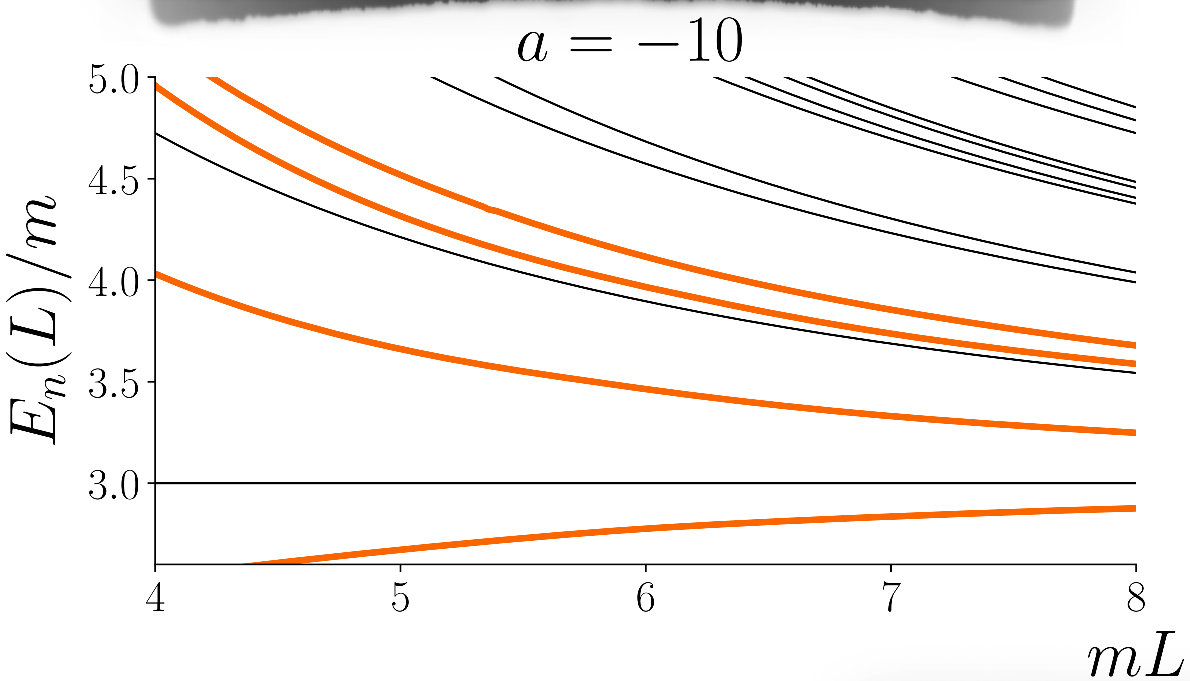
Including sub-channel resonances + ***different isospins*** + ***non-degenerate***

$$\pi\pi\pi \rightarrow \rho\pi \rightarrow \omega \rightarrow \rho\pi \rightarrow \pi\pi\pi$$

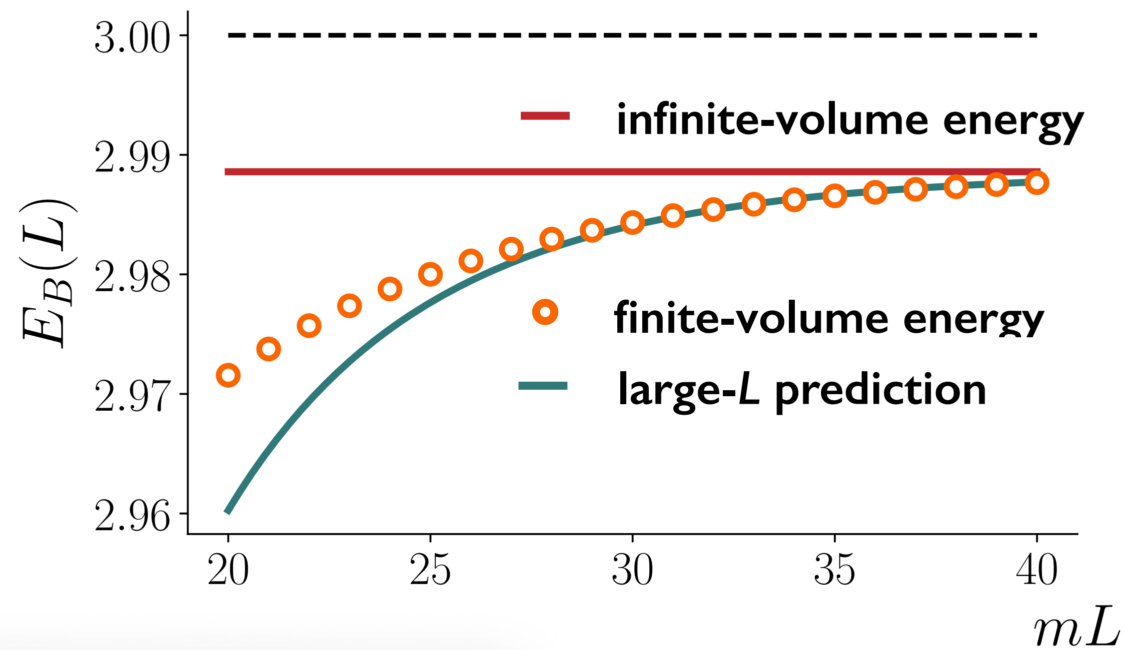
• Briceño, MTH, Sharpe (2018) • MTH, Romero-López, Sharpe (2020) • Blanton, Sharpe (2020)

Many toy results

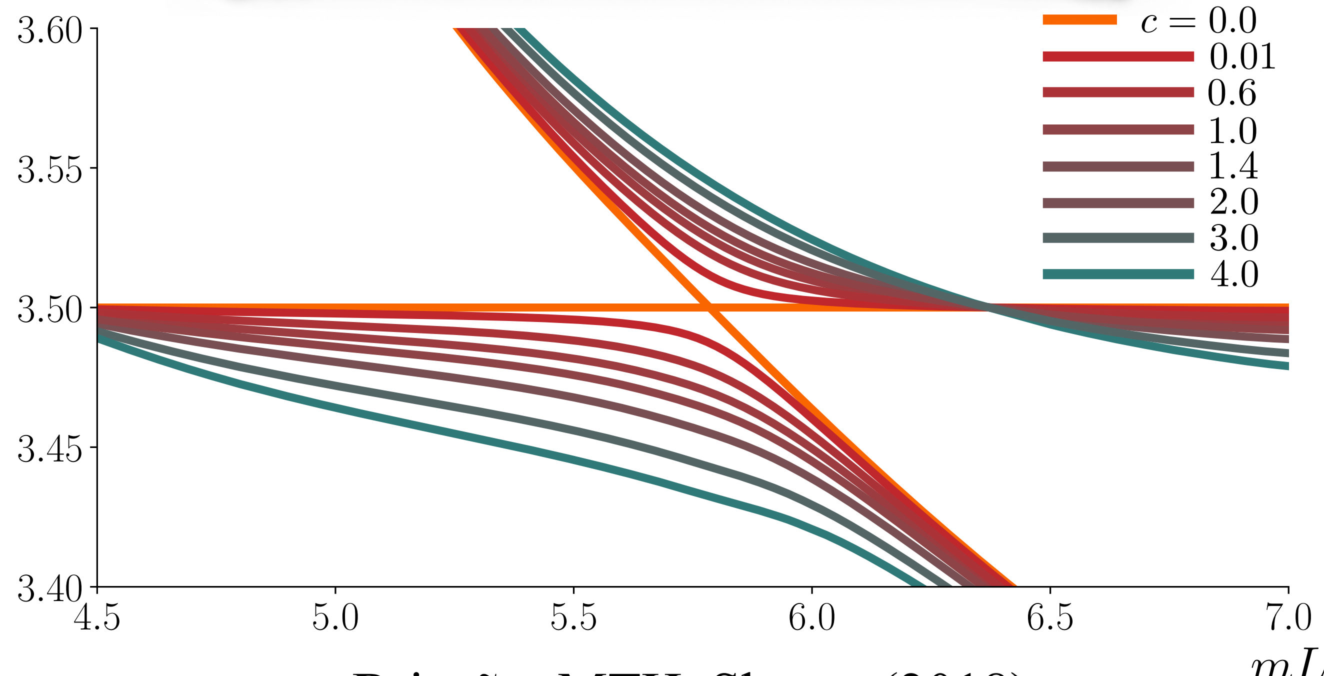
Spectrum with no 3-particle interaction



Finite-volume effects on a 3-particle bound state



Model of a 3-particle resonance



• Briceño, MTH, Sharpe (2018) •

Related work

- Finite-volume unitarity method

  Döring, Mai (2016,2017)

  Gives connection to unitarity relations

- Non-relativistic EFT method

  Hammer, Pang, Rusetsky (2017)

  Simplified derivation + integral equations

Do not yet include non-identical and non-generate, angular momentum mixing, 2-to-3

- All methods

  Rely on intermediate, scheme-dependent quantity

  Hold up to e^{-mL} and for $E_3^\star < 5m_\pi$

  Equivalent where comparable

Review: ***Lattice QCD and Three-particle Decays of Resonances***
MTH and Sharpe, *1901.00483*

Not covered here

- Activity extracting and fitting three-hadron energies

• Hörz, Hanlon •
• Blanton, Romero-López, Sharpe •
• Alexandru, Brett, Culver, Döring, Guo, Lee, Mai •
• Fischer, Kostrzewa, Liu, Romero-López, Ueding, Urbach •

- Activity connecting and extending formalisms

Relating infinite-volume equations • Jackura *et al.* (2019) •

Alternative derivations • Blanton, Sharpe (2020) •

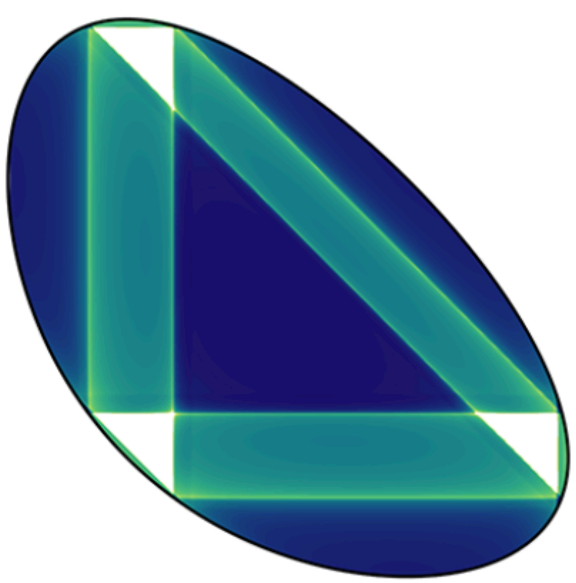
Equivalence of formalisms (where comparable) • Blanton, Sharpe (2020) •

Three-hadron decays • Müller, Rusetsky (2021) • MTH, Romero-López, Sharpe (2021) •

Energy-Dependent $\pi^+\pi^+\pi^+$ Scattering Amplitude from QCD

Maxwell T. Hansen,[1,2,*] Raul A. Briceño,[3,4,†] Robert G. Edwards,[3,‡]
Christopher E. Thomas,[5,§] and David J. Wilson[5,‖]

(for the Hadron Spectrum Collaboration)



EDITORS' SUGGESTION

Energy-Dependent $\pi^+ \pi^+ \pi^+$ Scattering Amplitude from QCD

A three-hadron scattering amplitude is computed using lattice QCD for the first time.

Maxwell T. Hansen *et al.*

Phys. Rev. Lett. **126**, 012001 (2021)

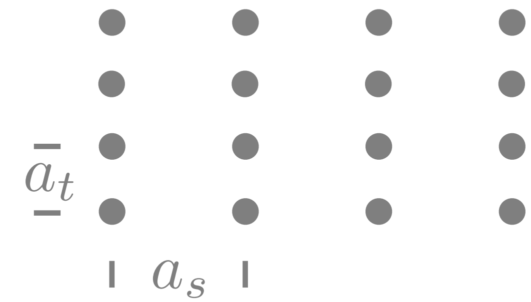
$\pi^+\pi^+\pi^+ \rightarrow \pi^+\pi^+\pi^+$

lattice details

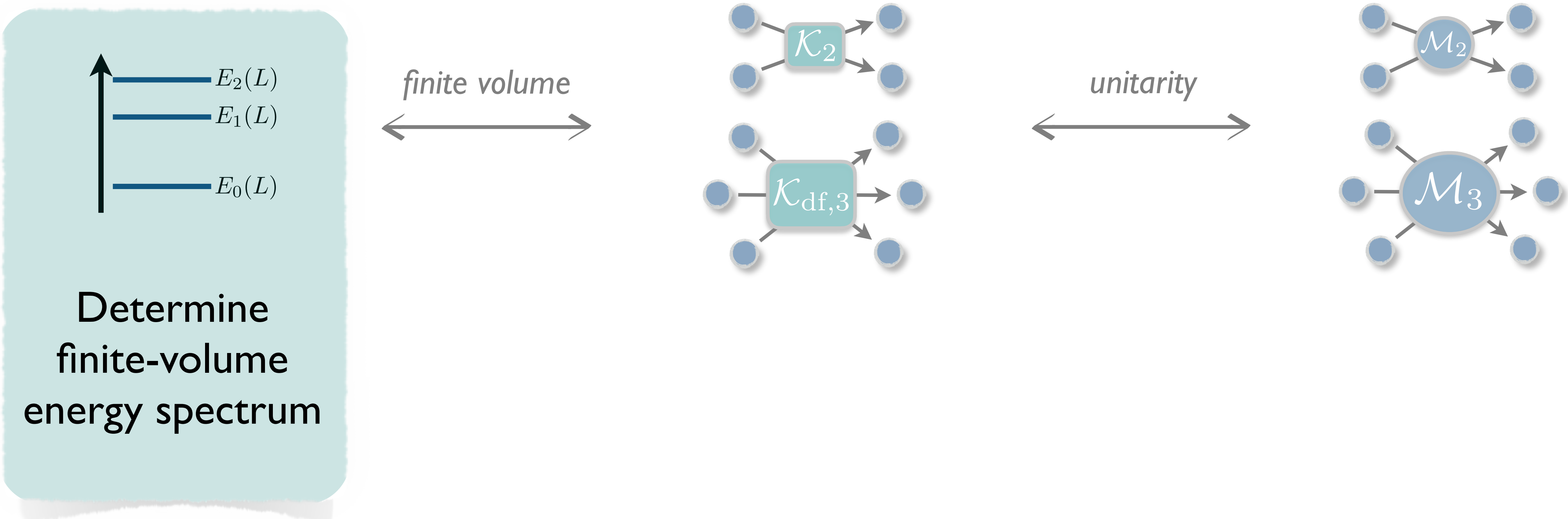
$N_f = 2+1 \qquad a_s/a_t = 3.444(6)$

$m_\pi \approx 400\text{MeV} \qquad a_s \approx 0.12\text{fm}$

$L_s/a_s = 20, 24$



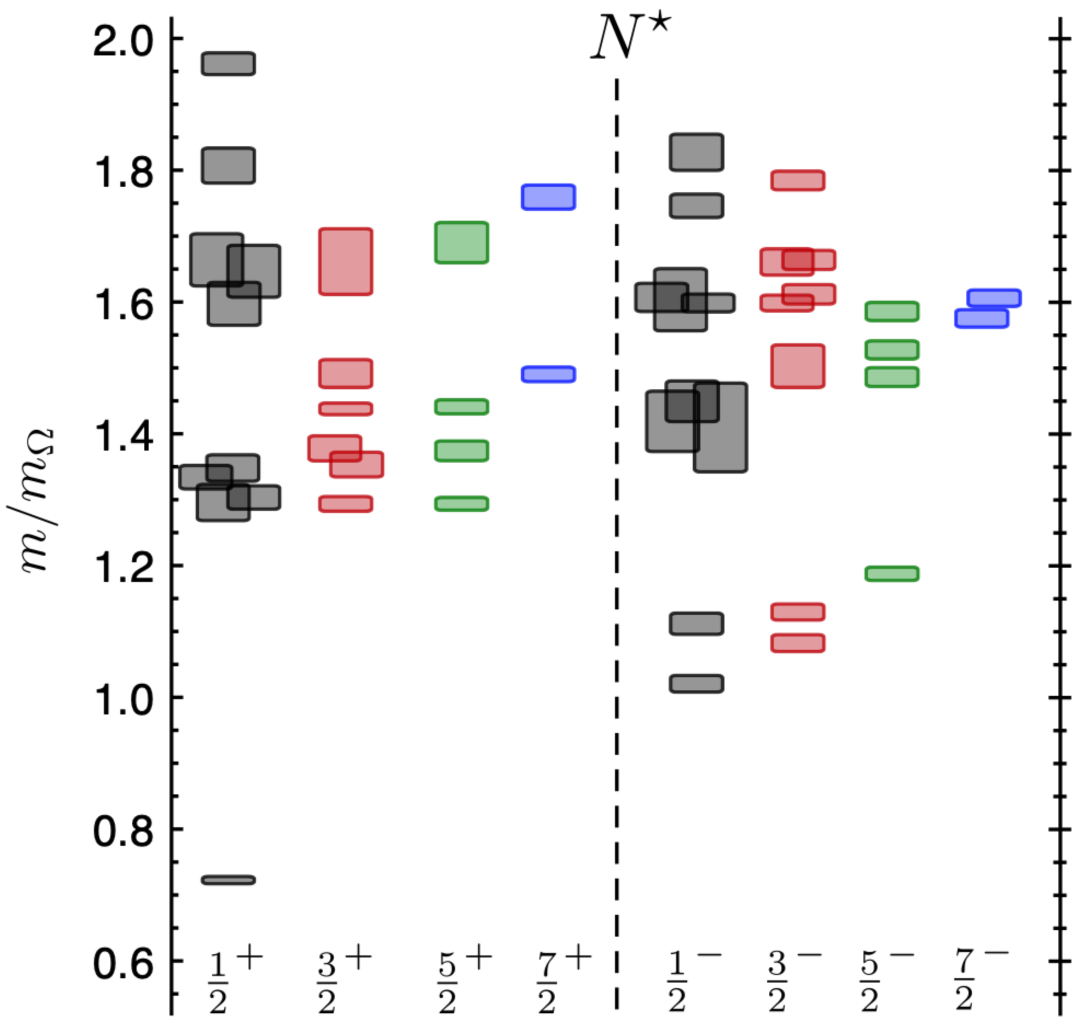
Workflow outline



MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001

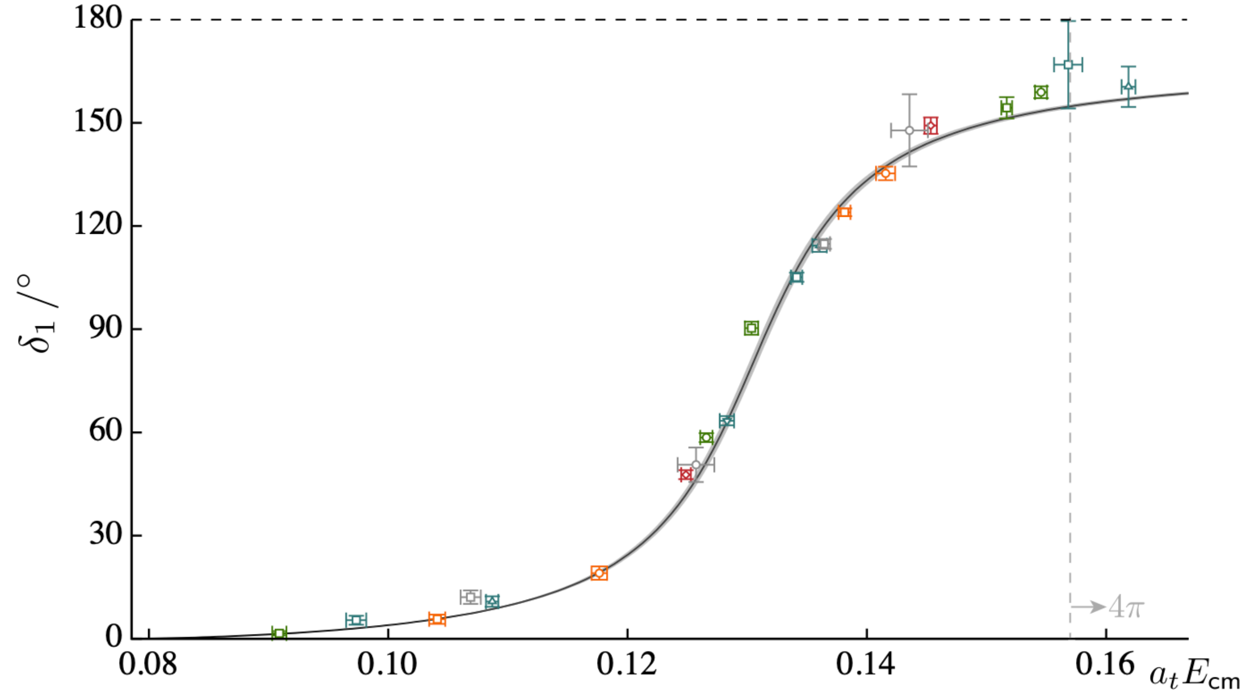
Two types of spectroscopy

Explore the spectrum of compact
QCD excited states
(via quark-model inspired operators)



Edwards, Dudek, Richards, Wallace (2011)

Extract the honest finite-volume
energy spectrum



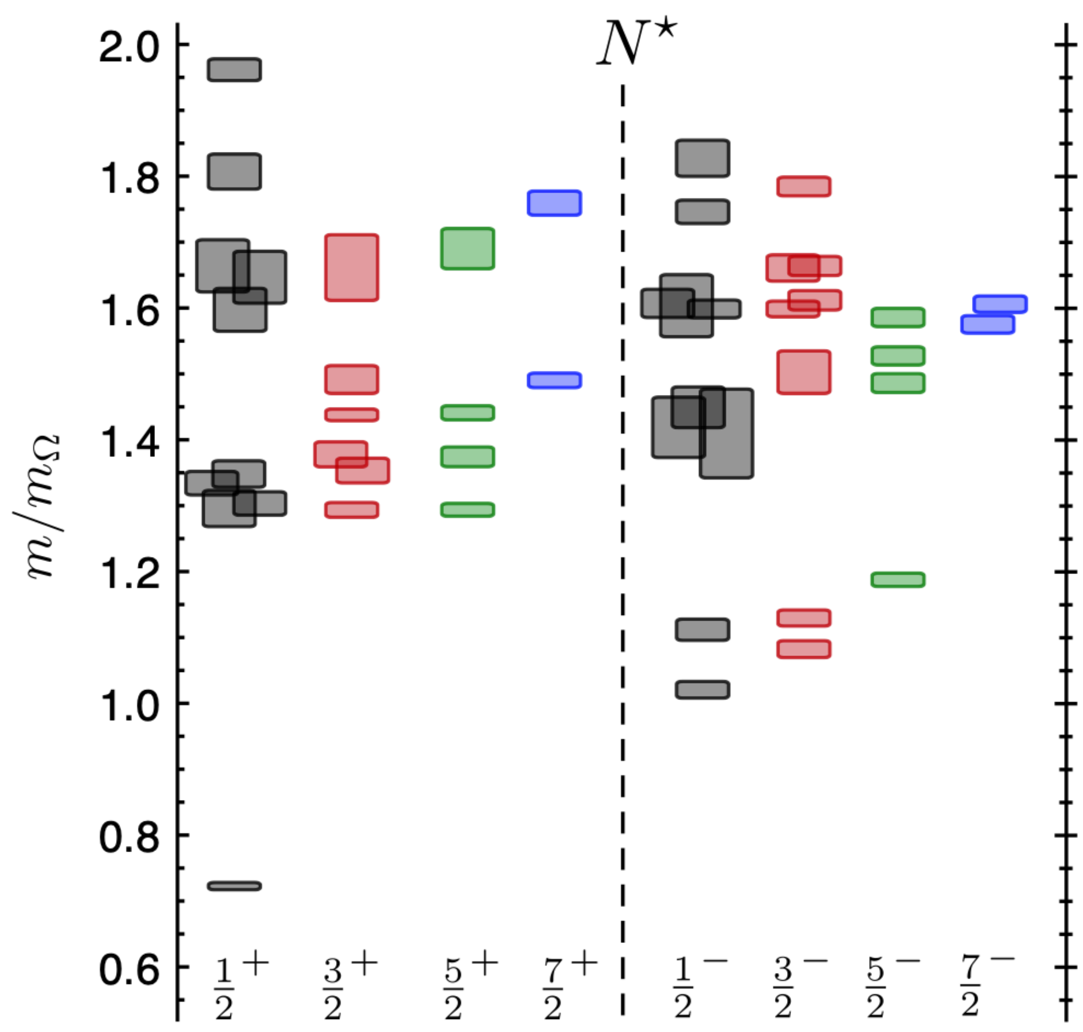
Wilson, Briceño, Dudek, Edwards, Thomas (2015)

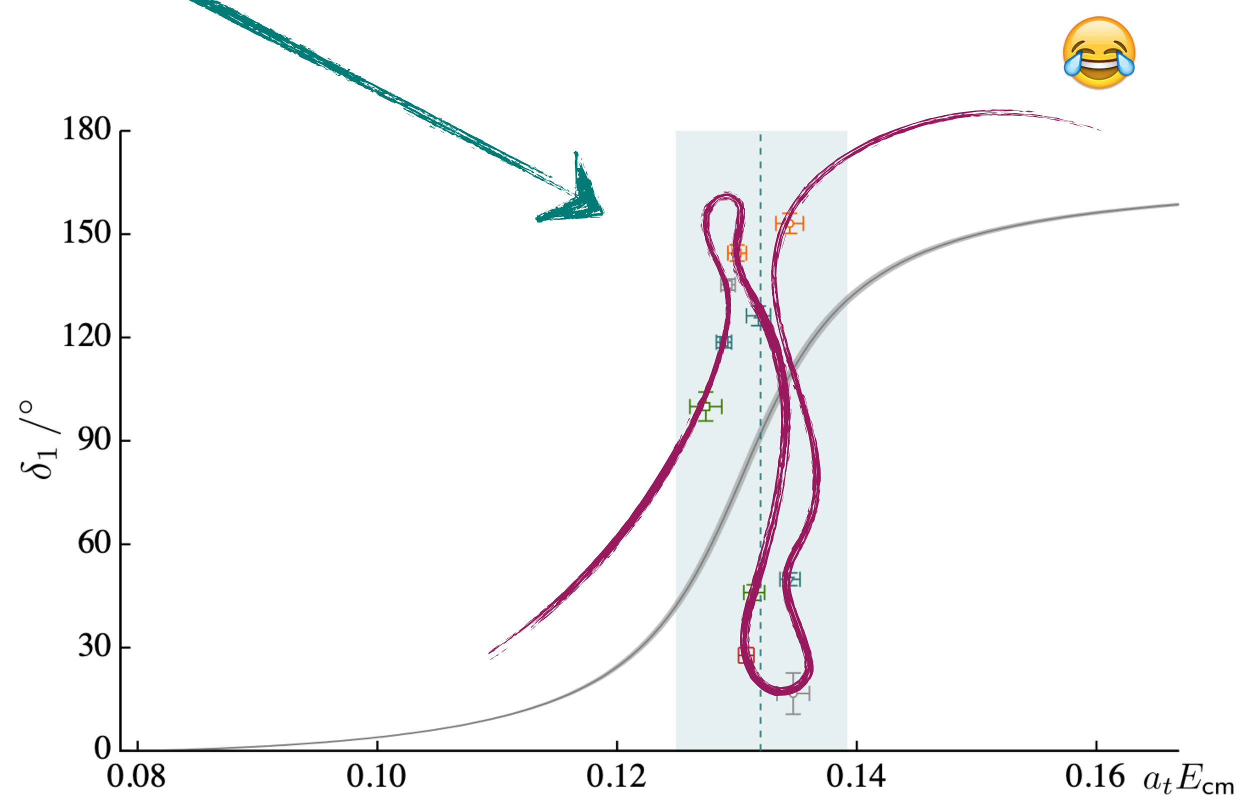
Two types of lattice spectroscopy

Explore the spectrum of compact QCD excited states (via quark-model inspired operators)

Extract the honest finite-volume energy spectrum

local operator spectrum = *not suitable for phase shift extraction*





Edwards, Dudek, Richards, Wallace (2011)

Wilson, Briceño, Dudek, Edwards, Thomas (2015)

Extracting the finite-volume spectrum

- Derivatives + gamma matrices + smearing → *basis of single-hadron operators*

$$\overline{q}\Gamma q, \quad \overline{q}\Gamma D q, \quad \overline{q}\Gamma D \cdots D q$$

- Variational method → *optimized single hadron*

$$\pi = c_1\, \overline{q}\Gamma q + c_2\, \overline{q}\Gamma D q + \cdots$$

- Group theory + individual momentum projection → *two- and three-pion operators*

$$(\pi\pi\pi)(\boldsymbol{P}, \Lambda) = \sum \mathrm{CG}\, \pi(\boldsymbol{p}_1)\pi(\boldsymbol{p}_2)\pi(\boldsymbol{p}_3)$$

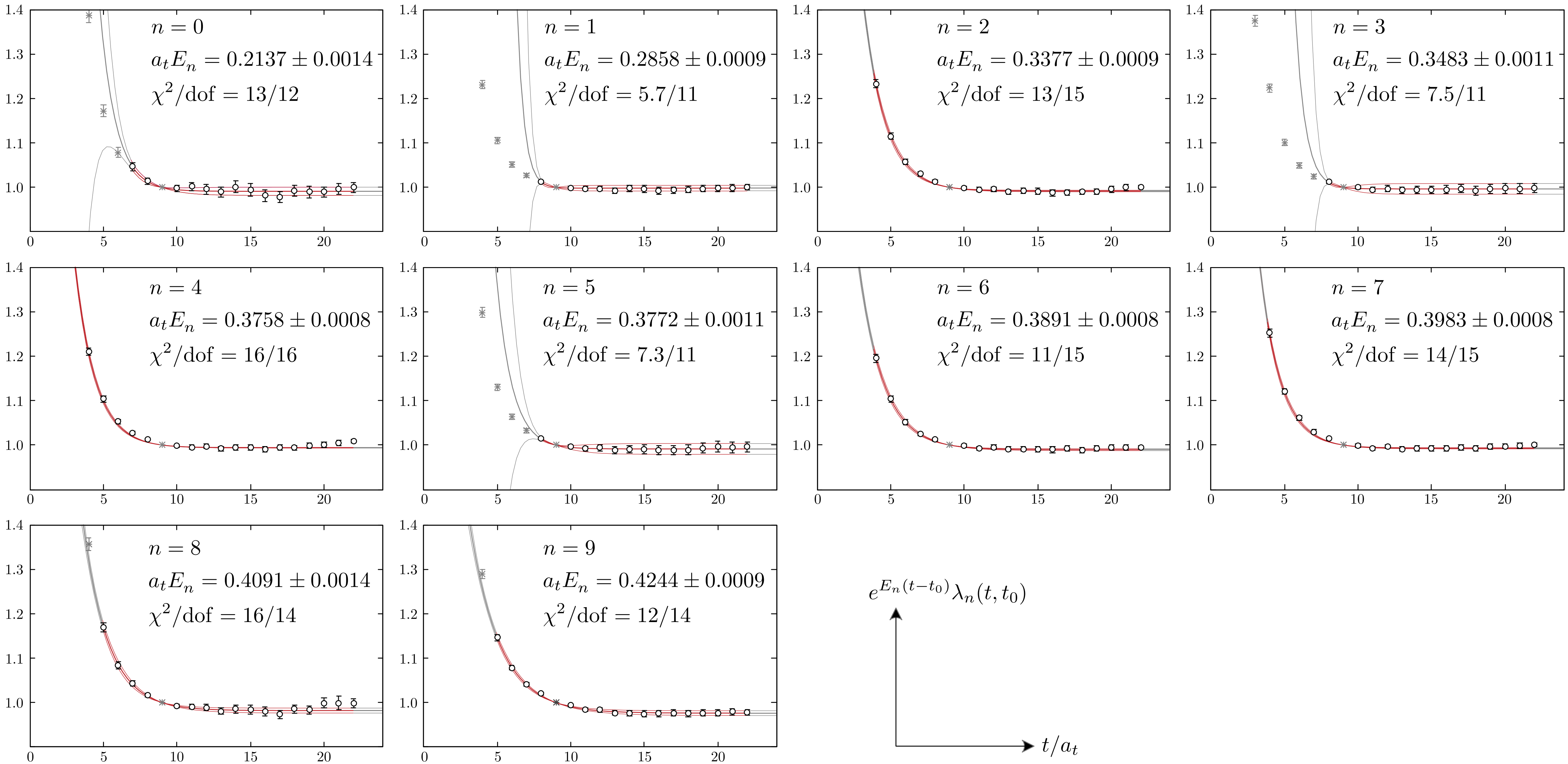
- Second variational method → *multi-pion finite-volume energies*

- Validate extraction…
  - Quality of energy plateaus
  - Stability under change of operators
  - Consistent with finite-volume formalism

Brought to you by distillation!

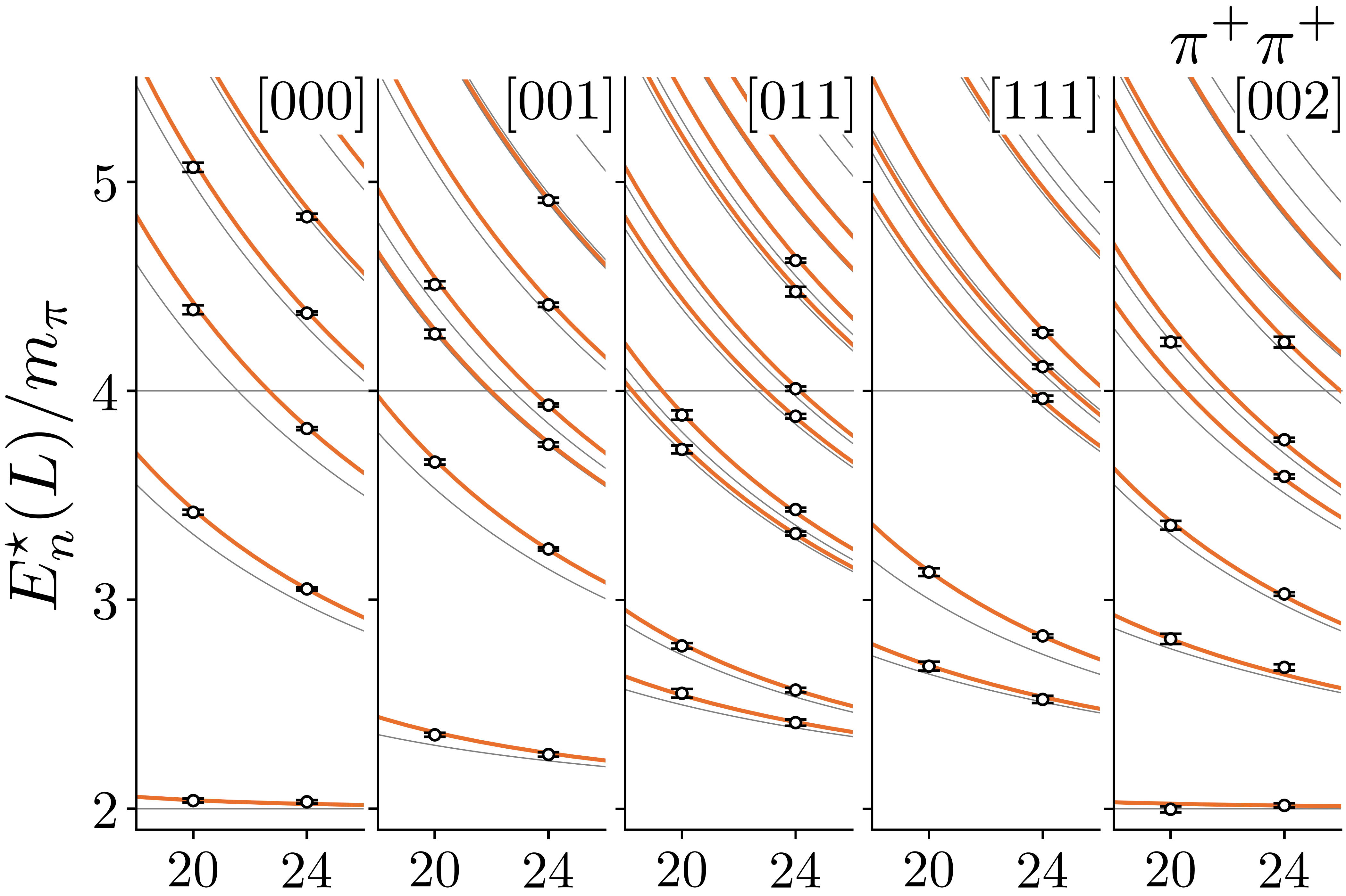
Peardon *et al.* (2009)

$I = 3\,(\pi^+\pi^+\pi^+),\ \boldsymbol{P} = [000],\ \Lambda = A_1^-,\ L/a_\mathsf{s} = 24$



MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001

$\pi^+\pi^+$ energies



MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001

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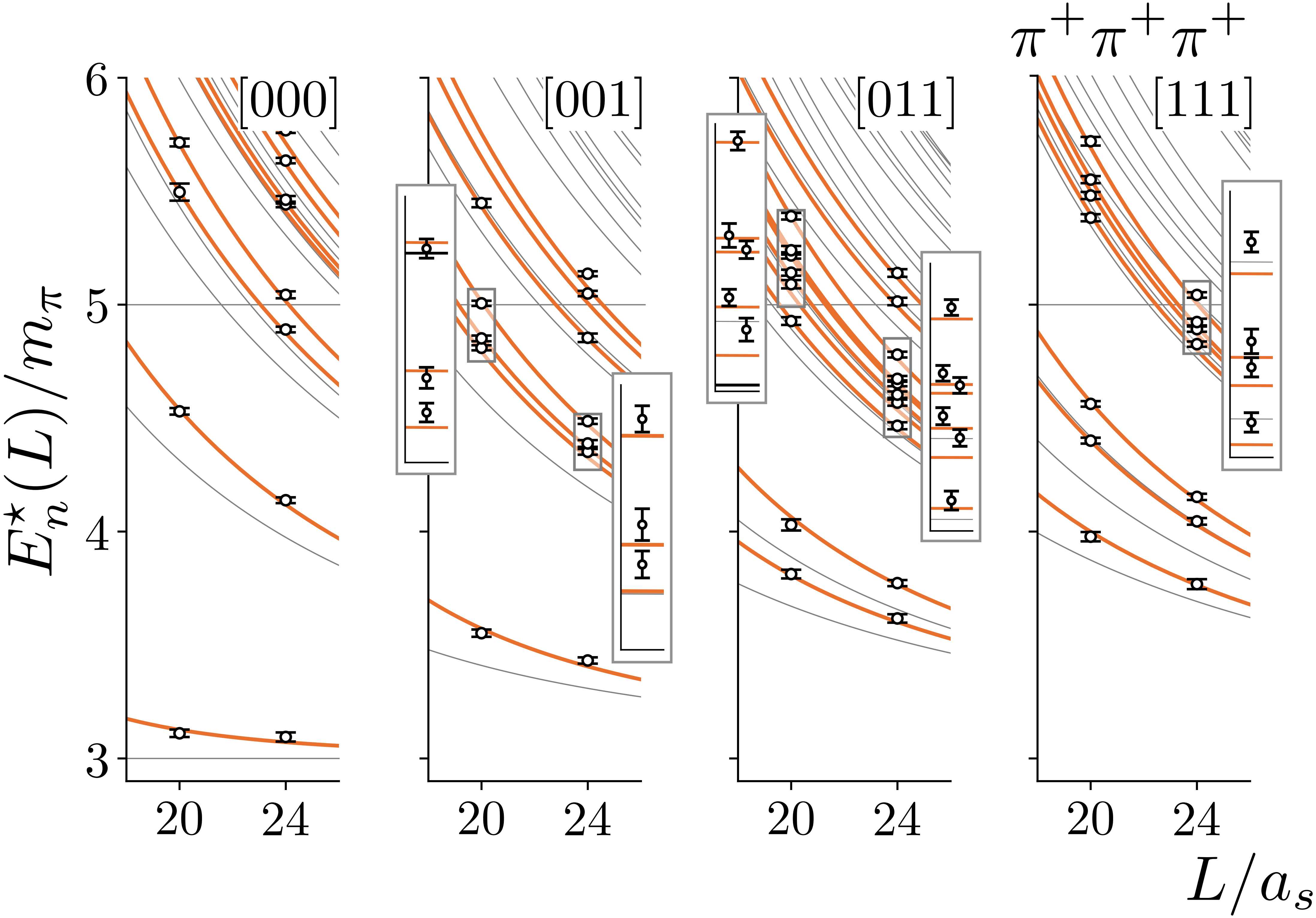


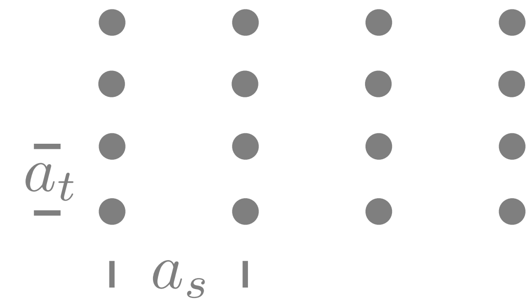
MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001

$$\pi^+\pi^+\pi^+ \to \pi^+\pi^+\pi^+$$

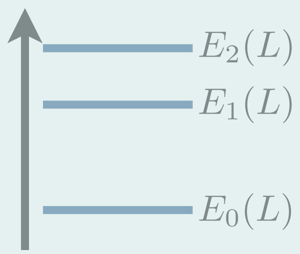
lattice details

$N_f = 2+1$ $\quad a_s/a_t = 3.444(6)$ $\quad L_s/a_s = 20, 24$

$m_\pi \approx 400\text{MeV}$ $\quad a_s \approx 0.12\text{fm}$



- [ ] Workflow outline

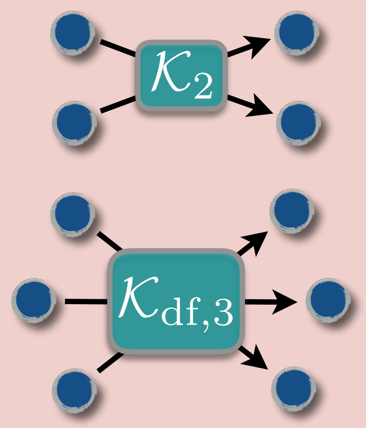


Determine finite-volume energy spectrum

✓

finite volume

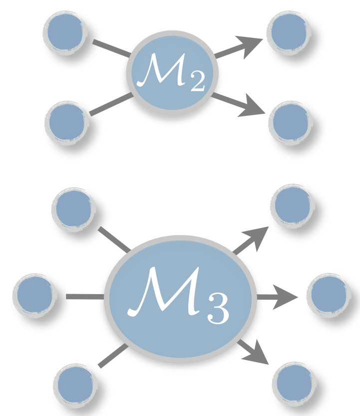
↔



Fit to constrain two- and three-body K matrices

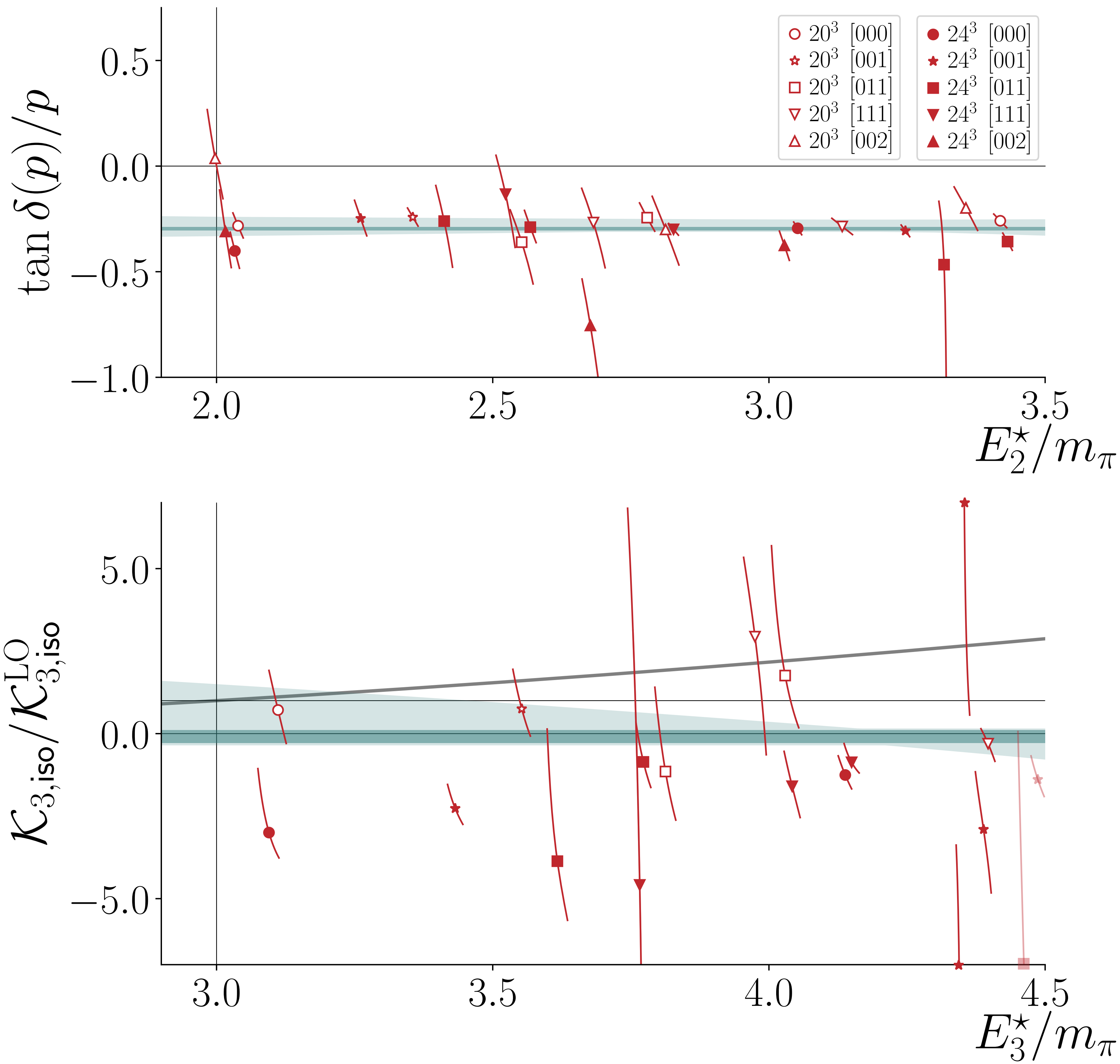
unitarity

↔



MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001

K matrix fits



Finite-volume formalism relates energies to K matrices

One-to-one for $K_{\mathrm{df},3}$ depending only on $E_{\mathrm{cm}} = E^\star$

Fit both two and three-body K to various polynomials

Cut on the CM energy in the fits

$K_{\mathrm{df},3}$ is scheme dependent (removed upon converting to $\mathscr{M}_3$)

$\pi^+\pi^+\pi^+ \to \pi^+\pi^+\pi^+$

lattice details

$N_f = 2+1$ $\quad a_s/a_t = 3.444(6)$

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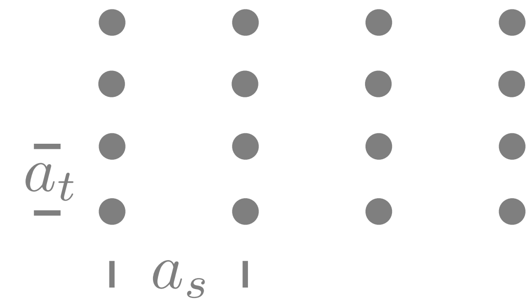

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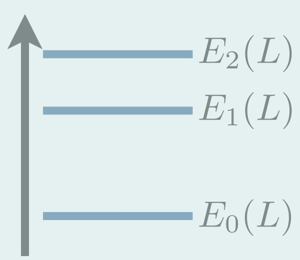
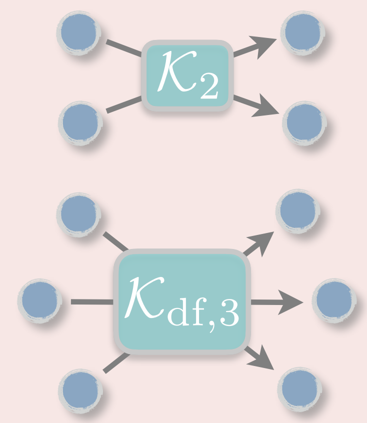


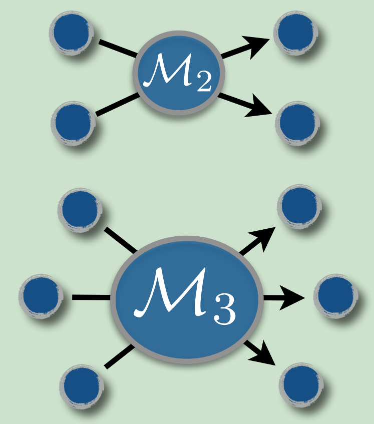
Determine finite-volume energy spectrum

finite volume



Fit to constrain two- and three-body K matrices

unitarity

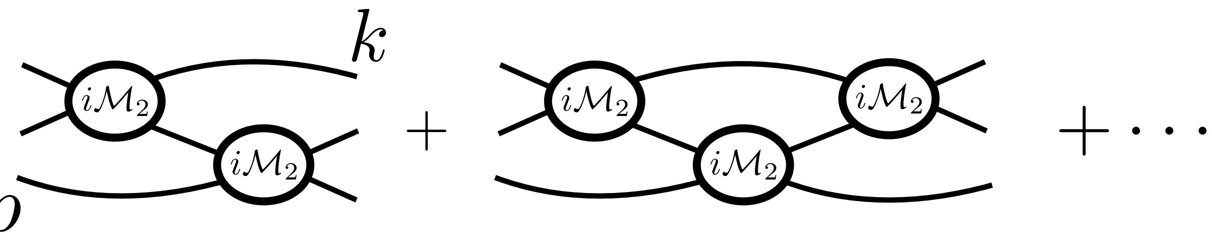


Solve integral equations to extract $3 \to 3$ scattering

MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001

Integral equation

$$\mathcal{M}_3^{\mathsf{un}}(E_3^\star,\boldsymbol{p},\boldsymbol{k}) = \mathcal{D}^{\mathsf{un}}(E_3^\star,\boldsymbol{p},\boldsymbol{k}) + \mathcal{E}^{\mathsf{un}}(E_3^\star,\boldsymbol{p})\mathcal{T}(E_3^\star)\mathcal{E}^{\mathsf{un}}(E_3^\star,\boldsymbol{k})$$



Vanishes for $K_{\text{df},3} = 0$

$$\boldsymbol{D}(N,\epsilon) = -\boldsymbol{\mathcal{M}}\cdot\boldsymbol{G}(\epsilon)\cdot\boldsymbol{\mathcal{M}} - \boldsymbol{\mathcal{M}}\cdot\boldsymbol{G}(\epsilon)\cdot\boldsymbol{P}\cdot\boldsymbol{D}(N,\epsilon)$$

$$\mathcal{D}^{\mathsf{un}}(E_3^\star,\boldsymbol{p},\boldsymbol{k}) = \lim_{\epsilon\to 0}\lim_{N\to\infty}\boldsymbol{D}_{pk}(N,\epsilon)$$

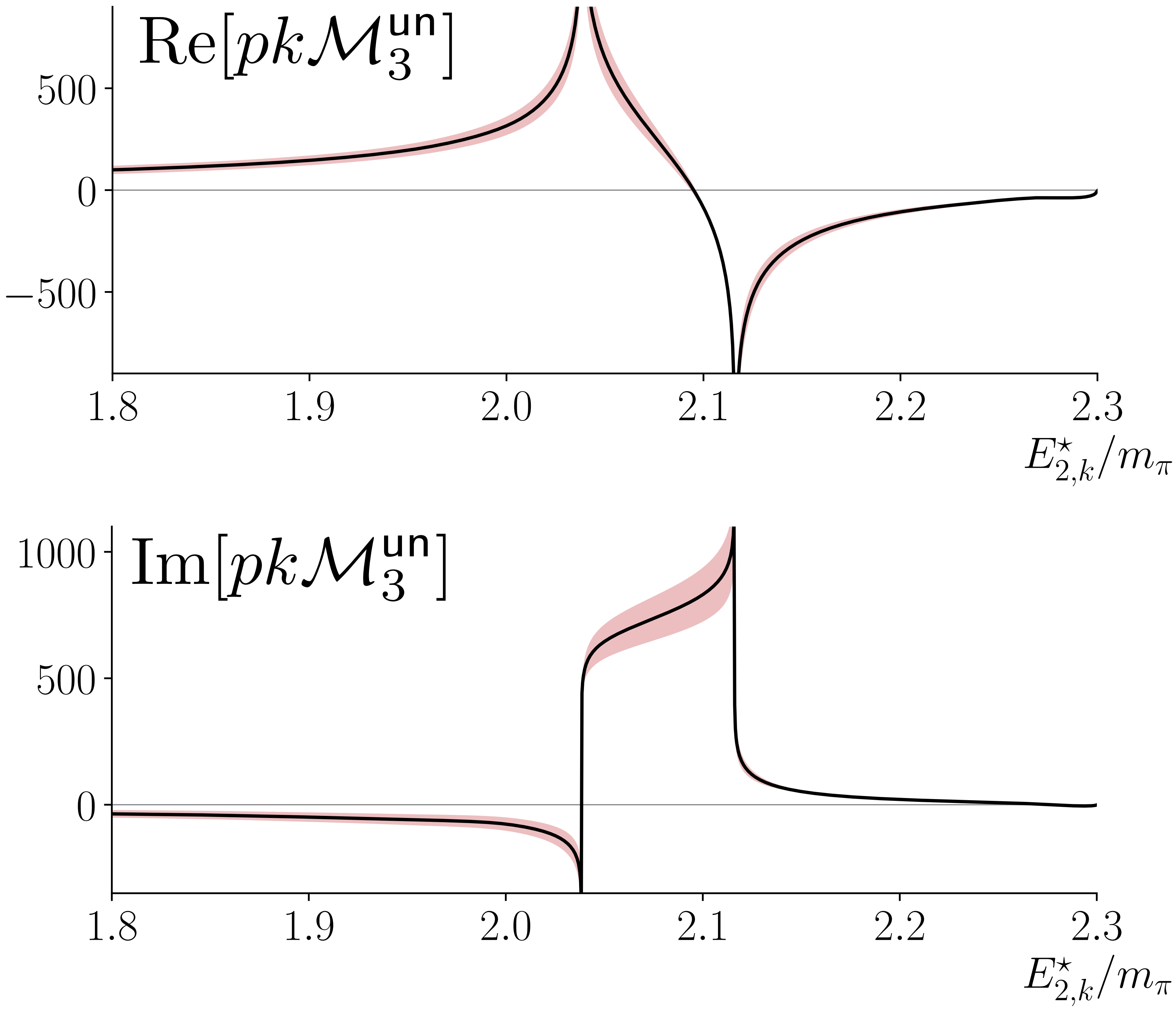
- See also…

Solving relativistic three-body integral equations in the presence of bound states

Andrew W. Jackura,[1,2,*] Raúl A. Briceño,[1,2,†] Sebastian M. Dawid,[3,4,‡] Md Habib E Islam,[2,§] and Connor McCarty[5,¶]

arXiv: 2010.09820

Integral equation



Total angular momentum = 0

Two-particle sub-system angular momentum = 0

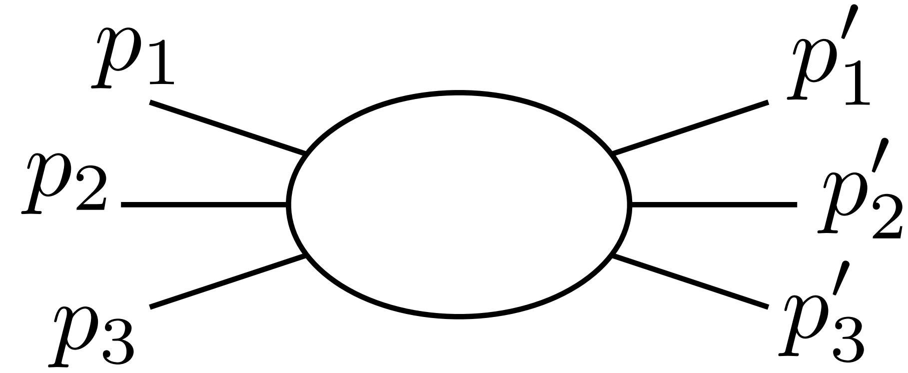
Plot at fixed $E_3^\star$ and p

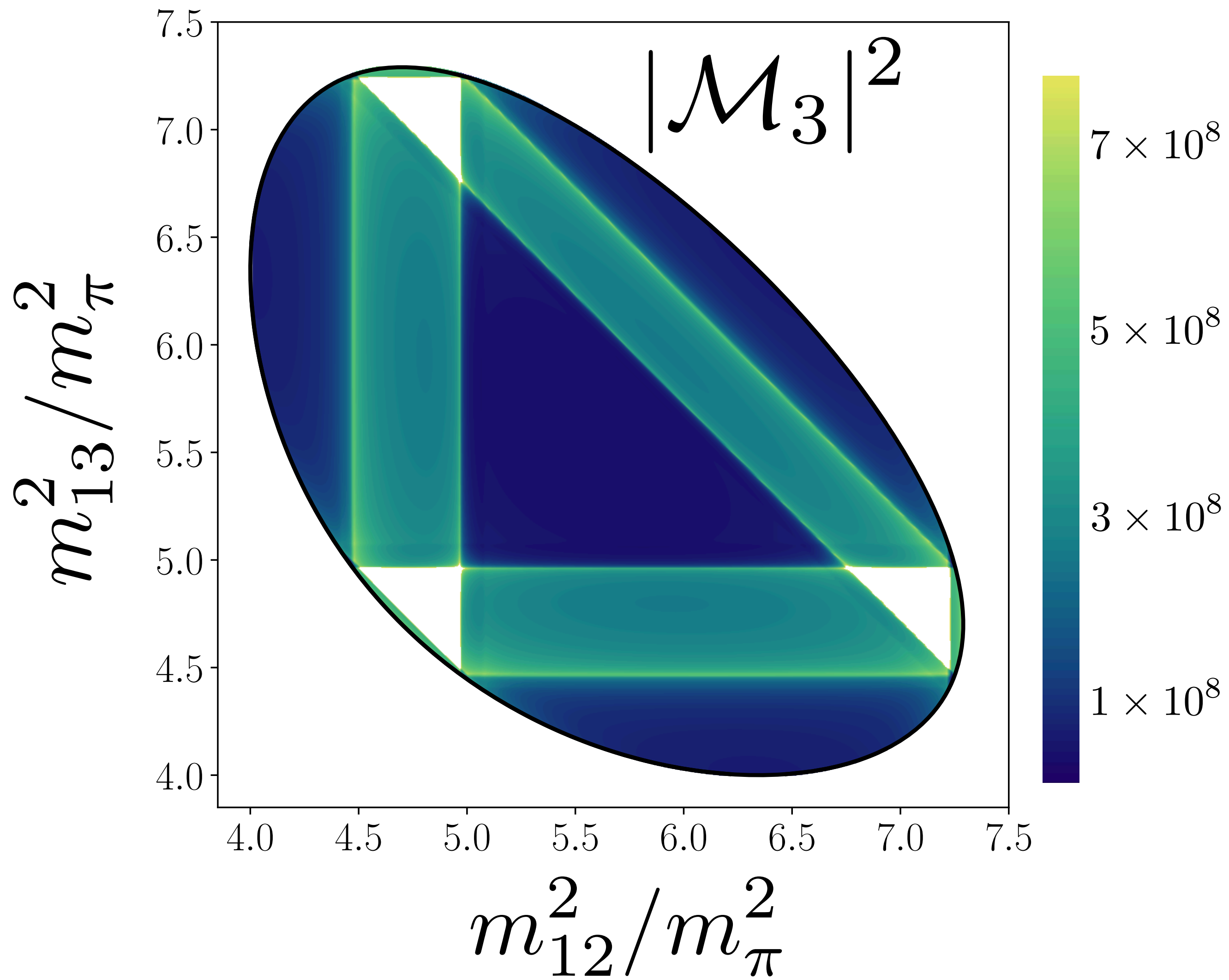
Both two- and three-body uncertainties estimated

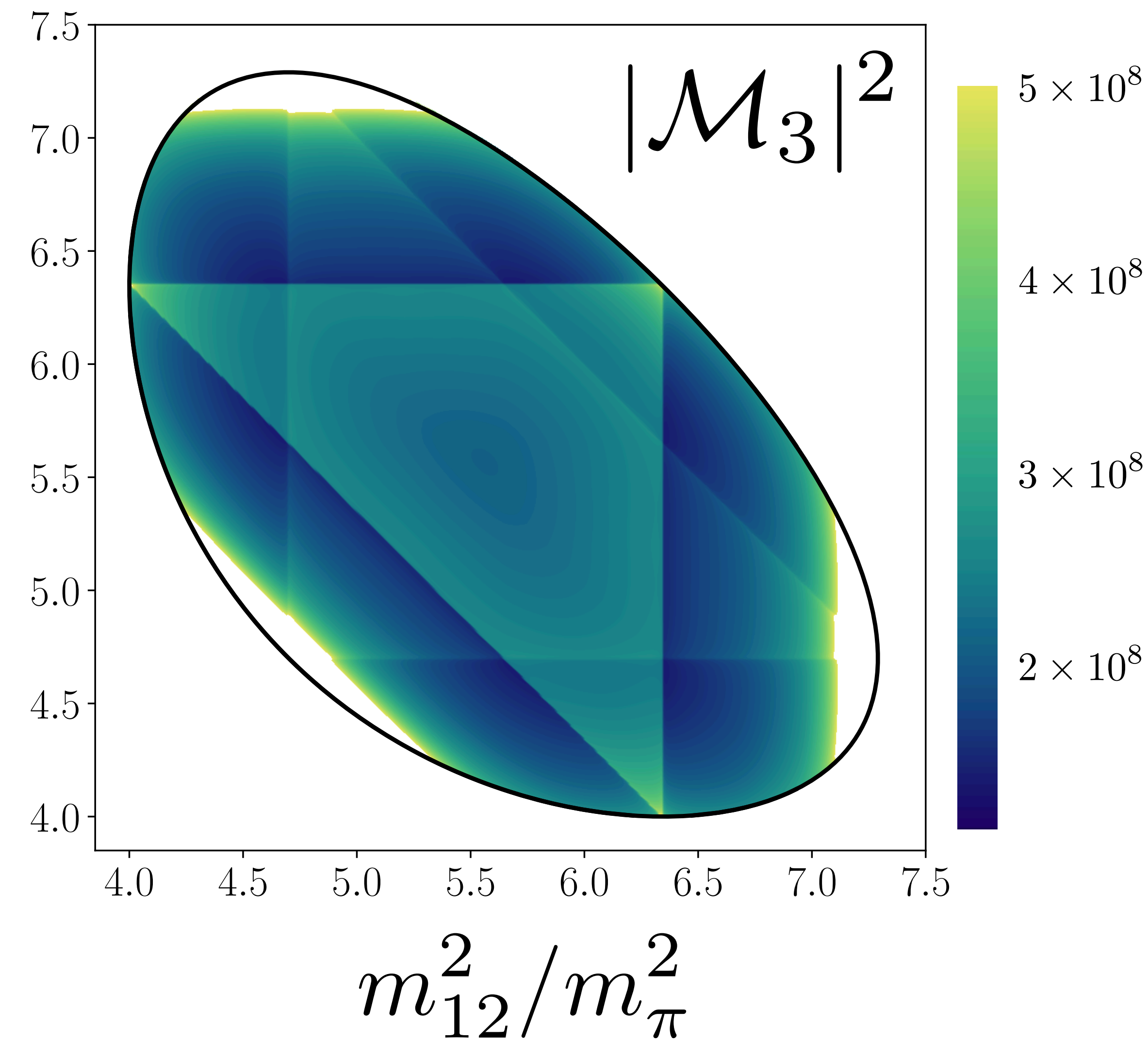
Still need to symmetrize

MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001

$$\mathcal{M}_3 = \sum_{i,j\in\{1,2,3\}} \mathcal{M}_3^{\mathsf{un}}(p_i', p_j)$$







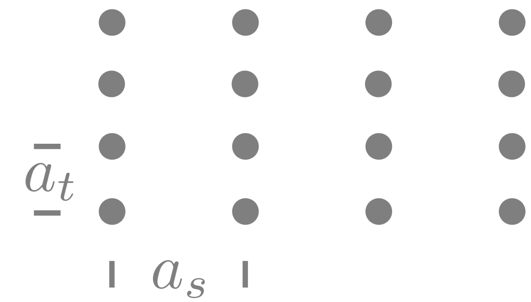
MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001

$\pi^+\pi^+\pi^+ \rightarrow \pi^+\pi^+\pi^+$

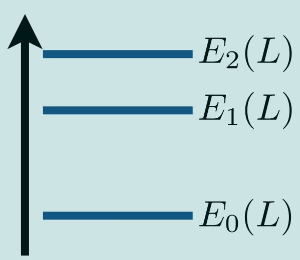
lattice details

$N_f = 2+1$ $a_s/a_t = 3.444(6)$ $L_s/a_s = 20, 24$

$m_\pi \approx 400\text{MeV}$ $a_s \approx 0.12\text{fm}$



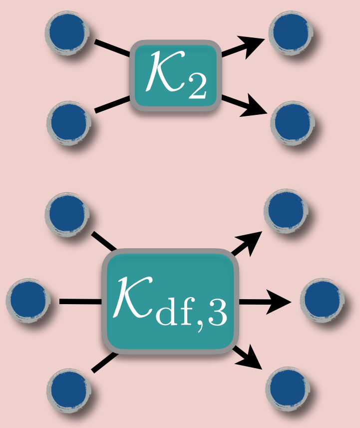
❑ Workflow outline



Determine finite-volume energy spectrum

✓

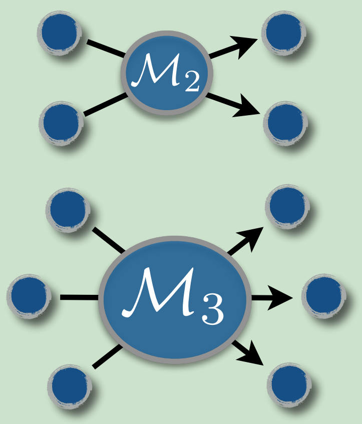
finite volume ⟷



Fit to constrain two- and three-body K matrices

✓

unitarity ⟷



Solve integral equations to extract $3 \rightarrow 3$ scattering

✓

MTH, Briceño, Edwards, Thomas, Wilson, *Phys.Rev.Lett.* 126 (2021) 012001

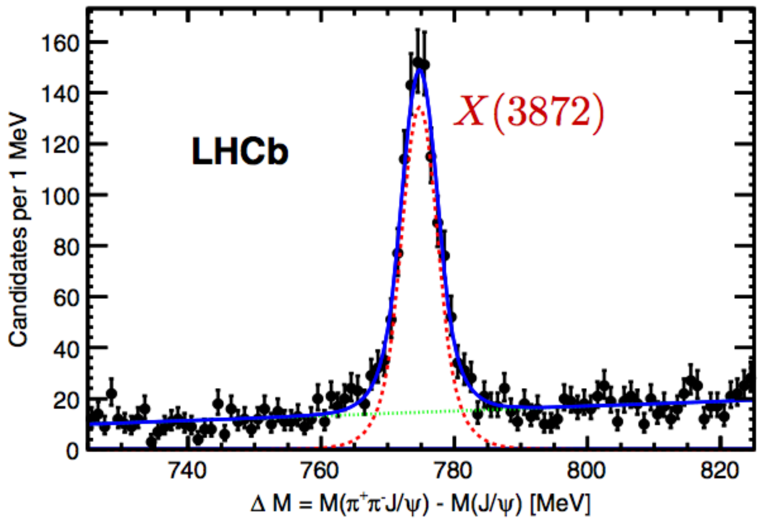
Outlook for resonances

- Exotics are challenging for many reasons

  incredible hierarchy of scales

$$M_X - M_{D^0 D^{0*}} = 0.01 \pm 0.18\,\mathrm{MeV}$$

$$\Gamma(X) < 1.2\mathrm{MeV}$$



• LHCb (PRD92, 2015) •

  many open channels

$$X \to \omega J/\psi,\ \pi\pi J/\psi,\ D^{*0}\bar{D}^0,\ \cdots$$

- Baryonic resonances are challenging as well

  need spin in multi-particle formalism

  signal-to noise degradation

- My speculative outlook

  should be possible to derive N-particle volume formalism

  then relate K-matrices to spectra to identify LQCD strategy

  hierarchy of scales = not always a problem $\mathcal{K}_2 \propto a \to \infty$

Conclusions

- LQCD is in the era of 'rigorous resonance spectroscopy'
- The finite-volume = ***a useful tool***
- Challenges and progress

  formal analysis was technical → *ground work is now set*

  scattering demands high precision excited states → *advanced algorithms make this possible*

  3-body amplitude is highly singular → *intermediate K matrix is not*

- Next steps…

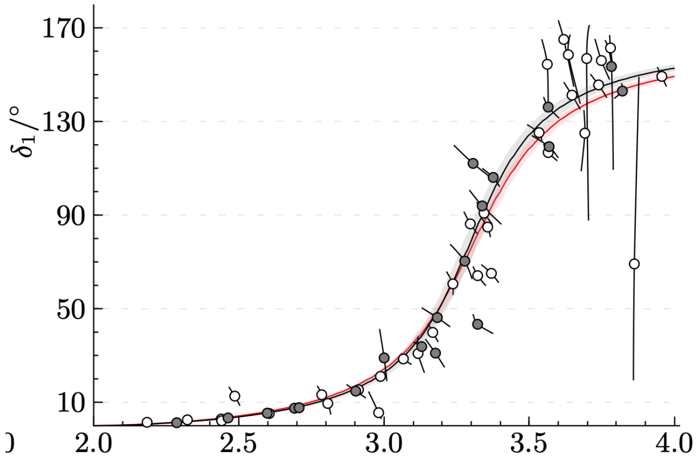
  complete 3-particle formalism → *extend to N-particle formalism*

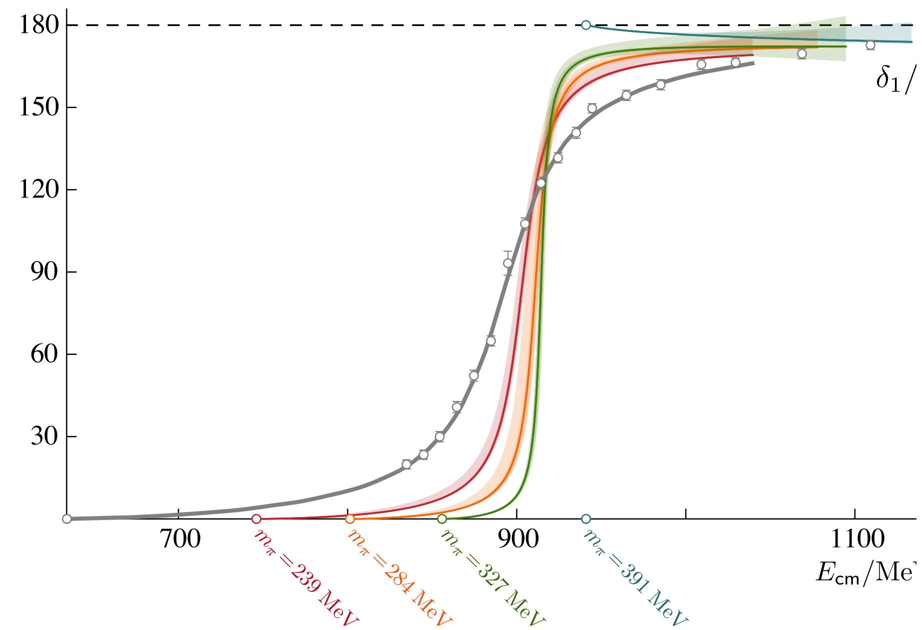
  extend studies involving an external current

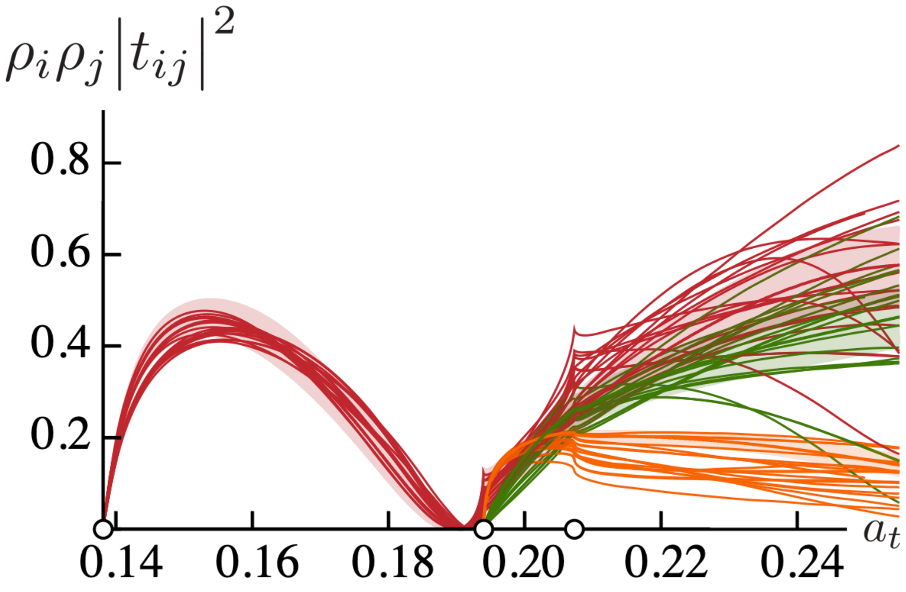
  push more channels into the precision regime

  The Roper! 😍 *(frontier requiring 2-to-3, non-degenerate, spin)*

Big Picture







A thriving field, with much more to come…

Thanks for listening!