Quantum States and Quantum Operations

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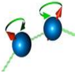
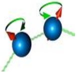
A quantum computing model



→ Quantum Computing Unit
Optical lattices, NMR, Anyons →



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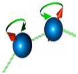
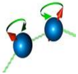
 ⟶ Quantum Computing Unit
Optical lattices, NMR, Anyons ⟶ 

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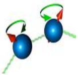
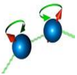
 → **Quantum Computing Unit** / Optical lattices, NMR, Anyons → 

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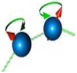

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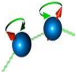

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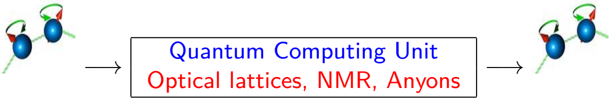

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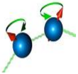 →

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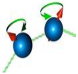
→ 

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- Choose a suitable quantum system to build the hardware.
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- Apply suitable measurement to extract useful information.

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$$\Phi(\rho) = F_1 \rho F_1^\dagger + \cdots + F_r \rho F_r^\dagger$$

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- So, one can do QIS research if one knows positive semi-definite matrices and the sum of linear maps of the form $\rho \mapsto F\rho F^\dagger$!

Quantum Channels with special properties

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In other words, given $\rho_1, \ldots, \rho_k \in D_n$ and $\sigma_1, \ldots, \sigma_k \in D_m$, find $m \times n$ matrices $F_1, \ldots, F_r$ such that $F_1^\dagger F_1 + \cdots + F_r^\dagger F_r = I_n$ and

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So, just solve the matrix equations for the unknowns $F_1, \ldots, F_r$.

Some results

Theorem [Li and Y. Poon, 2011]

Suppose $\{\rho_1, \ldots, \rho_k\}$ and $\{\sigma_1, \ldots, \sigma_k\}$ are commuting families. Then with a suitable choice of orthonormal bases, we may assume that

$$\rho_j = \begin{bmatrix} \rho_{j1} & & \\ & \ddots & \\ & & \rho_{jn} \end{bmatrix} \quad \text{and} \quad \sigma_j = \begin{bmatrix} \sigma_{j1} & & \\ & \ddots & \\ & & \sigma_{jm} \end{bmatrix}, \quad j = 1, \ldots, k.$$

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From D, one can construct $F_1, \dots, F_r$ to get the desired quantum channel.

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- The results were extended to compact operators in:

  M.H. Hsu, L.W. Kuo, M.C. Tsai, Completely positive interpolations of compact, trace-class and Schatten-p class operators. J. Funct. Anal. 267 (2014), no. 4, 1205–1240.

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One can use the matrix C to construct the matrices $F_1, \dots, F_r$ in the operator sum representation of the TPCP map.

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- This is known as the completion problem for psd matrices in matrix theory research.
- One can use positive semi-definite programming method to solve the problem numerically.

A general result

Theorem [Huang, Li, E.Poon, Sze, 2012]

Suppose $\rho_1, \ldots, \rho_k \in M_n$ and $\sigma_1, \ldots, \sigma_k \in M_m$ are density matrices with spectral decomposition:

$$\rho_i = X_i D_i^2 X_i^\dagger \quad \text{and} \quad \sigma_i = Y_i \tilde{D}_i^2 Y_i^\dagger, \quad i = 1, \ldots, k,$$

for some diagonal matrices $D_i \in M_{r_i}$, $\tilde{D}_i \in M_{s_i}$ with positive diagonal entries.

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There is a TPCP map $\Phi : M_n \to M_m$ such that $\Phi(\rho_i) = \sigma_i$ for all i if and only if:

For each $i = 1, \ldots, k$ and $j = 1, \ldots, r_i$, there are $s_i \times s$ matrices V_{ij} such that

$$[V_{i1} \cdots V_{r_i}][V_{i1} \cdots V_{r_i}]^\dagger = I_{s_i}$$

and

$$[D_i X_i^\dagger X_j D_j] = [\mathrm{tr}\,(V_{ip}^\dagger \tilde{D}_i^\dagger Y_i^\dagger Y_j \tilde{D}_j V_{jq})]_{1 \leq p \leq r_i, 1 \leq q \leq r_j}.$$

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But, SDP is inefficient even for moderate size problems.

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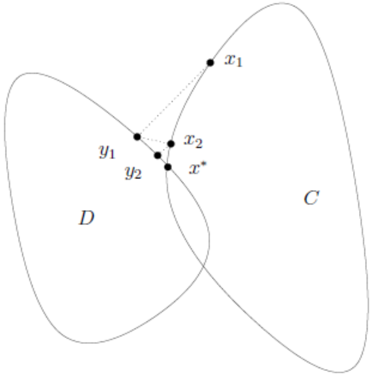

Figure 1: First few iterations of alternating projection algorithm. Both sequences are converging to the point $x^* \in C \cap D$.

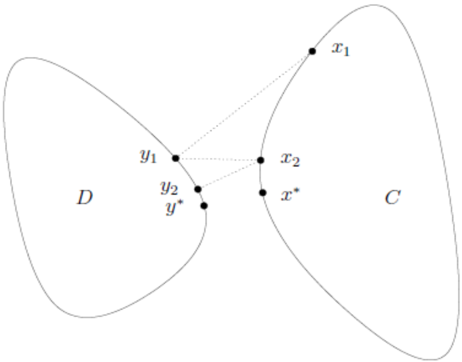

Figure 2: First few iterations of alternating projection algorithm, for a case in which $C \cap D = \emptyset$. The sequence x_k is converging to $x^* \in C$, and the sequence y_k is converging to $y^* \in D$, where $\|x^* - y^*\|_2 = \mathbf{dist}(C, D)$.

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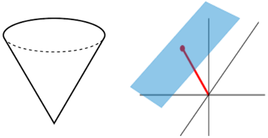

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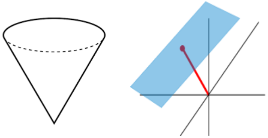

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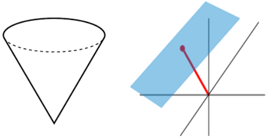

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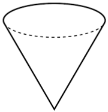

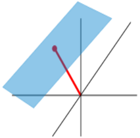

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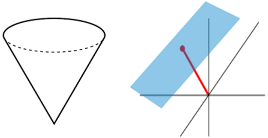

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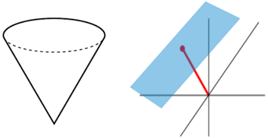

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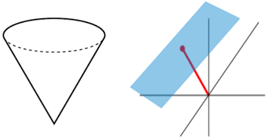

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Dmitriy Drusvyatskiy, Chi-Kwong Li, Diane Pelejo, Yuen-Lam Voronin, Henry Wolkowicz, Projection Methods for Quantum Channel Construction, Quantum Inf Process (2015) 14:30753096 DOI 10.1007/s11128-015-1024-y.

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- By the result of [Leung, Li, Poon, Watrous, 2010+],

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If such a T exists, then Φ is mixed unitary.

Additional problems

- Check whether the Werner-Holevo channel $\Phi : M_n \to M_n$ defined by

$$\Phi(\rho) = \frac{1}{n+1}((\mathrm{tr}\, \rho) I_n + \rho^t)$$

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