Quantum Gauge Theories with Quantum Computers
Part 1

Yannick Meurice

The University of Iowa

yannick-meurice@uiowa.edu

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Objectives and Recent Progress

- Quantum computations/simulations for high energy physics?
- Strategy:
  - big goals with enough intermediate steps
  - explore as many paths as possible
  - leave room for serendipity
- Tensor tools: QC friends and competitors (RG)
- Symmetry preserving truncations (YM, arxiv:1903.01918, PRD 100, 014506 and arxiv preprint in progress)
- Abelian Higgs model with cold atoms (PRL 121)
- Quantum computations (digital): IBM, IonQ, Rigetti, ...
- Real time scattering (PRD 99 094503 with Erik Gustafson and Judah Unmuth-Yockey; arXiv arXiv:1910.09478 with P. Dreher and IBM-Q; work in progress with N. Linke trapped ion lab)
- Quantum Joule experiments (arXiv:1903.01414, PRA 101.033608, with Jin Zhang and Shan-Wen Tsai)

Quantum computations/simulations for high energy physics?

Problems where perturbation theory and classical sampling (Monte Carlo) are challenged:

- Real-time evolution for QCD (-> PDF and GPD)
- Jet Physics (crucial for the LHC program)
- Finite density QCD (sign problem)
- Near conformal systems (BSM, needs very large lattices)
- Early cosmology
- Strong gravity

Computing with quantum devices (Feynman 82)?

- The number of transistors on a chip doubled almost every two years for more than 30 years
- At some point, the miniaturization involves quantum mechanics
- Capacitors are smaller but they are still on (charged) or off (uncharged)
- qubits: $|\Psi\rangle = \alpha|0\rangle + \beta|1\rangle$ is a superposition of the two possibilities
- Can we use quantum devices to explore large (2^{N_q})Hilbert spaces? (Feynman 82)
- Yes, if the interactions are localized (generalization of Trotter product formula, Lloyd 96)

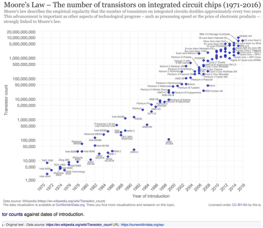


Figure: Moore's law, source: Wikipedia

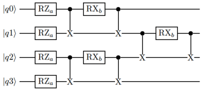


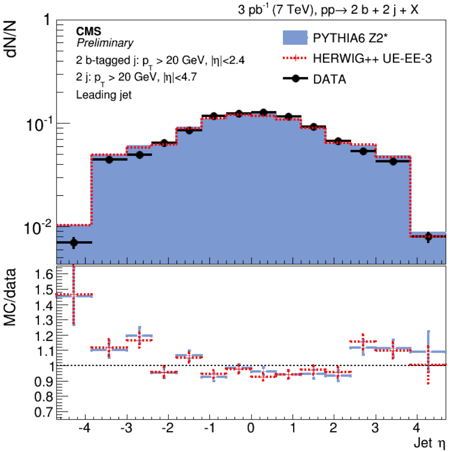
Figure 1: Circuit for 4 qubits with open boundary conditions

Figure: Quantum circuit for the quantum Ising model

Strategy: many intermediate steps towards big goals

- High expectations for quantum computing (QC): new materials, fast optimization, security, ...
- Risk management: theoretical physics is a multifaceted landscape
- Lattice gauge theory lesson: big goals can be achieved with small steps
- Example of big goals: ab-initio jet physics, PDF, ...
- Examples of small steps: real-time evolution in 1+1 Ising model, 1+1 Abelian Higgs model, Schwinger model, 2+1 $U(1)$ gauge theory ,....
- Many possible paths: quantum simulations (trapped ions, cold atoms,...), quantum computations (IBM, Rigetti,...)
- Small systems are interesting: use Finite Size Scaling (data collapse, Luscher's formula,....)

Pythia, Herwig, and other jet simulation models encapsulate QCD ideas, empirical observations and experimental data. It is crucial for the interpretation of collider physics experiments. Could we recover this understanding from scratch (ab-initio lattice QCD)?



Lessons from lattice gauge theory

We need to start with something simple!

Figure: Mike Creutz's calculator used for a Z_2 gauge theory on a 3^4 lattice (circa 1979).

Following the “Kogut sequence" for the quantum theories

An introduction to lattice gauge theory and spin systems*

John B. Kogut

Department of Physics, University of Illinois at Urbana-Champaign, Urbana, Illinois 61801

This article is an interdisciplinary review of lattice gauge theory and spin systems. It discusses the fundamentals, both physics and formalism, of these related subjects. Spin systems are models of magnetism and phase transitions. Lattice gauge theories are cutoff formulations of gauge theories of strongly interacting particles. Statistical mechanics and field theory are closely related subjects, and the connections between them are developed here by using the transfer matrix. Phase diagrams and critical points of continuous transitions are stressed as the keys to understanding the character and continuum limits of lattice theories. Concepts such as duality, kink condensation, and the existence of a local, relativistic field theory at a critical point of a lattice theory are illustrated in a thorough discussion of the two-dimensional Ising model. Theories with exact local (gauge) symmetries are introduced following Wegner's Ising lattice gauge theory. Its gauge-invariant “loop” correlation function is discussed in detail. Three – dimensional Ising gauge theory is studied thoroughly. The renormalization group of the two dimensional planar model is presented as an illustration of a phase transition driven by the condensation of topological excitations. Parallels are drawn to Abelian lattice gauge theory in four dimensions. Non-Abelian gauge theories are introduced and the possibility of quark confinement is discussed. Asymptotic freedom of O(n) Heisenberg spin systems in two dimensions is verified for $n \geq$ and is explained in . simple terms. The direction of present-day research is briefly reviewed.

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* Supported in part by the National Science Foundation under grant number NSF PHY77-25279.

I. INTRODUCTION—AN OVERVIEW OF THIS ARTICLE

This article consists of a series of introductory lectures on lattice gauge theory and spin systems. It is intended to explain some of the essentials of these subjects to students interested in the field and research physicists whose expertise lies in other domains. The expert in lattice gauge theory will find little new in the following pages aside from the author's personal perspective and overview. The style of this presentation is informal. The article grew out of a half-semester graduate course on lattice physics presented at the University of Illinois during the fall semester of 1978.

Lattice spin systems are familiar to most physicists because they model solids that are studied in the laboratory. These systems are of considerable interest in these lectures, but we shall also be interested in more abstract questions. In particular, we shall be using space-time lattices as a technical device to define cutoff field theories. The eventual goal of these studies is to construct solutions of cutoff theories so that field theories defined in real continuum Minkowski space-time can be understood. The lattice is mere scaffolding—an intermediate step used to analyze a difficult nonlinear system of an infinite number of degrees of freedom. Different lattice formulations of the same

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Goals of the lectures

- Introduce lattice models with continuous Abelian symmetries: $U(1)$ gauge theory and $O(2)$ spin model (section VI and VII in the “Kogut ladder"). Note: these two models can be combined in the Abelian Higgs model.
- Reformulate these models with the tensor method and provide a completely discrete and gauge-invariant Hamiltonian formulation using the transfer matrix.
- Show the equivalence to gauge-fixing in the temporal gauge $A_0 = 0$ (we can recover Gauss's law).
- Discuss implementation with quantum computers (Ising versions) and cold atom ladders (spin-2 approximation for the Abelian Higgs model).
- If time permits: phase shifts, geometrical correspondence between the lattice equations of motion and the discrete Noether equations (selection rules for tensors), topological sectors in the continuum limit.

Discretization of problems intractable with classical computing

Quantum computing (QC) requires a complete discretization of QFT

- Discretization of space: lattice gauge theory formulation
- Discretization of field integration: tensor methods for compact fields (as in Wilson lattice gauge theory and nonlinear sigma models, the option followed here)
- QC methods for scattering in ϕ^4 (non-compact) theories are discussed by JLP (Jordan Lee Preskill)
- JLP argue that QC is necessary because of the asymptotic nature of perturbation theory (PT) in λ for ϕ^4 and propose to introduce a field cut (but this makes PT convergent! YM PRL 88 (2002))
- Non compact fields methods ($\lambda\phi^4$) see: Macridin, Spentzouris, Amundson, Harnik, PRA 98 042312 (2018) (fermions+bosons) and Klco and Savage arXiv:1808.10378 and 1904.10440.
- In the following we focus on compact fields.

Important ideas of the tensor reformulation

- In most lattice simulations, the variables of integration are compact and character expansions (such as Fourier series) can be used to rewrite the partition function and average observables as discrete sums of contracted tensors. (Pontryagin duality, Peter-Weyl theorem).
- The “hard" integrals are done exactly and then field integrations provide Kronecker deltas. Example: the $O(2)$ model (I_n : Bessel)

$$e^{\beta \cos(\theta_i - \theta_j)} = \sum_{n_{ij}=-\infty}^{+\infty} e^{i n_{ij}(\theta_i - \theta_j)} I_{n_{ij}}(\beta)$$

- This reformulations have been used for RG blocking but they are also suitable for quantum computations/simulations when combined with truncations.
- Important features:
  - Truncations do not break global symmetries
  - Standard boundary conditions can be implemented
  - Matrix Product State ansatzs for operators are exact

From compact to discrete

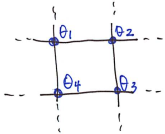
O(2) model

$$Z = \prod_x \int_{-\pi}^{\pi} d\theta_x \, e^{\beta \sum_{\langle xy \rangle} \cos(\theta_x - \theta_y)} = I_0^V(\beta) \, \mathrm{Tr}_{\{n,m\ldots\}} \prod_x T^{(x)}_{n_x n'_x m_x m'_x}$$

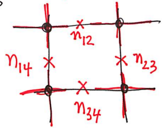
↳ nearest neighbor

$$e^{\beta \cos(\theta_x - \theta_y)} = \sum_{n_{xy}} e^{i n_{xy}(\theta_x - \theta_y)} I_{n_{xy}}(\beta)$$

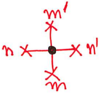
$$\prod_x \int_{-\pi}^{\pi} d\theta_x \longrightarrow \sum_{\{n\}}$$



continuous
compact



discrete
infinite



$$: T_{nn'mm'} = \sqrt{t_n t_{n'} t_m t_{m'}} \, \delta_{n+m, n'+m'}$$

$$t_n = I_n(\beta)/I_0(\beta)$$

Preview: the Compact Abelian Higgs Model CAHM

$$Z_{CAHM} = \prod_x \int_{-\pi}^{\pi} \frac{d\varphi_x}{2\pi} \prod_{x,\mu} \int_{-\pi}^{\pi} \frac{dA_{x,\mu}}{2\pi} e^{-S_{gauge}-S_{matter}},$$

$$S_{gauge} = \beta_{pl.} \sum_{x,\mu<\nu} (1 - \cos(A_{x,\mu} + A_{x+\hat{\mu},\nu} - A_{x+\hat{\nu},\mu} - A_{x,\nu})),$$

$$S_{matter} = \beta_{l.} \sum_{x,\mu} (1 - \cos(\varphi_{x+\hat{\mu}} - \varphi_x + A_{x,\mu})).$$

The CAHM is a gauged version of the $O(2)$ model where the global symmetry under a φ shift becomes local

$$\varphi'_x = \varphi_x + \alpha_x$$

and these local changes in S_{matter} are compensated by the gauge field changes

$$A'_{x,\mu} = A_{x,\mu} - (\alpha_{x+\hat{\mu}} - \alpha_x),$$

which also leave S_{gauge} invariant.

Pure gauge and spin-model limits

The matter fields can be decoupled by simply setting $\beta_{l.} = 0$. As they don't appear in the action, their integration yields a factor 1 and we are left with the pure gauge (PG) $U(1)$ lattice model with partition function

$$Z_{PG} = \prod_{x,\mu} \int_{-\pi}^{\pi} \frac{dA_{x,\mu}}{2\pi} e^{-S_{gauge}}.$$

The decoupling of the gauge fields is less straightforward. Strictly speaking, the O(2) spin model is obtained by removing the gauge fields introduced to make the global symmetry a local one and the partition function of the O(2) model reads

$$Z_{O(2)} = \prod_{x} \int_{-\pi}^{\pi} \frac{d\varphi_x}{2\pi} e^{-S_{O(2)}},$$

with

$$S_{O(2)} = \beta_{l.} \sum_{x,\mu} (1 - \cos(\varphi_{x+\hat{\mu}} - \varphi_x)).$$

Clock and Ising versions

- In the 3 models discussed φ_x and $A_{x,\mu}$ are continuous angles with the compact identification $0 \sim 2\pi$ (circle).
- We can also have discrete "clock" versions where φ_x and $A_{x,\mu}$ take only q values

$$0,\ \frac{2\pi}{q}, 2\frac{2\pi}{q}, \ldots, (q-1)\frac{2\pi}{q}$$

- These can be seen as approximations of the $U(1)$ model or just for their own sake
- The Ising case is of special interest (qbits) and usually kept in multiplicative notation ($\sigma = \pm 1$)

After Fourier transform, all the models can be represented with discrete states ($D = 2$)

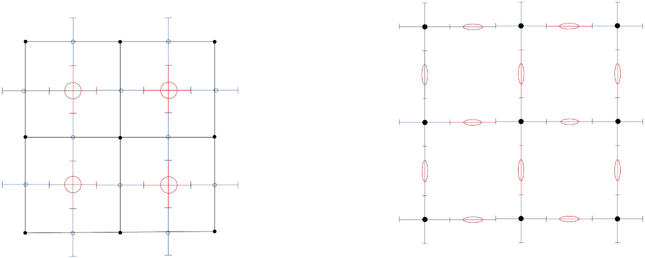


Figure: The red and blue dots carry integer quantum numbers. Ising: 0,1; spin-2 approximation of $U(1)$: -2,-1,0,1,2

After Fourier transform, all the models can be represented with discrete states ($D = 3$)

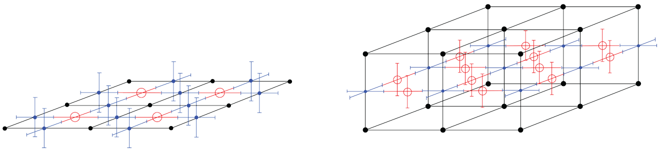


Figure: The red and blue dots carry integer quantum numbers

"Profound messages"

- Tensorial truncations are compatible with the general identities derived from the symmetries of these models (universal properties of these models can be reproduced with highly simplified formulations desirable for implementations with quantum computers or for quantum simulations experiments. Y. M., PHYSICAL REVIEW D 100, 014506 (2019))
- There is a geometrical analogy between the lattice equations of motion and the selection rules of the tensor formulation
- Noether theorem: for each symmetry there is a corresponding tensor redundancy (and we can "gauge fix" the corresponding integration variable)
- arxiv xxxxx: We discuss the transfer matrix in the gauge-invariant tensor reformulation of lattice . We show the equivalence to a gauge-fixed version in a maximal temporal gauge and explain how Gauss's law is enforced. We discuss semi-classical approximations and their weak coupling correspondence for the solvable cases.

Quantum circuit for the quantum Ising model

Quantum circuit with 3 Trotter steps (arXiv:1901.05944 E. Gustafson, YM and J. Unmuth-Yockey, PRD 99 094503)

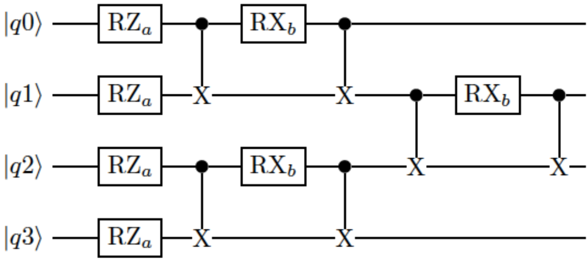


Figure 1: Circuit for 4 qubits with open boundary conditions

Trotter Fidelity

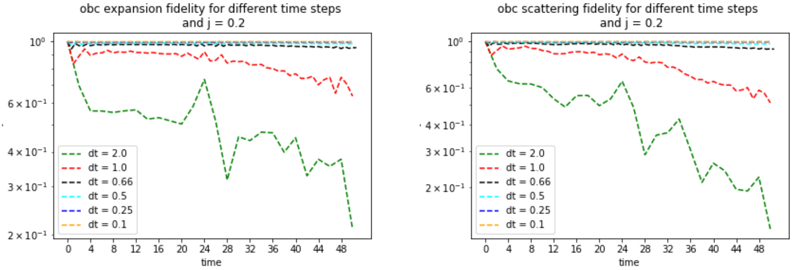
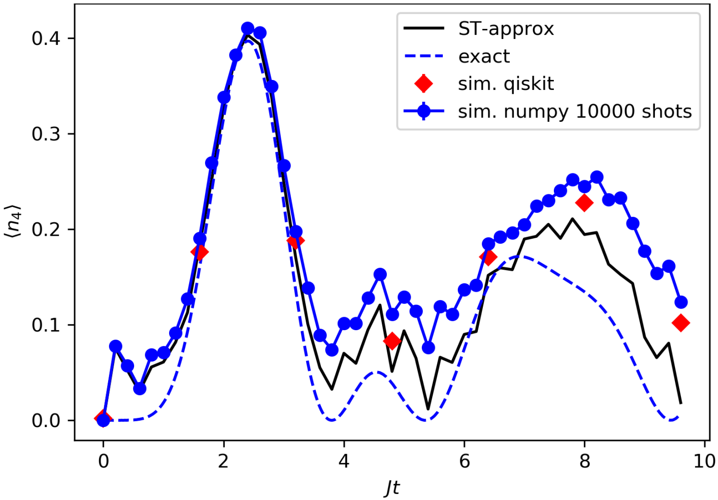


Figure: fidelity of Trotter operator at multiple different Trotter steps for (left to right) expansion and scattering with open boundary conditions (E. Gustafson, YM and J. Unmuth-Yockey arXiv:1901.05944, PRD 99 094503)

Systematic and statistical errors



Optical lattice implementation of the compact Abelian Higgs Model with a physical ladder(PRL 121, 223201)

After taking the time continuum limit:

$$\bar{H} = \frac{\tilde{U}_g}{2}\sum_i \left(\bar{L}^z_{(i)}\right)^2 + \frac{\tilde{Y}}{2}\sum_i (\bar{L}^z_{(i)} - \bar{L}^z_{(i+1)})^2 - \tilde{X}\sum_i \bar{L}^x_{(i)}$$

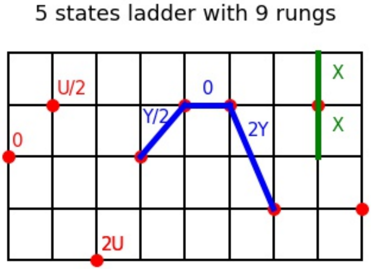


Figure: Ladder with one atom per rung: tunneling along the vertical direction, no tunneling in the the horizontal direction but short range attractive interactions. A parabolic potential is applied in the spin (vertical) direction.

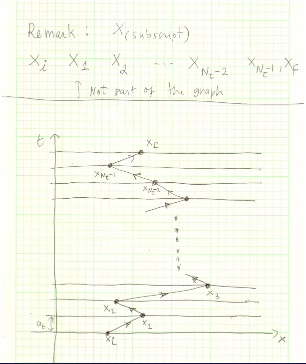
Nuts and Bolts

1. Path integral and Euclidean time
2. Classical gauge-invariance and Gauss's law
3. Lattice models (spin and gauge)

1. The path integral (Overview)

The path integral (R. Feynman) is a way to rewrite the quantum evolution as a sum over paths joining some initial and final positions $\mathbf{x}_i$ and $\mathbf{x}_f$ weighted by $\mathrm{e}^{iS/\hbar}$ where S is the action corresponding to a specific path joining $\mathbf{x}_i$ and $\mathbf{x}_f$. ($\int[\mathcal{D}\mathbf{x}]_{\mathbf{x}_i\rightarrow\mathbf{x}_f}$:"sum over the paths")

$$\langle \mathbf{x}_f|\mathrm{e}^{-i(t_f-t_i)\hat{H}/\hbar}|\mathbf{x}_i\rangle = \int[\mathcal{D}\mathbf{x}]_{\mathbf{x}_i\rightarrow\mathbf{x}_f}\mathrm{e}^{iS[\mathbf{x}]/\hbar},$$



Small step evolution

First express $\mathrm{e}^{-i(t_f-t_i)\hat{H}/\hbar}$ as a product of N_t short evolutions by time steps a_t with $N_t a_t = (t_f - t_i)$ and then inserting the identity

$$\int d\mathbf{x}|\mathbf{x}\rangle\langle\mathbf{x}|$$

between each of the successive steps.

$$\langle\mathbf{x}_f|\mathrm{e}^{-i(t_f-t_i)\hat{H}/\hbar}|\mathbf{x}_i\rangle = \tag{1}$$

$$\int d\mathbf{x}_{N_t-1}\cdots\int d\mathbf{x}_1\langle\mathbf{x}_f|\mathrm{e}^{-ia_t\hat{H}/\hbar}|\mathbf{x}_{N_t-1}\rangle\ldots\langle x_2|\mathrm{e}^{-ia_t\hat{H}/\hbar}|\mathbf{x}_1\rangle\langle\mathbf{x}_1|\mathrm{e}^{-ia_t\hat{H}/\hbar}|\mathbf{x}_i\rangle.$$

The point of using small time steps is that in this limit, the evolution operator matrix elements can be approximated by phases which will contribute to an overall $\mathrm{e}^{iS/\hbar}$ factor. The product of integrals

$$\int d\mathbf{x}_{N_t-1}\cdots\int d\mathbf{x}_1,$$

has a “sum over the paths" interpretation

Euclidean time

- It is convenient to think of the time parameter as a complex number and to use the Euclidean time τ (Wick rotation)

$$(t_f - t_i) \rightarrow -i\tau.$$

- The use of Euclidean time does not affect the spectrum of $\hat{H}$.
- For large positive τ, $e^{-\tau\hat{H}}$ makes the contributions of the high energy states negligible.
- Euclidean time in the path integral brings the substitution

$$\mathrm{e}^{iS/\hbar} \rightarrow \mathrm{e}^{-S/\hbar}.$$

- This positive Boltzmann weight allows to use importance sampling methods developed in statistical mechanics (Monte Carlo).

Quick review of path integral

$U(t) = e^{-i\frac{\hat{H}t}{\hbar}}$ evolution operator

$U(t) = \left(U\left(\frac{t}{n}\right)\right)^n$ evolution by n small time steps $\frac{t}{n} = \varepsilon$

$\int_{-\infty}^{+\infty} dx\, |x\rangle\langle x| = \mathbb{1}$ $\langle x|y\rangle = \delta(x-y)$

$$\langle x_n| e^{-i\frac{\hat{H}t}{\hbar}} |x_0\rangle = \prod_{i=1}^{n-1} \int_{-\infty}^{+\infty} dx_i \langle x_n| e^{-i\frac{\hat{H}}{\hbar}\frac{t}{n}} |x_{n-1}\rangle \cdots \langle x_1| e^{-i\frac{\hat{H}}{\hbar}\frac{t}{n}} |x_0\rangle$$

$x_n \to x_{final}$, $x_0 \to x_{in}$

The problem is exactly solvable for $\hat{H}_0 = \frac{p^2}{2m}$

$e^{-\frac{i}{\hbar}(\hat{H}_0+\hat{V})\varepsilon} = e^{-\frac{i}{\hbar}\hat{H}_0\varepsilon} e^{-\frac{i}{\hbar}\hat{V}\varepsilon} + O(\varepsilon^2)$ due to $[\hat{H}_0, \hat{V}] \neq 0$

Feynman - Kac - Trotter

$$\left(e^{-\frac{i}{\hbar}\frac{(\hat{H}_0+\hat{V})t}{n}}\right)^n \xrightarrow[n\to\infty]{} \left(e^{-\frac{i}{\hbar}\frac{\hat{H}_0 t}{n}} e^{-\frac{i}{\hbar}\frac{\hat{V}t}{n}}\right)^n$$

$\hat{H}_0$: $\langle x_f | e^{-i\frac{\hat{p}^2 t}{2m\hbar}} | x_i \rangle = \int_{-\infty}^{+\infty} dp \, \langle x_f | e^{-i\frac{p^2 t}{2m\hbar}} | p \rangle \langle p | x_i \rangle$

$\langle x | p \rangle = \frac{1}{\sqrt{2\pi\hbar}} e^{i\frac{p \cdot x}{\hbar}}$ + complete the square

$$= \frac{1}{2\pi\hbar} \sqrt{\frac{\pi 2 m \hbar}{i t}} \, e^{+\frac{i}{\hbar} m \frac{(x_f - x_i)^2}{2t}}$$

adding V

$$\langle x_n | e^{-i\frac{\hat{H} t}{\hbar}} | x_0 \rangle \underset{n\to\infty}{\simeq} \prod_{i=1}^{n-1} \int_{-\infty}^{+\infty} dx_i \; \langle x_n | e^{-i\frac{\hat{H}_0 t}{\hbar n}} | x_{n-1} \rangle e^{-\frac{i V(x_{n-1})}{\hbar}\frac{t}{n}}$$

($\hat{H} = \frac{\hat{p}^2}{2m} + \hat{V}$)

$$\cdots \langle x_1 | e^{-i\frac{\hat{H}_0 t}{\hbar n}} | x_0 \rangle e^{-i\frac{V(x_0) t}{\hbar n}}$$

$$\simeq \prod_{i=1}^{n} \int dx_i \sqrt{\frac{m}{i 2\pi \Delta t \hbar}} \; e^{i S/\hbar}$$

$$S \simeq \sum_{i=1}^{n-1} \Delta t \left[\frac{m}{2} \left(\frac{x_{i+1} - x_i}{\Delta t} \right)^2 - V(x_i) \right] \underset{n\to\infty}{\longrightarrow} \int dt \Big[\underbrace{\frac{m}{2} \dot{x}^2 - V(x)}_{L} \Big]$$

Euclidean time ($\hbar = 1$)

$$t \rightarrow -i\tau \qquad e^{-it\hat{H}} \rightarrow e^{-\tau\hat{H}}$$

$$iS = \int dt \left[\frac{m}{2}\dot{x}^2 - V(x)\right] \rightarrow i(-i)\int d\tau \left[-\frac{m}{2}\left(\frac{dx}{d\tau}\right)^2 - V(x)\right]$$

$$= -\underbrace{\int d\tau \left[\frac{m}{2}\left(\frac{dx}{d\tau}\right)^2 + V(x)\right]}_{S_E}$$

Quantum mechanics

$$S_E \simeq \sum_{ij} x_i K_{ij} x_j$$ for quadratic potential

(i, j → time index)

Quantum field theory!

$$S_E \simeq \sum_{x,y} \phi_x K_{xy} \phi_y$$

(ϕ → fields; x, y → space time indices)

$$\simeq \int d^D x \left[\frac{1}{2}(\partial_\mu \phi)(\partial^\mu \phi) + \frac{1}{2} m^2 \phi^2\right] \quad \rightarrow \text{free fields}$$

2. Classical gauge invariance and Gauss's law. Sign conventions for Maxwell's fields ($c = 1$ units)

$$
\begin{array}{rcl}
g_{\mu\nu} & = & diagonal(+ - - -) \\
A^{\mu} & : & (\phi, +\mathbf{A}) \\
\partial_{\mu} & : & (\partial/\partial t, +\boldsymbol{\nabla}) \\
\Box & = & \partial_{\mu}\partial^{\mu} \\
F^{i0} & = & E^{i} \\
F^{ij} & = & -\epsilon^{ijk}B^{k} \ (D = 4) \\
F^{12} & = & -B \ (D = 3)
\end{array}
$$

These are consistent with the standard $D - 1$-vector form:

$$\mathbf{E} = -\boldsymbol{\nabla}\phi - \dot{\mathbf{A}},$$

for any D, and the dimension-dependent relations

$$\boldsymbol{\nabla} \times \mathbf{A} = \begin{cases} \mathbf{B} \ (D = 4) \\ B \ (D = 3). \end{cases}$$

Manifestly relativistic form of Maxwell's equations

The basic object is the antisymmetric field-strength tensor

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu .$$

In MKSA units, the Lagrangian density is

$$\mathcal{L}^{Maxwell} = -(1/4)F_{\mu\nu}F^{\mu\nu} - \mu_0 J^\mu A_\mu .$$

The action is $S = \int d^4x \mathcal{L}^{Maxwell}$.

Requiring $\delta S = 0$ yields Maxwell's equations with charge and currents

$$\partial_\mu F^{\mu\nu} = \mu_0 J^\nu .$$

If we identify $F^{0i} = -E^i/c$, $F^{ij} = -\epsilon^{ijk}B^k$, $J^0 = \rho c$, $A^0 = \phi/c$ while the 3-vectors of J^μ and A^μ are the usual currents **J** and potentials **A**,we recover $\nabla \times \mathbf{B} = \mu_0(\mathbf{J} + \epsilon_0\dot{\mathbf{E}})$ and $\nabla \cdot \mathbf{E} = \rho/\epsilon_0$ with $\mu_0\epsilon_0 = 1/c^2$.

Dual equations in D-dimensions ($c = 1$ units).

$F^{\mu\nu}$ can be defined in arbitrary space-time dimensions D

The identification of the electric field $F^{0i} = -E^i$ remains valid for any D.

The identification of the magnetic field is D-dependent.

- $D = 2$: no magnetic field
- $D = 3$ it is just a parity odd (pseudo)scalar density: $B = F^{21}$

The homogeneous Maxwell equations are D-dependent.

For $D = 4$, we can define a *dual* tensor

$$\tilde{F}^{\mu\nu} \equiv \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma} = \epsilon^{\mu\nu\rho\sigma}\partial_\rho A_\sigma,$$

where $\epsilon^{\mu\nu\rho\sigma}$ is the totally antisymmetric Levi-Civita tensor ($\epsilon^{0123} = +1$). Since two gradients commute,

$$\partial_\mu \tilde{F}^{\mu\nu} = 0$$

Exercises: Check that this equation implies $\dot{\mathbf{B}} = -\nabla \times \mathbf{E}$ and $\nabla \cdot \mathbf{B} = 0$. Consider $D = 3$ with $\tilde{F}^{\mu} \equiv \frac{1}{2}\epsilon^{\mu\rho\sigma}F_{\rho\sigma}$.

Gauge invariance in classical electrodynamics

The description of the **E** and **B** fields in terms of A_μ is not one to one.
The gauge transformation

$$A_\mu \rightarrow A_\mu - \partial_\mu \alpha$$

leaves $F_{\mu\nu}$ unchanged.
For a manifestly relativistic formulation, the Lorenz gauge

$$\partial_\mu A^\mu = 0,$$

is Lorentz-invariant and plays a special role (the two last names are very similar but different). The Lorenz gauge condition has a residual invariance: under a gauge transformation

$$\partial_\mu A^\mu \rightarrow -\Box \alpha,$$

has no effect if α is a solution of the massless Klein-Gordon equation.
In the Lorenz gauge, Maxwell equations with charges and currents are simply

$$\Box A^\nu = \mu_0 J^\nu.$$

Sourceless solutions in the Lorenz gauge

$$A^\mu(x) = \epsilon^\mu(p)\exp(-ip\cdot x),$$

with the condition $p_\mu p^\mu = 0$. The Lorenz gauge condition implies that

$$p_\mu \epsilon^\mu(p) = 0.$$

The residual gauge symmetry allows:

$$\epsilon^\mu(p) \rightarrow \epsilon^\mu(p) + \lambda p^\mu,$$

and we are left with $D-2$ transverse polarizations. As an example in $D=4$, if p^μ represents the motion in the z direction $(E,0,0,E)$, the polarization ϵ^μ can be linear combinations of $(1,0,0,1)$, $(0,1,0,0)$ and $(0,0,1,0)$. The first possibility can be eliminated with the residual gauge transformation and we are left with two transverse polarizations ϵ^1 and ϵ^2. After this is done, we end up with a plane wave solution satisfying the conditions $A^0 = 0$ and $\boldsymbol{\nabla}\cdot\mathbf{A} = 0$.

Gauge fixing term

It is possible to add a gauge symmetry breaking term

$$\mathcal{L}^{gsb} = -\frac{\lambda}{2}(\partial_\mu A^\mu)^2$$

With this extra term, the equations of motion become

$$\Box A^\nu + (\lambda - 1)\partial^\nu(\partial_\mu A^\mu) = \mu_0 J^\nu.$$

By picking $\lambda = 1$, we recover

$$\Box A^\nu = \mu_0 J^\nu.$$

This choice is called the “Feynman gauge".
Physical processes should not depend on λ

The Hamiltonian formalism for Maxwell's gauge fields

The time derivative of A_0 does not appear in $\mathcal{L}^{Maxwell}$ and the canonically conjugate momentum is naively zero, but the variation of A_0 is needed to obtain the equation of motion (in absence of sources):

$$\partial_\mu F^{\mu 0} = 0,$$

which in any dimensions is equivalent to Gauss's law

$$\mathbf{\nabla} \cdot \mathbf{E} = 0.$$

With a symmetry breaking term

$$\mathcal{L}^{MGF} = -(1/4)F_{\mu\nu}F^{\mu\nu} - \frac{\lambda}{2}(\partial_\mu A^\mu)^2,$$

This introduces terms including the time derivative of A_0.

Conjugate momenta

$$\frac{\partial \mathcal{L}^{MGF}}{\partial \dot{\mathbf{A}}} = \dot{\mathbf{A}} + \boldsymbol{\nabla} A^0 = -\mathbf{E} \equiv \boldsymbol{\pi},$$

and

$$\frac{\partial \mathcal{L}^{MGF}}{\partial \dot{A}^0} = -\lambda \partial_\mu A^\mu \equiv \pi^0.$$

We can eliminate the time derivatives using the conjugate momenta:

$$\dot{\mathbf{A}} = \boldsymbol{\pi} - \boldsymbol{\nabla} A^0, \tag{2}$$

$$\dot{A}^0 = -\frac{\pi^0}{\lambda} - \boldsymbol{\nabla} \cdot \mathbf{A}. \tag{3}$$

Hamiltonian density

$$\mathcal{H}^{MGF} = \boldsymbol{\pi} \cdot \dot{\mathbf{A}} + \pi^0 \dot{A}^0 - \mathcal{L}^{MGF}$$

After eliminating the time derivatives, we obtain for $D = 4$

$$\mathcal{H}^{MGF} = \frac{1}{2}\boldsymbol{\pi} \cdot \boldsymbol{\pi} + \frac{1}{2}\mathbf{B} \cdot \mathbf{B} - \boldsymbol{\pi} \cdot \boldsymbol{\nabla} A^0 - \frac{1}{2\lambda}(\pi^0)^2 - \pi^0 \boldsymbol{\nabla} \cdot \mathbf{A}.$$

$\boldsymbol{\pi} = -\mathbf{E}$ (any D); $\mathbf{B} = \boldsymbol{\nabla} \times \mathbf{A}$ ($D = 4$);
$\mathbf{A}$ and $\boldsymbol{\pi}$ are conjugate variables (like x and p)
For $D = 3$, we just replace $\mathbf{B} \cdot \mathbf{B}$ by B^2.
For $D = 2$, there is no magnetic field.

Hamilton equations.

$$\dot{\mathbf{A}} = \frac{\partial \mathcal{H}^{MGF}}{\partial \boldsymbol{\pi}} = \boldsymbol{\pi} - \boldsymbol{\nabla} A^0.$$

This just the definition of the electric field in terms of the potentials.

Hamilton equations. (continued)

The second Hamilton equation is dimension dependent.
For $D = 4$, we have

$$\dot{\boldsymbol{\pi}} = -\frac{\partial \mathcal{H}^{MGF}}{\partial \mathbf{A}} = -\boldsymbol{\nabla} \times \mathbf{B} - \boldsymbol{\nabla}\pi^0.$$

Maxwell equation involving the current **j** (set to zero) but with an extra term $-\boldsymbol{\nabla}\pi^0$.

For $D = 2$ this Maxwell equation reads $\dot{E}_x = 0$.

For $D = 3$ we have $\dot{E}_x = \frac{\partial}{\partial y}B$ and $\dot{E}_y = -\frac{\partial}{\partial x}B$

For $D = 4$, we have the usual form $\dot{\mathbf{E}} = \boldsymbol{\nabla} \times \mathbf{B}$.

Hamilton's equations for A^0

$$\dot{A}^0 = \frac{\partial \mathcal{H}^{MGF}}{\partial \pi^0} = -\frac{\pi^0}{\lambda} - \nabla \cdot \mathbf{A},$$

(relation between velocities and momenta used to transition from Lagrangian to Hamiltonian: $\pi^0 = -\lambda(\partial_\mu A^\mu)$).

$$\dot{\pi}^0 = -\frac{\partial \mathcal{H}^{MGF}}{\partial A^0} = -\nabla \cdot \boldsymbol{\pi} = \nabla \cdot \mathbf{E}.$$

This is Gauss's law ($\nabla \cdot \mathbf{E} = 0$) with an extra term $\dot{\pi}^0$.

We recover Maxwell equations if we impose $\pi^0 = 0$ or equivalently the Lorenz condition $\partial_\mu A^\mu = 0$.

The quantization of the classical model using an Hilbert space of particles carrying momenta and polarization is non-trivial (see Weinberg, The Quantum Theory of Fields I). It is much simpler in the path integral formalism.

Euclidean time

Minkowskian	Euclidean
t	-iτ
e^{-itH}	$\mathrm{e}^{-\tau H}$
$g^{\mu\nu}$	$\delta^{\mu\nu}$
d^Dx	$-id^Dx_E$
$\mathbf{x}$	$\mathbf{x}$
∇	∇
$\mathbf{A}$	$\mathbf{A}$
$\mathbf{B}$	$\mathbf{B}$
ϕ	$-iA^D$
$\frac{\partial \mathbf{E}_M}{\partial t}$	$-\frac{\partial \mathbf{E}_E}{\partial \tau}$

The last two lines of the table require some explanations.

Euclidean signs

At Euclidean time, we keep the Minkowskian definition of the field strenght tensor

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu,$$

but all the indices can be raised or lowered without changing the sign. We define

$$E^j_E = F^{jD}_E,$$

which should transform like $\partial/\partial t \to i\partial/\partial\tau$ under $t \to -i\tau$. This can be accomplished with $\phi \to -iA^D$ because we can lower the index of the spatial index without changing the sign as done in Minkowski space. As we keep the standard relation $\boldsymbol{\nabla} \times \mathbf{A} = \mathbf{B}$ which involves only 3-vectors, we have for the same reason

$$F^{jk}_E = +\epsilon^{jkl} B^l_E.$$

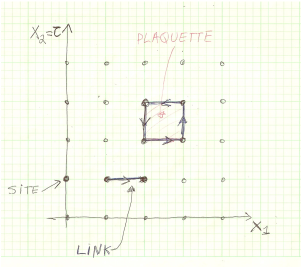
With these definitions, the sign in Maxwell equation changes.

$$\dot{\mathbf{E}}_E = -\boldsymbol{\nabla} \times \mathbf{B}_E.$$

3. Lattice quantization of spin and gauge models

Euclidean time: treat space and time on the same footing.

We discretize space and time using a D-dimensional (hyper) cubic Euclidean lattice. For instance, for $D = 2$, we use a square lattice. The *sites* are denoted $x = (x_1, x_2, \ldots x_D)$, with $x_D = \tau$, the Euclidean time direction. In lattice units, the space-time sites are labelled with integers $\frac{x}{a} = n_1\hat{1} + \cdots + n_D\hat{D}$, where $n_1 \ldots, n_D$ are integers and $\hat{1}, \ldots, \hat{D}$ units vectors in the the D positive directions.



Notations

The *links* between two nearest neighbor lattice sites x and $x + \hat{\mu}$ are labelled by (x, μ) and the *plaquettes* delimited by four sites x, $x + \hat{\mu}$, $x + \hat{\mu} + \hat{\nu}$ and $x + \hat{\nu}$ are labelled by (x, μ, ν). By convention, we start with the lowest index when introducing a circulation at the boundary of the plaquette as shown in Fig. 11.

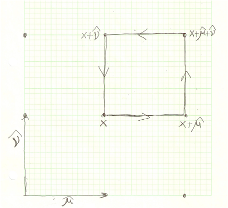


Figure: Plaquette associated with (x, μ, ν).

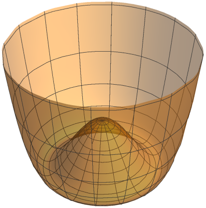
The total number of sites is denoted V. Unless specified periodic boundary conditions are assumed. They preserve a discrete translational symmetry.

Spin models

Euclidean Lagrangian density for N real scalar fields with a $O(N)$ global symmetry.

$$\mathcal{L}^{O(N)}_{Euclidean} = \frac{1}{2}\partial_\mu\vec{\phi}\cdot\partial_\nu\vec{\phi}\delta^{\mu\nu} + \lambda(\vec{\phi}\cdot\vec{\phi} - v^2)^2,$$

with $\vec{\phi}$ a N-dimensional vector.
The potential has degenerate minima on a $N-1$-dimensional hyper-sphere S_{N-1} and a local maximum at $\vec{\phi}=0$.
For $N=2$, the potential has the following shape



Nambu-Goldstone and Brout-Englert-Higgs modes

- The degenerate minima form a circle at the very bottom.
- We can study the small fluctuations about a given minimum on the circle. Note that the choice of a minimum breaks the $O(2)$ symmetry.
- There are “soft" fluctuations along the circle (Nambu-Goldstone modes) that somehow restore the symmetry and "hard" fluctuations in the radial direction (Brout-Englert-Higgs modes).
- We can extend this analysis for arbitrary N.
- In the large λ limit we obtain a Lagrangian which is just the kinetic term to be supplemented with the constraint that $\vec{\phi} \cdot \vec{\phi} = v^2$ everywhere. This keeps the NG modes and decouples the BEH mode.

Spin models

Euclidean action for the NG modes on a D-dimensional lattice with isotropic lattice spacing a

$$S_E = \frac{1}{2}\sum_{x,\mu} a^{D-2}(\vec{\phi}_{x+\hat{\mu}} - \vec{\phi}_x)\cdot(\vec{\phi}_{x+\hat{\mu}} - \vec{\phi}_x).$$

The constraint $\vec{\phi}_x \cdot \vec{\phi}_x = v^2$ can be expressed by introducing unit vectors: $\vec{\phi}_x = v\vec{\sigma}_x$ such that

$$\vec{\sigma}_x \cdot \vec{\sigma}_x = 1.$$

Redefining $a^{D-2}v^2 \equiv \beta$, we get the simple action

$$S = \beta\sum_{x,\mu}(1 - \vec{\sigma}_{x+\hat{\mu}} \cdot \vec{\sigma}_x).$$

These models are often called spin models or nonlinear sigma models. The case $N = 1$ is the well-known Ising model with $\sigma_x = \pm 1$.

Terminology

The case $N = 1$ is the well-known Ising model with $\sigma_x = \pm 1$.
For $N = 2$, the terminology $O(2)$-model, “planar model" or “classical XY model" is common. If we use the circle parametrization

$$\sigma^1_x = \cos(\varphi_x), \text{and } \sigma^2_x = \sin(\varphi_x),$$

then

$$\vec{\sigma}_{x+\hat{\mu}} \cdot \vec{\sigma}_x = \cos(\varphi_{x+\hat{\mu}} - \varphi_x).$$

For $N = 3$, the symmetry becomes non-Abelian and the model is sometimes called the “classical Heisenberg model".
In the large-N limit, the model becomes solvable if we take limit in such a way that $N/\beta(N) = \lambda_t$ remains constant.

Complex generalizations and local gauge invariance

I$O(2)$ model using the complex form $\Phi_x = \mathrm{e}^{i\varphi_x}$. Droping the constant term, the $O(2)$ action reads

$$S = -\frac{\beta}{2}\sum_{x,\mu}(\Phi_x^\star \Phi_{x+\hat{\mu}} + h.c) = -\beta \sum_{x,\mu} \cos(\varphi_{x+\hat{\mu}} - \varphi_x).$$

The $O(2)$ model has a global symmetry $\varphi_x \rightarrow \varphi_x + \alpha$. With the complex notation, this transformation becomes $\Phi_x \rightarrow \mathrm{e}^{i\alpha}\Phi_x$. We would like to promote this symmetry to a local one

$$\Phi_x \rightarrow \mathrm{e}^{i\alpha_x}\Phi_x.$$

This can be achieved by inserting a phase $U_{x,\mu}$ between $\Phi_x^\star$ and $\Phi_{x+\hat{\mu}}$ which transforms like

$$U_{x,\mu} \rightarrow \mathrm{e}^{i\alpha_x} U_{x,\mu} \mathrm{e}^{-i\alpha_{x+\hat{\mu}}}.$$

Non-Abelian generalizations

The procedure can be extended for arbitrary N-dimensional complex vectors $\mathbf{\Phi}_x$ with a local transformation involving a $U(N)$ matrix V_x:

$$\mathbf{\Phi}_x \to V_x \mathbf{\Phi}_x$$

In addition, we introduce $U(N)$ matrices $\mathbf{U}_{x,\hat{\mu}}$ transforming like

$$\mathbf{U}_{x,\hat{\mu}} \to V_x \mathbf{U}_{x,\hat{\mu}} V^{\dagger}_{x+\hat{\mu}}.$$

The action

$$S = -\frac{\beta}{2} \sum_{x,\mu} (\mathbf{\Phi}^{\dagger}_x \mathbf{U}_{x,\hat{\mu}} \mathbf{\Phi}_{x+\hat{\mu}} + h.c) =$$

has a local $U(N)$ gauge invariance. By taking the product of a set of $\mathbf{U}_{x,\mu}$ attached to the links of closed loops, we can construct gauge-invariant quantities.

Replacing Φ_x by a $SU(N)$ matrix $\mathbf{U}_x$ we get the principal chiral model.

$$S = -\frac{\beta}{2} \sum_{x,\mu} \left[\text{tr} \left[\mathbf{U}^{\dagger}_{x+\hat{\mu}} \mathbf{U}_x \right] + h.c. \right]$$

Pure gauge theories

If we consider two successive links in positive directions, then the local transformation in the middle site cancel and

$$\mathbf{U}_{x,\mu}\mathbf{U}_{x+\hat{\mu},\nu} \rightarrow V_x\mathbf{U}_{x,\mu}\mathbf{U}_{x+\hat{\mu},\nu}V^{\dagger}_{x+\hat{\mu}+\hat{\nu}}.$$

If the second link goes in the negative direction, we use the Hermitian conjugate and a similar property holds

$$\mathbf{U}_{x,\mu}\mathbf{U}^{\dagger}_{x+\hat{\mu}-\hat{\nu},\nu} \rightarrow V_x\mathbf{U}_{x,\mu}\mathbf{U}^{\dagger}_{x+\hat{\mu}-\hat{\nu},\nu}V^{\dagger}_{x+\hat{\mu}-\hat{\nu}}.$$

We can pursue this process for an arbitrary path connecting x to some x_{final}. The transformation on the right will be $V^{\dagger}_{x_{final}}$. If we close the path and take the trace, we obtain a gauge-invariant quantity. We call these traces of products of gauge matrices over closed loops “Wilson loops". In the case where the loop goes around the imaginary time direction, we call it a “Polyakov loop"

Wilson action (non-Abelian)

On a square, cubic or hypercubic lattice, the smallest path that give a non-trivial Wilson loop is a square. We call this square a plaquette. Claude Itzykson coined this terminology after Ken Wilson's seminar in Orsay in 1973. We explained that a plaquette is specified by (x, μ, ν). The corresponding matrix is

$$\mathbf{U}_{plaquette} = \mathbf{U}_{x,\mu,\nu} = \mathbf{U}_{x,\mu}\mathbf{U}_{x+\hat{\mu},\nu}\mathbf{U}^{\dagger}_{x+\hat{\nu},\mu}\mathbf{U}^{\dagger}_{x,\nu}$$

The simplest gauge-invariant lattice model has an action, called Wilson's action:

$$S_{Wilson} = \beta \sum_{(x,\mu,\nu)} (1 - \frac{1}{2N}(\mathrm{Tr}\,\mathbf{U}_{x,\mu,\nu} + h.c))$$

Wilson action (Abelian)

In the Abelian case ($N = 1$), the matrix reduces to a phase

$$U_{x,\mu} = \mathrm{e}^{iA_{x,\mu}},$$

and there is no need to take the trace. $A_{x,\mu}$ is called the gauge field. The Abelian action reads

$$S_{Abelian} = \beta \sum_{x,\mu<\nu} (1 - \cos(A_{x,\mu} + A_{x+\hat{\mu},\nu} - A_{x+\hat{\nu},\mu} - A_{x,\nu})).$$

Under gauge transformation, the gauge field transformation is

$$A_{x,\mu} \to A_{x,\mu} + \alpha_x - \alpha_{x+\hat{\mu}}$$

In the Ising case, we have $\sigma_{x,\mu} = \pm 1$ at every link and the action reads

$$S_{Ising} = \beta \sum_{x,\mu<\nu} (1 - \sigma_{x,\mu}\sigma_{x+\hat{\mu},\nu}\sigma_{x+\hat{\nu},\mu}\sigma_{x,\nu}).$$

Compact Abelian Higgs Model CAHM

$$Z_{CAHM} = \prod_x \int_{-\pi}^{\pi} \frac{d\varphi_x}{2\pi} \prod_{x,\mu} \int_{-\pi}^{\pi} \frac{dA_{x,\mu}}{2\pi} e^{-S_{gauge}-S_{matter}},$$

$$S_{gauge} = \beta_{pl.} \sum_{x,\mu<\nu} (1 - \cos(A_{x,\mu} + A_{x+\hat{\mu},\nu} - A_{x+\hat{\nu},\mu} - A_{x,\nu})),$$

$$S_{matter} = \beta_{l.} \sum_{x,\mu} (1 - \cos(\varphi_{x+\hat{\mu}} - \varphi_x + A_{x,\mu})).$$

The CAHM is a gauged version of the $O(2)$ model where the global symmetry under a φ shift becomes local

$$\varphi'_x = \varphi_x + \alpha_x$$

and these local changes in S_{matter} are compensated by the gauge field changes

$$A'_{x,\mu} = A_{x,\mu} - (\alpha_{x+\hat{\mu}} - \alpha_x),$$

which also leave S_{gauge} invariant.

The matter fields can be decoupled by simply setting $\beta_{l.} = 0$. As they don't appear in the action, their integration yields a factor 1 and we are left with the pure gauge (PG) $U(1)$ lattice model with partition function

$$Z_{PG} = \prod_{x,\mu} \int_{-\pi}^{\pi} \frac{dA_{x,\mu}}{2\pi} e^{-S_{gauge}}.$$

The decoupling of the gauge fields is less straightforward. Strictly speaking, the O(2) spin model is obtained by removing the gauge fields introduced to make the global symmetry a local one and the partition function of the O(2) model reads

$$Z_{O(2)} = \prod_{x} \int_{-\pi}^{\pi} \frac{d\varphi_x}{2\pi} e^{-S_{O(2)}},$$

with

$$S_{O(2)} = \beta_{l.} \sum_{x,\mu} (1 - \cos(\varphi_{x+\hat{\mu}} - \varphi_x)).$$

Character expansions

From the point of view of quantum computing, we see that using compact fields for models with Abelian symmetries guarantees that we have a discrete set of states. For functions over the multiplicative group $\sigma = \pm 1$, we will need

$$\exp(\beta\sigma) = \cosh(\beta) + \sigma \sinh(\beta).$$

Another useful formula is the Fourier expansion

$$\exp(\beta \cos(\theta)) = \sum_{n=-\infty}^{+\infty} I_n(\beta) \exp(in\theta),$$

where $I_n(\beta)$ is the modified Bessel function of order n. Generalizations of Pontryagin duality to compact non-Abelian groups appear in the Peter-Weyl theorem. This will translate into expansions in spherical harmonics in the following. The elegance and practical implications of having compact fields suggest that we should try to identify physical signatures of compactness that could allow us to tested this hypothesis experimentally. This is a subject of study for the future.

Ising model

For the Ising model, at each link (x, μ) coming out of x in the positive μ-th direction, we use the expansion (56)

$$e^{\beta\sigma_{x+\hat{\mu}}\sigma_x} = \cosh(\beta) \sum_{n_{x,\mu}=0,1} [\sigma_{x+\hat{\mu}}\sqrt{\tanh(\beta)}\sigma_x\sqrt{\tanh(\beta)}\,]^{n_{x,\mu}},$$

This attaches an index $n_{x,\mu}$ at each link (x, μ). It is then possible to pull together various $(\sqrt{\tanh(\beta)}\sigma_x)^n$ and sum over σ_x. Using

$$\sum_{\sigma=\pm 1} \sigma^n = 2\delta(mod[n, 2]),$$

we can rewrite the partition function as the trace of a tensor product:

$$Z = (\cosh(\beta))^{VD}\mathrm{Tr}\prod_x T^{(x)}_{(n_{x-\hat{1},1}, n_{x,1}, \ldots, n_{x,D})}.$$

The local tensor $T^{(x)}$ has $2D$ indices.

Tensor explicit form

$$T^{(x)}_{(n_{x-\hat{1},1},n_{x,1},\dots,n_{x-\hat{D},D},n_{x,D})} = \tag{4}$$

$$\sqrt{t_{n_{x-\hat{1},1}} t_{n_{x,1}}, \dots, t_{n_{x-\hat{D},D}} t_{n_{x,D}}} \times \delta(mod[n_{x,out} - n_{x,in}, 2]),$$

with the definitions

$$\begin{aligned} t_n &\equiv (\tanh(\beta))^n \\ n_{x,in} &\equiv \sum_{\mu} n_{x-\hat{\mu},\mu} \\ n_{x,out} &\equiv \sum_{\mu} n_{x,\mu}, \end{aligned} \tag{5}$$

where the sums over μ run from 1 to D.