\( \Lambda(1405) \) mediated triangle singularity in the \( K^-d \rightarrow p\Sigma^- \): a window to \( K\bar{N} \) subthreshold amplitudes.

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Motivation: $\bar{K}N$ interaction background

Aim:

Study of the $K^-d \rightarrow p\Sigma^- (p\Sigma^- \rightarrow K^-d)$ reactions close to threshold for the first time.

- Process driven by a triangle singularity (TS).
- This reaction have access to $\bar{K}N$ subthreshold amplitudes

$\bar{K}N$ Interaction:

Perturbative QCD is inappropriate to treat low energy hadron interactions.

Chiral Perturbation Theory (ChPT) is an effective theory with hadrons as degrees of freedom which respects the symmetries of QCD.

- limited to a moderate range of energies above threshold
- not applicable close to a resonance (singularity in the amplitude)

But it is not so straight forward ...
Motivation: $\bar{K}N$ interaction background

$\bar{K}N$ interaction is dominated by the presence of the $\Lambda(1405)$ resonance, located only 27 MeV below the $\bar{K}N$ threshold.

- In 1995 Kaiser, Siegel and Weise reformulated the problem in terms of a Unitary extension of ChPT (UChPT) in coupled channels.

The pioneering work -- Kaiser, Siegel, Weise, NP A594 (1995) 325


All of them obtaining in general similar features:
- $\bar{K}N$ scattering data reproduced very satisfactorily
- Two-pole structure of $\Lambda(1405)$
Motivation: $\bar{K}N$ interaction background

This topic has experienced a renewed interest after recent experimental advances:

The energy shift and width of the $1s$ state in kaonic hydrogen measured by SIDDHARTA@DAΦNE fixes the $K^{-}p$ scattering length with a 20% precision!!!

Photoproduction $\gamma p \rightarrow K^{+}\pi \Sigma$ data by the CLAS@Jlab provided detailed line shape results of the $\Lambda(1405)$


Motivation: $\bar{K}N$ interaction background

$K^-p \rightarrow MB\ (S = -1)$ total cross sections from different groups:


Motivation: $\bar{K}N$ interaction background

**Threshold observables obtained from recent studies:**

<table>
<thead>
<tr>
<th>Method</th>
<th>$\gamma$</th>
<th>$R_n$</th>
<th>$R_c$</th>
<th>$a_p(K^-p \rightarrow K^-p)$</th>
<th>$\Delta E_{1s}$</th>
<th>$\Gamma_{1s}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ikeda-Hyodo-Weise (NLO) [23]</td>
<td>2.37</td>
<td>0.19</td>
<td>0.66</td>
<td>$-0.70 + i 0.89$</td>
<td>306</td>
<td>591</td>
</tr>
<tr>
<td>Guo-Oller (fit I + II) [25]</td>
<td>$2.36^{+0.24}_{-0.23}$</td>
<td>$0.188^{+0.028}_{-0.029}$</td>
<td>$0.661^{+0.012}_{-0.011}$</td>
<td>($-0.69 \pm 0.16$) + i ($0.94 \pm 0.11$)</td>
<td>308 ± 56</td>
<td>619 ± 73</td>
</tr>
<tr>
<td>Mizutani et al (Model s) [26]</td>
<td>2.40</td>
<td>0.189</td>
<td>0.645</td>
<td>$-0.69 + i 0.89$</td>
<td>304</td>
<td>591</td>
</tr>
<tr>
<td>Mai-Meissner (fit 4) [29]</td>
<td>$2.38^{+0.09}_{-0.10}$</td>
<td>$0.191^{+0.013}_{-0.017}$</td>
<td>$0.667^{+0.006}_{-0.005}$</td>
<td>$-0.73 + i 0.85$</td>
<td>288$^{+34}_{-32}$</td>
<td>572$^{+39}_{-38}$</td>
</tr>
<tr>
<td>Cieply-Smejkal (NLO) [76]</td>
<td>2.37</td>
<td>0.191</td>
<td>0.660</td>
<td>$-0.73 + i 0.85$</td>
<td>310</td>
<td>607</td>
</tr>
<tr>
<td>Shevchenko (two-pole Model) [77]</td>
<td>2.36</td>
<td>0.191</td>
<td>0.660</td>
<td>$-0.73 + i 0.85$</td>
<td>308</td>
<td>607</td>
</tr>
<tr>
<td>WT+Born+NLO</td>
<td>$2.36^{+0.03}_{-0.03}$</td>
<td>$0.188^{+0.010}_{-0.011}$</td>
<td>$0.659^{+0.005}_{-0.002}$</td>
<td>$-0.65^{+0.02}<em>{-0.08} + i 0.88^{+0.02}</em>{-0.05}$</td>
<td>288$^{+23}_{-8}$</td>
<td>588$^{+9}_{-40}$</td>
</tr>
<tr>
<td>WT+NLO+Born+RES</td>
<td>2.36</td>
<td>0.189</td>
<td>0.661</td>
<td>$-0.64 + i 0.87$</td>
<td>283</td>
<td>587</td>
</tr>
</tbody>
</table>

Exp.  \[2.36 \pm 0.04 \quad 0.189 \pm 0.015 \quad 0.664 \pm 0.011 \quad (-0.66 \pm 0.07) + i (0.81 \pm 0.15) \quad 283 \pm 36 \quad 541 \pm 92\]


\[
\gamma = \frac{\Gamma(K^-p \rightarrow \pi^+\Sigma^-)}{\Gamma(K^-p \rightarrow \pi^-\Sigma^+)} = 2.36 \pm 0.04
\]

\[
R_n = \frac{\Gamma(K^-p \rightarrow \pi^0\Lambda)}{\Gamma(K^-p \rightarrow \text{neutral states})} = 0.664 \pm 0.011
\]

\[
R_c = \frac{\Gamma(K^-p \rightarrow \pi^+\Sigma^-, \pi^-\Sigma^+)}{\Gamma(K^-p \rightarrow \text{inelastic channels})} = 0.189 \pm 0.015
\]
Motivation: $\bar{K}N$ interaction background

$K^-p \rightarrow K^-p$ scattering amplitudes generated by recent chirally motivated approaches:

Motivation: $\overline{K}N$ interaction background

Pole positions of the $\Lambda(1405)$ for some state-of-the-art models:

Motivation: $\overline{K}N$ interaction background

Many efforts have been made in order to extract information about subthreshold amplitudes…

Fit to photoproduction data from CLAS

$K^-n \rightarrow \pi^-\Lambda$ amplitude (pure $I = 1$ process)

AMADEUS collaboration, KLOE detector at DAFNE
**Motivation: \(\bar{K} N\) interaction background**

*Exciting results reported by GlueX Collaboration in the \(\pi^0\Sigma^0\) invariant mass distribution from the \(\gamma p \rightarrow K^+\pi^0\Sigma^0\) process!!!*

- The fitting procedure supports the composite state nature of the \(\Lambda(1405)\)
- Very valuable information can be extracted from further experimental and theoretical analysis of these data.

Nilanga Wickramaarachchi et al., oral presentation at HYP2022, Prague.
\[ p \Sigma^- \rightarrow K^- d \] reaction proceeds via these 2 mechanisms:

\[ p(P^0, \vec{P}) \]

\[ \Lambda^*(P^0 - q^0, \vec{P} - \vec{q}) \]

\[ K^- \]

\[ \Sigma^- (P^0, -\vec{P}) \]

\[ n(P^0 + q^0, -\vec{P} + \vec{q}) \]

\[ (a) \]

\[ \Sigma^- (P^0, -\vec{P}) \]

\[ \Lambda^*(P^0 - q^0, -\vec{P} - \vec{q}) \]

\[ K^- \]

\[ p(P^0, \vec{P}) \]

\[ \pi^-(q) \]

\[ n(P^0 + q^0, \vec{P} + \vec{q}) \]

\[ (b) \]

\[ -it^{(a)} = (-i)g_{\Lambda^*K^-p}(-i)g_{\Lambda^*K^-p}D - F \]

\[ \frac{1}{2f} \int \frac{d^4q}{(2\pi)^4} \frac{\bar{\sigma}_2 q\cdot k}{q^2 - m_K^2 + i\epsilon} \frac{M_{\Lambda^*}}{E^*_{\Lambda^*}} P^0 - q^0 - E^*_{\Lambda^*}(\vec{P} - \vec{q}) + i\Gamma_{\Lambda^*}/2 \]

\[ \times \frac{M_N}{E_N} P^0 - q^0 - k^0 - E_N(\vec{P} - \vec{q} - \vec{k}) + i\epsilon \frac{M_N}{E^*_{\Lambda^*}} P^0 + q^0 - E^*_{\Lambda^*}(\vec{P} + \vec{q}) + i\epsilon \theta(q_{\text{max}} - |\vec{P} - \vec{q} - \vec{k}/2|), \]

\[ -it^{(b)} = -g_{\Lambda^*K^-p}g_{\Lambda^*\pi} - g_{\Sigma^-g_{\Lambda^*\pi}} \frac{f_{\pi NN}}{m_\pi} \int \frac{d^4q}{(2\pi)^4} \frac{1}{q^2 - m_\pi^2 + i\epsilon} \frac{M_{\Lambda^*}}{E^*_{\Lambda^*}} P^0 - q^0 - E^*_{\Lambda^*}(\vec{P} - \vec{q}) + i\Gamma_{\Lambda^*}/2 \]

\[ \times \bar{\sigma}_1 \cdot \vec{q} \frac{M_N}{E_N} P^0 - q^0 - k^0 - E_N(-\vec{P} - \vec{q} - \vec{k}) + i\epsilon \frac{M_N}{E^*_{\Lambda^*}} P^0 + q^0 - E^*_{\Lambda^*}(\vec{P} + \vec{q}) + i\epsilon \theta(q_{\text{max}} - | - \vec{P} - \vec{q} - \vec{k}/2|), \]
Formalism I: Mechanisms + Amplitudes

\[ -it_{ij}^{(a)} = g_{\Lambda^+,K^-p} \, g_{\Lambda^+,K^-p} \, g_d \frac{D-F}{2f} \int \frac{d^3q}{(2\pi)^3} V_{ij}(q) F(P^0, P'^0, \bar{q}, \omega_K(q), \vec{P}, \vec{k}) F^2(\Lambda, m_K) \]

\[ -it_{ij}^{(b)} = -g_{\Lambda^+,K^-p} \, g_{\Lambda^+,\pi^+} \, g_d \, \frac{f_{\pi NN}}{m_\pi} \int \frac{d^3q}{(2\pi)^3} W_{ij}(q) F(P'^0, P^0, \bar{q}, \omega_\pi(q), -\vec{P}, \vec{k}) F^2(\Lambda, m_\pi) \]

\[
F(P^0, P'^0, \bar{q}, \omega, \vec{P}, \vec{k}) = \frac{1}{2\omega(q)} \frac{M_N}{E_N(\vec{P} - \vec{q} - \vec{k})} \frac{M_N}{E_N(-\vec{P} + \vec{q})} \frac{M_{\Lambda^+}}{E_{\Lambda^+}(\vec{P} - \vec{q})} \\
\times \frac{\theta(q_{max} - |\vec{P} - \vec{q} - \vec{k}|)}{\sqrt{s - k^0} - E_N(-\vec{P} + \vec{q}) - E_N(\vec{P} - \vec{q} - \vec{k}) + i\epsilon} \\
\times \left\{ \begin{array}{c}
1 \\
\frac{1}{P^0 - \omega(q) - E_{\Lambda^+}(\vec{P} - \vec{q}) + i\frac{\Gamma_{\Lambda^+}}{2}} \frac{1}{P^0 - \omega(q) - k^0 - E_N(\vec{P} - \vec{q} - \vec{k}) + i\epsilon} \\
\frac{1}{P^0 - E_{\Lambda^+}(\vec{P} - \vec{q}) - \omega(q) + i\frac{\Gamma_{\Lambda^+}}{2}} \frac{1}{\sqrt{s - E_{\Lambda^+}(\vec{P} - \vec{q})} - E_N(-\vec{P} + \vec{q}) + i\frac{\Gamma_{\Lambda^+}}{2}} \\
+ \frac{1}{P'^0 - E_N(-\vec{P} + \vec{q}) - \omega(q) + i\epsilon} \frac{1}{\sqrt{s - E_{\Lambda^+}(\vec{P} - \vec{q})} - E_N(-\vec{P} + \vec{q}) + i\frac{\Gamma_{\Lambda^+}}{2}} \\
+ \frac{1}{P'^0 - E_N(-\vec{P} + \vec{q}) - \omega(q) + i\epsilon} \frac{1}{\sqrt{s - E_{\Lambda^+}(\vec{P} - \vec{q})} - E_N(-\vec{P} + \vec{q}) + i\frac{\Gamma_{\Lambda^+}}{2}} \end{array} \right\}
\]
Formalism I: Triangle singularity

*TS can be developed when the 3 intermediate particles \( \Lambda(1405) \) (1), \( n \) (2), \( p \) (3):*

\[ A \xrightarrow{1} B \]
\[ A \xrightarrow{2} C \]
\[ A \xrightarrow{3} \]

1, 2, 3 particles are simultaneously placed on Shell and they are colinear fulfilling Norton-Coleman theorem

These conditions are encoded in the following equation:

\[ q_{on} = q_{a^-} \]


\[ \rightarrow \text{For this study, \( TS \) should appear at } \sqrt{s} \approx 2380 \text{MeV} \]
Formalism I:

Differential cross section for the $K^−d \rightarrow p\Sigma^−$ reaction.

$$\frac{d\sigma}{d\cos \theta_p} = \frac{1}{4\pi} \frac{1}{s} M_p M_{\Sigma^-} M_d \frac{p}{k} \sum \sum |t|^2$$

$$\sum \sum |t|^2 = \frac{1}{3} \sum_{i,j} |t_{ij}(a) + t_{ij}(b)|^2$$

$d (S = 1)$ polarizations

$$j = \uparrow\uparrow, \frac{1}{\sqrt{2}}(\uparrow\downarrow + \downarrow\uparrow), \downarrow\downarrow$$

$p\Sigma^−$ spin configurations

$$i = \uparrow\uparrow, \uparrow\downarrow, \downarrow\uparrow, \downarrow\downarrow$$

Pole couplings and coordinates needed to compute the cross section:

<table>
<thead>
<tr>
<th>State</th>
<th>$g_{\Lambda^* \bar{K}N}$</th>
<th>$g_{\Lambda^* \pi \Sigma}$</th>
<th>(Mass, $\Gamma$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Lambda(1390)$</td>
<td>$1.2 + i 1.7$</td>
<td>$-2.5 - i 1.5$</td>
<td>(1390, 66)</td>
</tr>
<tr>
<td>$\Lambda(1426)$</td>
<td>$-2.5 + i 0.94$</td>
<td>$0.42 - i 1.4$</td>
<td>(1426, 16)</td>
</tr>
</tbody>
</table>

Results I: spin transitions

Contribution of several spin transitions to the \( K^-d \rightarrow p\Sigma^- \) cross section.
Results I: role of the mechanisms and poles in the total cross section

Contribution of the high and low mass poles and the mechanisms (a) and (b) to $K/p \cdot \sigma(K^- d \rightarrow p\Sigma^-)$.
Formalism II: Incorporation of the explicit $\overline{K}N$ amplitudes and $\psi$ Bonn deuteron wave function

Deuteron wave function replacement:

$$
g_d \frac{M_N}{E(\vec{P} - \vec{q} - \vec{k})} \frac{M_N}{E_N(-\vec{P} + \vec{q})} \frac{\theta(q_{\text{max}} - |\vec{P} - \vec{q} - \vec{k}|)}{\sqrt{s} - k^0 - E_N(-\vec{P} + \vec{q}) - E_N(\vec{P} - \vec{q} - \vec{k}) + i\epsilon} \rightarrow -(2\pi)^{3/2} \psi(\vec{P} - \vec{q} - \vec{k}/2)
$$


Formal equivalence between Breit-Wigner amplitudes and theoretical amplitudes:

$$
\sum_{i=1}^{2} \frac{M^{(i)}_{\Lambda^*}}{E^{(i)}_{\Lambda^*}(\vec{P} - \vec{q})} \frac{g^{(i)}_{\Lambda^*, K-p} g^{(i)}_{\Lambda^*, K-p}}{\sqrt{s} - E_N(-\vec{p} + \vec{q}) - E^{(i)}_{\Lambda^*}(\vec{P} - \vec{q}) + i\frac{\Gamma_{\Lambda^*}^{(i)}}{2}} \equiv t_{K-p, K-p}(M_{\text{inv}})
$$

$$
\sum_{i=1}^{2} \frac{M^{(i)}_{\Lambda^*}}{E^{(i)}_{\Lambda^*}(\vec{P} - \vec{q})} \frac{g^{(i)}_{\Lambda^*, K-p} g^{(i)}_{\Lambda^*, \pi+\Sigma^-}}{\sqrt{s} - E_N(\vec{p} + \vec{q}) - E^{(i)}_{\Lambda^*}(-\vec{P} - \vec{q}) + i\frac{\Gamma_{\Lambda^*}^{(i)}}{2}} \equiv t_{K-p, \pi+\Sigma^-}(M'_{\text{inv}})
$$

$$
M_{\text{inv}}^2 = s + M_N^2 - 2\sqrt{s}E_N(-\vec{P} + \vec{q}) \quad M'_{\text{inv}}^2 = s + M_N^2 - 2\sqrt{s}E_N(\vec{P} + \vec{q})
$$
Results II: using explicit $\bar{K}N$ amplitudes + $\psi$ Bonn deuteron wave function

Energy dependence of the real and the imaginary parts of the $K^-d \rightarrow p\Sigma^-$ and $K^-p \rightarrow \pi^+\Sigma^-$ amplitudes.
Results II: using explicit $\bar{K}N$ amplitudes + $\psi$ Bonn deuteron wave function

$K^-d \rightarrow p\Sigma^-$ cross sections for different considered models.
CONCLUSIONS

We have studied the $K^-d \rightarrow p\Sigma^- (p\Sigma^- \rightarrow K^-d)$ reaction via a triangular topology (with two possible mechanisms) that embeds a TS.

- The peak associated to the TS shows up few MeV above $K^-d$ threshold, being clearly visible in the case of the narrow (high mass) $\Lambda(1405)$ state.

- The mechanism involving the pion exchange has shown to be the dominant one.

- We have seen that the particular dependence of the $K^-d \rightarrow p\Sigma^-$ transition on the $\bar{K}N$ amplitudes below threshold weighted by the structures tied to TS makes this process very sensitive to the different models.

The measurement of this reaction will provide valuable information for $\bar{K}$ bound states in nuclei as well as it will help to narrow the uncertainty around the location of the lower mass pole of the $\Lambda(1405)$. 
\[-it_{ij}^{(a)} = g_d \frac{D - F}{2f} \int \frac{d^3q}{(2\pi)^3} V_{ij}(q) F'(P^0, P^{\pi 0}, \bar{q}, \omega_K(q), \bar{P}, \bar{k}) F^2(\Lambda, m_K)\]

\[-it_{ij}^{(b)} = -g_d \frac{D + F}{2f} \int \frac{d^3q}{(2\pi)^3} W_{ij}(q) G'(P^0, P^{\pi 0}, \bar{q}, \omega_\pi(q), \bar{P}, \bar{k}) F^2(\Lambda, m_\pi)\]

\[
F'(P^0, P^{\pi 0}, \bar{q}, \omega_K, \bar{P}, \bar{k}) = \frac{1}{2 \omega(q)} \frac{M_N}{E_N(\bar{P} - \bar{q} - \bar{k})} \frac{M_N}{E_N(-\bar{P} + \bar{q})} \frac{1}{\sqrt{s - k^0 - E_N(-\bar{P} + \bar{q}) - E_N(\bar{P} - \bar{q} - \bar{k}) + i\epsilon}}
\times \left\{ \sum_{i=1,2} \frac{M_{\Lambda^*}^{(i)}}{(g_{\Lambda^* - K - p}^{(i)})^2} \frac{1}{P^0 - \omega_K(q) - E^{(i)}_{\Lambda^*}(\bar{P} - \bar{q}) + i\frac{\Gamma_{\Lambda^*}^{(i)}}{2}} \frac{1}{P^0 - E_N(-\bar{P} + \bar{q}) + i\epsilon} \right\}
\times \left( \frac{1}{P^0 - E_N(-\bar{P} + \bar{q}) - \omega_K(q) + i\epsilon} \right) t_{K^- p, K^- p}(M_{\text{inv}}) \right\}
\]

\[
G'(P^0, P^{\pi 0}, \bar{q}, \omega_\pi, \bar{P}, \bar{k}) = \frac{1}{2 \omega(q)} \frac{M_N}{E_N(-\bar{P} - \bar{q} - \bar{k})} \frac{M_N}{E_N(\bar{P} + \bar{q})} \frac{1}{\sqrt{s - k^0 - E_N(\bar{P} + \bar{q}) - E_N(-\bar{P} - \bar{q} - \bar{k}) + i\epsilon}}
\times \left\{ \sum_{i=1,2} \frac{M_{\Lambda^*}^{(i)}}{g_{\Lambda^* - p_\pi - \Sigma^-}^{(i)} g_{\Lambda^* - p_\pi - \Sigma^-}^{(i)}} \frac{1}{P^0 - \omega_\pi(q) - E^{(i)}_{\Lambda^*}(-\bar{P} - \bar{q}) + i\frac{\Gamma_{\Lambda^*}^{(i)}}{2}} \frac{1}{P^0 - E_N(\bar{P} + \bar{q}) + i\epsilon} \right\}
\times \left( \frac{1}{P^0 - E_N(-\bar{P} - \bar{q}) - \omega_\pi(q) + i\epsilon} \right) t_{K^- p, \pi^+ \Sigma^-}(M_{\text{inv}}') \right\}
\]
Form Factor \[ F(\Lambda, m_i) = \frac{\Lambda^2 - m_i^2}{\Lambda^2 + q^2} \]
Backup slides

S+D-wave contribution in the deuteron WF

\[ \frac{k}{p} \sigma(K_d \rightarrow p\Sigma^-) \text{ (mb)} \]

- S-wave contr. in deuteron WF
- S+D-wave contr. in deuteron WF

\[ s^{1/2} \text{ [MeV]} \]
Motivation: $\bar{K}N$ interaction background

**Effective Chiral Lagrangian:**

\[
\mathcal{L}^{\text{eff}}(B, U) = \mathcal{L}^{(1)}_{MB}(B, U) + \mathcal{L}^{(2)}_{MB}(B, U)
\]

\[\rightarrow \text{derive an interaction kernel } V_{ij}\]

- **Leading order (LO)**

\[
\mathcal{L}^{(1)}_{MB} = \langle \bar{B}(i\gamma_\mu D^\mu - M_0)B \rangle + \frac{1}{2} D\langle \bar{B}\gamma_\mu \gamma_5 \{u^\mu, B\} \rangle + \frac{1}{2} F\langle \bar{B}\gamma_\mu \gamma_5 [u^\mu, B] \rangle
\]
Motivation: $\bar{K}N$ interaction background

Effective Chiral Lagrangian:

$$\mathcal{L}^{\text{eff}}(B, U) = \mathcal{L}^{(1)}_{MB}(B, U) + \mathcal{L}^{(2)}_{MB}(B, U) \Rightarrow \text{derive an interaction kernel } V_{ij}$$

• Leading order (LO)

$$\mathcal{L}^{(1)}_{MB} = \left( \bar{B}(i\gamma_\mu D^\mu - M_0)B \right) + \frac{1}{2}D\langle \bar{B}\gamma_\mu \gamma_5 \{u^\mu, B\} \rangle + \frac{1}{2}F\langle \bar{B}\gamma_\mu \gamma_5[u^\mu, B] \rangle$$

Tomozawa-Weinberg term (WT)

1. Dominant contribution.
2. Interaction mediated, basically, by the constant $f$ of the leptonic decay of the pseudoscalar meson

$$V^{\text{WT}}_{ij} = -\frac{N_i N_j}{4f^2} C_{ij} \left\{ (2\sqrt{s} - M_i - M_j)X^s_f \chi^s_0 + \frac{2\sqrt{s} + M_i + M_j}{(E_i + M_i)(E_j + M_j)} X^{s'}_f \left[ q_j \cdot q_i + i(\vec{q}_j \times \vec{q}_i) \cdot \vec{\sigma} \right] \chi^s_0 \right\}$$
Motivation: $\bar{K}N$ interaction background

Lagrangian:

$$\mathcal{L}^{\text{eff}}(B, U) = \mathcal{L}^{(1)}_{MB}(B, U) + \mathcal{L}^{(2)}_{MB}(B, U)$$

\[ \rightarrow \text{derive an interaction kernel } V_{ij} \]

• Leading order (LO)

$$\mathcal{L}^{(1)}_{MB} = \langle \bar{B}(i\gamma_\mu D^\mu - M_0)B \rangle + \frac{1}{2} D\langle \bar{B}\gamma_\mu \gamma_5\{u^\mu, B\} \rangle + \frac{1}{2} F\langle \bar{B}\gamma_\mu \gamma_5[u^\mu, B] \rangle$$

1. Direct diagram (s-channel Born term)

$$V_{ij}^D = V_{ij}^D(D, F)$$

2. Cross diagram (u-channel Born term)

$$V_{ij}^C = V_{ij}^C(D, F)$$
Motivation: $\bar{K}N$ interaction background

1. Direct diagram (s-channel Born term)

$$V_{ij}^D = \frac{N_i N_j}{12 f^2} \sum_k \frac{C_{ii,k}^{(\text{Born})} C_{jj,k}^{(\text{Born})}}{s - M_k^2} \left\{ \left( \sqrt{s} - M_k \right) (s + M_i M_j - \sqrt{s} (M_i + M_j)) \chi_j^{s'} \chi_i^s \right. \\
+ \left. \frac{(s + \sqrt{s} (M_i + M_j) + M_i M_j) \left( \sqrt{s} + M_k \right)}{(E_i + M_i)(E_j + M_j)} \chi_j^{s'} \left[ \vec{q}_j \cdot \vec{q}_i + i(\vec{q}_j \times \vec{q}_i) \cdot \vec{\sigma} \right] \chi_i^s \right\}$$

2. Cross diagram (u-channel Born term)

$$V_{ij}^C = -\frac{N_i N_j}{12 f^2} \sum_k \frac{C_{j,k,i}^{(\text{Born})} C_{i,k,j}^{(\text{Born})}}{u - M_k^2} \left\{ u(\sqrt{s} + M_k) + \sqrt{s} (M_j (M_i + M_k) + M_i M_k) \\
- M_j (M_i + M_k)(M_i + M_j) - M_i^2 M_k \right\} \chi_j^{s'} \chi_i^s + \left[ u(\sqrt{s} - M_k) + \sqrt{s} (M_j (M_i + M_k) + M_i M_k) \\
+ M_j (M_i + M_k)(M_i + M_j) + M_i^2 M_k \right] \chi_j^{s'} \frac{\vec{q}_j \cdot \vec{q}_i + i(\vec{q}_j \times \vec{q}_i) \cdot \vec{\sigma}}{(E_i + M_i)(E_j + M_j)} \chi_i^s \left\}$$
Motivation: $\bar{K}N$ interaction background

- **Next to leading order (NLO), just considering the contact term**

\[
\mathcal{L}_{\phi B}^{(2)} = b_D \langle \bar{B}\{\chi_+, B\} \rangle + b_F \langle \bar{B}[\chi_+, B] \rangle + b_0 \langle \bar{B}B \rangle \langle \chi_+ \rangle + d_1 \langle \bar{B}\{u_\mu, [u^\mu, B]\} \rangle \\
+ d_2 \langle \bar{B}[u_\mu, [u^\mu, B]] \rangle + d_3 \langle \bar{B}u_\mu \rangle \langle u^\mu B \rangle + d_4 \langle \bar{B}B \rangle \langle u^\mu u_\mu \rangle \\
- \frac{g_1}{8 M_N^2} \langle \bar{B}\{u_\mu, [u_\nu, \{D^\mu, D^\nu\} B]\} \rangle - \frac{g_2}{8 M_N^2} \langle \bar{B}[u_\mu, [u_\nu, \{D^\mu, D^\nu\} B]\rangle \\
- \frac{g_3}{8 M_N^2} \langle \bar{B}u_\mu \rangle \langle [u_\nu, \{D^\mu, D^\nu\} B]\rangle - \frac{g_4}{8 M_N^2} \langle \bar{B}\{D^\mu, D^\nu\} B \rangle \langle u_\mu u_\nu \rangle \\
- \frac{h_1}{4} \langle \bar{B}[\gamma^\mu, \gamma^\nu] B u_\mu u_\nu \rangle - \frac{h_2}{4} \langle \bar{B}[\gamma^\mu, \gamma^\nu] u_\mu [u_\nu, B] \rangle - \frac{h_3}{4} \langle \bar{B}[\gamma^\mu, \gamma^\nu] u_\mu \{u_\nu, B\} \rangle \\
- \frac{h_4}{4} \langle \bar{B}[\gamma^\mu, \gamma^\nu] u_\mu \rangle \langle u_\nu, B \rangle + h.c.
\]

- Contributions with $g_3$ get cancelled
- $b_0, b_D, b_F, d_1, d_2, d_3, d_4, g_1, g_2, g_4, h_1, h_2, h_3, h_4$ are not well established, so they should be treated as parameters of the model!
Motivation: $\bar{K}N$ interaction background

- **Next to leading order (NLO), just considering the contact term**

\[
V_{ij}^{NLO} = \frac{N_i N_j}{f^2} \left[ D_{ij} - 2L_{ij} q_j^\mu q_i^\mu + \frac{1}{2M_N^2} g_{ij}(p_i^\mu q_j^\nu p_i^\nu q_i^\nu + p_j^\mu q_j^\nu p_j^\nu q_i^\nu) \right] \left( \chi_j^{s'} \chi_i^s \right. \\
\left. - \chi_j^{s'} \bar{q}_j \cdot \bar{q}_i + i(\bar{q}_j \times \bar{q}_i) \cdot \vec{\sigma} \chi_i^s \right) + \frac{N_i N_j}{f^2} h_{ij} \left[ - \left( \frac{q_{i0}}{E_i + M_i} + \frac{q_{j0}}{E_j + M_j} \right) \chi_j^{s'} \chi_i^s \\
+ \left( \frac{q_{i0}}{E_i + M_i} + \frac{q_{j0}}{E_j + M_j} \right) \chi_j^{s'} \bar{q}_j \cdot \bar{q}_i \chi_i^s \right] \left[ \frac{q_{i0}}{E_i + M_i} + \frac{q_{j0}}{E_j + M_j} \right. \\
\left. + \frac{\bar{q}_j \cdot \bar{q}_i}{(E_i + M_i)(E_j + M_j)} - 1 \right) i\chi_j^{s'} (\bar{q}_j \times \bar{q}_i) \cdot \vec{\sigma} \chi_i^s
\]
**Motivation:** $\overline{K}N$ interaction background

Unitarization via the Bethe-Salpeter equation which it is solved by factorizing $V$ and $T$ matrices on-shell out the internal integrals

\[
T_{ij} = V_{ij} + V_{il} G_l V_{lj} + V_{il} G_l V_{lk} G_k V_{kj} + \ldots
\]

\[
T_{ij} = V_{ij} + V_{il} G_l T_{lj} \quad \rightarrow \quad T_{ij} = (1 - V_{il} G_l)^{-1} V_{lj}
\]

**Pure algebraic equation**

\[
G_l = \frac{2M_l}{(4\pi)^2} \left\{ a_l(\mu) + \ln \frac{M_l^2}{\mu^2} + \frac{m_i^2 - M_l^2 + s}{2s} \ln \frac{m_i^2}{M_l^2} + \frac{q_{cm}}{\sqrt{s}} \ln \left[ \frac{(s+2\sqrt{s}q_{cm})^2 - (M_l^2 - m_i^2)^2}{(s-2\sqrt{s}q_{cm})^2 - (M_l^2 - m_i^2)^2} \right] \right\}
\]

Subtraction constants for the dimensional regularization scale $\mu$ in all the $k$ channels.

\[
V_{ij} = V_{ij}^{WT} + V_{ij}^{D} + V_{ij}^{C} + V_{ij}^{NLO} \quad \Rightarrow \quad T = (1 - VG)^{-1} V \quad \Rightarrow \quad T_{ij}
\]
Motivation: $\overline{K}N$ interaction background

Fitting parameters:

- Decay constant $f$ either partially constrained $1.12 f^\text{exp}_\pi \leq f \leq 1.26 f^\text{exp}_\pi$, $f^\text{exp}_\pi = 93$ MeV or taking fixed values depending on the process $f_\pi, f_K, f_\eta$.

- Axial vector couplings $D, F$ varying around the experimental values yet imposing $g_A = D + F = 1.26$. Most of the models fix them at the exp. values $D = 0.80, F = 0.46$.

- 7 (14) coefficients of the NLO lagrangian terms $b_0, b_D, b_F, d_1, d_2, d_3, d_4$ ($+g_1, g_2, g_4, h_1, h_2, h_3, h_4$).

- Parameters from the regularization method.

  The vast majority employs dim. reg. with 6 subtracting constants (isospin symmetry):

\[
\begin{align*}
  a_{K^-p} &= a_{K^0n} = a_{\overline{K}N} \\
  a_{\pi^+\Sigma^-} &= a_{\pi^-\Sigma^+} = a_{\pi^0\Sigma^0} = a_{\pi\Sigma} \\
  a_{\eta\Lambda} \\
  a_{\eta\Sigma} \\
  a_{K^+\Xi^-} &= a_{K^0\Xi^0} = a_{K\Xi}
\end{align*}
\]
Motivation: $\bar{K}N$ interaction background

Available experimental data used (or not) to constrain the parameters present in the chirally motivated models:

<table>
<thead>
<tr>
<th>Observable</th>
<th>Points</th>
<th>Observable</th>
<th>Points</th>
</tr>
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<tbody>
<tr>
<td>$\sigma_{K^-p\to K^-p}$</td>
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<td>$\sigma_{K^-p\to K^0n}$</td>
<td>9</td>
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<tr>
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<td>$\sigma_{K^-p\to K^0\Xi^0}$</td>
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<td>$\Delta E_{1s}$</td>
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</tr>
<tr>
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<td>$\Gamma_{1s}$</td>
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</tr>
<tr>
<td>$R_c$</td>
<td>1</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Data commonly used in the fitting procedures:

\[
\gamma = \frac{\Gamma(K^-p \to \pi^+\Sigma^-)}{\Gamma(K^-p \to \pi^-\Sigma^+)} = 2.36 \pm 0.04
\]

\[
R_n = \frac{\Gamma(K^-p \to \pi^0\Lambda)}{\Gamma(K^-p \to \text{neutral states})} = 0.664 \pm 0.011
\]

\[
R_c = \frac{\Gamma(K^-p \to \pi^+\Sigma^-, \pi^-\Sigma^+)}{\Gamma(K^-p \to \text{inelastic channels})} = 0.189 \pm 0.015
\]

$K^-p$ scattering data at energies close to the production threshold (obviously above $\bar{K}N$ threshold)

Observables at $\bar{K}N$ threshold

CLAS Photoproduction processes provide subthreshold information (barely used!!!)

\[
d\sigma\left(\gamma p \to \pi \Sigma K^+ \right)
\]

\[
dM_{\pi \Sigma}
\]
Formalism I: Triangle singularity

Explicit integral of the intermediate loop containing the 3 propagators:

\[ I_1 = i \int \frac{d^4 q}{(2\pi)^4} \frac{1}{(q^2 - m_2^2 + i\epsilon)[(P - q)^2 - m_1^2 + i\epsilon][(P - q - p_{13})^2 - m_3^2 + i\epsilon]} \]

Integrating over \( q^0 \) … and taking only the part of the integral containing the singularity structure

\[ I(m_{23}) = \int \frac{d^3 q}{(P^0 - \omega_1(\vec{q}) - \omega_2(\vec{q}) + i\epsilon)(E_{23} - \omega_2(\vec{q}) - \omega_3(\vec{k} + \vec{q}) + i\epsilon)} \]

\[ = 2\pi \int_0^{\infty} dq \frac{q^2}{P^0 - \omega_1(q) - \omega_2(q) + i\epsilon} f(q), \quad f(q) = \int_{-1}^{1} dz \frac{1}{E_{23} - \omega_2(q) - \sqrt{m_3^2 + q^2 + k^2 + 2qkz + i\epsilon}} \]

\[ \omega_{1,2}(q) = \sqrt{m_{1,2}^2 + q^2}, \quad \omega_3(\vec{q} + \vec{k}) = \sqrt{m_3^2 + (\vec{q} + \vec{k})^2}, \quad E_{23} = P^0 - k^0 \]

\[ q = \vec{q}, \quad k = |\vec{k}| = \sqrt{\lambda(M^2, m_{13}^2, m_{23}^2)}, \quad M = \sqrt{P^2}, \quad m_{13,23} = \sqrt{p_{13,23}^2} \]
Formalism I: Triangle singularity

\[ P^0 - \omega_1(\vec{q}) - \omega_2(\vec{q}) + i\epsilon = 0 \]

\[ q_{on^+} = q_{on} + i\epsilon, \quad q_{on} = \frac{1}{2M} \sqrt{\lambda(M^2, m_1^2, m_2^2)} \]
Formalism I: Triangle singularity

\[ P^0 - \omega_1(q) - \omega_2(q) + i\epsilon = 0 \]

\[ q_{on+} = q_{on} + i\epsilon, \quad q_{on} = \frac{1}{2M} \sqrt{\lambda(M^2, m_1^2, m_1^2)} \]

\[ f(q) \] contains end-point singularities (logarithmic branch points) for \( z = \pm 1 \)

\[ E_{23} - \omega_2(q) - \sqrt{m_3^2 + q^2 + k^2 \pm 2qk + i\epsilon} = 0 \]

\[ z = -1 \]

\[ q_{a+} = \gamma(vE_2^* + p_2^*) + i\epsilon \quad q_{b+} = \gamma(-vE_2^* + p_2^*) + i\epsilon \]

\[ q_{a-} = \gamma(vE_2^* - p_2^*) - i\epsilon \quad q_{b-} = -\gamma(vE_2^* + p_2^*) - i\epsilon \]
Formalism I: Triangle singularity

$q_{b+}$ and $q_{a-}$ are mutually exclusive as solutions that are simultaneously in the $q$ (positive) integration range. The interesting casuistry for TS is given by $q_{a-} q_{a+} q_{o^+}$:

$I(m_{23})$ analytic in this Kinematic region

threshold singularity

Triangle singularity (TS)

\[
\lim_{\epsilon \to 0} (q_{o^+} - q_{a^-}) = 0
\]

This is only fulfilled when all three intermediate particles are placed on shell and when:

\[ z = -1 \quad \text{Momentum of part. 2 is anti-parallel to that of (2,3) system from the decaying particle rest system} \]

\[
\omega_1(q_{o^+}) - p_{13}^0 - \sqrt{m_3^2 + (q_{o^+} - k)^2} = 0
\]

→ For this study, TS should appear at $\sqrt{s} \approx 2380\, MeV$