# Phase structure of pure Yang-Mills theory in an anisotropic system: A new extreme condition of QCD

Daiki Suenaga (RIKEN in Japan)

Masakiyo Kitazawa (Osaka U. in Japan)
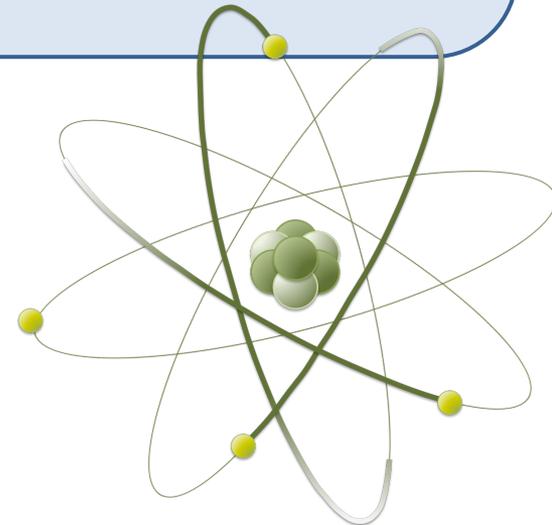
• QCD at finite temperature

- QCD at finite temperature contains fruitful phases and is important toward further understanding of QCD: QGP formation, chiral restoration, etc.


heavy-ion collisions
(at RHIC, LHC, etc.)

lattice simulations

Fukushima-Hatsuda (2017)
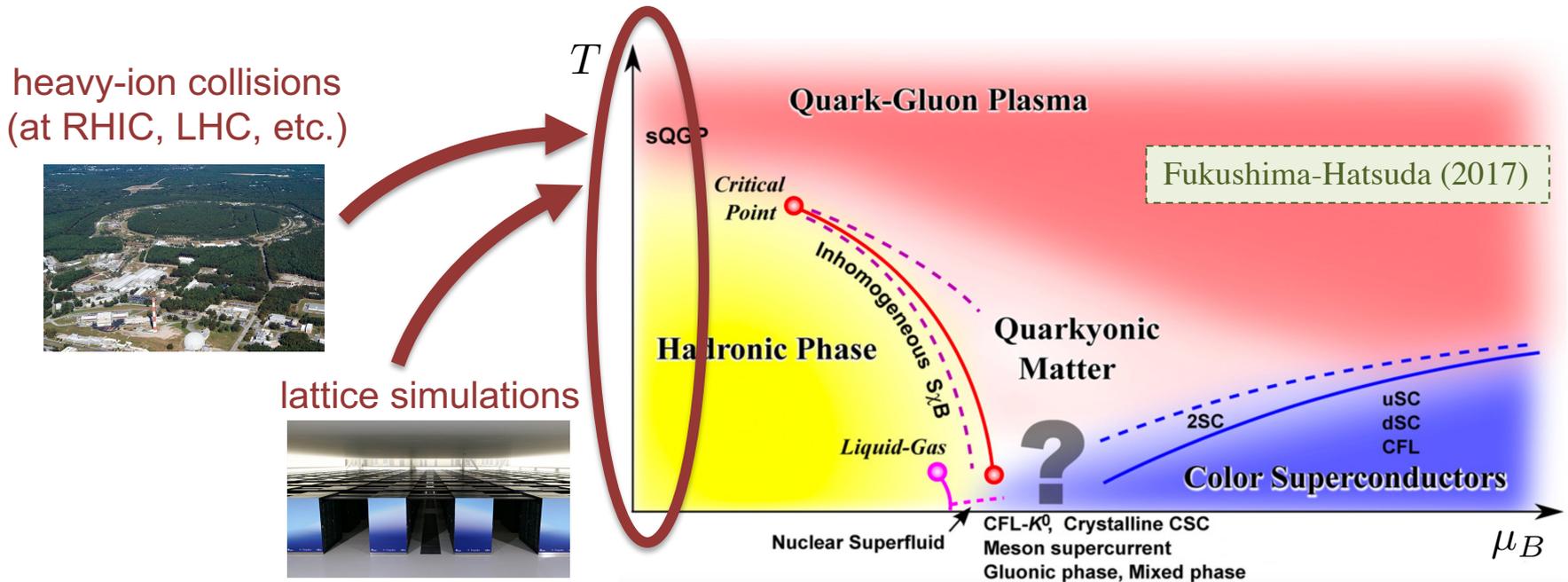
- Many investigations from heavy-ion collision experiments and numerical lattice simulations, and so on

• Compactification

- Finite temperature system is realized by compactifying imaginary time

$$t \to -i\tau$$

$$\tau = 1/T$$
$$\tau = 0$$

(Matsubara's method)

$$A_\mu(0, \vec{x}) = A_\mu(1/T, \vec{x}) \text{ (periodic)}$$
$$\psi(0, \vec{x}) = -\psi(1/T, \vec{x}) \text{ (antiperiodic)}$$

compactification with period $1/T$

- What happened when another spatial axis is also compactified?

$$\infty$$
$$-\infty$$
$$t \quad x \quad y \quad z$$

$$\tau$$
$$x$$
$$y \quad z$$

$$=$$

$$\tau \text{ (finite)}$$
$$x \text{ (finite)}$$
$$y, z$$

anisotropic system
→ new extreme condition!

• Compactification

- Finite temperature system is realized by compactifying imaginary time

∞

t

$t \to -i\tau$

τ

$\tau = 1/T$

$\tau = 0$

$-\infty$

compactification with period $1/T$
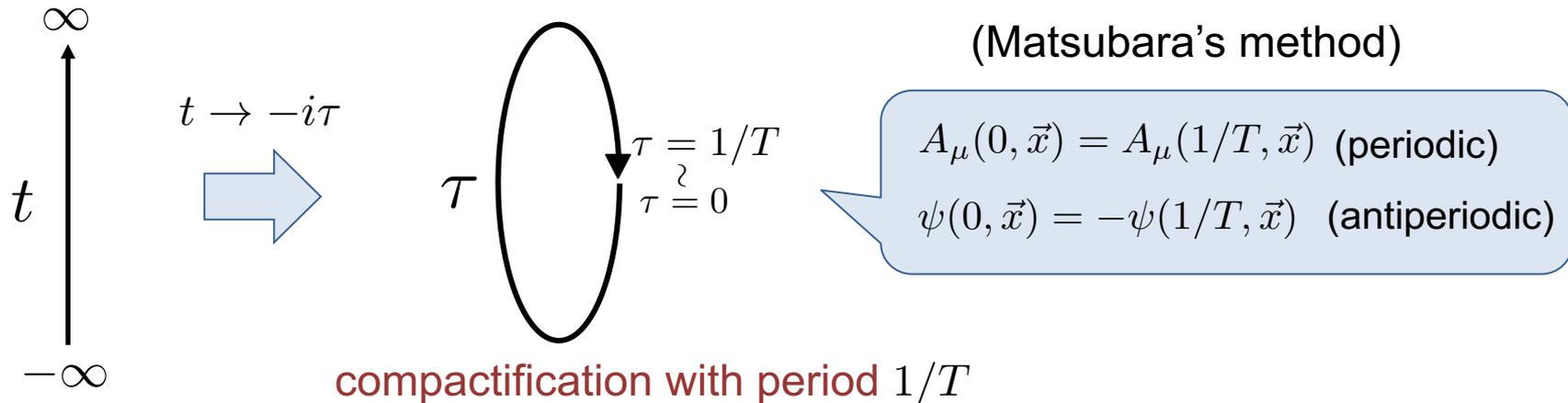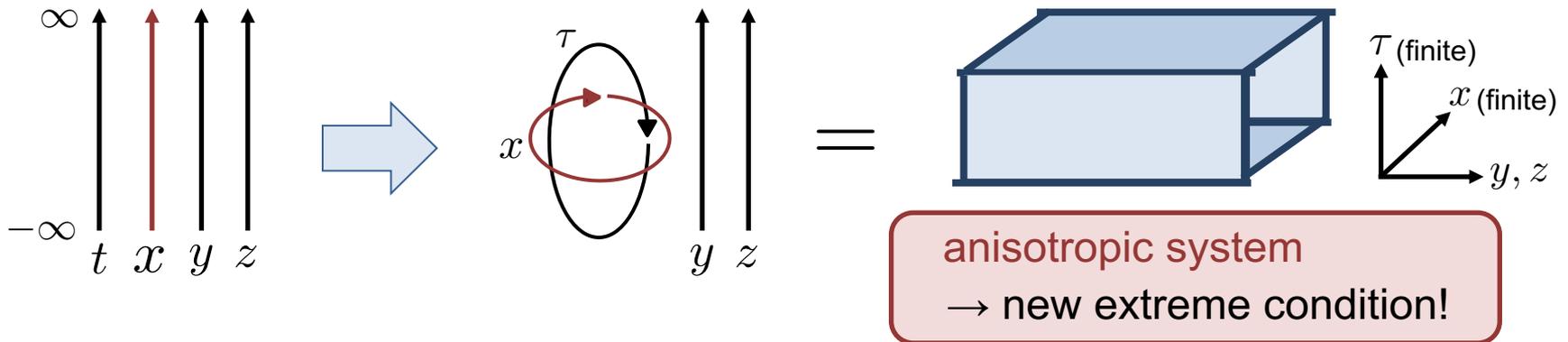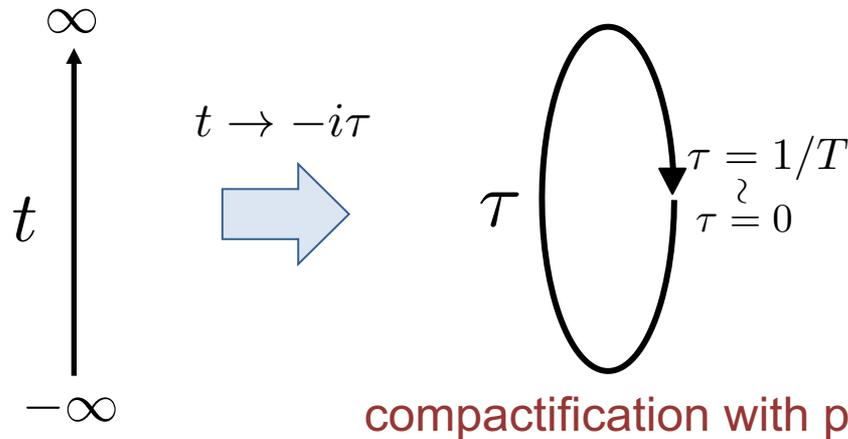
$$\mathbb{S}^1 \times \mathbb{R}^3$$

- What happened when another spatial axis is also compactified?

∞

$-\infty$

$t\ x\ y\ z$

τ

x

$y\ z$
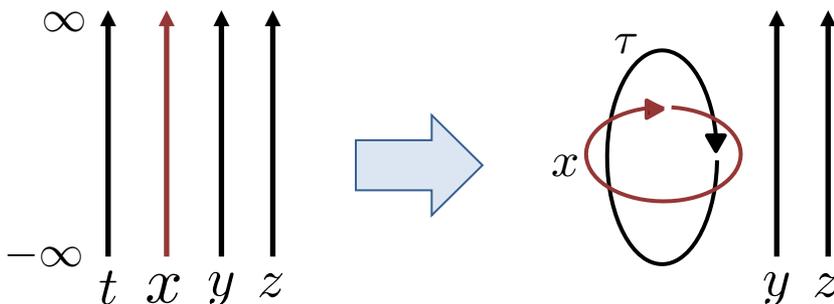
$$\mathbb{T}^2 \times \mathbb{R}^2$$

• Anisotropic system as a new QCD environment

- Lattice simulation is nothing but a finite-volume study

lattice setup

≠ infinitely large

straightforward

τ (finite)

x (finite)

y, z
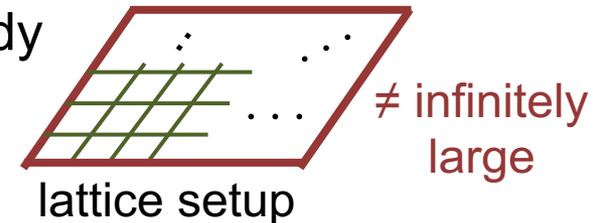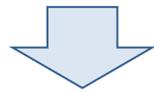
- Simulation in anisotropic system is done by choosing boundary conditions and lattice size
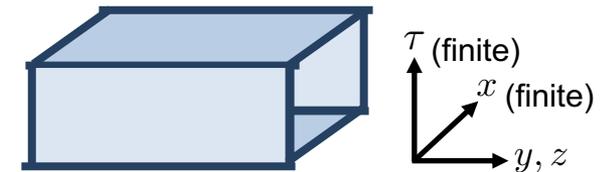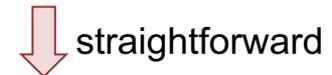
For example

- Polyakov loops in pure YM theory in anisotropic system were simulated

    Chernodub-Goy-Molochkov, PRD (2019)

- Pressure/energy in pure YM theory in anisotropic system were simulated

   Kitazawa-Mogliacci-Kolbe-Horowitz, PRD (2019)

- We explored phase structures of pure YM theory in anisotropic system ($\mathbb{T}^2 \times \mathbb{R}^2$) by means of an effective theory

- ## Anisotropic system as a new QCD environment

lattice simulations

$\mathbb{T}^2 + \mathbb{R}^2$

?.?.

Quark-Gluon Plasma

sQGP

Critical Point

Inhomogeneous $S_\chi B$

Hadronic Phase

Liquid-Gas

?

Quarkyonic Matter

2SC

uSC
dSC
CFL

Color Superconductors

μ_B

Nuclear Superfluid

CFL-K^0, Crystalline CSC
Meson supercurrent
Gluonic phase, Mixed phase
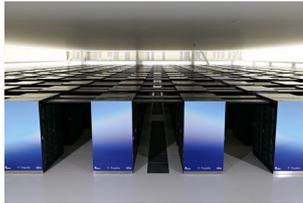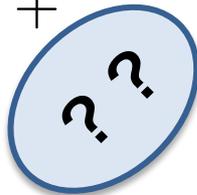
new axis

? ?

$1/L_x$ (L_x is extent of x direction)
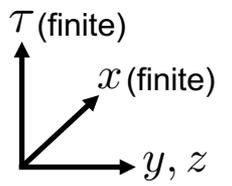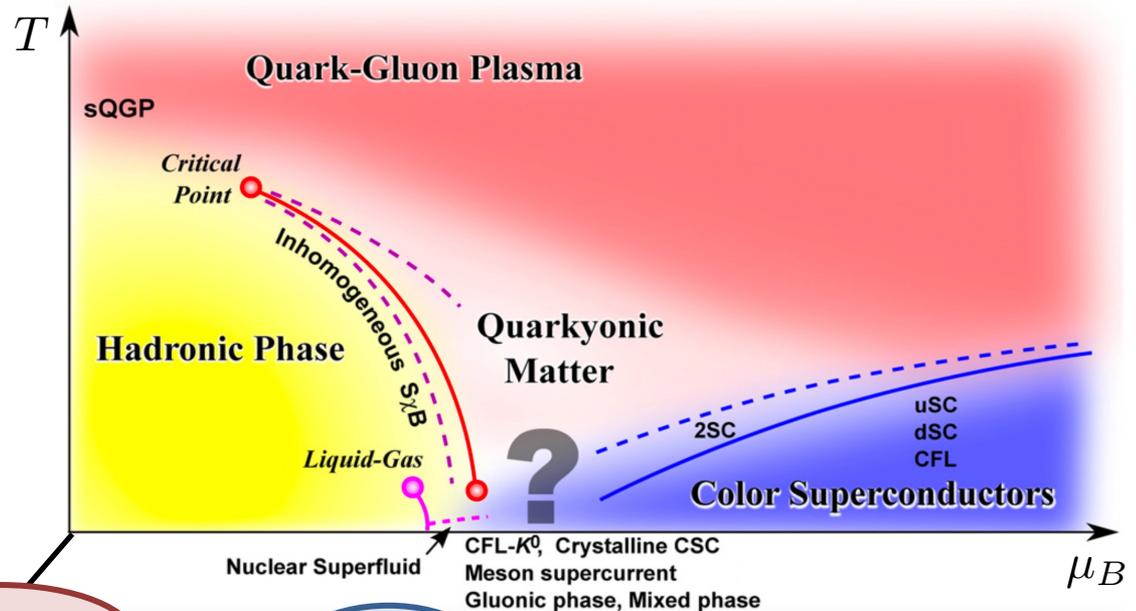
- We focus on exploring phase structures of pure YM theory in this domain

τ (finite)

x (finite)

y, z

Suenaga-Kitazawa, in preparation

• Our approach

[1] Meisinger-Miller-Ogilvie, PRD65, 034009 (2002)

- We employ model at finite T ($\mathbb{S}^1 \times \mathbb{R}^3$) given in Ref.[1] for study on $\mathbb{T}^2 \times \mathbb{R}^2$

$$\Rightarrow f = f_{\text{pert}} + f_{\text{pot}} \quad \text{(free energy density)}$$

I) $f_{\text{pert}} = \dfrac{2}{L_\tau} \displaystyle\sum_{j,k=1}^{N_c} \left(1 - \dfrac{\delta_{jk}}{N_c}\right) \sum_{l_\tau} \int \dfrac{d^3 p}{(2\pi)^3} \ln\left[\left(\omega_\tau - \dfrac{(\Delta\theta_\tau)_{jk}}{L_\tau}\right)^2 + \boldsymbol{p}^2\right]$

$L_\tau = 1/T, \ \omega_\tau = (2\pi l_\tau)/L_\tau$
$(\Delta\theta_\tau)_{jk} = (\theta_\tau)_j - (\theta_\tau)_k$

- from perturbative calculation with $A_\tau = \dfrac{1}{L_\tau}\text{diag}\left[(\theta_\tau)_1, \cdots, (\theta_\tau)_{N_c}\right]$ and $\displaystyle\sum_{i=1}^{N_c}(\theta_\tau)_i = 0$

($\rightarrow$ Polyakov loop)

II) $f_{\text{pot}} = -\dfrac{1}{L_\tau R^3} \ln\left[\displaystyle\prod_{j<k} \sin^2\left(\dfrac{(\Delta\theta_\tau)_{jk}}{2}\right)\right]$ $\leftarrow$ Haar measure potential by strong-coupling expansion

- R is size of colorful domain

Polonyi-Szlachanyi (1982)

• Our approach

[1] Meisinger-Miller-Ogilvie, PRD65, 034009 (2002)

- We employ model at finite T ($\mathbb{S}^1 \times \mathbb{R}^3$) given in Ref.[1] for study on $\mathbb{T}^2 \times \mathbb{R}^2$

$$\Rightarrow f = f_{\mathrm{pert}} + f_{\mathrm{pot}} \quad \text{(free energy density)}$$

I) $$f_{\mathrm{pert}} = \frac{2}{L_\tau} \sum_{j,k=1}^{N_c} \left(1 - \frac{\delta_{jk}}{N_c}\right) \sum_{l_\tau} \int \frac{d^3 p}{(2\pi)^3} \ln \left[\left(\omega_\tau - \frac{(\Delta\theta_\tau)_{jk}}{L_\tau}\right)^2 + \boldsymbol{p}^2\right]$$

$$L_\tau = 1/T, \ \omega_\tau = (2\pi l_\tau)/L_\tau$$
$$(\Delta\theta_\tau)_{jk} = (\theta_\tau)_j - (\theta_\tau)_k$$

- from perturbative calculation with $A_\tau = \frac{1}{L_\tau}\mathrm{diag}\left[(\theta_\tau)_1, \cdots, (\theta_\tau)_{N_c}\right]$ and $\sum_{i=1}^{N_c}(\theta_\tau)_i = 0$

($\rightarrow$ Polyakov loop)

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$\leftarrow$ Haar measure potential by strong-coupling expansion

- R is size of colorful domain

Polonyi-Szlachanyi (1982)

f_{pot} dominates for $R \ll L_\tau$ f_{pert} dominates for $R \gg L_\tau$
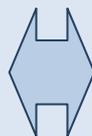
Polyakov loop
$$P_\tau \equiv \frac{1}{N_c}\mathrm{Tr}\left[e^{iA_\tau L_\tau}\right]$$

$R \updownarrow$ **confined** $P_\tau \rightarrow 0$ $R \updownarrow$ **deconfined** $P_\tau \rightarrow 1$

• Our approach

[1] Meisinger-Miller-Ogilvie, PRD65, 034009 (2002)

- We employ model at finite T ($\mathbb{S}^1 \times \mathbb{R}^3$) given in Ref.[1] for study on $\mathbb{T}^2 \times \mathbb{R}^2$

⇒ $f = f_{\mathrm{pert}} + f_{\mathrm{pot}}$ (free energy density)

- Determine values of Polyakov loop from $\frac{\partial f}{\partial (\theta_\tau)_j} = 0$

- Results of energy density ϵ/T^4 and pressure p/T^4 for $SU(3)$
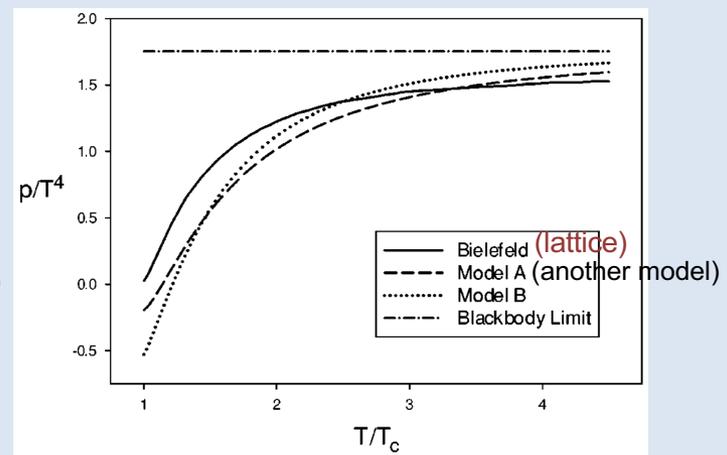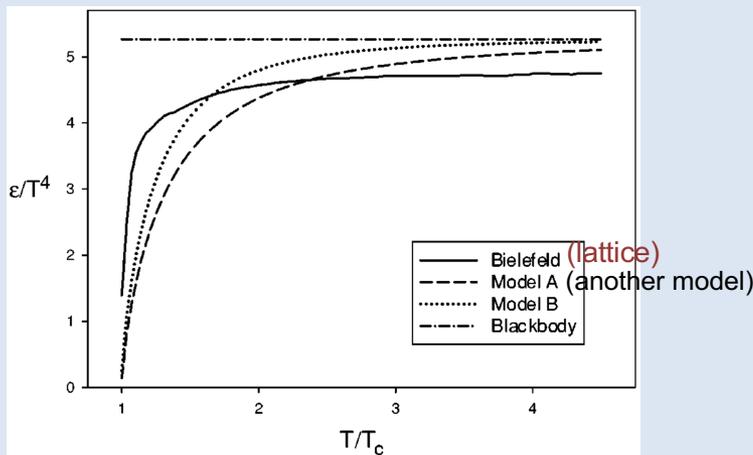
- **Our model on** $\mathbb{T}^2 \times \mathbb{R}^2$ [1] Meisinger-Miller-Ogilvie, PRD65, 034009 (2002)

  - Our model on $\mathbb{T}^2 \times \mathbb{R}^2$ is given by a straightforward extension of Ref.[1]

$$f = f_{\text{pert}} + f_{\text{pot}} \text{ (free energy density)}$$

I) $f_{\text{pert}} = \dfrac{2}{L_\tau L_x} \sum_{j,k=1}^{N_c} \left(1 - \dfrac{\delta_{jk}}{N_c}\right) \sum_{l_\tau, l_x} \int \dfrac{d^2 p_L}{(2\pi)^2} \ln \left[\left(\omega_\tau - \dfrac{(\Delta\theta_\tau)_{jk}}{L_\tau} \right)^2 + \left(\omega_x + \dfrac{(\Delta\theta_x)_{jk}}{L_x} \right)^2 + \boldsymbol{p}_L^2 \right]$

with $A_\tau = \dfrac{1}{L_\tau} \text{diag}\left((\theta_\tau)_1, \cdots, (\theta_\tau)_{N_c} \right)$ and $A_x = \dfrac{1}{L_x} \text{diag}\left((\theta_x)_1, \cdots, (\theta_x)_{N_c} \right)$

II) $f_{\text{pot}} = -\dfrac{1}{L_\tau R^3} \ln \left[\prod_{j<k} \sin^2 \left(\dfrac{(\Delta\theta_\tau)_{jk}}{2} \right) \right] - \dfrac{1}{L_x R^3} \ln \left[\prod_{j<k} \sin^2 \left(\dfrac{(\Delta\theta_x)_{jk}}{2} \right) \right]$

  - Two "Polyakov loops" can be defined: $P_\tau \equiv \dfrac{1}{N_c} \text{Tr} \left[e^{iA_\tau L_\tau} \right]$, $P_x \equiv \dfrac{1}{N_c} \text{Tr} \left[e^{iA_x L_x} \right]$

  - We explore phase structure on two $Z_{N_c}^{(\tau)} \times Z_{N_c}^{(x)}$ symmetries by evaluating values of P_τ and P_x on $L_x - L_\tau$ plane

2. Model

• Techniques

- f_{pert} includes complicated "Matsubara summations $\sum\limits_{l_\tau, l_x}$ "

$$
f_{\mathrm{pert}} = \frac{2}{L_\tau L_x} \sum_{j,k=1}^{N_c} \left(1 - \frac{\delta_{jk}}{N_c}\right) \sum_{l_\tau, l_x} \int \frac{d^2 p_L}{(2\pi)^2} \ln\left[\left(\omega_\tau - \frac{(\Delta\theta_\tau)_{jk}}{L_\tau}\right)^2 + \left(\omega_x + \frac{(\Delta\theta_x)_{jk}}{L_x}\right)^2 + \boldsymbol{p}_L^2\right]
$$

with $\omega_\tau = (2\pi l_\tau)/L_\tau, \;\; \omega_x = (2\pi l_x)/L_x \;\; (l_\tau, l_x \in \mathbb{Z})$

- Handled by Epstein-Hurwitz zeta function

textbook: Elizalde, "Ten physical applications of spectral zeta functions" (LNPMGR, volume 35)

⟹ ⎧ - Vacuum contributions including UV divergences are easily subtracted

    ⎨ - The very nonlinear sums are translated into sums with
    ⎩   modified Bessel functions $K_\nu(x)$

→ easy to be evaluated!

- ## **Results for $L_x = \infty$**

- When $L_x = \infty$, the model reduced to that in finite temperature $\mathbb{S}^1 \times \mathbb{R}^3$

- Find the stationary points of the free energy

$SU(2) : T_c = 1/(0.873R)$

$SU(3) : T_c = 1/(0.733R)$
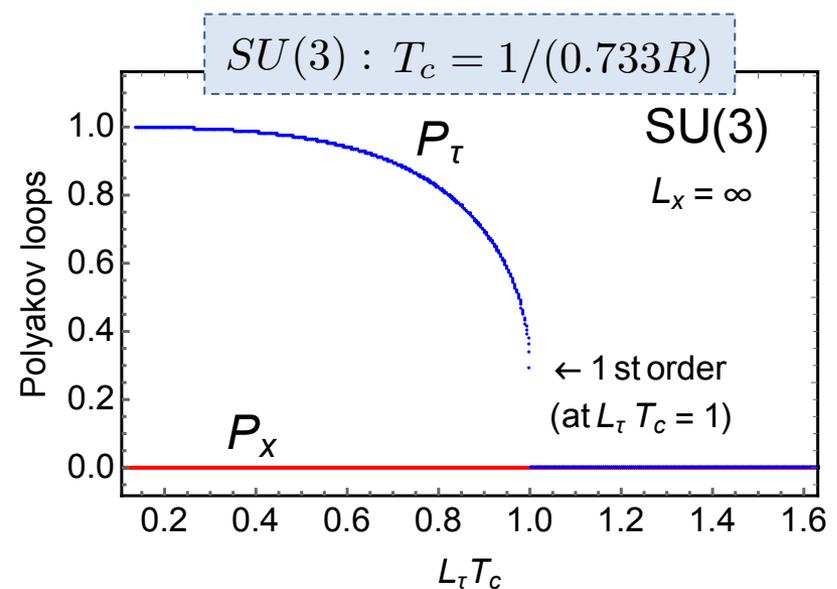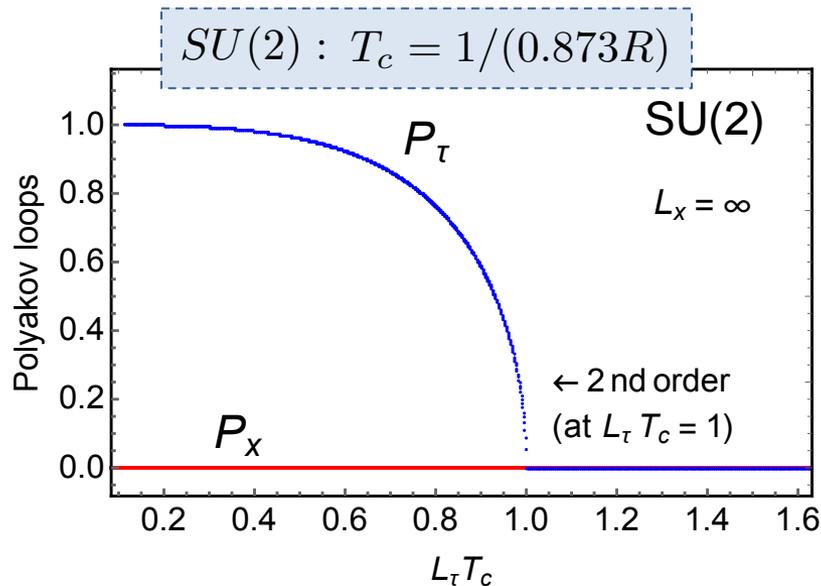
T_c is deconfinement temperature

- These results are consistent with analysis in Ref.[1]

[1] Meisinger-Miller-Ogilvie, PRD65, 034009 (2002)

- ## Results for $SU(2)$ in $\mathbb{T}^2 \times \mathbb{R}^2$

$$T_c = 1/(0.873R)$$

P_τ for $SU(2)$

P_x for $SU(2)$

finite T system ($\mathbb{S}^1 \times \mathbb{R}^3$) for $L_x = \infty$

$P_\tau \neq 0$
$P_x = 0$

$P_\tau = 0, \ P_x = 0$

2nd

2nd

2nd

$P_\tau \neq 0, \ P_x \neq 0$

2nd

$P_\tau = 0$
$P_x \neq 0$

2nd
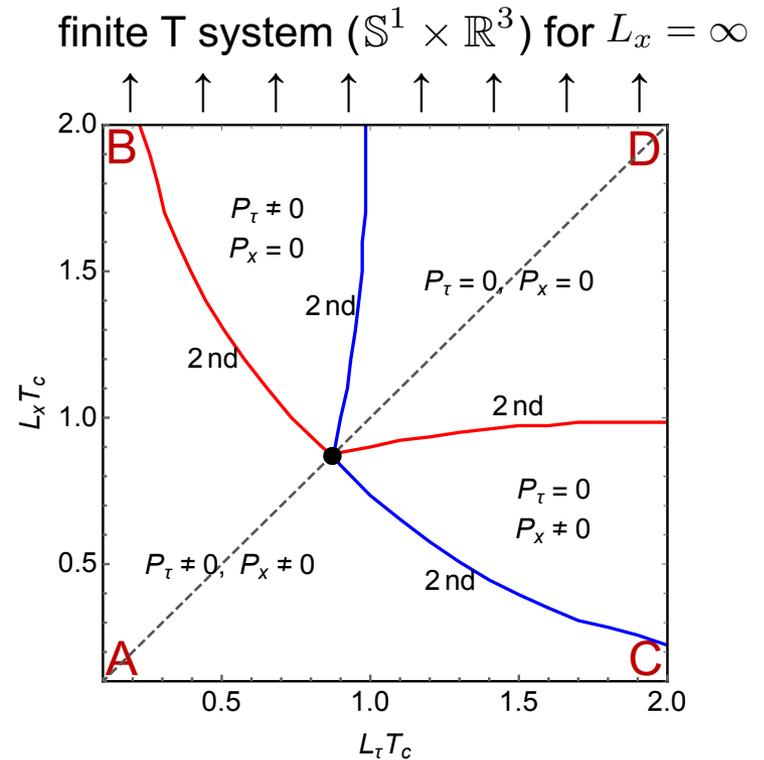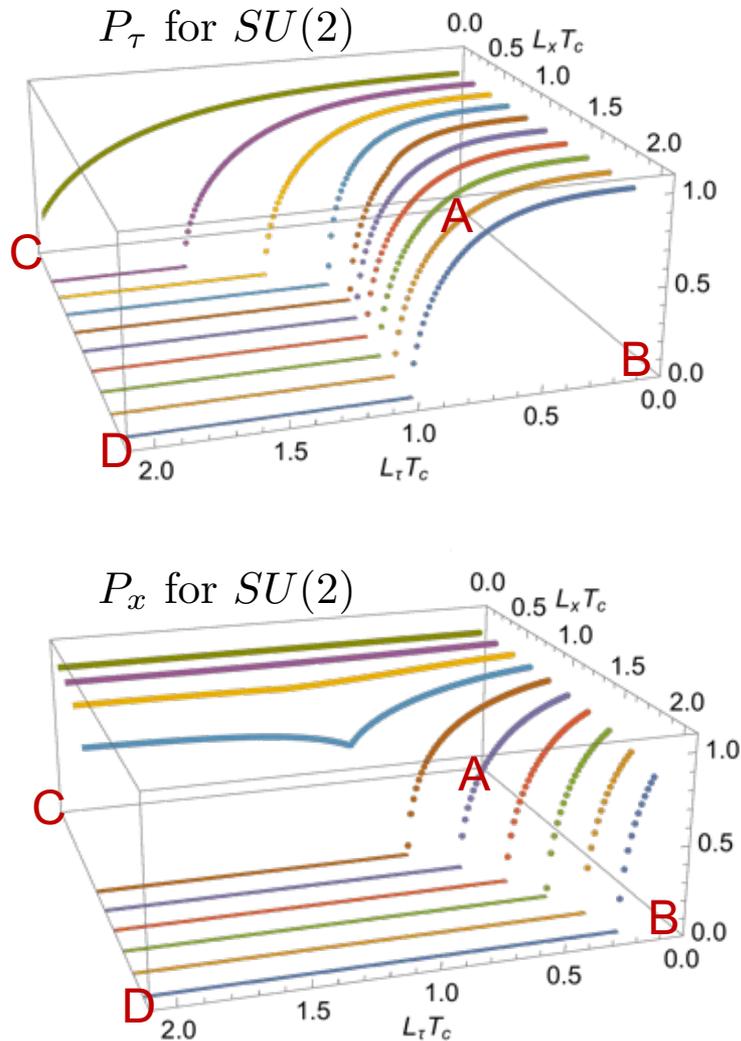
phase diagram for $SU(2)$

- 2nd order for all transitions

- **Results for $SU(2)$ in $\mathbb{T}^2 \times \mathbb{R}^2$**

$$T_c = 1/(0.873R)$$

P_τ for $SU(2)$

finite T system ($\mathbb{S}^1 \times \mathbb{R}^3$) for $L_x = \infty$

approaches $L_x T_c$ axis

$P_\tau \neq 0$
$P_x = 0$

$P_\tau = 0$, $P_x = 0$

2nd

2nd

2nd

$P_\tau \neq 0$, $P_x \neq 0$

2nd

$P_\tau = 0$
$P_x \neq 0$

2nd
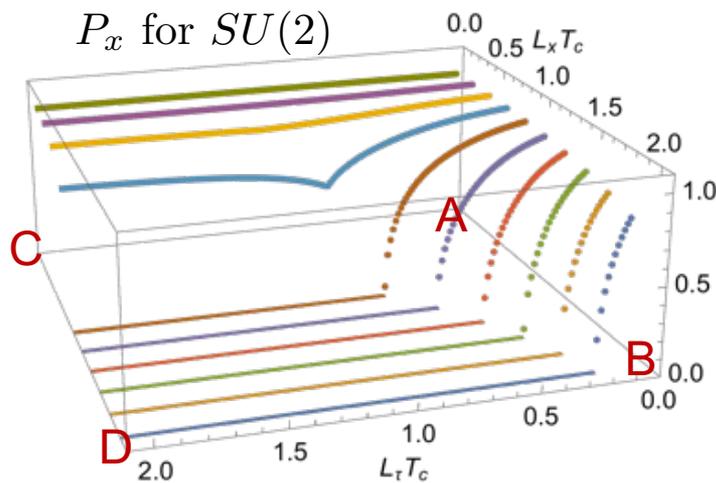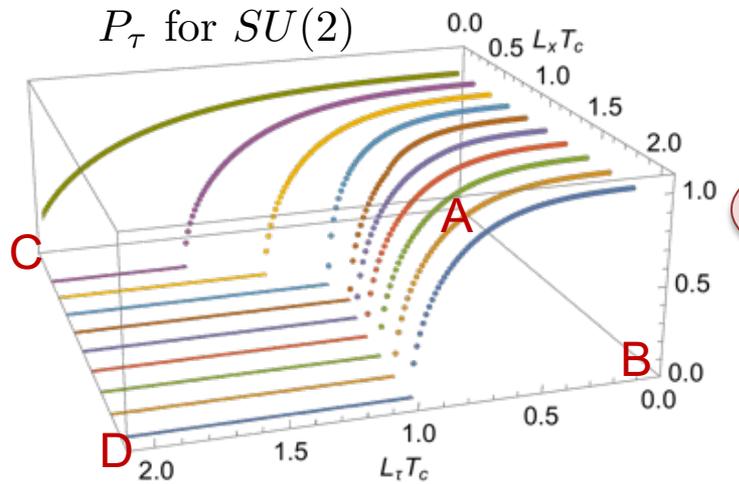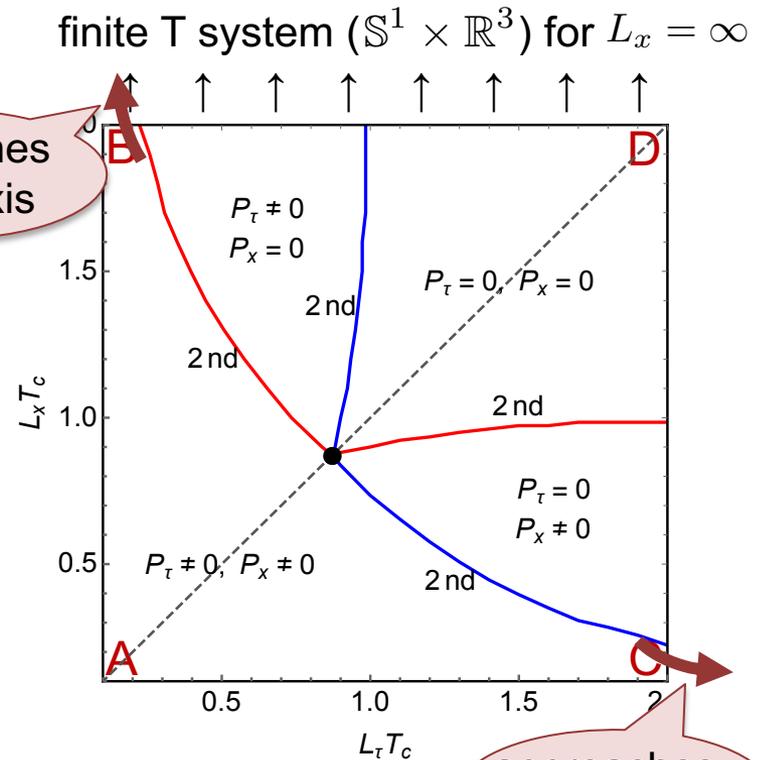
phase diagram

approaches $L_\tau T_c$ axis

P_x for $SU(2)$

- 2nd order for all transitions

Suenaga-Kitazawa, in preparation

• **Results for $SU(3)$ in $\mathbb{T}^2 \times \mathbb{R}^2$**

$$T_c = 1/(0.733R)$$
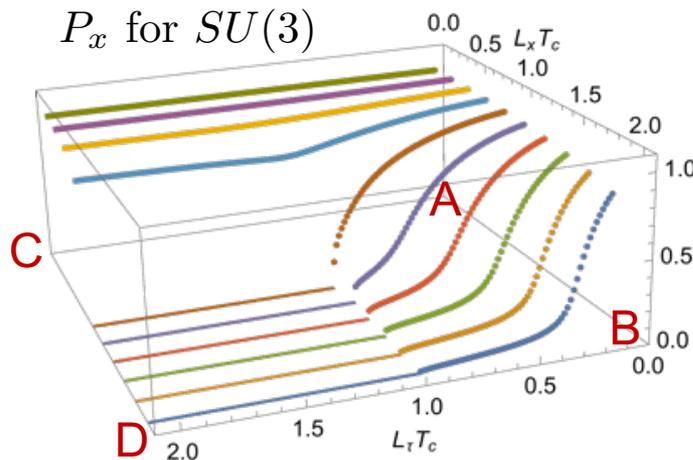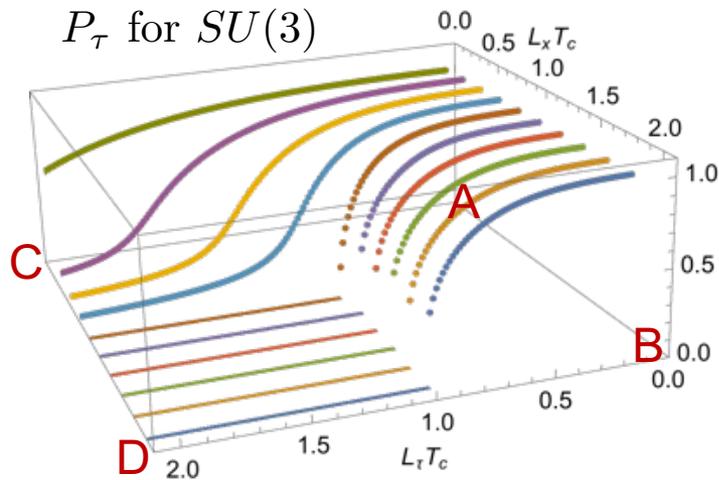
P_τ for $SU(3)$

P_x for $SU(3)$

finite T system ($\mathbb{S}^1 \times \mathbb{R}^3$) for $L_x = \infty$
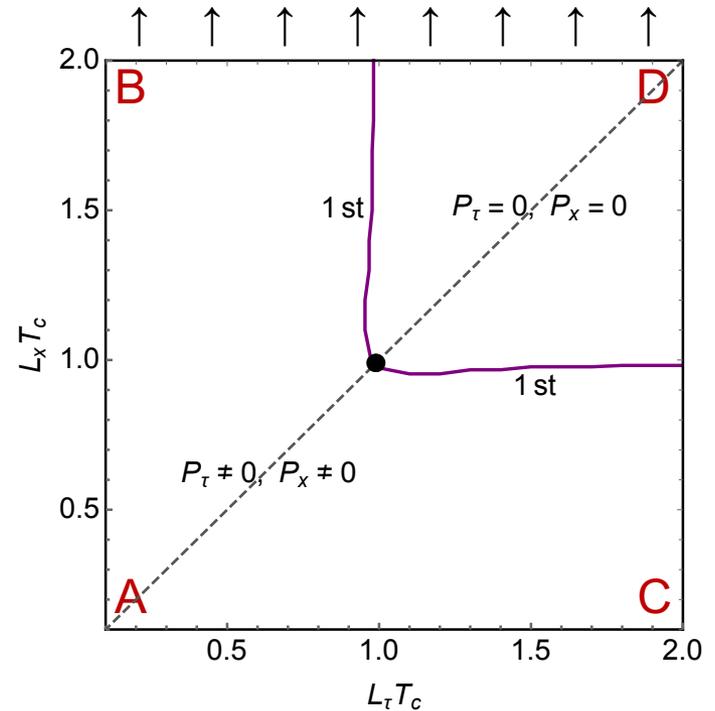
phase diagram for $SU(3)$

- 1nd order for all transitions

- **Thermodynamic quantities**

  - L_x dependence of energy density ϵ/T^4 and pressures p_x/T^4, p_z/T^4 for $SU(3)$

Lattice: Kitazawa-Mogliacci-Kolbe
-Horowitz, PRD (2019)

p_z/T^4

ϵ/T^4

p_x/T^4

$\Big\{$
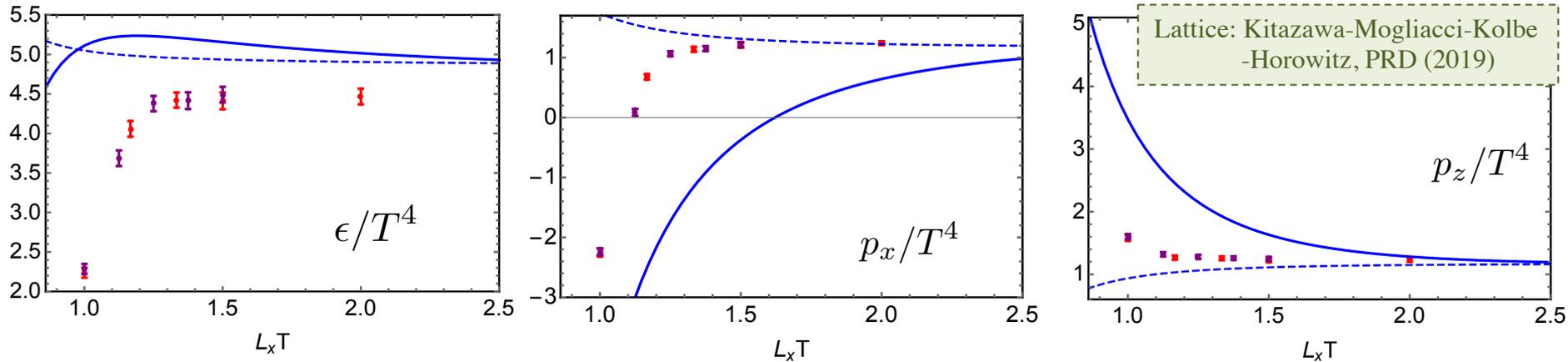
‒ ‒ ‒ Results by fixing $P_x = 0$ in our model

——— Results with full analysis of our model

$T = 2.10T_c$

• Thermodynamic quantities

- L_x dependence of energy density ϵ/T^4 and pressures p_x/T^4, p_z/T^4 for $SU(3)$

Lattice: Kitazawa-Mogliacci-Kolbe
-Horowitz, PRD (2019)

decrease

ϵ/T^4

decrease

p_x/T^4

increase

p_z/T^4

$T = 2.10 T_c$

{
- - - - Results by fixing $P_x = 0$ in our model
——— Results with full analysis of our model
}

- For larger volume $L_x T \gtrsim 1.3$, results with $P_x = 0$ (- - - -) seem to be better

- Qualitative behavior for smaller volume $L_x T \lesssim 1.3$ could be reproduced

⟹
- Importance of including two Polyakov loops P_τ, P_x was shown
- Improving the model is needed for quantitative fit

- We focused on pure YM theory on new environment of anisotropic system $\mathbb{T}^2 \times \mathbb{R}^2$

  lattice: Chernodub-Goy-Molochkov, PRD (2019)
 Kitazawa-Mogliacci-Kolbe-Horowitz, PRD (2019) etc.

$\mathbb{T}^2 \times \mathbb{R}^2$

lattice (OK!)

new axis!

? ?

- We drew phase diagram on $Z_{N_c}^{(\tau)} \times Z_{N_c}^{(x)}$ symmetry for $SU(2), SU(3)$ by employing a model motivated from Ref.[1]

  [1] Meisinger-Miller-Ogilvie, PRD65, 034009 (2002)

- Comparison with lattice results was qualitatively good but quantitatively unsatisfactory

  - Importance of two Polyakov loops were confirmed
  - Improving the model is necessary
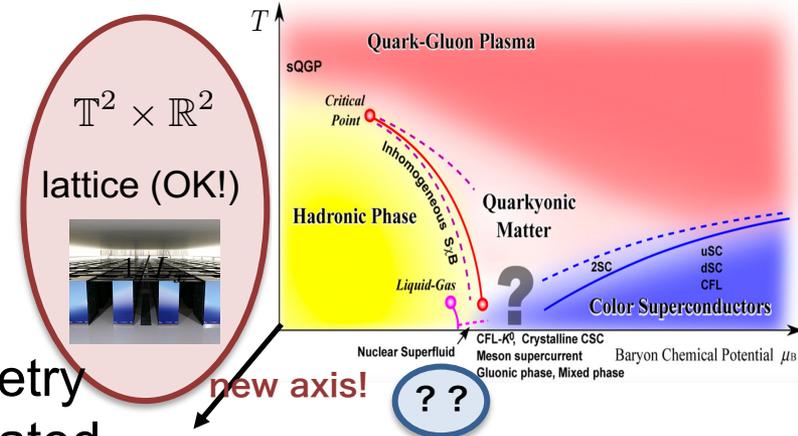
Investigation of QCD in anistropic system is frontier

join us!

- We need more study from both effective model and lattice QCD