Meson Spectroscopy

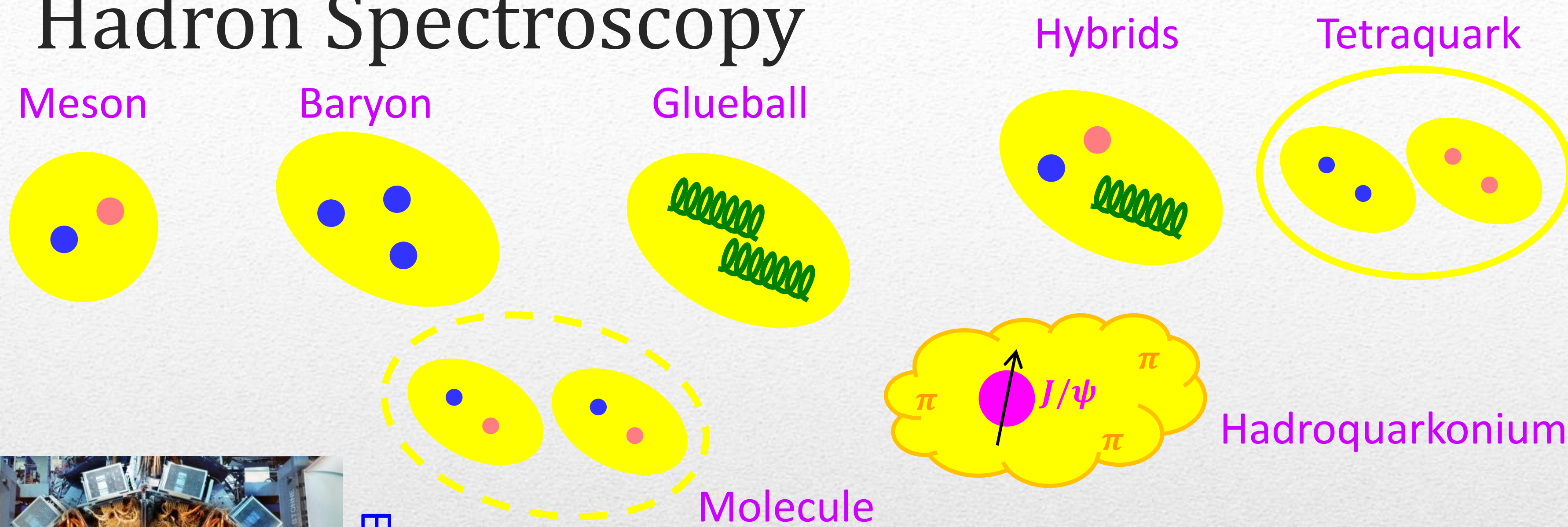
Alessandro Pilloni

QNP, September 6th, 2022

Università degli Studi di Messina

Istituto Nazionale di Fisica Nucleare

Hadron Spectroscopy



Experiment

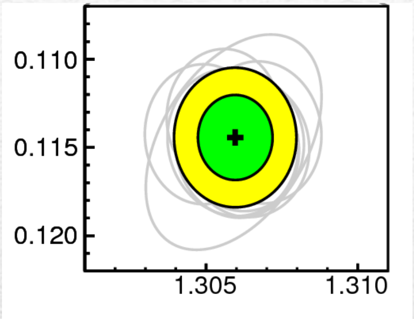
Lattice QCD

Data → Amplitude analysis → Properties, Model building

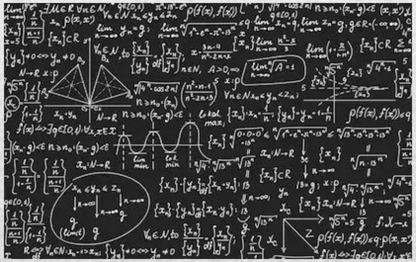
Interpretations on the spectrum leads to understanding fundamental laws of nature

Bottom-up approach

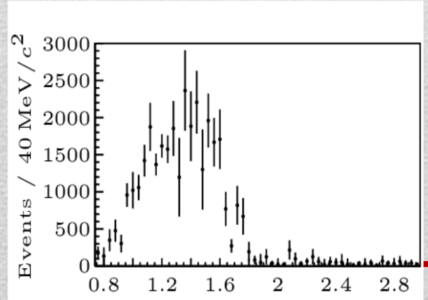
Less predictive power ✘
Some physical interpretation ✘
Minimally biased ✓



3) You extract physics

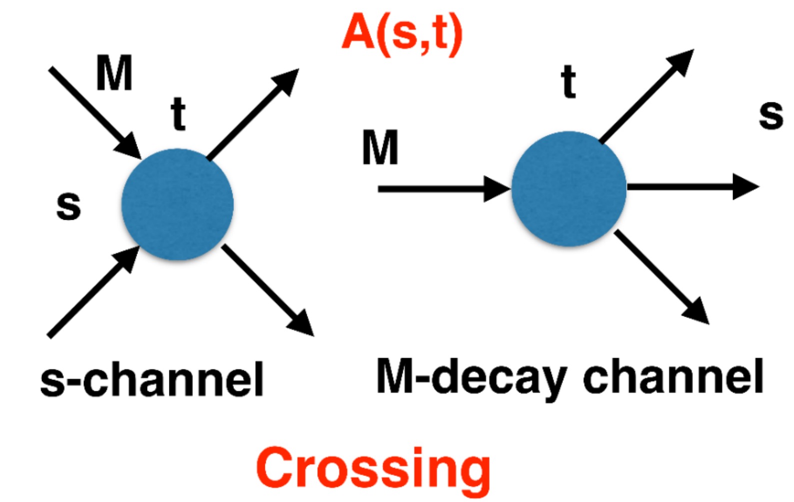


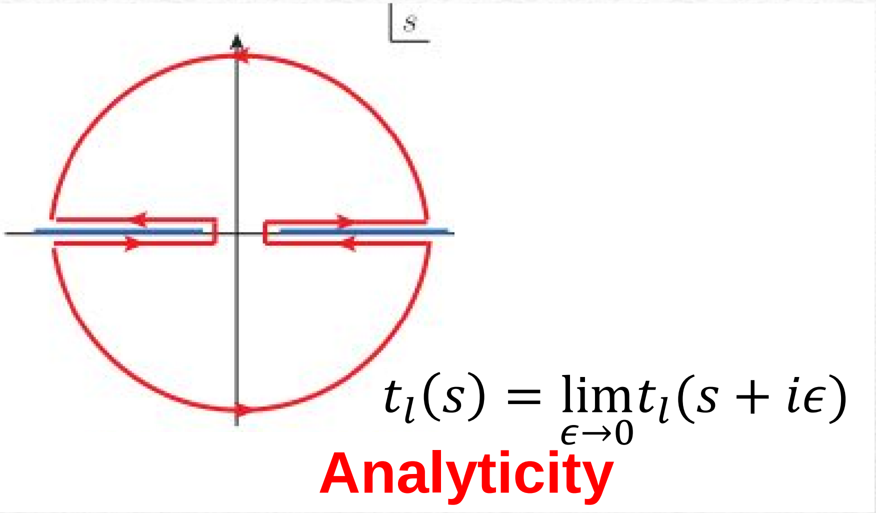
2) You choose a set of generic amplitudes

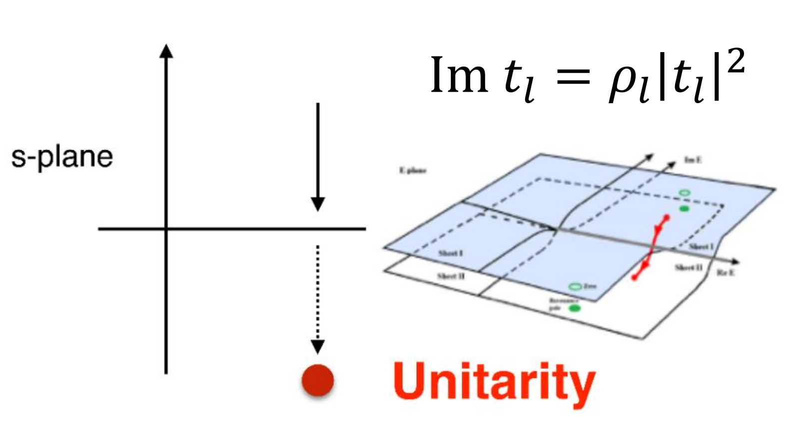


1) You start with data

S-Matrix principles







+ Lorentz, discrete & global symmetries

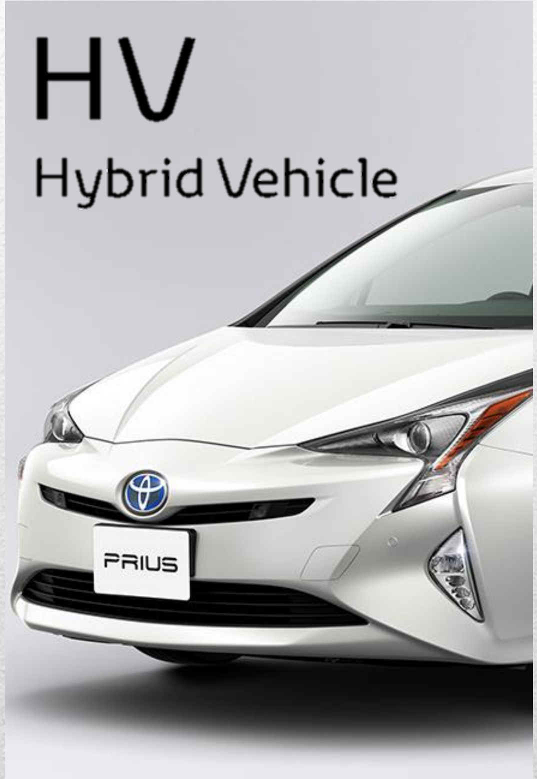
These are constraints the amplitudes have to satisfy, but do not fix the dynamics

They can be imposed with an increasing amount of rigor, to extract robust physics information

The «background» phenomena can be effectively parameterized in a controlled way

Meson Spectroscopy

The hybrid π_1



A. Rodas, AP *et al.*
PRL122, 042002

The scalar glueball

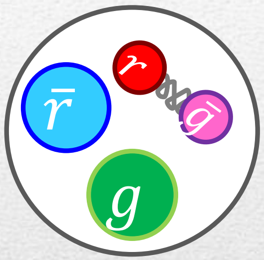
A. Rodas, AP *et al.*
EPJC82, 1, 80

XYZ photoprod.

M. Albaladejo et al., PRD102, 114010
D. Winney, AP et al., to appear

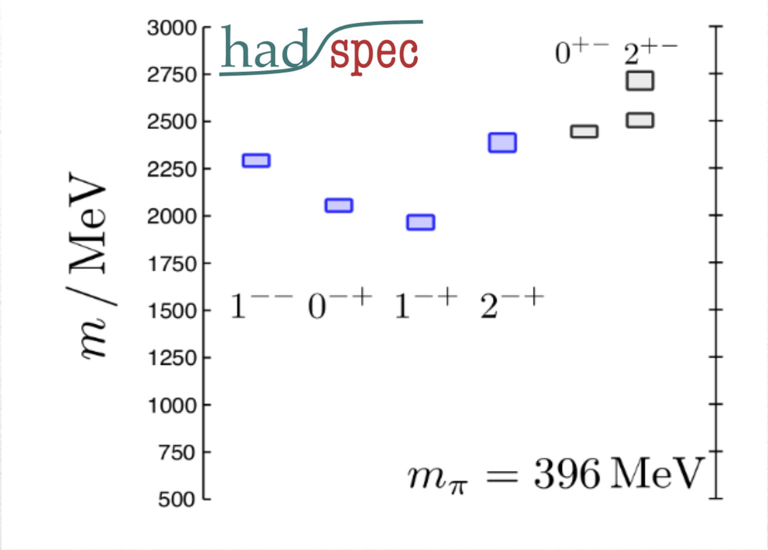
Hybrid hunting

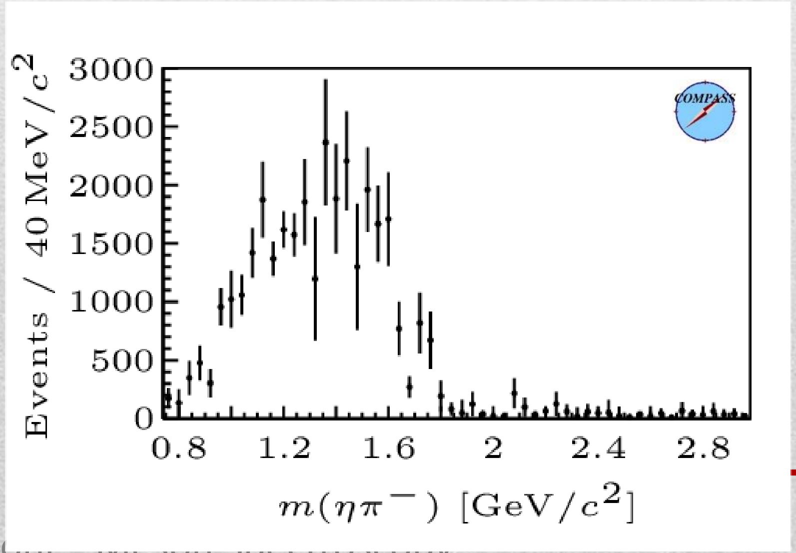
Constituent gluon (quasiparticle excitation),
$J^{PC} = 1^{+-}$, mass $\sim 1.0-1.5$ GeV

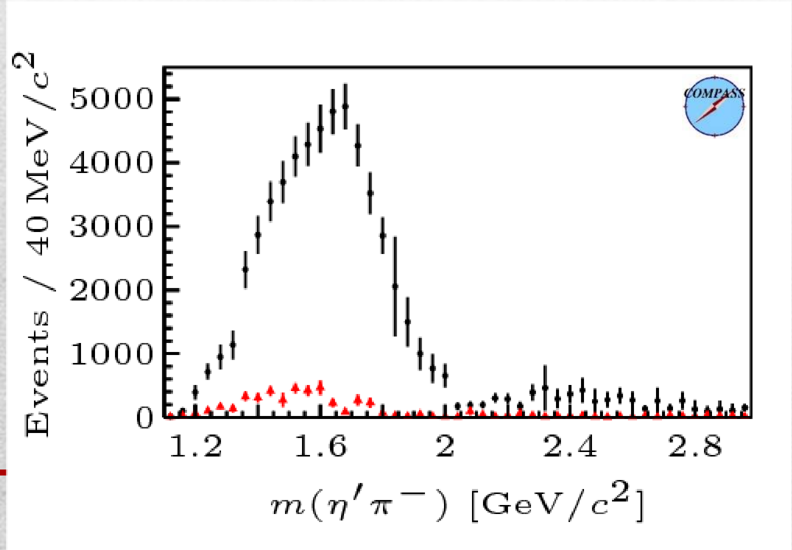


Look for a π_1 state with $\boldsymbol{J^{PC} = 1^{-+}}$

decaying into $\begin{cases} \eta\,\pi \text{ and } \eta'\pi \\ \rho\,\pi \to 3\pi \\ b_1\pi \to 5\pi \end{cases}$







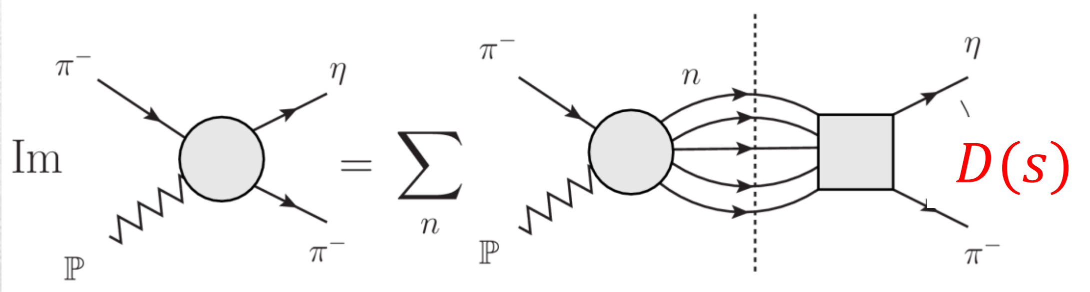
Amplitudes for $\eta^{(\prime)}\pi$

We build the partial wave amplitudes according to the N/D method

Jackura, Mikhasenko, AP *et al.* (JPAC & COMPASS), PLB

Rodas, AP *et al.* PRL

$$a(s) = \frac{n(s)}{D(s)}$$



The $D(s)$ contains all the Final State Interactions constrained by unitarity → universal

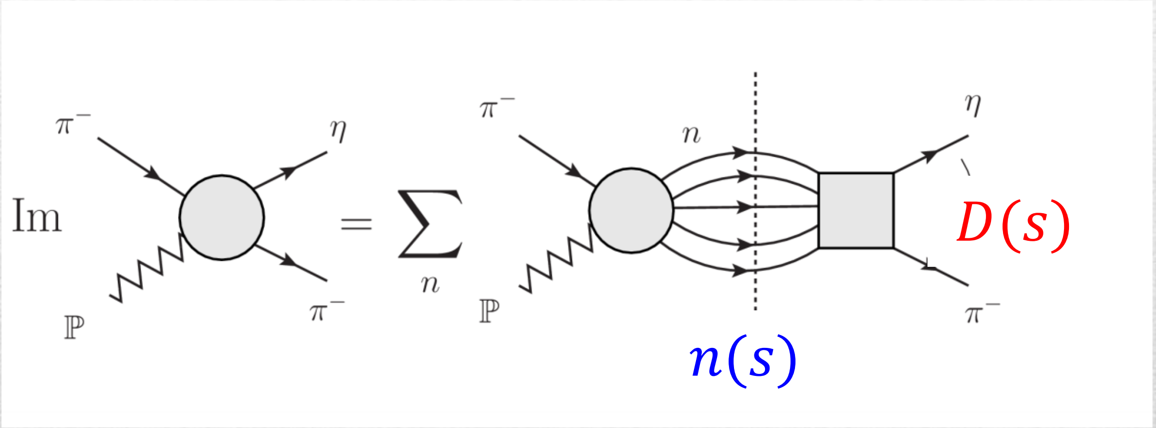
Amplitudes for $\eta^{(\prime)}\pi$

We build the partial wave amplitudes according to the N/D method

Jackura, Mikhasenko, AP *et al.* (JPAC & COMPASS), PLB

Rodas, AP *et al.* PRL

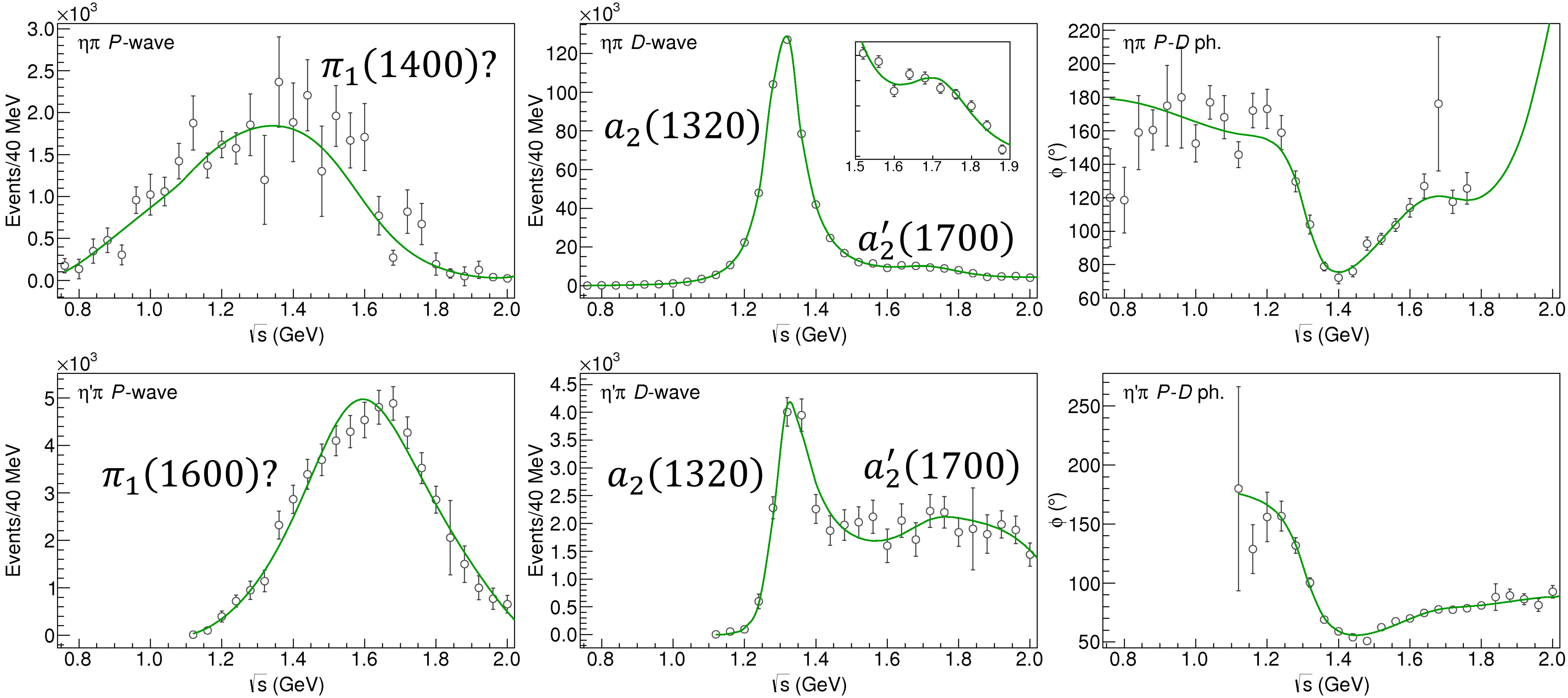
$$a(s) = \frac{n(s)}{D(s)}$$



The $n(s)$ → background physics, process-dependent, smooth

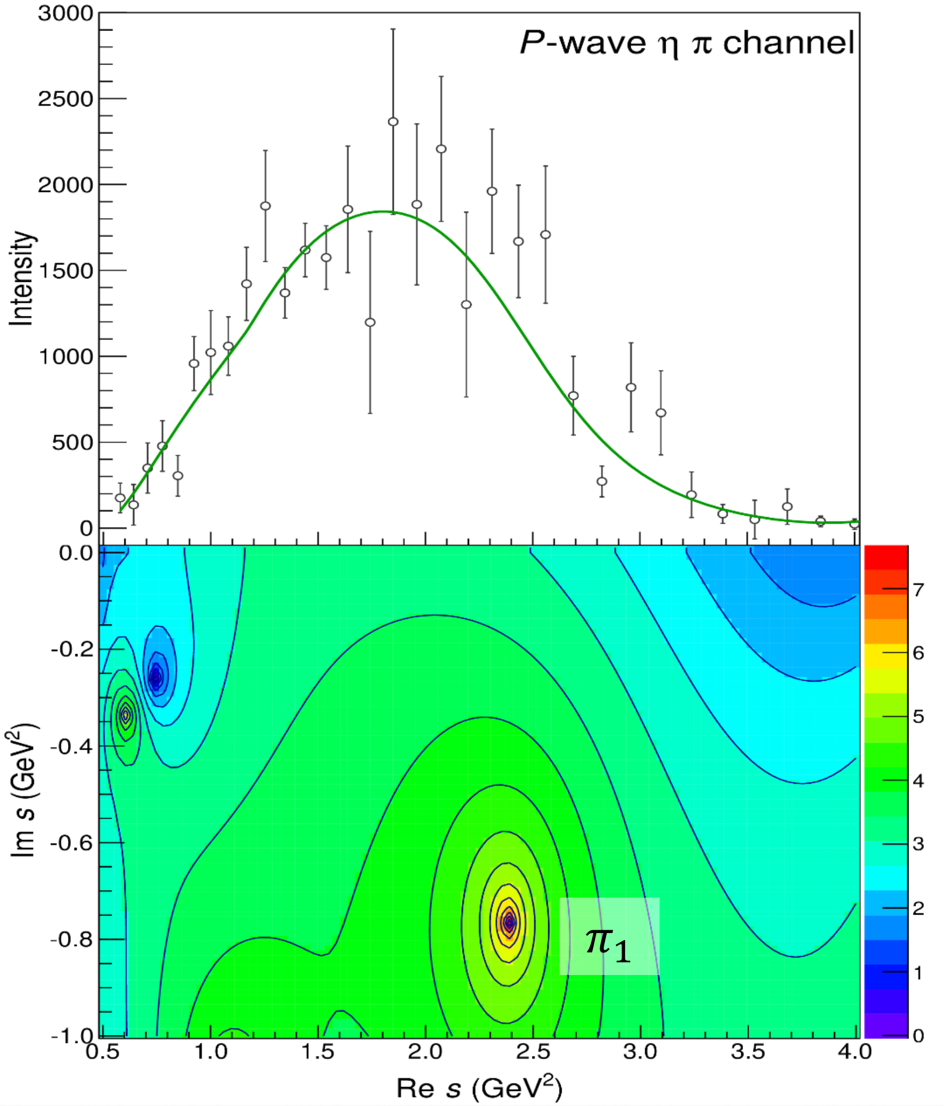
Fit to $\eta^{(\prime)}\pi$

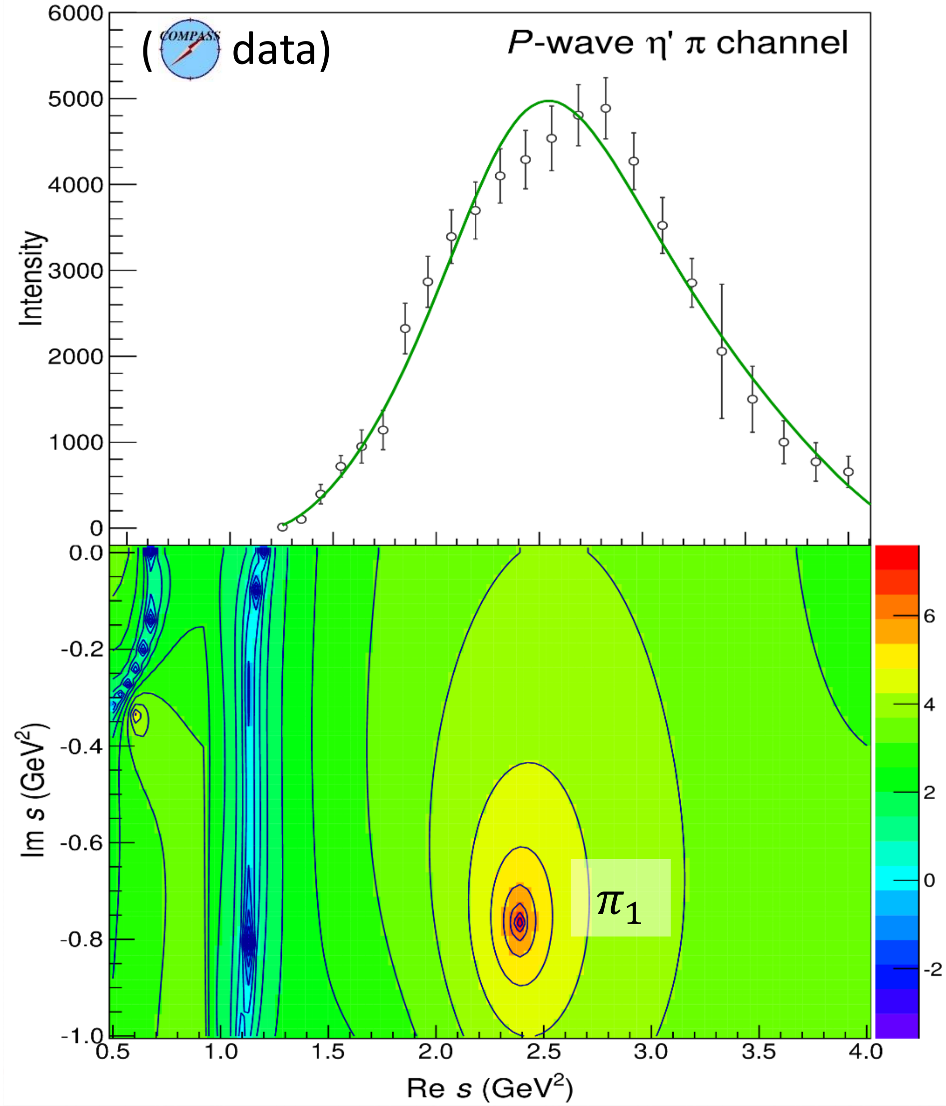
(data)



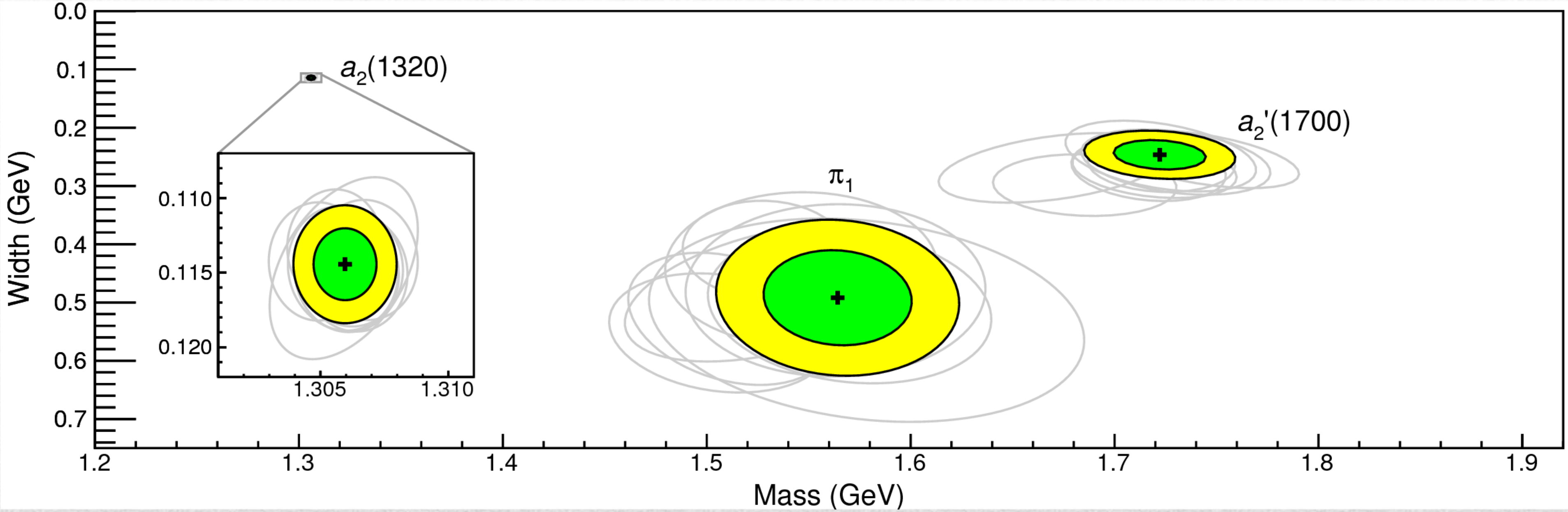
$J^{PC} = 1^{-+}$ $\qquad$ $J^{PC} = 2^{++}$

Pole hunting





Final results



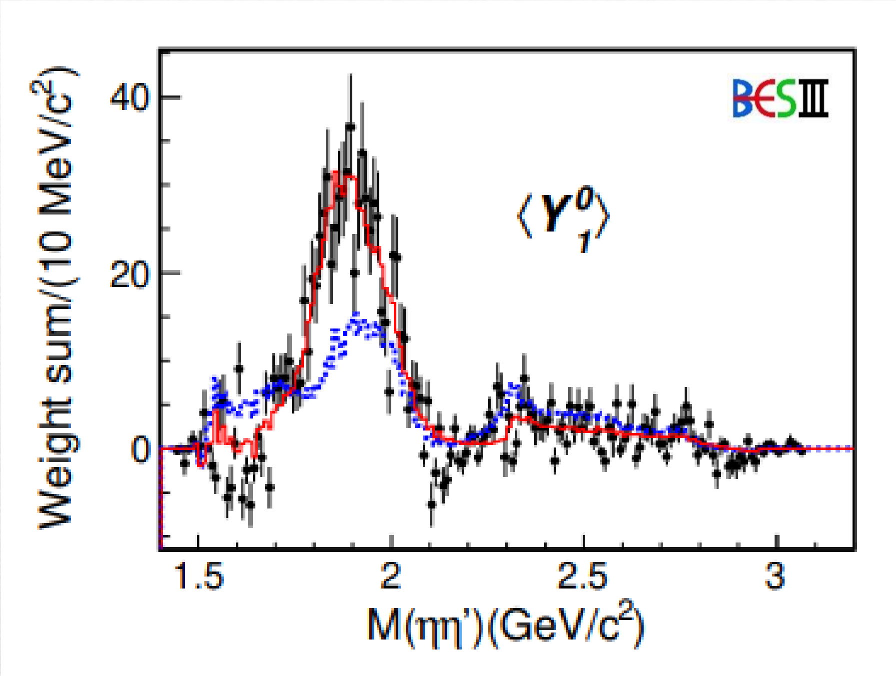
Poles	Mass (MeV)	Width (MeV)
$a_2(1320)$	$1306.0 \pm 0.8 \pm 1.3$	$114.4 \pm 1.6 \pm 0.0$
$a_2'(1700)$	$1722 \pm 15 \pm 67$	$247 \pm 17 \pm 63$
π_1	$1564 \pm 24 \pm 86$	$492 \pm 54 \pm 102$

Agreement with Lattice is restored

That's the most rigorous extraction of an exotic meson available so far!

An isoscalar η_1?

There is a recent claim by BESIII in $J/\psi \to \gamma\, \eta\, \eta'$ of resonant activity in *P*-wave



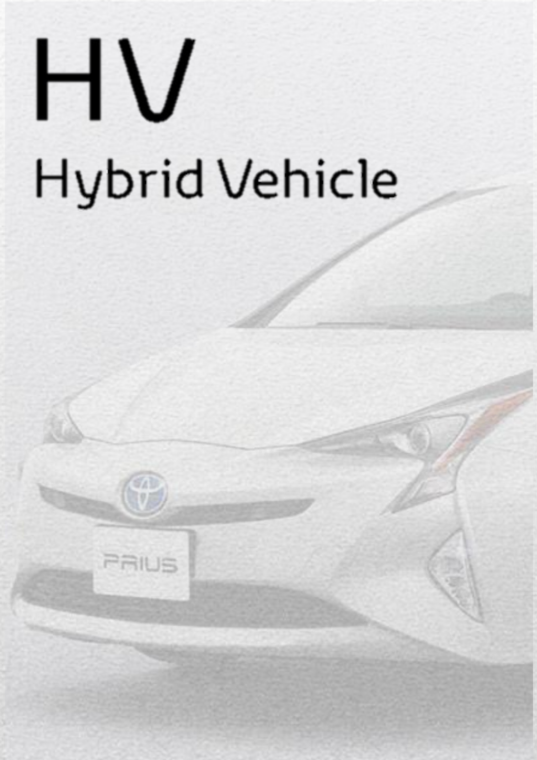
Not enough information
to perform a similar analysis but…
stay tuned!

BW parameters:

$$M = 1855 \pm 9^{+6}_{-1}\ \text{MeV},$$
$$\Gamma = 188 \pm 18^{+3}_{-8}\ \text{MeV}$$

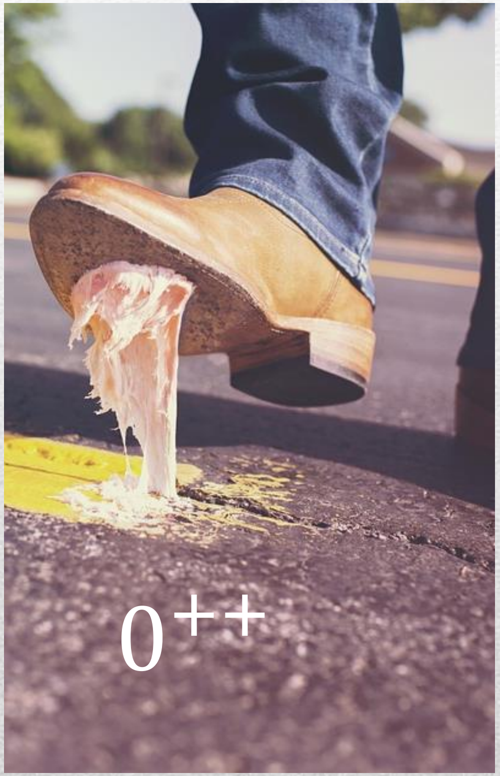
Meson Spectroscopy

The hybrid π_1



A. Rodas, AP *et al.*
PRL122, 042002

The scalar glueball



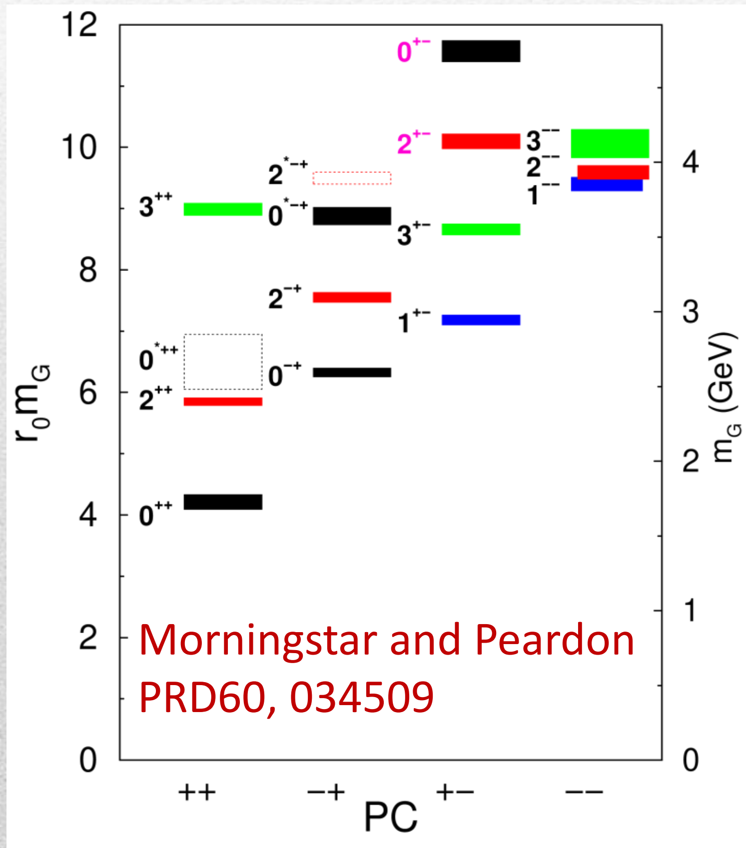
A. Rodas, AP *et al.*
EPJC82, 1, 80

XYZ photoprod.

M. Albaladejo et al., PRD102, 114010
D. Winney, AP et al., to appear

Glueballs

The clearest sign of confinement in pure Yang-Mills
The worst state to search in real life



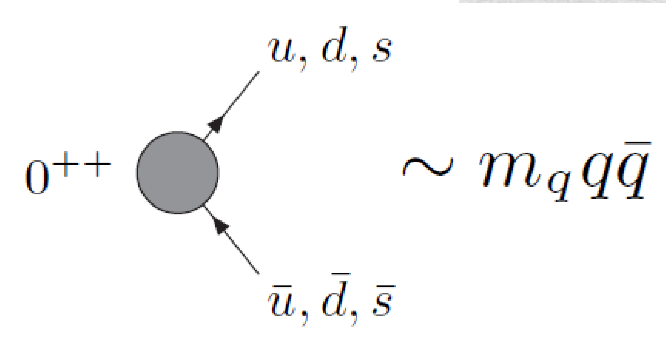
J^{PC}	Mass MeV			
	Unquenched	Quenched		
	This work	M&P	Ky	Meyer
0^{-+}		2590(40)(130)	2560(35)(120)	2250(60)(100)
2^{-+}	3460(320)	3100(30)(150)	3040(40)(150)	2780(50)(130)
0^{-+}	4490(590)	3640(60)(180)		3370(150)(150)
2^{-+}				3480(140)(160)
5^{-+}				3942(160)(180)
0^{--} (exotic)	5166(1000)			
1^{--}		3850(50)(190)	3830(40)(190)	3240(330)(150)
2^{--}	4590(740)	3930(40)(190)	4010(45)(200)	3660(130)(170)
2^{--}				3.740(200)(170)
3^{--}		4130(90)(200)	4200(45)(200)	4330(260)(200)
1^{+-}	3270(340)	2940(30)(140)	2980(30)(140)	2670(65)(120)
3^{+-}	3850(350)	3550(40)(170)	3600(40)(170)	3270(90)(150)
3^{+-}				3630(140)(160)
2^{+-} (exotic)		4140(50)(200)	4230(50)(200)	
0^{+-} (exotic)	5450(830)	4740(70)(230)	4780(60)(230)	
5^{+-}				4110(170)(190)
0^{++}	1795(60)	1730(50)(80)	1710(50)(80)	1475(30)(65)
2^{++}	2620(50)	2400(25)(120)	2390(30)(120)	2150(30)(100)
0^{++}	3760(240)	2670(180)(130)		2755(30)(120)
3^{++}		3690(40)(180)	3670(50)(180)	3385(90)(150)
0^{++}				3370(100)(150)
0^{++}				3990(210)(180)
2^{++}				2880(100)(130)
4^{++}				3640(90)(160)
6^{++}				4360(260)(200)

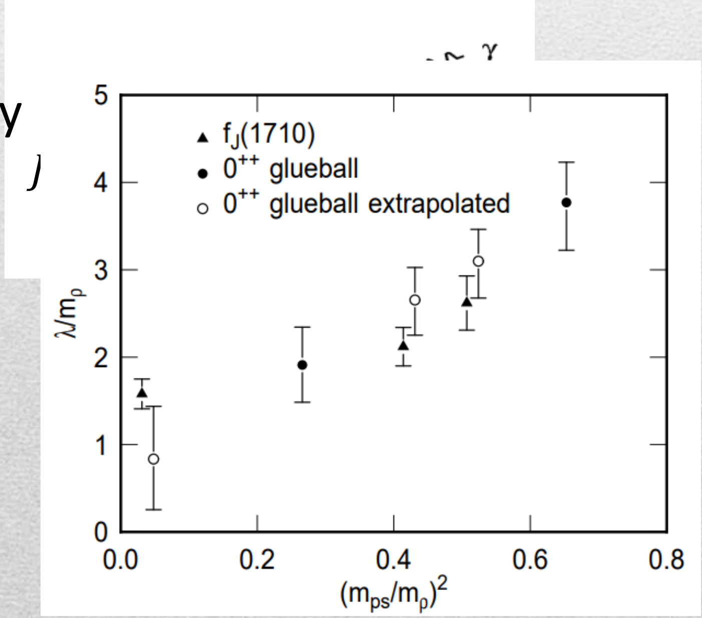
Gregory *et al.*
JHEP1210, 170

How to identify a glueball

You don't. Since it mixes with light isoscalars, there is no model-independent way of saying which state is (mostly) the glueball. Only suggestions:

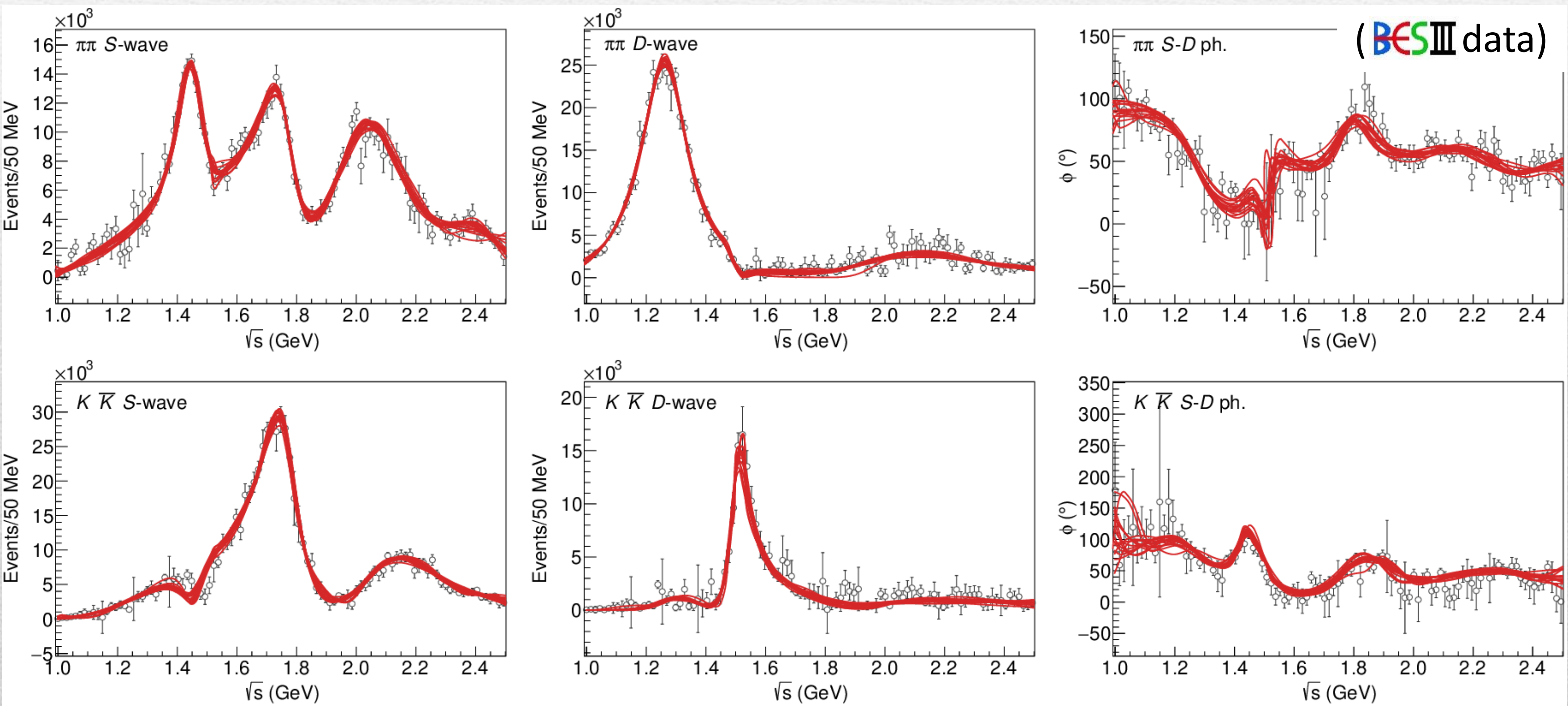
- There is one too many wrt QM. Indeed, $f_0(1370), f_0(1500), f_0(1710)$
- A glueball couples to photons only throughout mixing, so radiative widths should be small
- Their production is enhanced in gluon-rich processes, as J/ψ radiative decays
- It couples equally to mesons of all flavors (?)
  However, an argument based on chiral symmetry claims the coupling proportional to quark mass



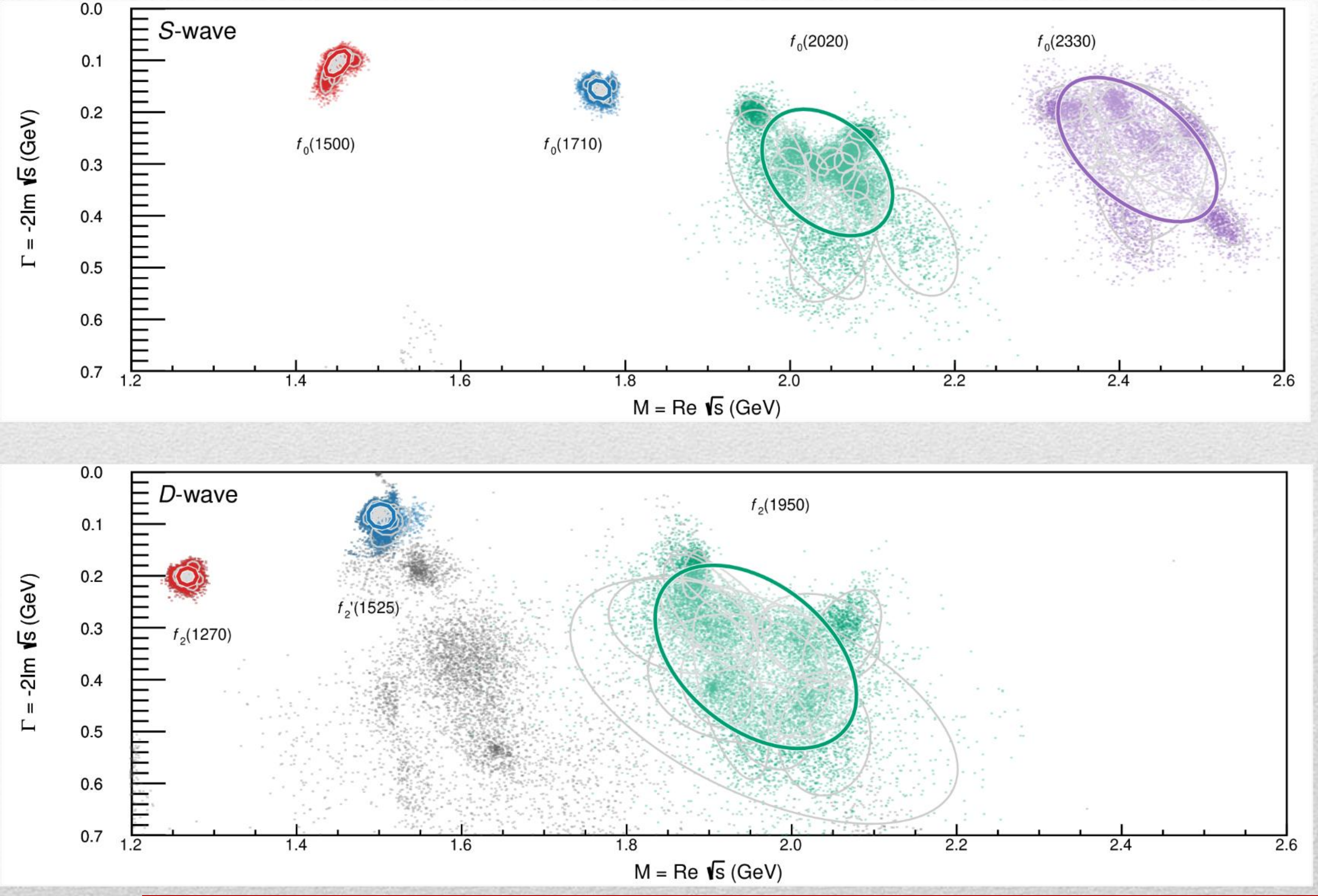


$J/\psi \to \gamma\, \pi^0\pi^0$ and $\to \gamma\, K_S^0 K_S^0$

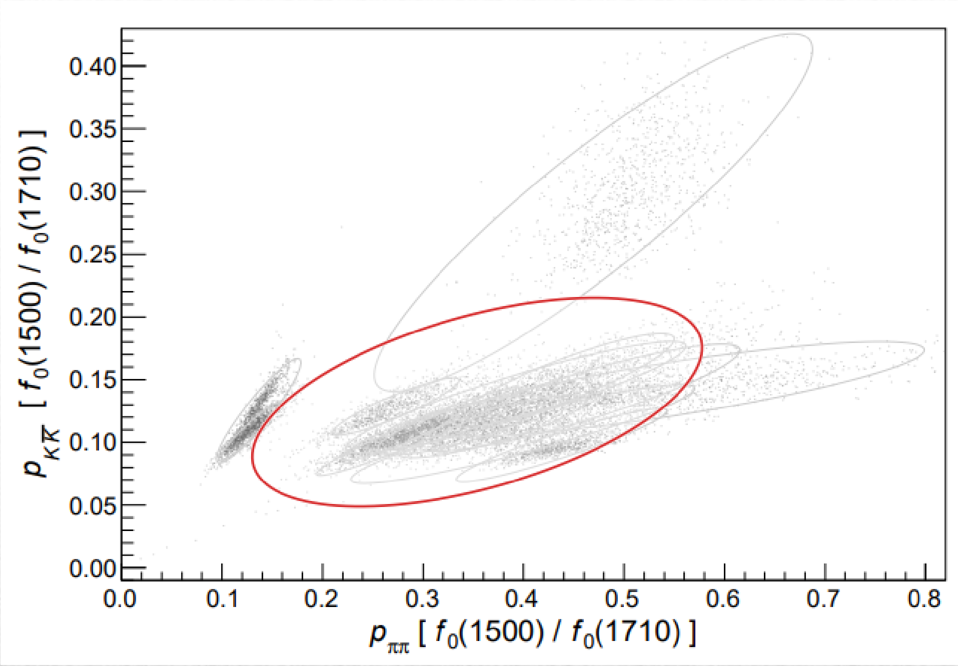
We consider the S and D wave by BESIII to use the information about their relative phase.
Fit quality and local description improve when a third $\rho\rho$ channel is added



Pole extraction



Looking at the residues



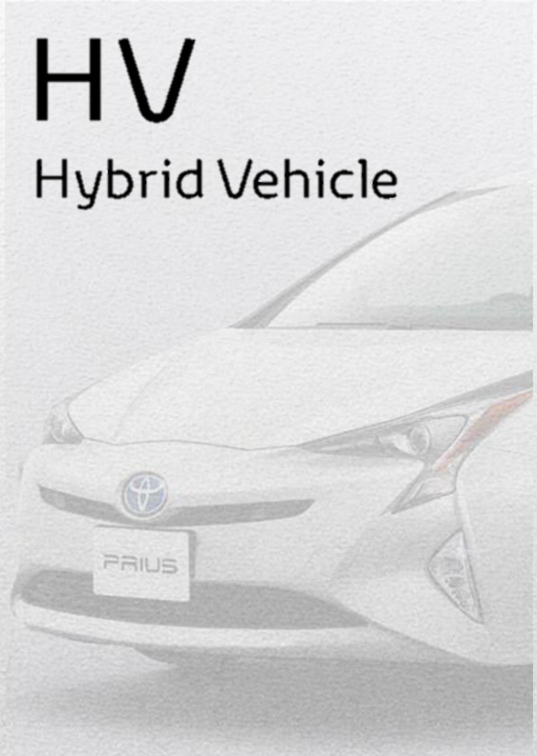
Despite the large systematics, the $f_0(1710)$ couples to kaons more than the $f_0(1500)$

Also, the $f_0(1710)$ couples to the initial gluon-rich state more than the $f_0(1500)$

Both these fact suggest a sizeable glueball component for the $f_0(1710)$

Meson Spectroscopy

The hybrid π_1



A. Rodas, AP *et al.*
PRL122, 042002

The scalar glueball

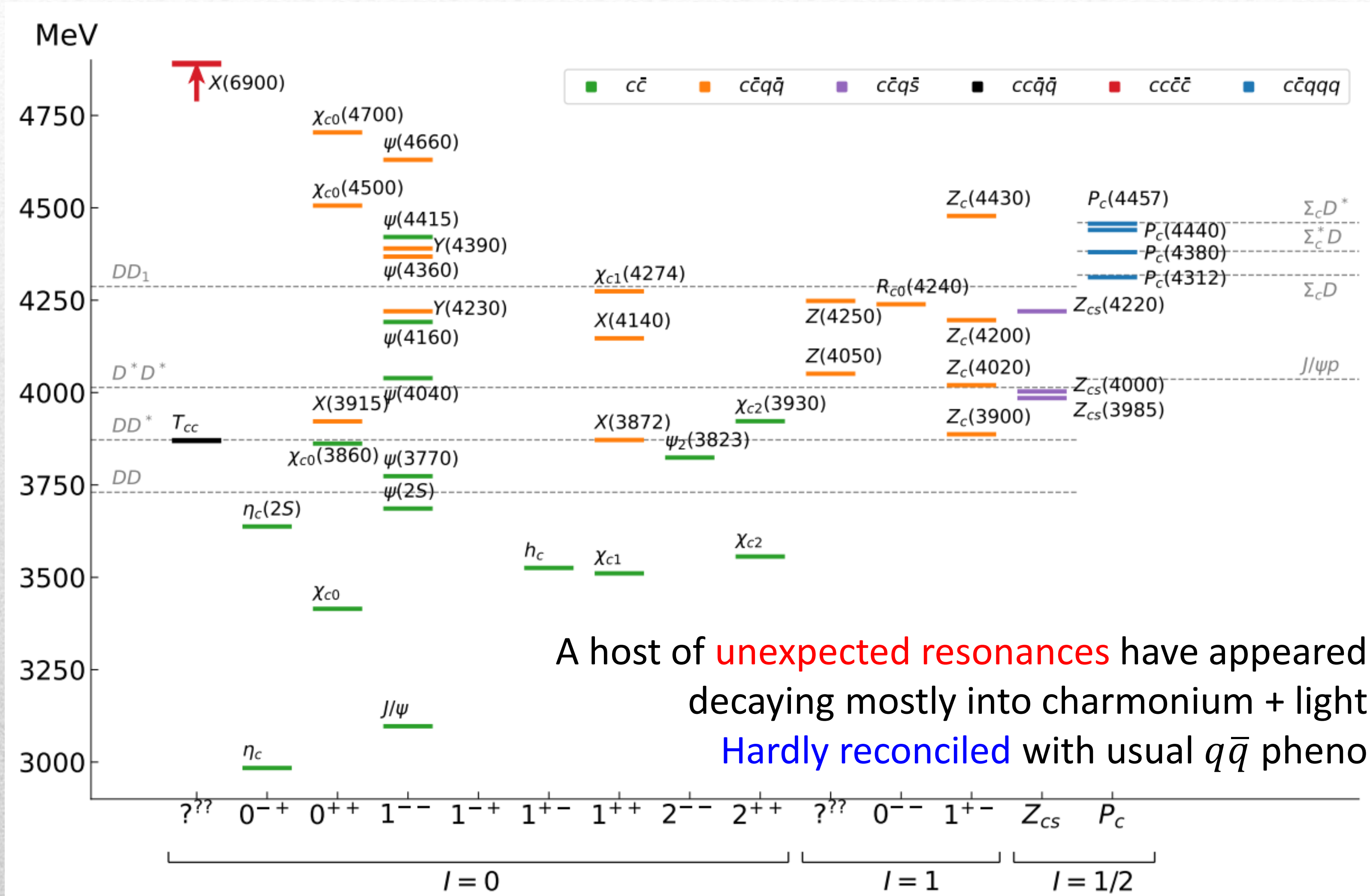
A. Rodas, AP *et al.*
EPJC82, 1, 80

XYZ photoprod.

M. Albaladejo et al., PRD102, 114010
D. Winney, AP et al., to appear

Exotic landscape at $c\bar{c}$

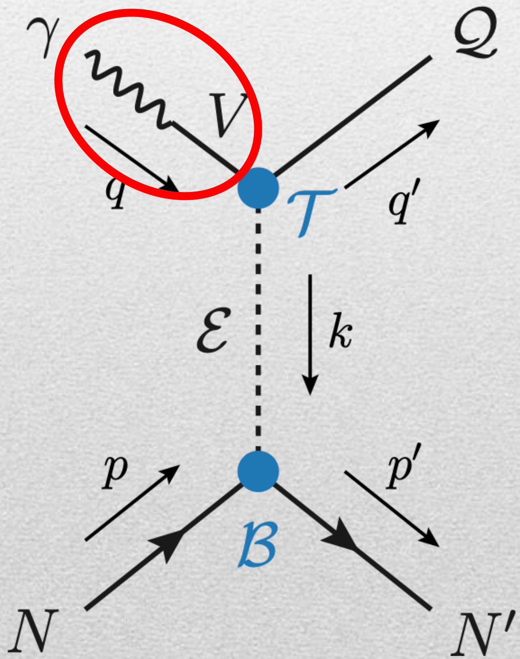
Esposito, AP, Polosa, Phys.Rept. 668
JPAC, arXiv:2112.13436



A host of unexpected resonances have appeared
decaying mostly into charmonium + light
Hardly reconciled with usual $q\bar{q}$ pheno

Exclusive (quasi-real) photoproduction

- XYZ have so far not been seen in photoproduction: independent confirmation
- Not affected by 3-body dynamics: determination of resonant nature
- Experiments with high luminosity in the appropriate energy range are promising
- We study near-threshold (LE) and high energies (HE)
- Couplings extracted from data as much as possible, not relying on the nature of XYZ



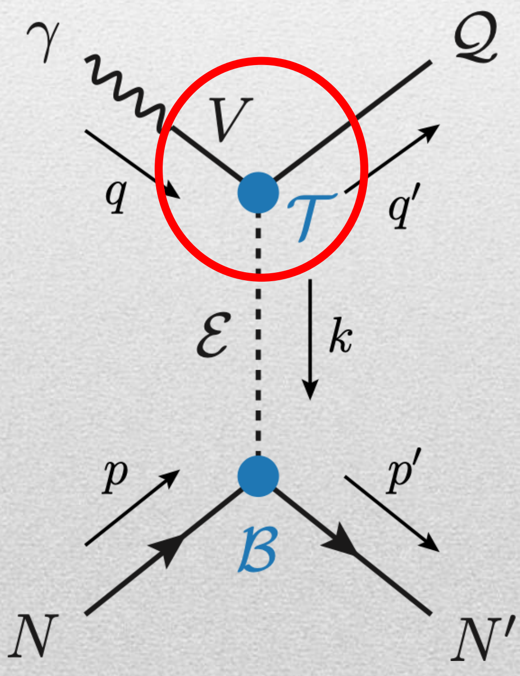
M. Albaladejo et al. [JPAC], PRD

$$< \lambda_Q \lambda'_N | T | \lambda_\gamma \lambda_N >= \sum_{V,\mathcal{E}} \frac{e f_V}{m_V} \mathcal{T}^{\alpha_1 \cdots \alpha_j}_{\lambda_V = \lambda_\gamma, \lambda_Q} \mathcal{P}_{\alpha_1 \cdots \alpha_j ; \beta_1 \cdots \beta_j} \mathcal{B}^{\beta_1 \cdots \beta_j}_{\lambda_N \lambda'_N}$$

VMD is used to couple the incoming photon to a vector quarkonium V

Exclusive (quasi-real) photoproduction

- XYZ have so far not been seen in photoproduction: independent confirmation
- Not affected by 3-body dynamics: determination of resonant nature
- Experiments with high luminosity in the appropriate energy range are promising
- We study near-threshold (LE) and high energies (HE)
- Couplings extracted from data as much as possible, not relying on the nature of XYZ



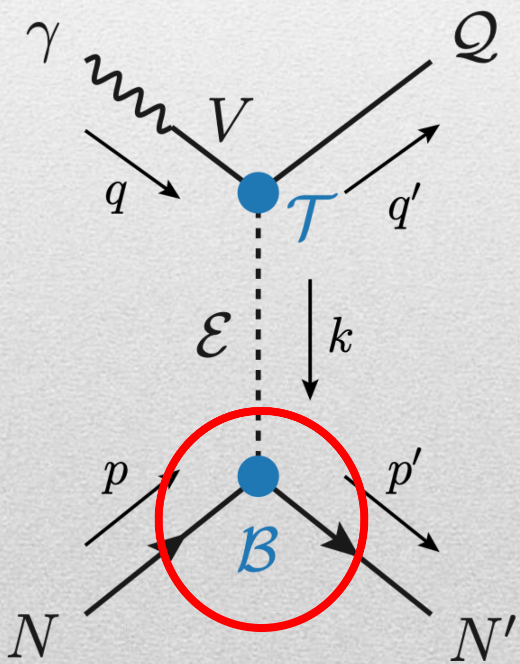
M. Albaladejo et al. [JPAC], PRD

$$< \lambda_{\mathcal{Q}} \lambda'_N | T | \lambda_\gamma \lambda_N > = \sum_{V,\mathcal{E}} \frac{e f_V}{m_V} \mathcal{T}^{\alpha_1 \cdots \alpha_j}_{\lambda_V = \lambda_\gamma, \lambda_{\mathcal{Q}}} \mathcal{P}_{\alpha_1 \cdots \alpha_j; \beta_1 \cdots \beta_j} \mathcal{B}^{\beta_1 \cdots \beta_j}_{\lambda_N \lambda'_N}$$

Top vertex from measured $\mathcal{Q} \to V\mathcal{E}$ decay width

Exclusive (quasi-real) photoproduction

- XYZ have so far not been seen in photoproduction: independent confirmation
- Not affected by 3-body dynamics: determination of resonant nature
- Experiments with high luminosity in the appropriate energy range are promising
- We study near-threshold (LE) and high energies (HE)
- Couplings extracted from data as much as possible, not relying on the nature of XYZ



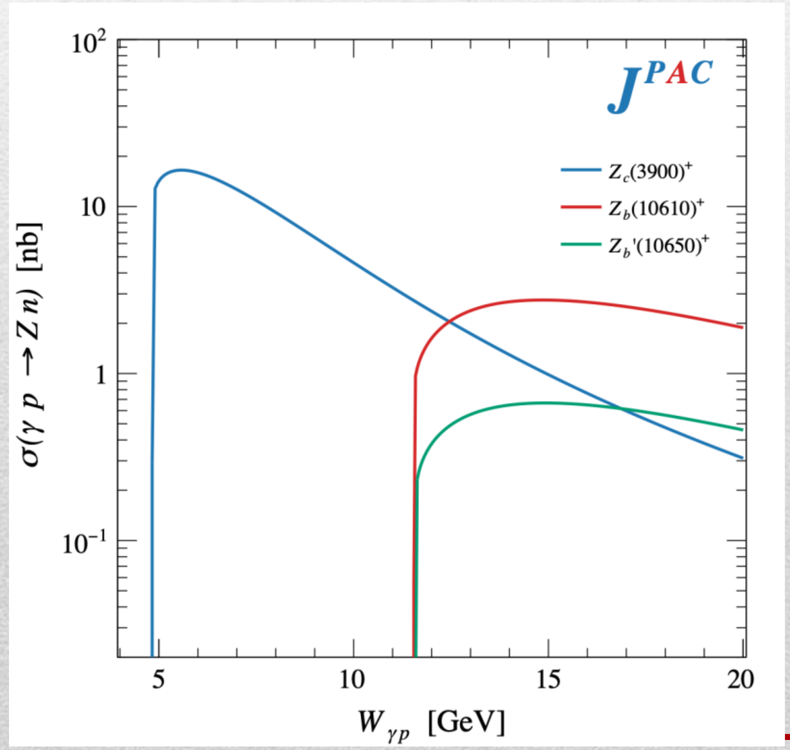
M. Albaladejo et al. [JPAC], PRD

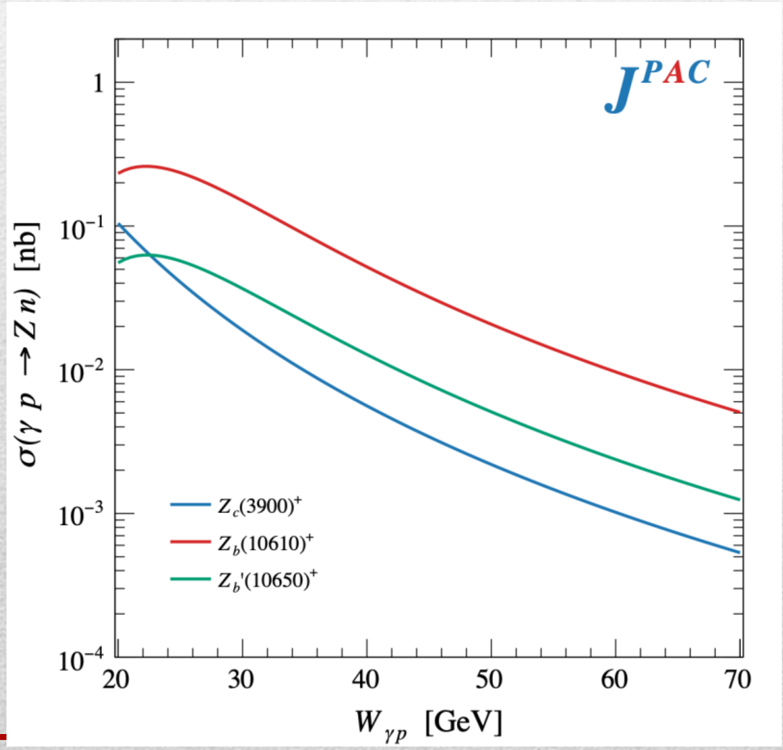
$$< \lambda_{\mathcal{Q}} \lambda'_N | T | \lambda_\gamma \lambda_N > = \sum_{V,\mathcal{E}} \frac{e f_V}{m_V} \mathcal{T}^{\alpha_1 \cdots \alpha_j}_{\lambda_V = \lambda_\gamma, \lambda_{\mathcal{Q}}} \mathcal{P}_{\alpha_1 \cdots \alpha_j; \beta_1 \cdots \beta_j} \mathcal{B}^{\beta_1 \cdots \beta_j}_{\lambda_N \lambda'_N}$$

Bottom vertex from standard photoproduction pheno, exponential form factors to further suppress large t

Z photoproduction

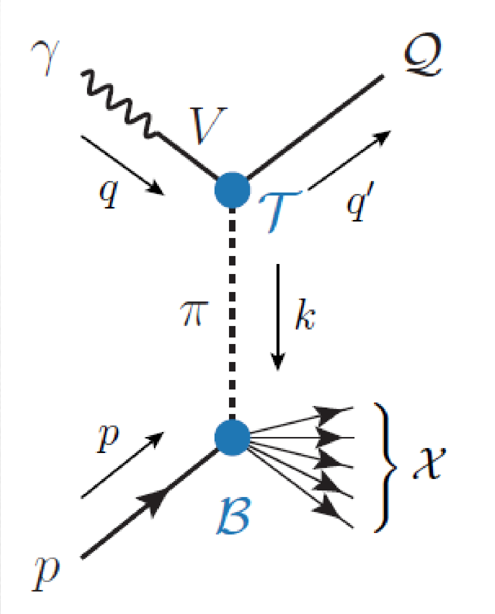
- The Zs are charged charmoniumlike 1^{+-} states close to open flavor thresholds
- Focus on $Z_c(3900)^+ \to J/\psi\,\pi^+$, $Z_b(10610)^+, Z_b'(10650)^+ \to \Upsilon(nS)\,\pi^+$
- The pion is exchanged in the t-channel
- Sizeable cross sections especially at low energy

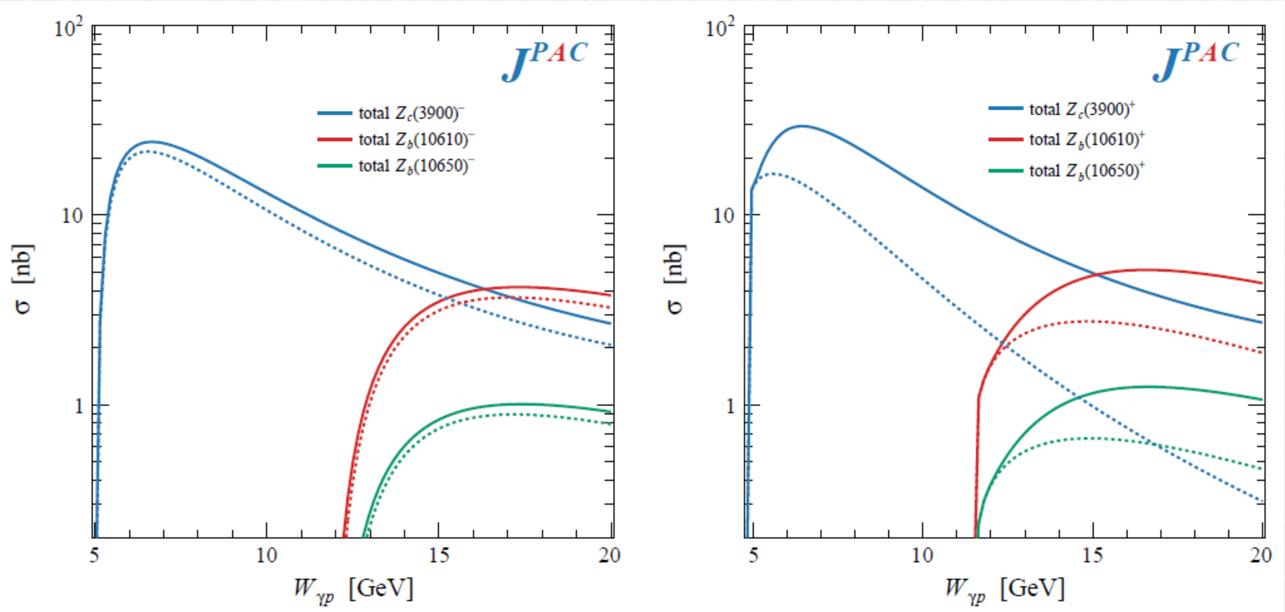




Semi-inclusive photoproduction

- Semi-inclusive cross sections are typically larger
- The bottom vertex depends on the (known) pion-proton total cross section
- The pion is exchanged in the t-channel



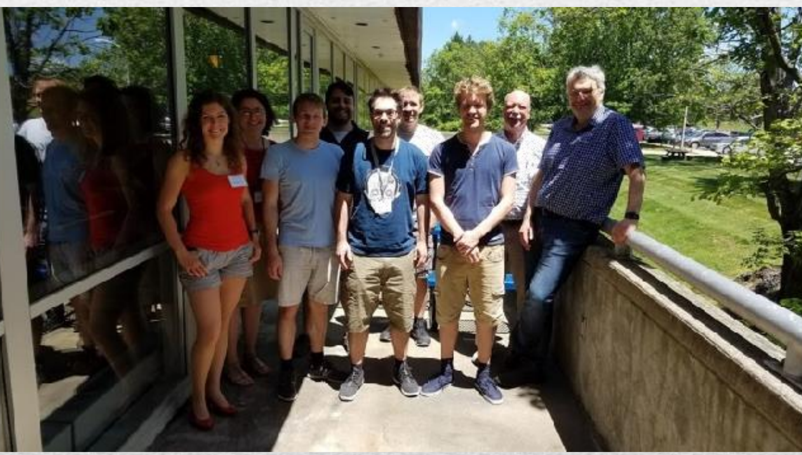


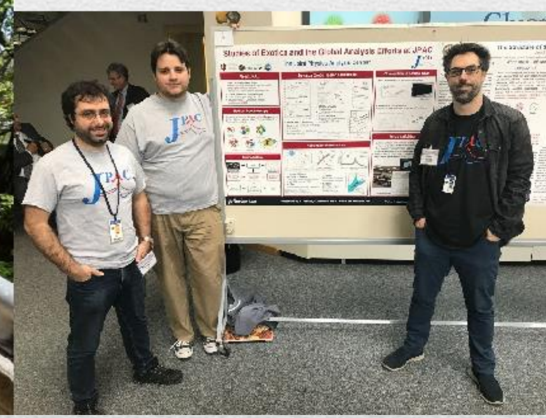
Joint Physics Analysis Center

Bottom-up approaches are important!

- Joint effort between theorists and experimentalists in support of experimental data from JLab12 and other accelerator laboratories
- Cooperation between JPAC and experiments: co-authoring papers

http://www.jpac-physics.org





Thank you!

BACKUP

Conclusions & prospects

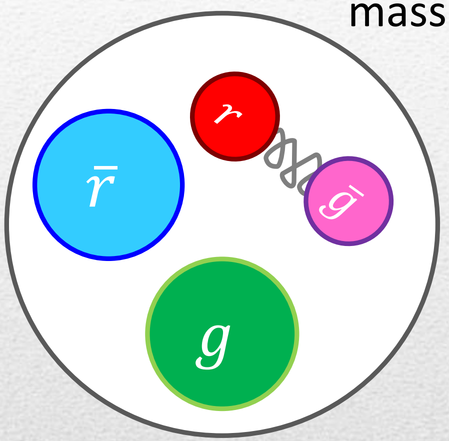
The study of exotic hadrons is a challenging task
Experiments are very prolific! Constant feedback on predictions

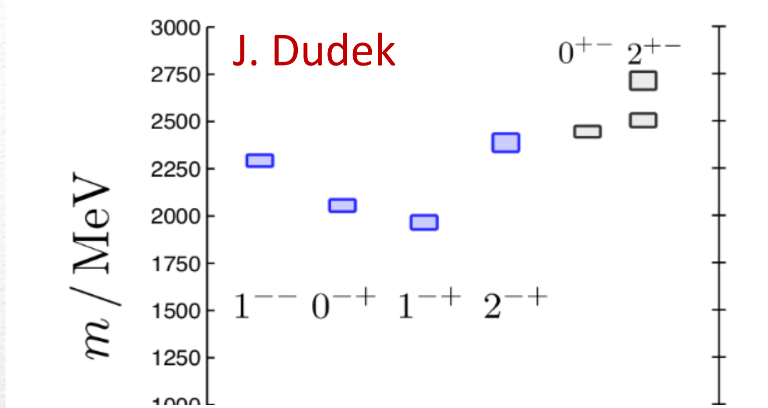
- Study of spectra and decay patterns will improve our understanding, new data expected by BESIII, LHCb, GlueX
- A more refined study of amplitudes provides a complimentary tool to give insights on the nature of some of the states
- Facilities to study photoproduction at low energies are very welcome to pursue this program

Thank you

Hybrid hunting

Excited gluon,
$J^{PC} = 1^{+-}$
mass $\sim 1.0-1.5$ GeV

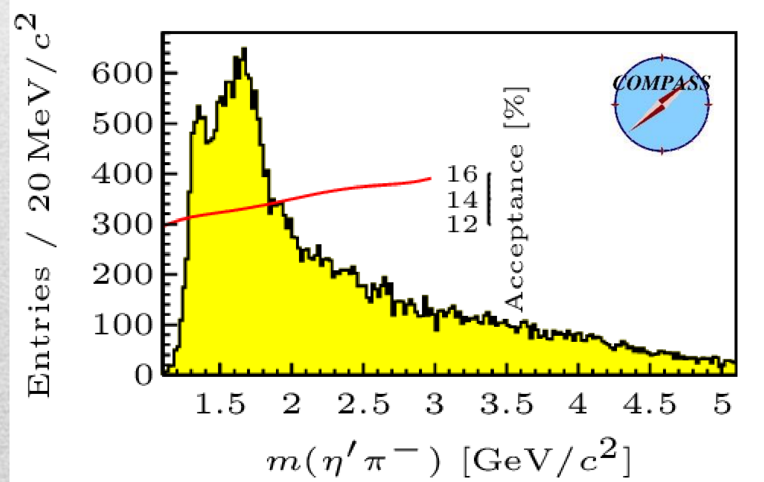




excitation of a quasiparticle?
→ degenerate $J^{PC} = (0, \mathbf{1}, 2)^{-+}, 1^{--}$

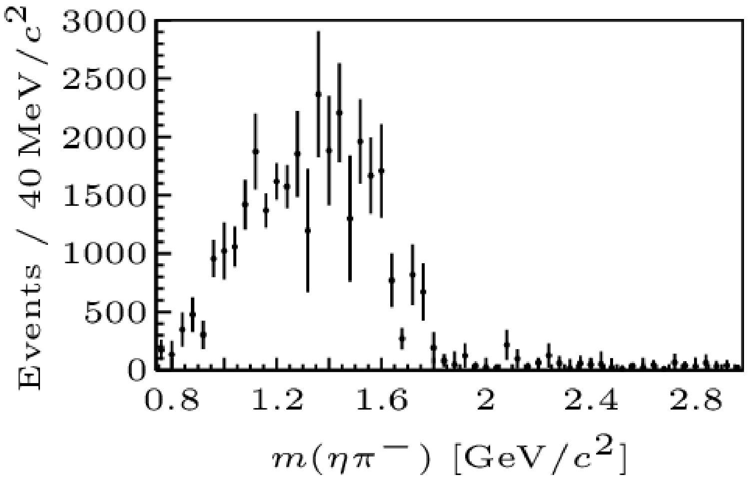
Look for a π_1 state with $\boldsymbol{J^{PC} = 1^{-+}}$

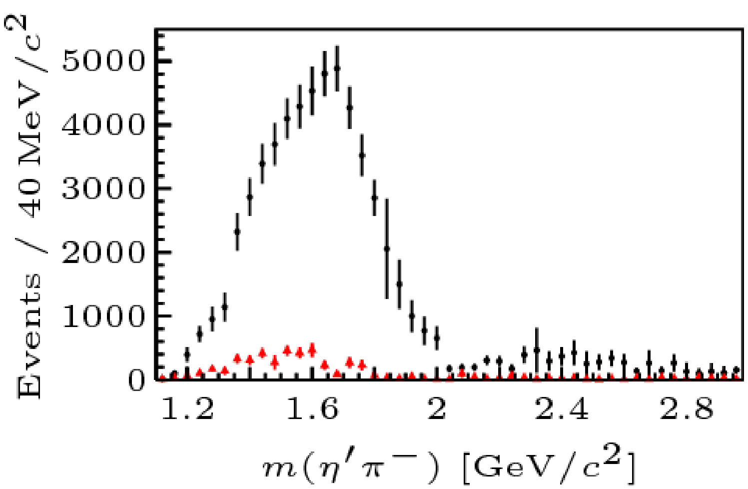
$$\text{decaying into} \begin{cases} \eta\,\pi \text{ and } \eta'\pi \\ \rho\,\pi \to 3\pi \\ b_1\pi \to 5\pi \end{cases}$$



Small signal in data

Two hybrid states???





$\pi_1(1400)$ $I^G(J^{PC}) = 1^-(1^{-+})$

See also the mini-review under non-$q\bar{q}$ candidates in PDG 2006 , Journal of Physics G33 1 (2006).

$\pi_1(1400)$ MASS	1354 ± 25 MeV (S = 1.8)
$\pi_1(1400)$ WIDTH	330 ± 35 MeV

Decay Modes

Mode		Fraction (Γ_i/Γ)	Scale Factor/ Conf. Level
Γ_1	$\eta\pi^0$	seen	
Γ_2	$\eta\pi^-$	seen	
Γ_3	$\eta'\pi$		

Neither lattice nor models predict two 1^{-+} states in this region!

$\pi_1(1600)$ $I^G(J^{PC}) = 1^-(1^{-+})$

$\pi_1(1600)$ MASS	1662^{+8}_{-9} MeV
$\pi_1(1600)$ WIDTH	241 ± 40 MeV (S = 1.4)

A hybrid meson ($\mathbf{8}$) cannot decay into $\eta\pi$ in the SU(3) limit

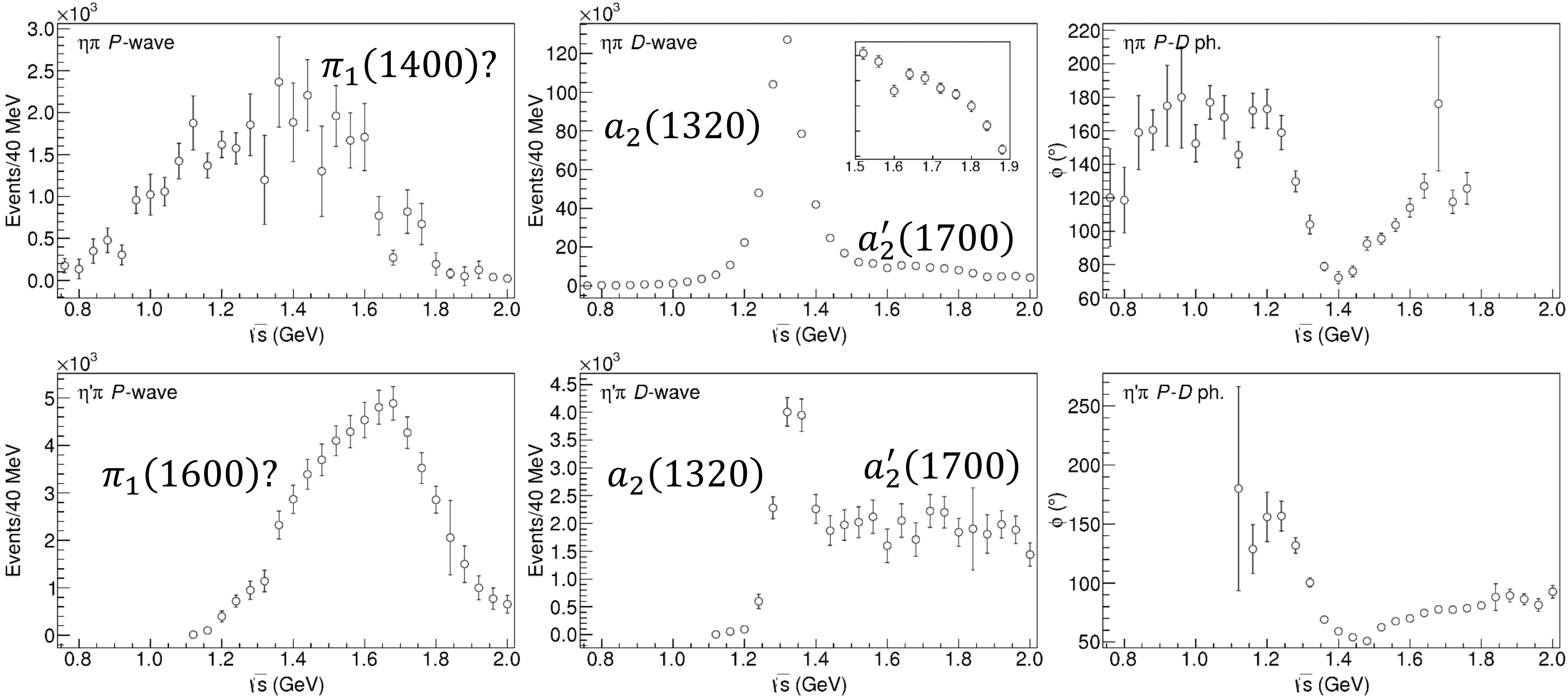
Decay Modes

Mode		Fraction (Γ_i/Γ)	Scale Factor/ Conf. Level
Γ_1	$\pi\pi\pi$	seen	
Γ_2	$\rho^0\pi^-$	seen	
Γ_3	$f_2(1270)\pi^-$	not seen	
Γ_4	[illegible]	[illegible]	
Γ_5	$\eta'(958)\pi^-$	seen	
Γ_6	$f_1(1285)\pi$	seen	

Tetraquark ($\mathbf{10} \oplus \overline{\mathbf{10}}$)? Requires doubly charged

Data in $\eta^{(\prime)}\pi$

(COMPASS data)



$J^{PC} = 1^{-+}$ $\qquad$ $J^{PC} = 2^{++}$

Coupled channel: the model

A. Rodas, AP *et al.* (JPAC) PRL122, 042002

Two channels, $i, k = \eta\pi, \eta'\pi$

Two waves, $J = P, D$

37 fit parameters

$$D^J_{ki}(s) = \left[K^J(s)^{-1}\right]_{ki} - \frac{s}{\pi}\int_{s_k}^{\infty} ds' \frac{\rho N^J_{ki}(s')}{s'(s' - s - i\epsilon)}$$

$$K^J_{ki}(s) = \sum_R \frac{g_k^{(R)} g_i^{(R)}}{m_R^2 - s} + c^J_{ki} + d^J_{ki}\, s$$

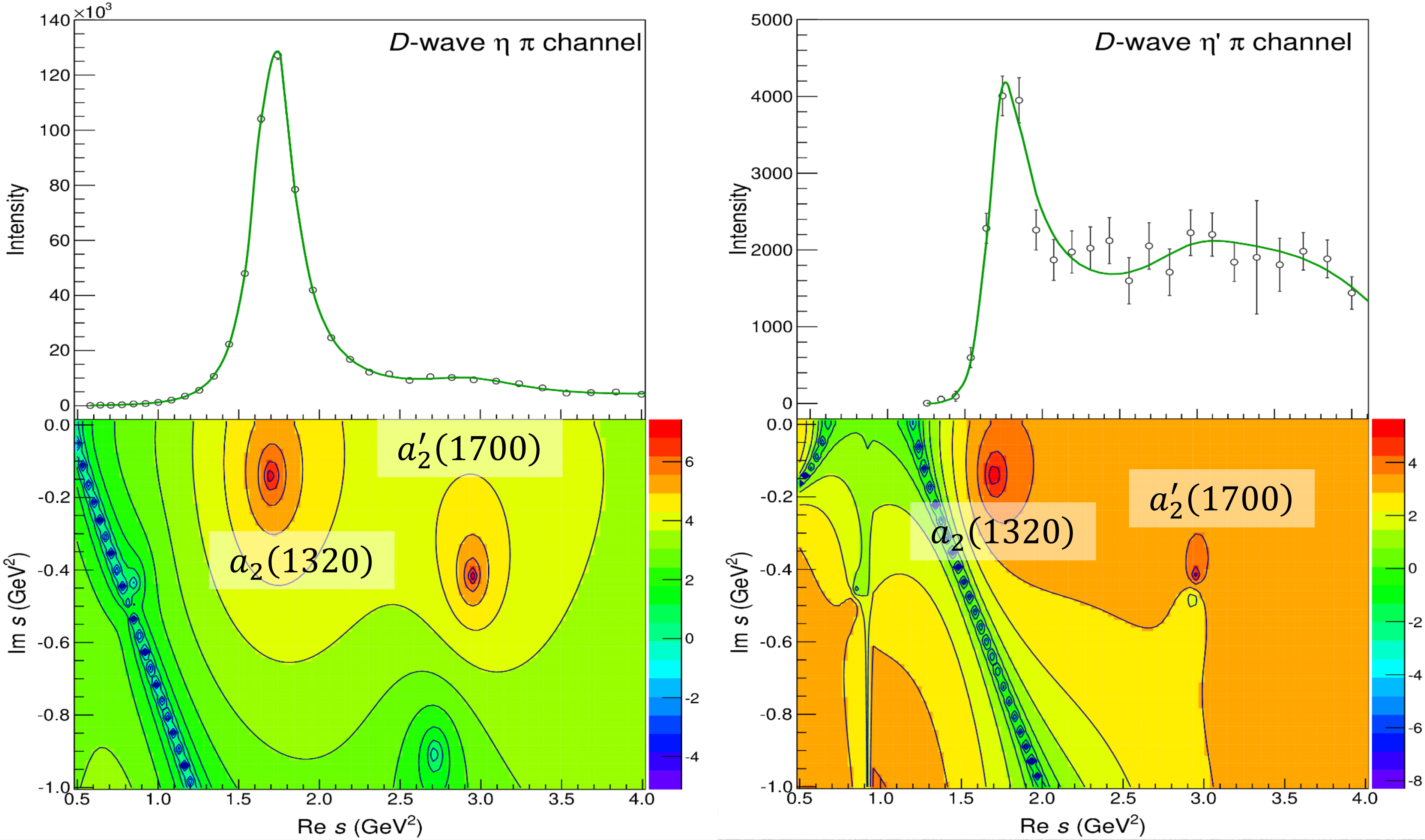
1 *K*-matrix pole for the P-wave
2 *K*-matrix poles for the D-wave

$$\rho N^J_{ki}(s') = \delta_{ki} \frac{\lambda^{J+1/2}\left(s', m^2_{\eta^{(\prime)}}, m^2_\pi\right)}{(s' + s_R)^{2J+1+\alpha}}$$

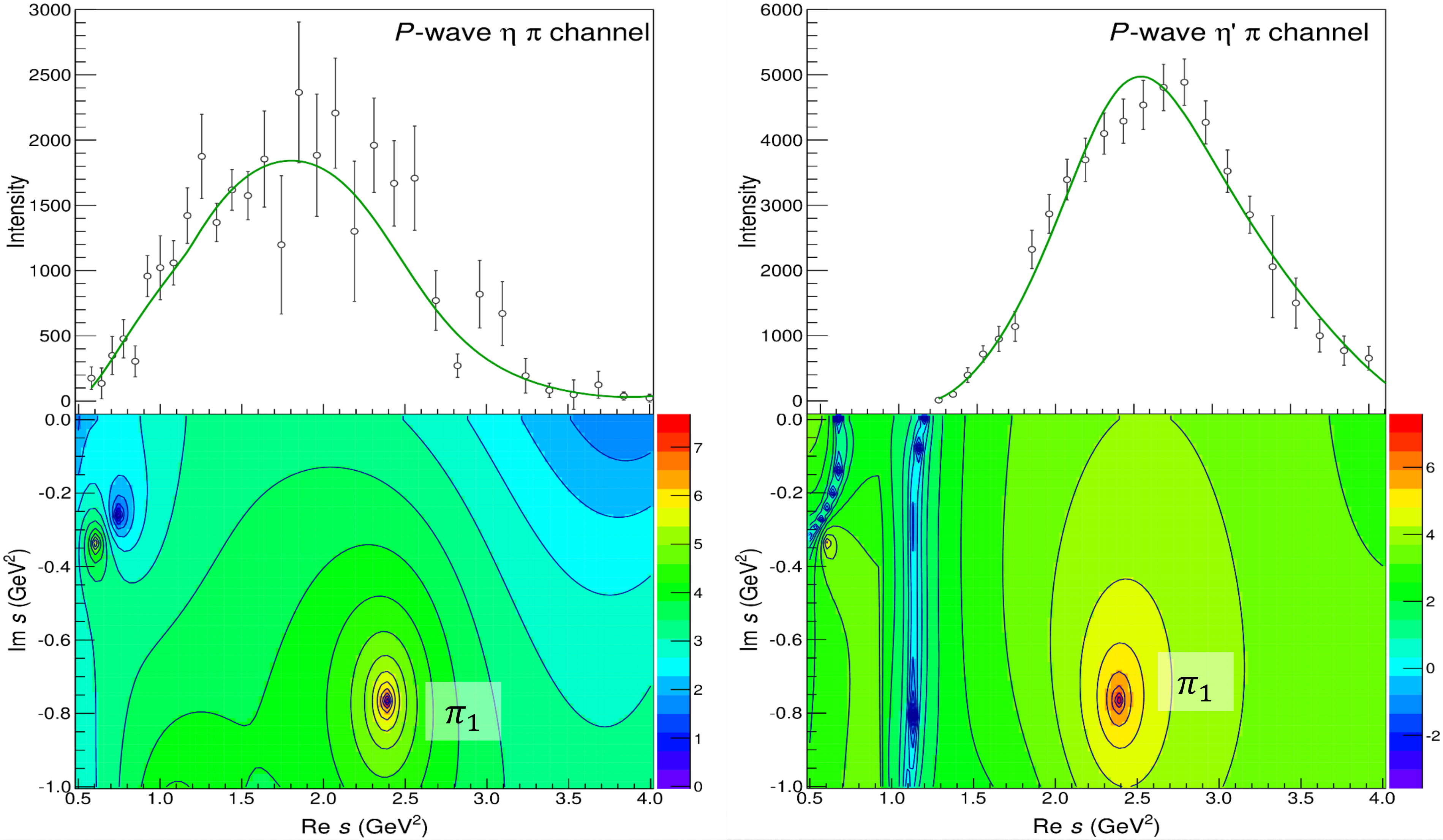
$$n^J_k(s) = \sum_{n=0}^{3} a^{J,k}_n T_n\left(\frac{s}{s+s_0}\right)$$

Left-hand scale (Blatt-Weisskopf radius) $s_R = s_0 = 1\ \mathrm{GeV}^2$
$\alpha = 2$, 3rd order polynomial for $n^J_k(s)$

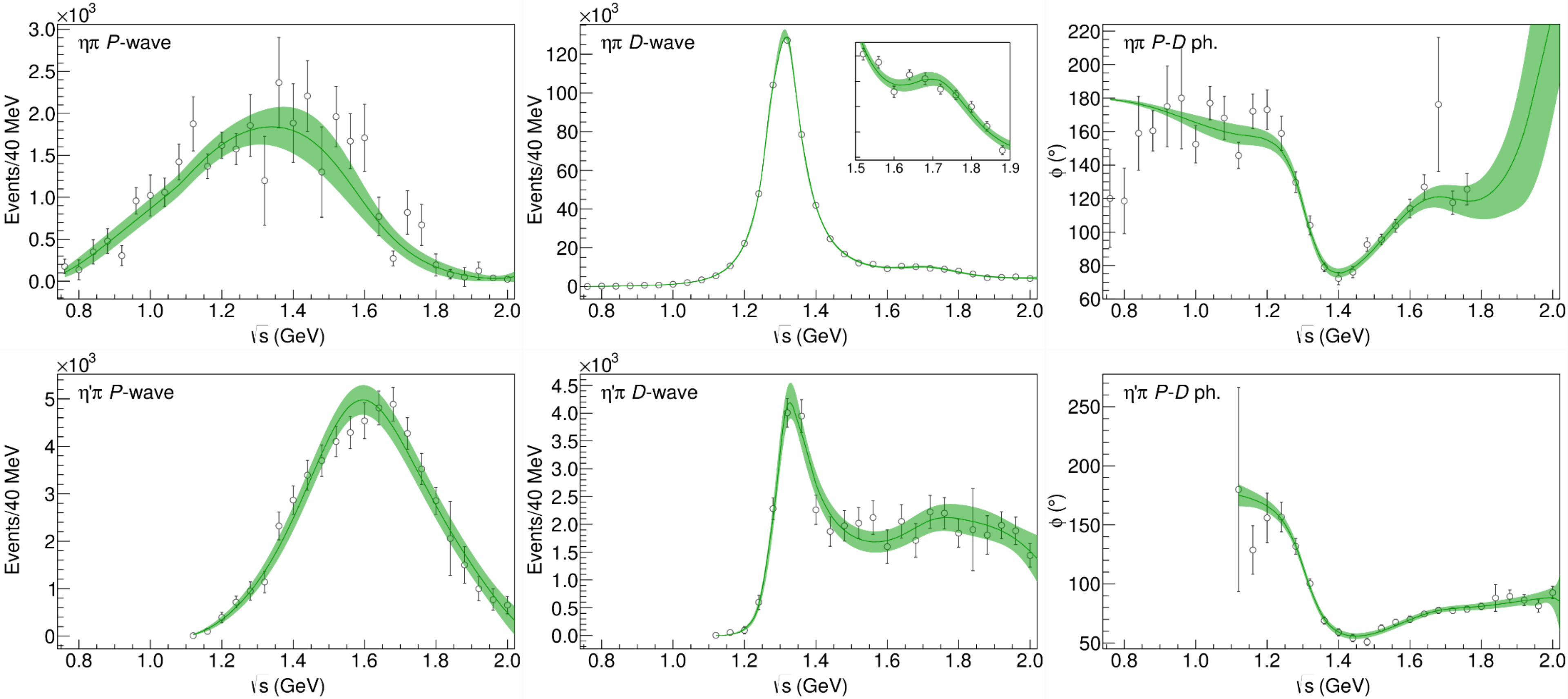
Pole hunting



Pole hunting



Statistical Bootstrap

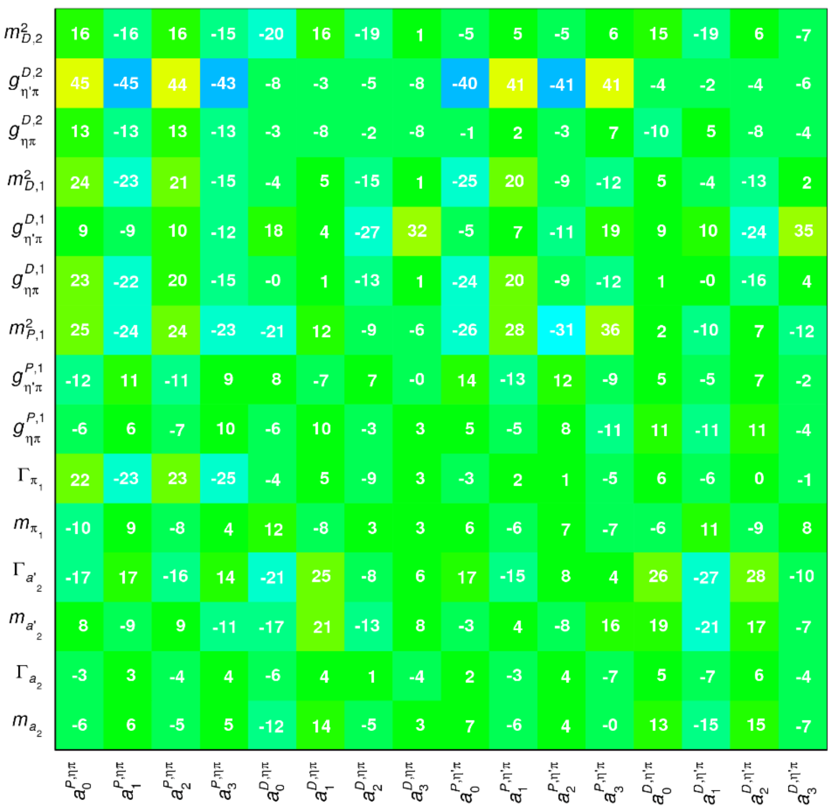


Correlations

Denominator parameters uncorrelated with the numerator ones ✓

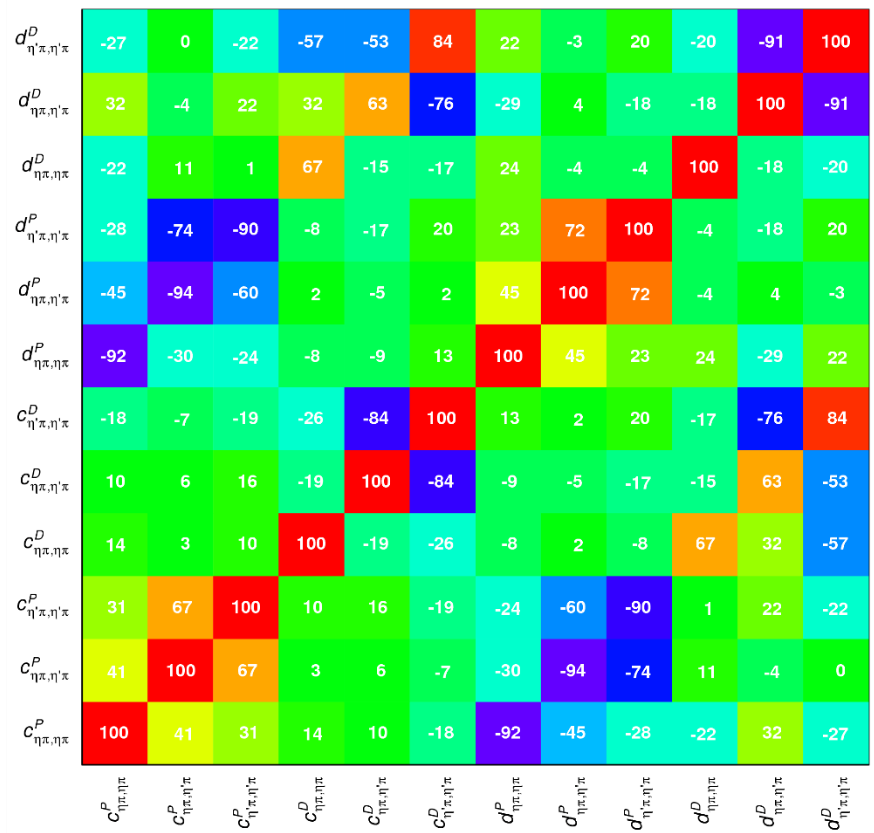
Denominator parameters uncorrelated between *P*- and *D*-wave ✓

Production (numerator) parameters



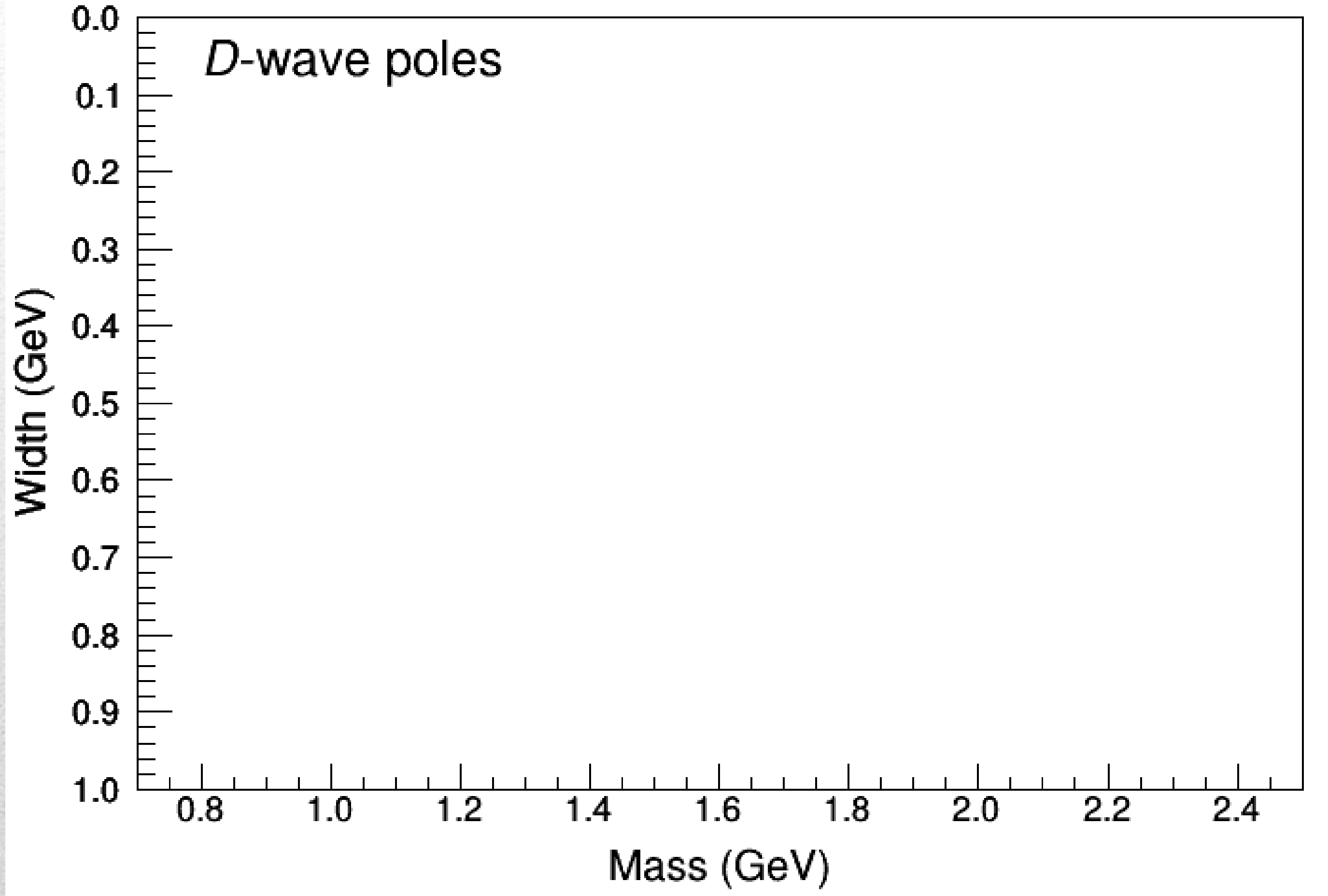
K-matrix «pole» parameters

K-matrix «bkg» parameters



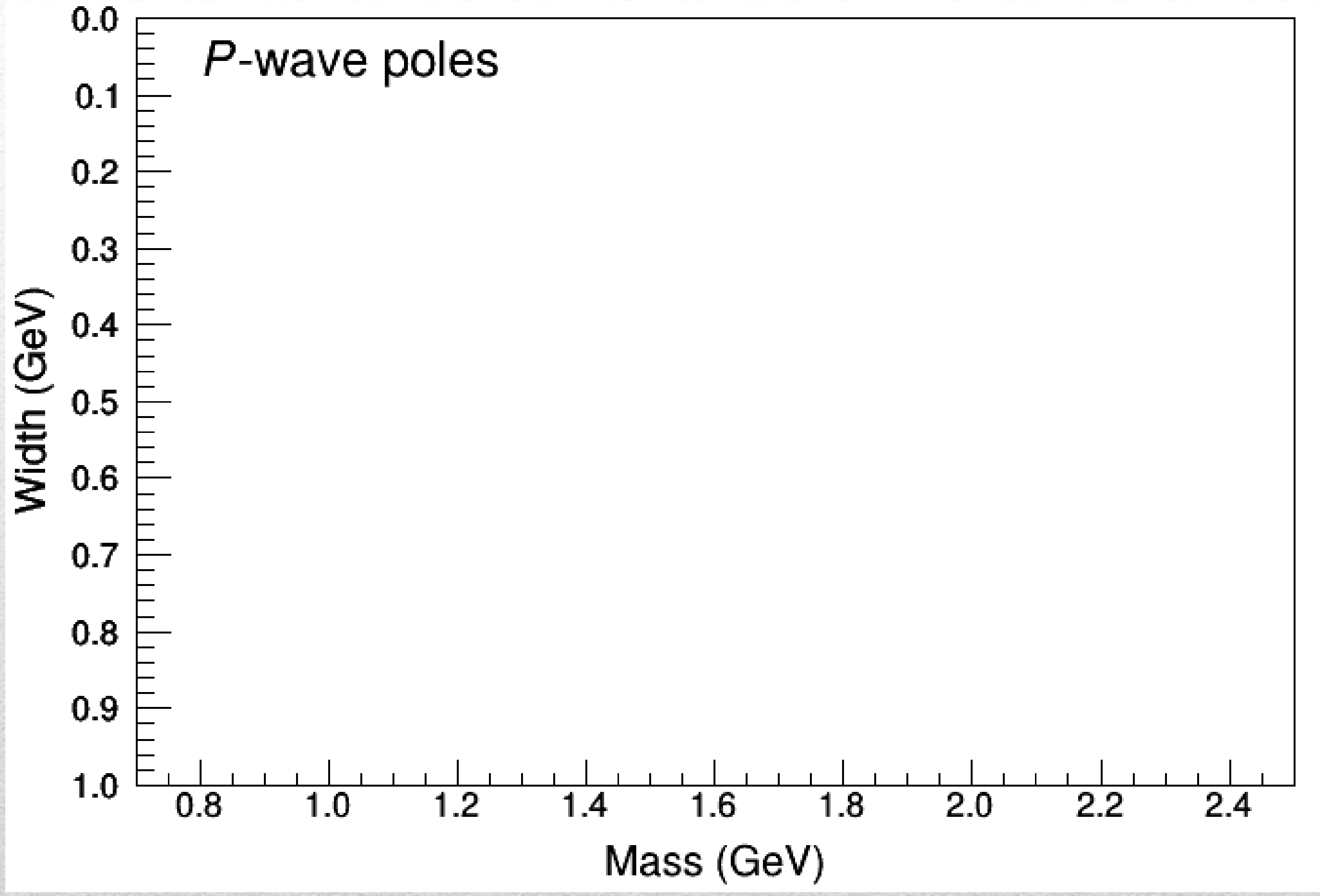
K-matrix «bkg» parameters

Statistical Bootstrap



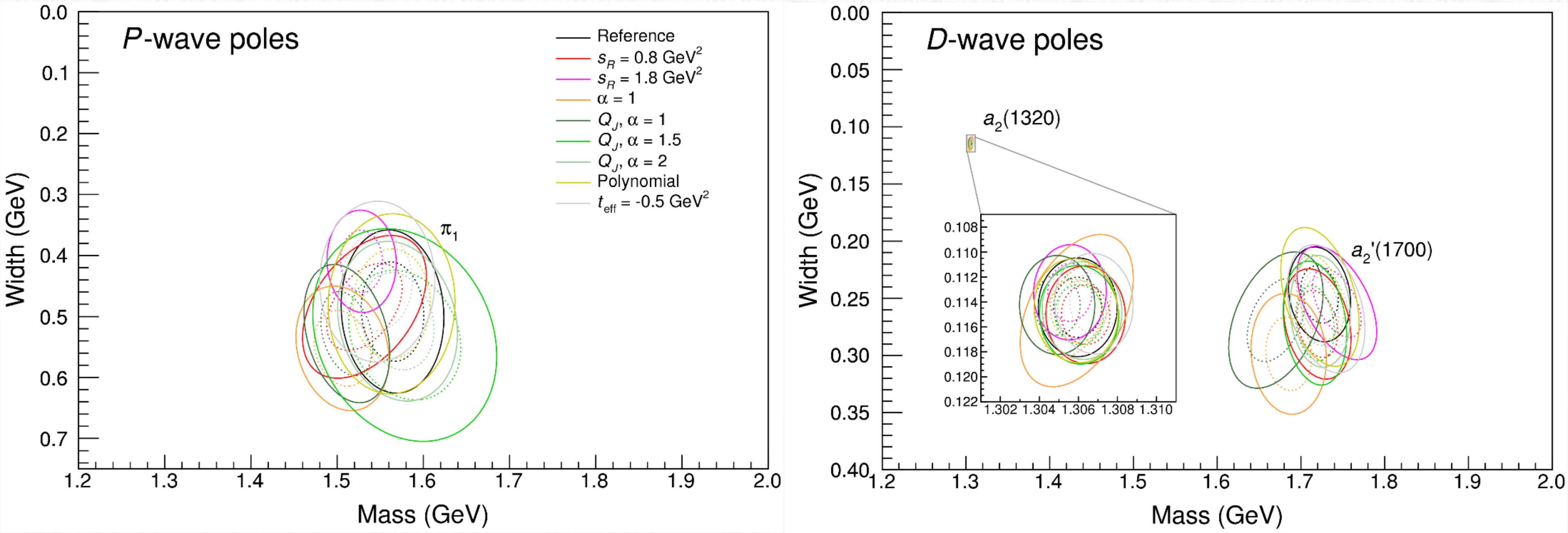
For each fit, we search poles: two clusters in *D*-wave: $a_2(1320)$ and $a_2'(1700)$

Statistical Bootstrap



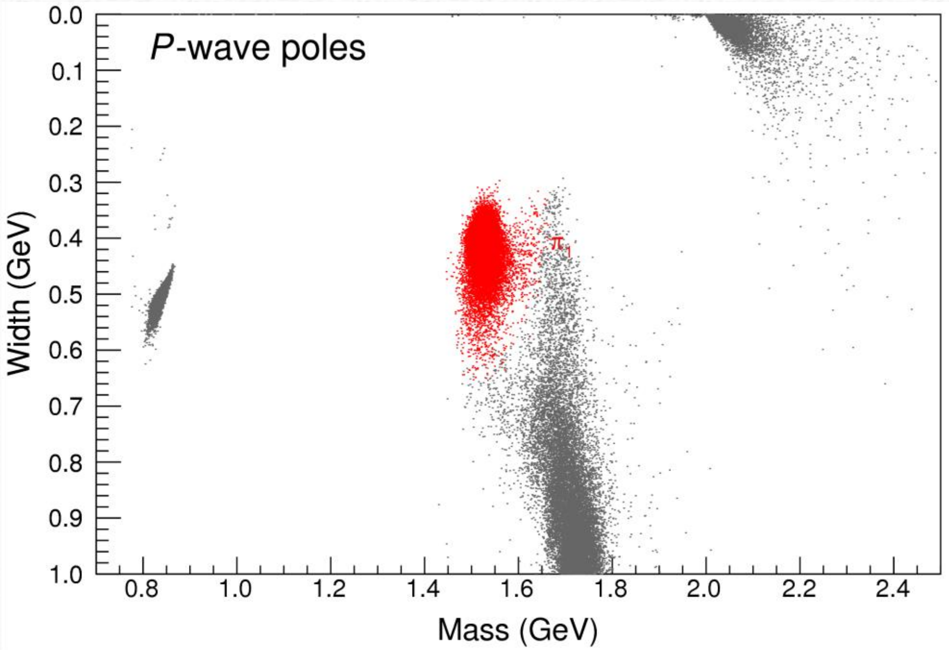
Only one stable cluster in P-wave: a single π_1

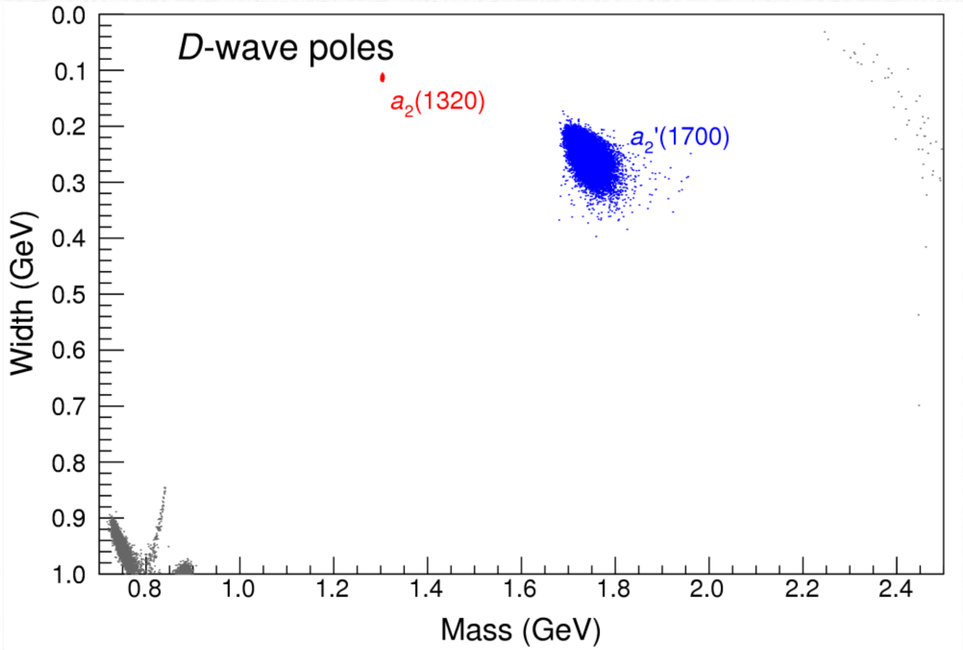
Systematic studies



For each class, the maximum deviation of mass and width is taken as a systematic error
Deviation smaller than the statistical error are neglected
Systematic of different classes are summed in quadrature

Bootstrap for $s_R = 1.8\ \mathrm{GeV}^2$





Our skepticism about a second pole in the relevant region is confirmed:
It is unstable and not trustable

Same model as before

A. Rodas, AP *et al.* (JPAC) EPJC82, 1, 80 (2022)

Two/three channels, $i,k = \pi\pi, KK, \rho\rho$ Two waves, $J = S, D$ 40-56 parameters

$$D^J_{ki}(s) = \left[K^J(s)^{-1}\right]_{ki} - \frac{s}{\pi}\int_{s_k}^{\infty} ds' \frac{\rho N^J_{ki}(s')}{s'(s'-s-i\epsilon)}$$

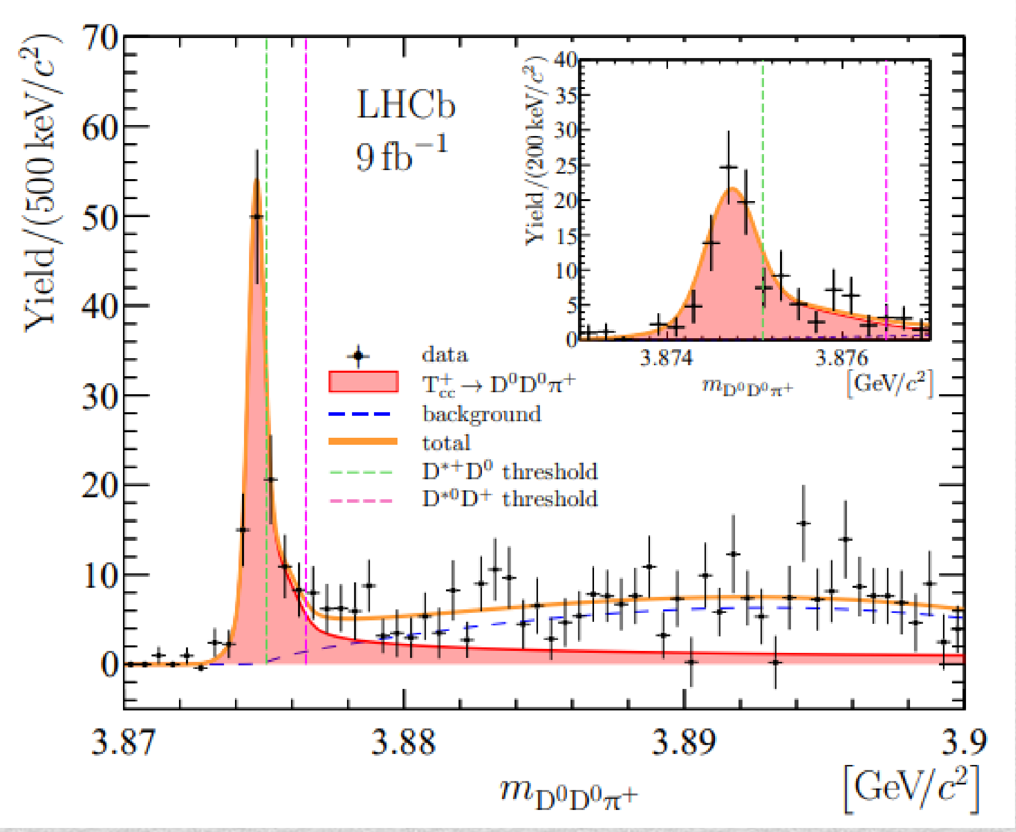
$$K^J_{ki}(s) = \sum_R \frac{g_k^{(R)} g_i^{(R)}}{m_R^2 - s} + c^J_{ki} + d^J_{ki}\, s$$

3 *K*-matrix pole for the S-wave
3 *K*-matrix poles for the D-wave

$$\rho N^J_{ki}(s') = \delta_{ki} \frac{\lambda^{J+1/2}\left(s', m^2_{\eta^{(\prime)}}, m^2_\pi\right)}{(s'+s_R)^{2J+1+\alpha}}$$

$$n^J_k(s) = \sum_{n=0}^{3} a_n^{J,k}\, T_n\left(\frac{s}{s+s_0}\right)$$

Very open charm: the T_{cc}^{+}



A supernarrow state is seen at the $D^{*+}D^0$ threshold

BW parameters give

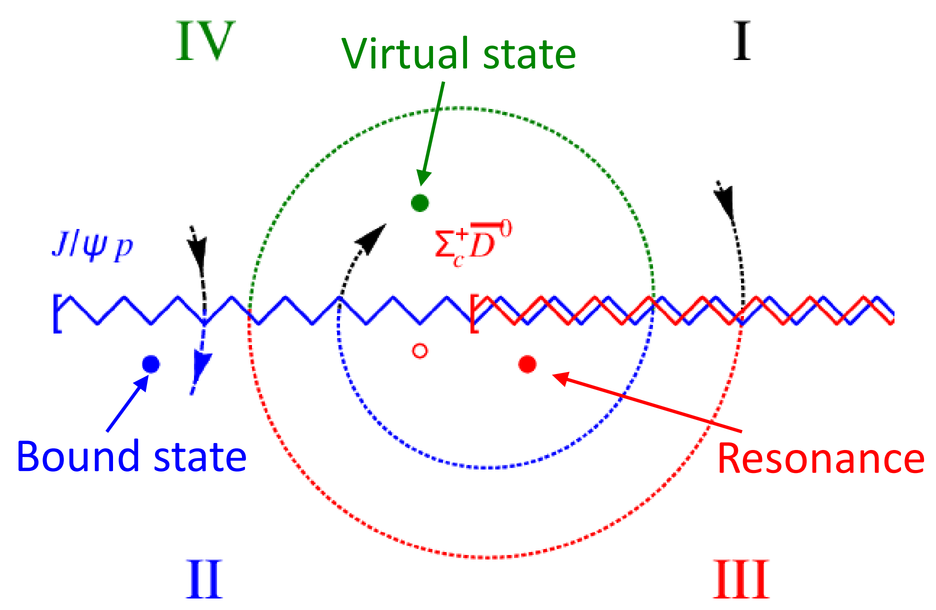
$$\delta m_{BW} = -273 \pm 61 \text{ keV}$$
$$\Gamma_{BW} = 410 \pm 65 \text{ keV}$$

It cannot mix with ordinary charmonia

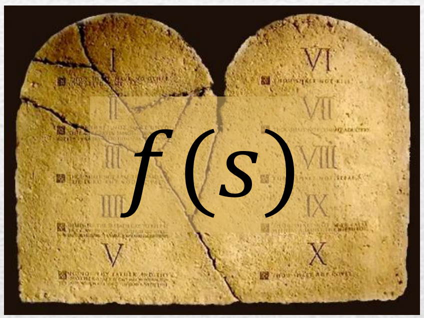
It does not have lighter open channels

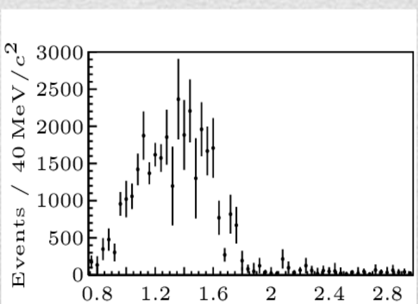
Bound and virtual states

The situation is more complicated when more channels are open



The flowchart(s)





1) You are given a model/theory

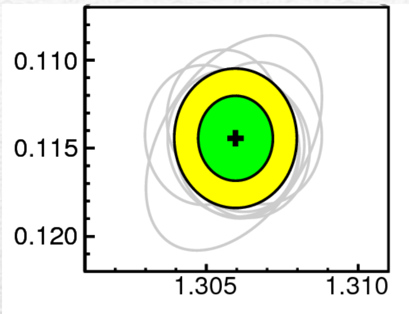
2) You calculate the amplitude

3) You compare with data.
   Or you don't.

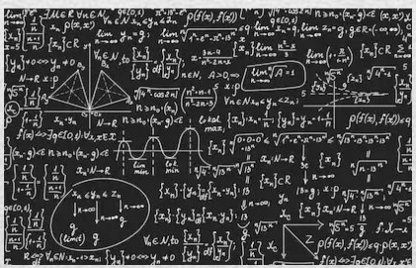
Predictive power ✓
Physical interpretation ✓
(within the model! ✘)
Biased by the input ✘

The flowchart(s)

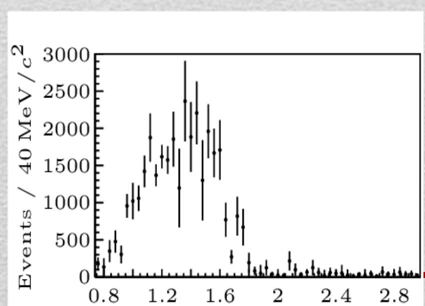
Less predictive power ✘
Some physical interpretation ✘
Minimally biased ✓



3) You extract physics



2) You choose a set of generic amplitudes

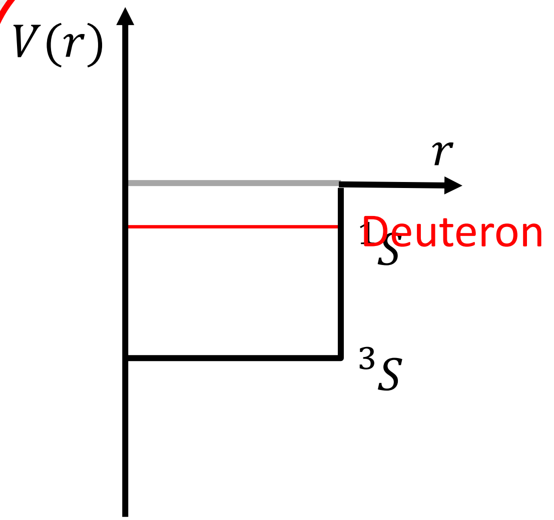


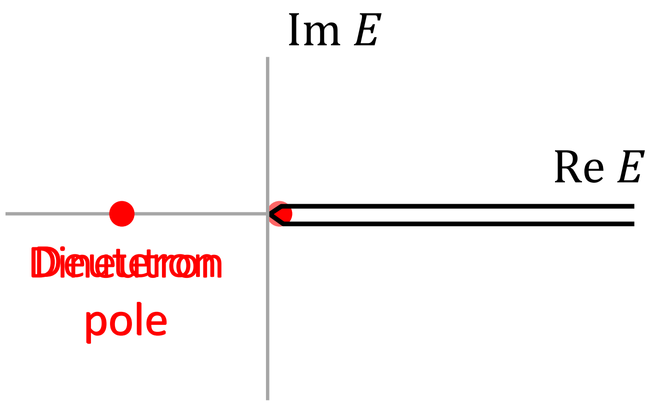
1) You start with data

Bound and virtual states

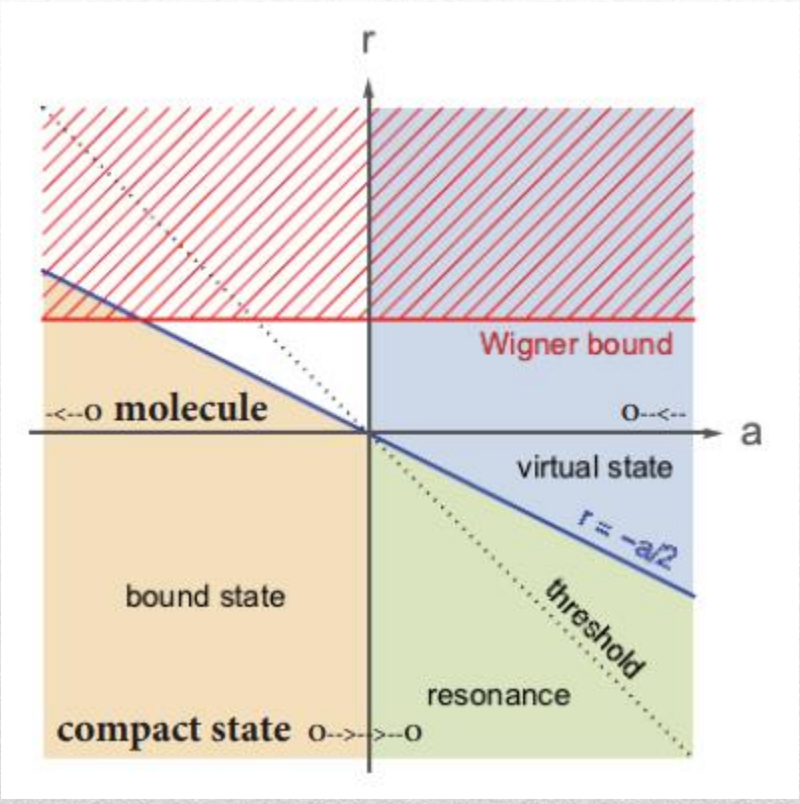
Decreasing the potential strength, the pole jumps on the 2nd sheet (dineutron),

the pole reaches threshold, bound state on the real axis 1st sheet (deuteron)





Bound and virtual states



Matuschek *et al.* EPJA57 (2021) 3, 101

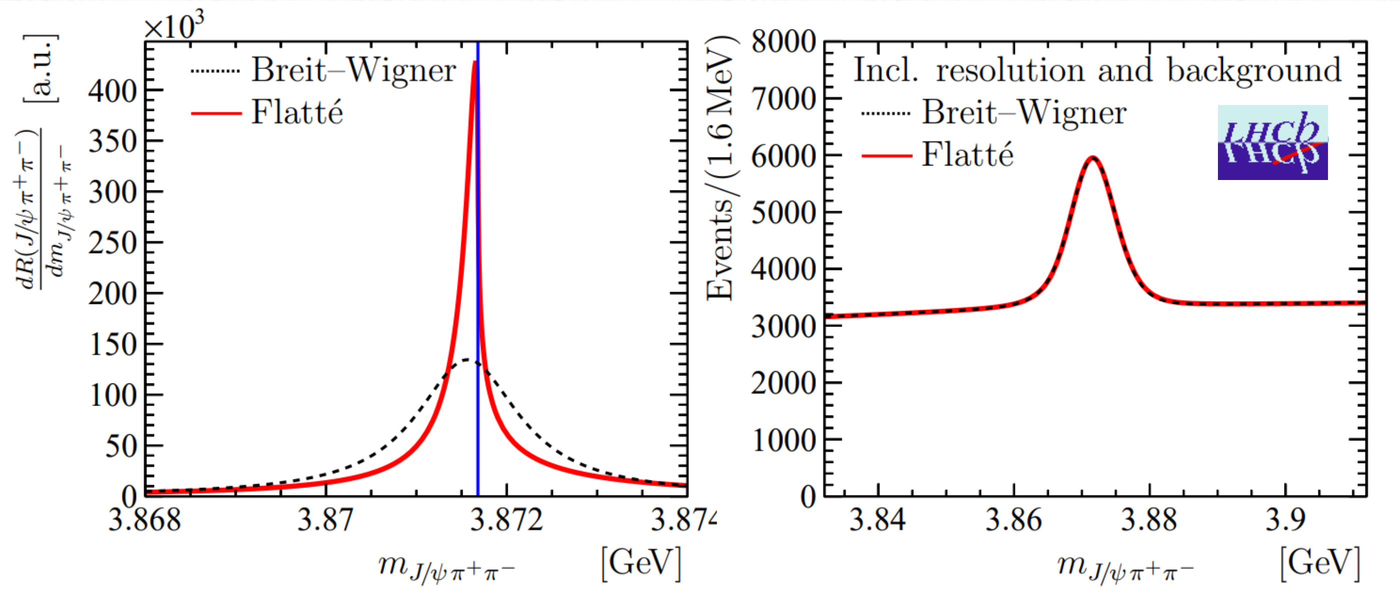
The amplitude close to threshold can be expanded as

$$A(E) = \frac{1}{\frac{1}{a} + \frac{1}{2} r_0 k^2 - ik + O(k^4)}$$

a is the scattering length
r_0 is the effective range

The sign of a controls whether we have a bound or virtual state

The lineshape of the $X(3872)$



LHCb, PRD 102, 092005

Blue line is $D^0\overline{D^{*0}}$, D^+D^{*-} is $\delta = 8.2$ MeV heavier

Because of experimental resolution, different lineshapes are indistinguishable

Unitary parametrizations tend to be narrower,
$\Gamma_{BW} = 1.39 \pm 0.24 \pm 0.10$ MeV, $\Gamma_{Fl} = 0.22^{+0.07+0.11}_{-0.06-0.13}$ MeV

The lineshape of the $X(3872)$

Esposito, Maiani, AP, Polosa, Riquer, PRD 105 (2022) 3, L031503

LHCb data is fitted with the Flatte' parametrization

Two options: a) set $\delta = 0$ b) I expand at threshold

The $D^0\overline{D}^{*0}$ threshold

The D^+D^{*-} threshold

$$t^{-1}(E) \propto E - m_X^0 + \frac{i}{2} g_{\mathrm{LHCb}} \left(\sqrt{2\mu E} + \sqrt{2\mu_+ (E - \delta)} \right)$$

$$+ \frac{i}{2} \cancel{\left(\Gamma_\rho^0(E) + \Gamma_\omega^0(E) + \Gamma_0^0 \right)}$$

The $J/\psi\ \rho, \omega$, and other unknown channels

This considers coupled channel, but Weinberg's criterion applies to single channel bound states only

The lineshape of the $X(3872)$

Esposito, Maiani, AP, Polosa, Riquer,
PRD 105 (2022) 3, L031503

Option b) $$-5.34\ \mathrm{fm} \lesssim r_0 \lesssim -1.56\ \mathrm{fm}$$

Option a) $$-3.78\ \mathrm{fm} \lesssim (r_0)_{\delta\to 0} \lesssim 0\ \mathrm{fm}$$

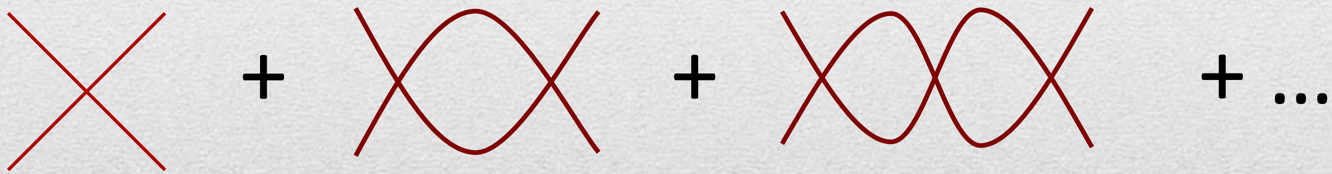
According to Weinberg’s, the first result points to a sizeable short-range structure of the X(3872)

Still disagreement on how to perform the extraction though

Weinberg’s criterion and lineshapes

Let us imagine to have a theory with a bound state with a binding momentum much smaller than the inverse of the range of the potential

The potential is just a delta function,
we calculate the $2 \to 2$ scattering amplitude

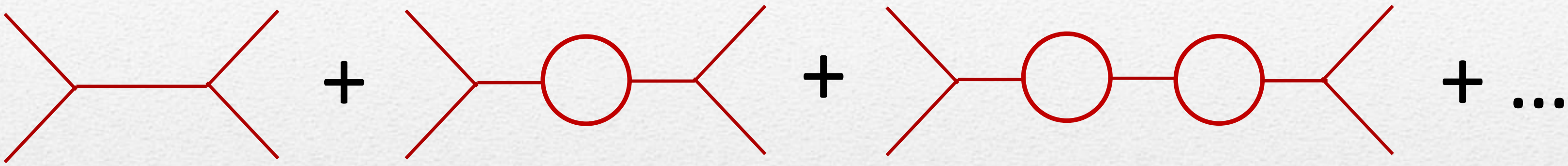


$$A(E) = \frac{1}{1/a - i\sqrt{2\mu E}}$$

This has a pole at $E_B = -\frac{1}{2\mu a^2}$ and residue $g^2 = \sqrt{\frac{2B}{\mu}}$

Weinberg's criterion and lineshapes

Now let us consider the propagation of a bare intermediate state



$$A(E) = \frac{g_0^2}{E_0 - E - i g_0^2 \sqrt{2\mu E}} = \frac{g_0^2}{-E_B - E - i g_0^2 \sqrt{2\mu E} - g_0^2 \sqrt{2\mu E_B}}$$

This has a pole at E_B and residue $g^2 = \sqrt{\frac{2B}{\mu}}(1 - Z)$ where Z is the w.f. renormalization,

$Z = \left(1 + g_0^2 \sqrt{\frac{\mu}{2B}}\right)^{-1}$ = overlap between the bare state and the continuum

Weinberg’s criterion and lineshapes

The amplitude can be rewritten as

$$A(E) = \frac{1}{\frac{1}{a} + \frac{1}{2} r_0 k^2 - ik}$$

Thus identifying $a = -2\,\frac{1-Z}{2-Z}\frac{1}{\sqrt{2\mu E_B}}, r_0 = -\frac{Z}{1-Z}\frac{1}{\sqrt{2\mu E_B}}$

So a negative r_0 points to a short range component in the wave function

This is true up to corrections of the order of the range of the potential, which btw are positive under general assumptions

Esposito, Maiani, Pilloni, Polosa, Riquer, 105 (2022) 3, L031503

A little theorem (Landau-Smorodinski)

- Consider the Schroedinger's equation for the radial wave function of the molecular constituents

$$u_k''(r) + \left[k^2 - U(r)\right] u_k(r) = 0$$

with $U(r) = 2\mu V(r)$, $V(r) < 0$ is the potential, assumed to be attractive everywhere.

- We consider the wave function for two values of the momentum: $u_{k_{1,2}} \equiv u_{1,2}$

With simple manipulations we find the identity

$$u_2 u_1' - u_2' u_1 \Big|_0^R = (k_2^2 - k_1^2) \int_0^R dr\, u_2 u_1 \quad \text{(A)}$$

R >> a_0, the range of the potential ($\simeq 1/m_\pi$).

- Consider now the free equation, $\psi_k''(r) + k^2 \psi_k(r) = 0$, from which we also obtain

$$\psi_2 \psi_1' - \psi_2' \psi_1 \Big|_0^R = (k_2^2 - k_1^2) \int_0^R dr\, \psi_2 \psi_1 \quad \text{(B)}$$

- Normalizing to unity at r=0, the general expression for ψ_k is

$$\psi_k(r) = \frac{\sin(kr + \delta(k))}{\sin \delta(k)}, \text{ and: } \psi_k'(0) = k \cot \delta(k).$$

- The radial wave function u_k vanishes at r=0, and we normalize so that it tends exactly to the corresponding ψ_k for large enough radii.
- Now, subtract (A) from (B) and let $R \to \infty$ (the integral now is convergent) to find

$$k_2 \cot \delta(k_2) - k_1 \cot \delta(k_1) = (k_2^2 - k_1^2) \int_0^\infty dr\, (\psi_2 \psi_1 - u_2 u_1)$$

L. Maiani

$$k_2 \cot\delta(k_2) - k_1 \cot\delta(k_1) = (k_2^2 - k_1^2)\int_0^\infty dr\,(\psi_2\psi_1 - u_2 u_1) \quad \text{(C)}$$

We compare (C) with the parameters of the scattering amplitude.

First we set $k_1 = 0$. Since $\lim_{k_1\to 0} k_1 \cot\delta(k_1) = -\kappa_0$

$$k_2 \cot\delta(k_2) = -\kappa_0 + k_2^2 \int_0^\infty dr\,\left(\psi_2\psi_0 - u_2 u_0\right)$$

For small momenta: $k_2 \cot\delta(k_2) = -\kappa_0 + \frac{1}{2} r_0 k_2^2 + \dots$ so that

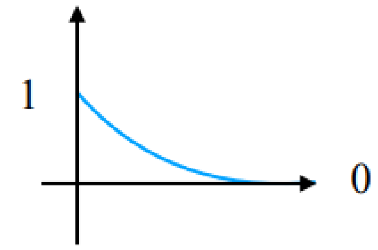
$$\boxed{r_0 = 2\int_0^\infty dr\,(\psi_0^2 - u_0^2)}$$

We know that $u_0(0) = 0$, $\psi_0(0) = 1$. Defining $\Delta(r) = \psi_0(r) - u_0(r)$ we have

$$\Delta(0) = +1,\ \Delta(\infty) = 0$$

The equations of motion imply $\Delta''(r) = -U(r)u_0(r)$. In presence of a single bound state, where u(r) has no nodes, we get

$$\Delta''(r) > 0 \to \psi_0(r) > u_0(r) \qquad \text{that is}$$

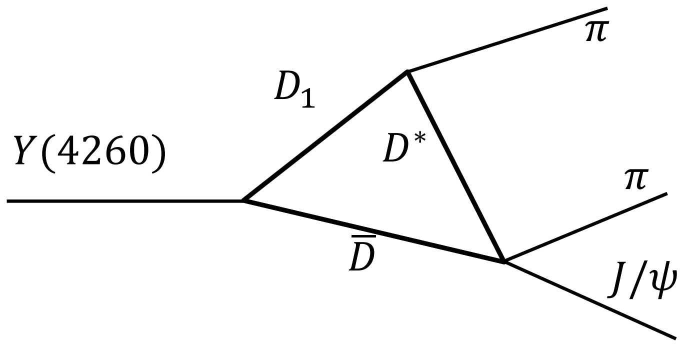


$$\boxed{r_0 > 0}$$

- reassuringly: $r_0(\text{deuteron}) = +1.75\ \text{fm}$,
- conversely a negative value of $r_0 > 0$ implies $Z > 0$

L. Maiani

Triangle singularity



Logarithmic branch points can simulate a resonant behavior, only in very special kinematical conditions

Coleman and Norton, Nuovo Cim. 38, 438

However, this effects cancels in Dalitz projections, no peaks

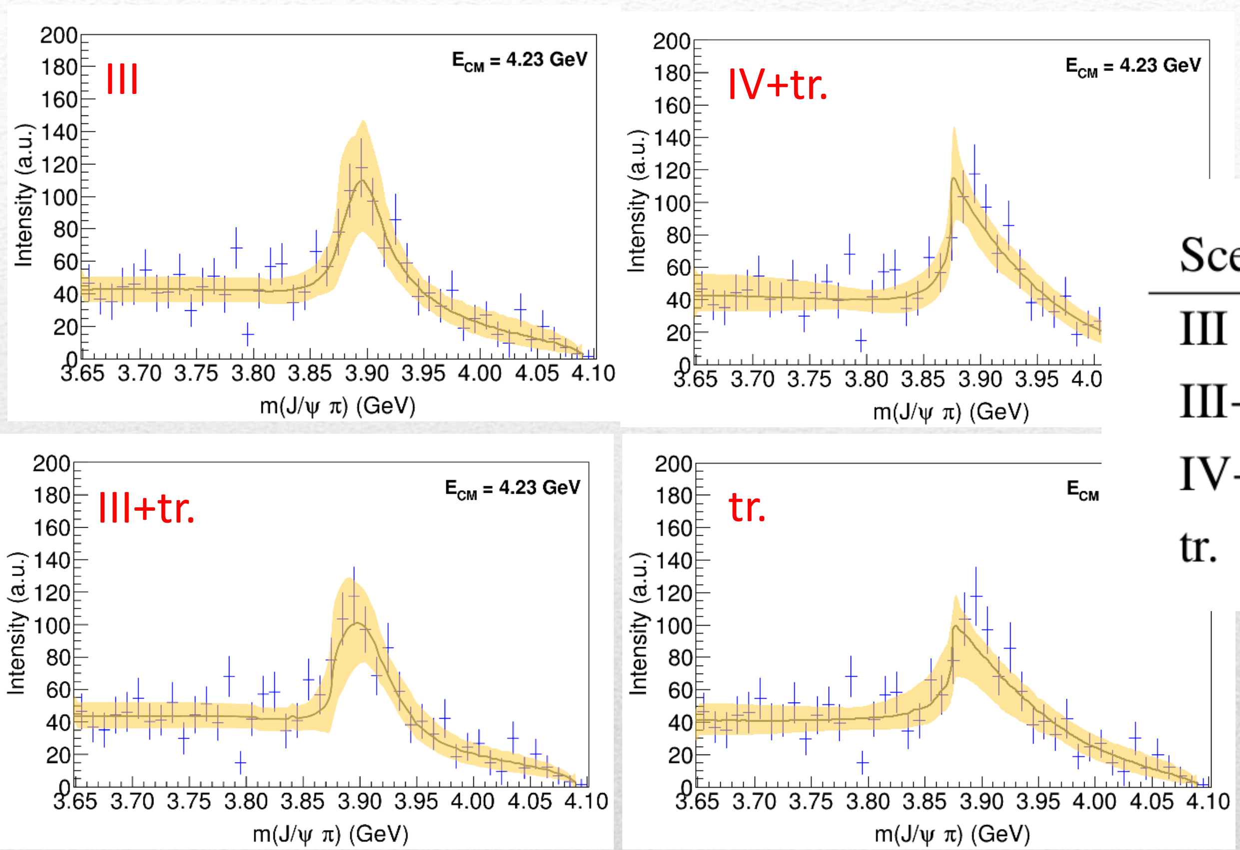
Schmid, Phys.Rev. 154, 1363

$$f_{0,i}(s) = b_{0,i}(s) + \frac{t_{ij}}{\pi} \int_{s_i}^{\infty} ds' \frac{\rho_j(s') b_{0,j}(s')}{s' - s}$$

...but the cancellation can be spread in different channels, you might still see peaks in other channels only!

Szczepaniak, PLB747, 410-416
Szczepaniak, PLB757, 61-64
Guo, Meissner, Wang, Yang PRD92, 071502

Fit summary



Scenario	χ^2	DOF	χ^2/DOF
III	644	532	1.21
III+tr.	642	532	1.21
IV+tr.	666	532	1.25
tr.	695	532	1.31

Naive loglikelihood ratio test give a $\sim 4\sigma$ significance of the scenario III+tr. over IV+tr., looking at plots it looks too much – better using some more solid test

Testing scenarios

- We approximate all the particles to be scalar – this affects the value of couplings, which are not normalized anyway – but not the position of singularities.
  This also limits the number of free parameters

$$f_i(s,t,u) = 16\pi \left[a_{0,i}^{(t)}(t) + a_{0,i}^{(u)}(u) + \sum_j t_{ij}(s) \left(c_j + \frac{s}{\pi} \int_{s_j}^{\infty} ds' \frac{\rho_j(s') b_{0,j}(s')}{s'(s'-s)} \right) \right],$$

The scattering matrix is parametrized as $(t^{-1})_{ij} = K_{ij} - i\,\rho_i\,\delta_{ij}$

Four different scenarios considered:

- «III»: the K matrix is $\frac{g_i\, g_j}{M^2 - s}$, this generates a pole in the closest unphysical sheet the rescattering integral is set to zero
- «III+tr.»: same, but with the correct value of the rescattering integral
- «IV+tr.»: the K matrix is constant, this generates a virtual state pole in the IV sheet
- «tr.»: same, but the pole is pushed far away by adding a penalty in the χ^2