New insights on the factorization of single-inclusive $e^+e^-$ annihilation

QNP2022 - The 9th International Conference on Quarks and Nuclear Physics
Extraction of TMD Fragmentation Functions

Access to the 3D-dynamics of confinement

Standard TMD factorization
(Collins factorization formalism)

- SIDIS  $d\sigma \sim H_{\text{SIDIS}} F D$
- DIA  $d\sigma \sim H_{\text{DIA}} D_1 D_2$

Always two TMDs that have to be extracted simultaneously

SIA would provide a much cleaner access to TMD FFs!  $d\sigma \propto D$


We will consider the process $e^+ e^- \rightarrow h X$
The transverse momentum of the detected hadron is measured w.r.t. the thrust axis

$$z_h = \frac{E}{Q/2}, \quad T = \frac{\sum_i |\vec{P}_{(c.m.)},i \cdot \hat{n}|}{\sum_i |\vec{P}_{(c.m.)},i|}, \quad P_T \text{ w.r.t } \tilde{n}$$
Physical intuition picture

Depending on where the hadron is located within the jet the underlying kinematics can be remarkably different, resulting in different factorization theorems.

1. The hadron is detected near the boundary of the jet:
   - Moderately small $P_T$
   - The hadron $P_T$ causes the spread of the jet affecting the topology of the final state (i.e. the value of thrust)

2. The hadron is detected very close to the axis of the jet:
   - Extremely small $P_T$
   - Soft radiation affects significantly the transverse deflection of the hadron from the thrust axis

3. The hadron is detected in the central region of the jet:
   - Most common scenario
   - Majority of experimental data fall into this case

M. Boglione, A. Simonelli, JHEP 02 (2022) 013
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M. Boglione, A. Simonelli, JHEP 02 (2022) 013
Radiation decomposition

\[ d\sigma \sim H \]

- Hard (far off-shell) radiation dresses $\gamma^* \rightarrow q\bar{q}$ vertex
Radiation decomposition

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Radiation decomposition

\[ d\sigma \sim H \times J(\tau) \times S(\tau) \]

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- Backward radiation irrelevant for the transverse motion of the detected hadron
- Soft radiation does not affect the transverse motion of the detected hadron

Hallmark of $R_2$ and $R_3$
Radiation decomposition

\[ d\sigma \sim H \times J(\tau) \times S(\tau) \times G_{h/j}^{asy}(\tau, z, P_T) \]

- Hard (far off-shell) radiation dresses  $\gamma^* \rightarrow q\bar{q}$ vertex
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- Collinear (forward) radiation necessarily contributes to TMD effects
Subtractions

\[ d\sigma \sim H \times J(\tau) \times \frac{S(\tau)}{Y_L(\tau)} \times G_{h/j}^{asy}(\tau, z, P_T) \]

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\[ d\sigma \sim H \times J(\tau) \times \frac{S(\tau)}{Y_L(\tau)} \times ??? \times G_{h/j}^{\text{asy}}(\tau, z, P_T) \]

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Two different interpretations that lead to two different factorization theorems!
Subtractions: approach I

1) Forward soft-collinear radiation is **TMD-irrelevant**

\[ d\sigma \sim H \times J(\tau) \times \frac{S(\tau)}{Y_L(\tau) \times Y_R(\tau)} \times G_{h/j}^{\text{asy}}(\tau, z, P_T) \]

\[ = S_{\text{thr}}(\tau) \]

The hadronization process is not described by a TMD FF

The underlying physics of \( R_2 \) and \( R_3 \) is almost indistinguishable

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The above scheme is much more close to a representation of just two kinematic regions, \( R_1 \) and \( R_3 \), and the “bulk” of the phase space exists just as a limit of its boundaries.

This limit is indeed a **matching region**, M, and not an independent kinematic configuration: the label “\( R_2 \)” does not fit it anymore.
Matching between what??

\[
R_3 \implies M \quad \ldots \quad \iff \quad R_1
\]

Low-\(P_T\) approximation of the generalized FJF

\[
G_{h/j}(\tau, z, P_T) \to G_{h/j}^{asym}(\tau, z, P_T)
\]

????

R1 seems to be too far

The previous forward-radiation scheme is \textit{incomplete} and there must be (at least) another kinematic region between M and \(R_1\)

The “bulk” of the phase space deserves its own independent kinematic region \(R_2\)
Subtractions: approach II

2) Forward soft-collinear radiation is **TMD-relevant**

\[ d\sigma \sim H \times J(\tau) \times \frac{S(\tau)}{Y_L(\tau)} \times \frac{G_{a/s}^{asy}(\tau, z, P_T)}{C_R(\tau, P_T)} = D_{h/j}(z, P_T) \]

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The hadronization process is described by a **TMD FF**

Totally "symmetric" structure among the three regions

Now **Region 2** is a truly independent kinematic region!
Kinematic structure of SIA$^{thr}$

Approach I leads to a *hybrid* btw R2 and R3 and indeed matches the two regions

\[ R_3 \quad \Rightarrow \quad M_{2,3} \quad \Leftarrow \quad R_2 \quad \cdots \quad R_1 \]

Approach I

Approach II

Having a well-defined factorization theorem in a matching region is a *unusual* and *remarkable* fact!

- Very helpful for phenomenological description of experimental data
- Extremely useful for constraining the non-perturbative behavior of generalized FJFs
Kinematic structure of SIA$^{thr}$

Approach I leads to a *hybrid* btw R2 and R3 and indeed *matches* the two regions

\[ R_3 \quad \Longrightarrow \quad M_{2,3} \quad \longleftrightarrow \quad R_2 \quad \Longrightarrow \quad M_{1,2} \quad \longleftrightarrow \quad R_1 \]

Having a well-defined factorization theorem in a matching region is a *unusual* and *remarkable* fact!

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- Extremely useful for constraining the non-perturbative behavior of generalized FJFs

Constraining the long-distance behavior of TMF FFs would require a factorization theorem $M_{12}$, but this has not been investigated, yet.
Kinematic structure of SIA$^{thr}$

\textbf{ii)} Approach I coincides with the result obtained in SCET

\[ R_3 \quad \text{Large } b_T \quad M_{2,3} \quad \text{Small } b_T \quad R_2 \quad \ldots \quad R_1 \]

Approach I

Approach II

- Boglione, Simonelli, JHEP 02 (2021) 076
- Boglione, Simonelli, JHEP 02 (2022) 013

\[ d\sigma^{(I)}_{M_{2,3}} \sim H \ J(\tau) \ S_{thr} \ G_{h/j}^{asy}(\tau, z, P_T) = H \ J(\tau) \ S_{thr} \ C_R(\tau, P_T) \]

\[ \frac{G_{h/j}^{asy}(\tau, z, P_T)}{C_R(\tau, P_T)} = D_{h/j}(z, P_T) \]

...and this coincides with Eq.(2.21) of Makris, Ringer, Waalewijn, JHEP 02 (2021) 070
Kinematic structure of SIA$^{thr}$

iii) The two approaches coincide in the small $b_T$ limit, i.e. they are totally equivalent at perturbative level

\[
\begin{align*}
R_3 & \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 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TMD UNIVERSALITY

The TMD FF appearing in the Region 2 factorized cross section is not defined as in SIDIS and DIA (standard TMD factorization).

The differences concern the non-perturbative region (large $b_T$) as they are due to the impact of long-distance soft radiation in standard TMD factorization theorems:

$$D^{\text{usual}}(z, b_T) = D^{R_2}(z, b_T) \sqrt{M_S(b_T)}$$

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Factorization Theorem of Region 2
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RAPIDITY DIVERGENCES TREATMENT

Standard treatment of rapidity divergences cannot be applied to Region 2.

$$\frac{\partial}{\partial y_1} \ldots S(\tau, y_1, \ldots) \ D(z, b_T, y_1) \neq 0$$

Boglione, Simonelli, JHEP 02 (2021) 076; JHEP 02 (2022) 013
SIA$^{thr}$ has a **double nature**:

Thrust dependent observable

TMD observable

The thrust $\tau$ **naturally** regularizes the rapidity divergences. The 2-jet limit $\tau \to 0$ corresponds to removing the regulator and to exposing the rapidity divergences in fixed order calculations.

So the final result depends on a regulator? Yes, but...

1) The thrust is *measured*.
2) When the regulator is removed the (factorized) cross section vanishes, as showed by resummation.

The rapidity cut-offs $y_{1,2}$ **artificially** regularize the rapidity divergences. The limits $y_{1,2} \to \pm \infty$ correspond to removing the regulator and to exposing the rapidity divergences in fixed order calculations.

So the final result depends on a regulator? Yes (in principle), but...

1) The rapidity cut-offs are just mathematical tools.
2) In standard TMD factorization they cancel among themselves before the limit $y_{1,2} \to \pm \infty$ is taken and the final cross section is automatically rapidity cut-offs independent.
SIA$^{thr}$ has a **double nature**: Thrust dependent observable

TMD observable

Both kind of regularization coexists in SIA$^{thr}$. Therefore, it should not be surprising that the two mechanisms intertwine themselves and hence that thrust and rapidity regulators are strictly related.

There are cases where only the thrust survives…

$$\frac{S(\tau, y_1, y_2)}{\mathcal{Y}_L(\tau, y_2) \mathcal{Y}_R(\tau, y_1)} = S_{\text{thr}}(\tau)$$

...and cases where only the rapidity cut-off…

$$\frac{G_{h/j}^{asy}(\tau, z, P_T)}{C_R(\tau, P_T, y_1)} = D_{h/j}(z, P_T, y_1)$$

...but only in Region 2 the factorized cross section depends both on $\tau$ and on $y_1$!

This signals a **redundancy** of regulators: one can be expressed in terms of the other. In particular, the rapidity cut-off $y_1$ should be a function of thrust, such that when it is removed, also $\tau$ is removed. In other words:

$$\tau \rightarrow 0 \iff y_1 \rightarrow +\infty$$
Kinematic argument: \[ y_h \geq -\log \sqrt{\tau} \quad \Rightarrow \quad y_1 \propto -\log \sqrt{\tau} \]

Rapidity of detected hadron
➢ Kinematic argument: \[ y_h \geq - \log \sqrt{\tau} \]

Rapidity of detected hadron

➢ Formally:

The double counting due to the overlap between soft and collinear (forward) radiation is cancelled only if the rapidity cut-off is fixed to a function of thrust and transverse momentum.

\[ k_T \lesssim \frac{c_1}{b_T}, \quad k_T \lesssim \frac{Q e^{y_1}}{u_E} \]

\[ y_1 = L_u - L_b \]

This is also the minimum of the factorized cross section as a function of \( y_1 \).

The value of the rapidity cut-off that factorizes the cross section of Region 2 is the solution of Collins-Soper evolution equation:

\[ \frac{\partial}{\partial y_1} d\sigma_{R2} = 0 \]

\[ \widehat{G}_R(u, y_1) = \widehat{K}(b_T) \]
The solution that we have found is valid ONLY at perturbative level

\[ \bar{y}_1 = L_u - L_b \]

Correct behavior as \( u \to +\infty \), i.e. \( \tau \to 0 \)

Improper behavior (even divergent!) as \( b_T \to +\infty \), i.e. \( P_T \to 0 \)

The rapidity cut-off must be large and positive for the factorization theorem to be valid
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The rapidity cut-off must be large and positive for the factorization theorem to be valid

**b* prescription:**

\[
b_T \to \frac{b_T}{\sqrt{1 + \frac{b_T^2}{b_{\text{MAX}}^2}}}\]

\[\mu_b^* = c_1/b_T^* \]
\[\lambda_b^* = 2\beta_0 a_S(\mu_b^*)L_b^* \]

Crucial and central role of \( g_K \) in correlating thrust and transverse momentum

Also consistent with kinematics:

\[ \hat{y}_1 = -\log\sqrt{\tau} + b_T\text{-logs} \]

\[ \left\{ \begin{array}{c}
    \rightarrow L_b, \ b_T \rightarrow 0 \\
    \rightarrow \text{const}, \ b_T \rightarrow \infty
\end{array} \right. \]
Factorization theorem in Region 2

\[ d\sigma_{R_2} \sim H \ J(u) \ \frac{S(u, \bar{y}_1, y_2)}{\mathcal{Y}_L(u, y_2)} \ \tilde{D}_{h/j}(z, b_T, \bar{y}_1) \]

\[ = H \ J \ \frac{S}{\mathcal{Y}_L} \bigg|_{\text{ref. scale}} \exp \left\{ \int_{\mu_J}^{Q} \frac{d\mu'}{\mu'} \gamma_J + \frac{1}{2} \int_{\mu_S}^{Q} \frac{d\mu'}{\mu'} \gamma_S \right\} \times \tilde{D}_{h/j}(z, b_T) \bigg|_{y_1=0} \]

\[ \times \exp \left\{ \frac{1}{2} \int_{\mu_S}^{\mu_S e \bar{y}_1} \frac{d\mu'}{\mu'} \left[ \hat{g} - \gamma_K \log \left( \frac{\mu'}{\mu_S} \right) \right] - \bar{y}_1 \ \tilde{K} \bigg|_{\mu_S} \right\} \]
Factorization theorem in Region 2

\[ d\sigma_{R2} \sim H J(u) \frac{S(u, \bar{y}_1, y_2)}{\mathcal{Y}_L(u, y_2)} \tilde{D}_{h/j}(z, b_T, \bar{y}_1) \]

\[ = H J \left. \frac{S}{\mathcal{Y}_L} \right|_{\text{ref. scale}} \exp \left\{ \int_{\mu_J}^{Q} \frac{d\mu'}{\mu'} \frac{\gamma_J}{2} + \frac{1}{2} \int_{\mu_S}^{Q} \frac{d\mu'}{\mu'} \frac{\gamma_S}{2} \right\} \times \tilde{D}_{h/j}(z, b_T) \big|_{\bar{y}_1=0} \]

\[ \times \exp \left\{ \frac{1}{2} \int_{\mu_S}^{\bar{y}_1} \frac{d\mu'}{\mu'} \left[ \frac{\tilde{g}}{\gamma K} \log \left( \frac{\mu'}{\mu_S} \right) - \bar{y}_1 \right] \big|_{\mu_S} \right\} \]

Genuinely thrust. Exponent is half of standard thrust distribution in e+e- annihilation.
Factorization theorem in Region 2

\[
d\sigma_{R_2} \sim H J(u) \frac{S(u, \bar{y}_1, y_2)}{\mathcal{V}_L(u, y_2)} \tilde{D}_{h/j}(z, b_T, \bar{y}_1)
\]

\[
= H J \left. \frac{S}{\mathcal{V}_L} \right|_{\text{ref. scale}} \exp \left\{ \int_{\mu_J}^{Q} \frac{d\mu'}{\mu'} \gamma_J + \frac{1}{2} \int_{\mu_S}^{Q} \frac{d\mu'}{\mu'} \gamma_S \right\} \times \left. \tilde{D}_{h/j}(z, b_T) \right|_{y_1=0}
\]

\[
\times \exp \left\{ \frac{1}{2} \int_{\mu_S}^{\mu_S e^{\bar{y}_1}} \frac{d\mu'}{\mu'} \left[ \hat{g} - \gamma_K \log \left( \frac{\mu'}{\mu_S} \right) \right] - \bar{y}_1 \left. \tilde{K} \right|_{\mu_S} \right\}
\]

Genuinely TMD. Reference scales as* in standard TMD factorization
Factorization theorem in Region 2

\[ d\sigma_{R_2} \sim H \cdot J(u) \cdot \frac{S(u, y_1, y_2)}{Y_L(u, y_2)} \cdot \tilde{D}_{h/j}(z, b_T, y_1) \]

\[ = H \cdot J \cdot \frac{S}{Y_L} \bigg|_{\text{ref. scale}} \exp \left\{ \int_{\mu_J}^{Q} \frac{d\mu'}{\mu'} \gamma_J + \frac{1}{2} \int_{\mu_S}^{Q} \frac{d\mu'}{\mu'} \gamma_S \right\} \times \tilde{D}_{h/j}(z, b_T) \bigg|_{y_1=0} \]

\[ \times \exp \left\{ \frac{1}{2} \int_{\mu_S}^{Q} \frac{d\mu'}{\mu'} \left[ \hat{g} - \gamma_K \log \left( \frac{\mu'}{\mu_S} \right) \right] - y_1 \tilde{K} \bigg|_{\mu_S} \right\} \]

**Correlation** part. It encodes the correlations between the measured variables.
Factorization theorem in Region 2

\[
d\sigma_{R_2} \sim H J(u) \frac{S(u, \bar{y}_1, y_2)}{\mathcal{Y}_L(u, y_2)} \tilde{D}_{h/j}(z, b_T, \bar{y}_1)
\]

\[
= H J \left. \frac{S}{\mathcal{Y}_L} \right|_{\text{ref. scale}} \exp \left\{ \int_{\mu_J}^{Q} \frac{d\mu'}{\mu'} \gamma_J + \frac{1}{2} \int_{\mu_S}^{Q} \frac{d\mu'}{\mu'} \gamma_S \right\} \times \tilde{D}_{h/j}(z, b_T) \bigg|_{y_1 = 0}
\]

\[
\times \exp \left\{ \frac{1}{2} \int_{\mu_S}^{\mu} \frac{d\mu'}{\mu'} \left[ \hat{g} - \gamma_K \log \left( \frac{\mu'}{\mu} \right) \right] - \bar{y}_1 \frac{\tilde{K}}{\mu_S} \right\}
\]

The function \(g_K\) is in both terms and not only into the TMD FF

Resummpation

\[
\frac{d\sigma_{R_2}}{dz \, dT \, dP_T} \bigg|_{\text{LL}} = - \frac{\sigma_B}{1 - T} N_C \int \frac{d^2 \vec{b}_T}{(2\pi)^2} \tilde{D}_{h/j}^{\text{LL}}(z, b_T) \bigg|_{y_1 = 0}
\]

\[
\times \exp \{ - \log (1 - T) f_1(\bullet) \} \gamma(\bullet)
\]

\[
\bullet = \{-a_S \beta_0 \log (1 - T), 2a_S \beta_0 L_b^*, g_K(b_T)\}
\]
Factorization theorem in Region 2

\[ d\sigma_{R_2} \sim H \ J(u) \ \frac{S(u, \bar{y}_1, y_2)}{Y_L(u, y_2)} \ \widetilde{D}_{h/j}(z, b_T, \bar{y}_1) \]

\[ = H \ J \ \frac{S}{Y_L} \bigg|_{\text{ref. scale}} \ \exp \left\{ \int_{\mu_J}^{Q} \frac{d\mu'}{\mu'} \gamma_J + \frac{1}{2} \int_{\mu_S}^{Q} \frac{d\mu'}{\mu'} \gamma_S \right\} \times \ \widetilde{D}_{h/j}(z, b_T) \bigg|_{y_1 = 0} \]

\[ \times \ \exp \left\{ \frac{1}{2} \int_{\mu_S}^{\mu} \frac{d\mu'}{\mu'} \left[ \tilde{g} - \gamma_K \log \left( \frac{\mu'}{\mu_S} \right) \right] - \bar{y}_1 \tilde{K} \bigg|_{\mu_S} \right\} \]

The function \( g_K \) is in both terms and not only into the TMD FF

Resummation

\[ \frac{d\sigma_{R_2}}{dz \ dT \ dP_T} \stackrel{\text{NLL}}{=} - \frac{\sigma_B}{1 - T} \ N_C \ \int \frac{d^2\vec{b}_T}{(2\pi)^2} \ \widetilde{D}_{h/j}^{\text{NLL}}(z, b_T) \bigg|_{y_1 = 0} \]

\[ \times \ \exp \left\{ - \log (1 - T) f_1(\bullet) + f_2(\bullet) \right\} \ \gamma(\bullet) \]

\( \bullet = \{-a_S \beta_0 \log (1 - T), 2a_S \beta_0 L_b^*, g_K(b_T)\} \)
Factorization theorem in Region 2

\[ d\sigma_{R2} \sim H J(u) \frac{S(u, \bar{y}_1, y_2)}{\mathcal{Y}_L(u, y_2)} \tilde{D}_{h/j}(z, b_T, \bar{y}_1) \]

\[ = H J \frac{S}{\mathcal{Y}_L} \bigg|_{\text{ref. scale}} \exp \left\{ \int_{\mu_J}^Q \frac{d\mu'}{\mu'} \gamma_J + \frac{1}{2} \int_{\mu_S}^Q \frac{d\mu'}{\mu'} \gamma_S \right\} \times \tilde{D}_{h/j}(z, b_T) \bigg|_{y_1=0} \]

\[ \times \exp \left\{ \frac{1}{2} \int_{\mu_S}^{\mu_S e^{\bar{y}_1}} \frac{d\mu'}{\mu'} \left[ \hat{g} - \gamma K \log \left( \frac{\mu'}{\mu_S} \right) \right] - \bar{y}_1 \tilde{K} \bigg|_{\mu_S} \right\} \]

**The function \( g_K \) is in both terms and not only into the TMD FF**

**Resummation**

\[ \frac{d\sigma_{R2}}{dz \, dT \, dP_T} \bigg|_{\text{NNLL}} = - \frac{\sigma_B}{1 - T} N_C \int \frac{d^2 b_T}{(2\pi)^2} \tilde{D}_{h/j}^{\text{NNLL}}(z, b_T) \bigg|_{y_1=0} \left( 1 + a_S C_1(b_T) \right) \]

\[ \times \exp \left\{ - \log (1 - T) f_1(\bullet) + f_2(\bullet) - \frac{1}{\log (1 - T)} f_3(\bullet) \right\} \left( \gamma(\bullet) - \frac{1}{\log (1 - T)} \rho(\bullet) \right) \]

\[ \bullet = \{-a_S \beta_0 \log (1 - T), 2a_S \beta_0 L^*_b, g_K(b_T)\} \]
A strongly simplified version of this theorem is possible

\[ \bar{y}_1 = L_u - L_b^* \left( 1 + \frac{1 - e^{-\frac{2\beta_0}{1} \left(g_K - K^*\right)}}{\lambda_b^*} \right) \]

In thrust space:
\[ \hat{y}_1 = -\log \sqrt{\tau} \]

1) Thrust is not resummed

These approximations lead to the result of Boglione, Simonelli, JHEP 02 (2021) 076

PHENOMENOLOGY: Boglione, Gonzalez-Hernandez, Simonelli, 2206.08876 [hep-ph]

Conclusions

- The factorization properties of SIA\textsuperscript{thr} have been deeply investigated.

- The tension between SCET and the formalism presented here has been clarified: the two approaches lead to two different factorization theorems, describing different kinematics. The very bulk of the phase space (Region 2) is addressed by this CSS-based treatment, while SCET result is relevant (and remarkable!) for matching.

\[
\begin{align*}
R_3 \quad \text{Large } b_T & \quad \longrightarrow \quad M_{2,3} \quad \iff \quad \text{Small } b_T \quad \longrightarrow \quad R_2
\end{align*}
\]

- Factorization in Region 2 exposes the double nature of SIA\textsuperscript{thr}, an observable which is both thrust-dependent and TMD. This is made manifest in the relation linking the TMD rapidity regulator with the thrust.

\[
\bar{y}_1 = L_u - L^*_b \left( 1 + \frac{1 - e^{-\frac{2\theta_0}{\gamma K}} (g_K - \tilde{K}^*)}{\lambda_b^*} \right)
\]

- First phenomenological results have been obtained in a strongly simplified version of this formalism, showing nevertheless very good agreement with BELLE data.

- The next step is to perform phenomenology on BELLE data without introducing simplifications to the formalism and provide the first cleanest extraction of a TMD FF.
THANK YOU